Generalized Israel junction conditions for a Gauss-Bonnet brane world

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In spacetimes of dimension greater than four it is natural to consider higher order (in R) corrections to the Einstein equations. In this paper generalized Israel junction conditions for a membrane in such a theory are derived. This is achieved by generalizing the Gibbons-Hawking boundary term. The junction conditions are applied to simple brane world models, and are compared to the many contradictory results in the literature.

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It is well known that for a boundaryless spacetime, the Einstein equations can be derived from the Einstein-Hilbert action. This is no longer true if the spacetime has a boundary. The problem is resolved by adding a Gibbons-Hawking boundary term to the action [1]. Varying the action then gives the correct field equations, as well as boundary conditions at the edge of the spacetime.

A slight variation of this is to consider an infinitely thin $(D-2)$-brane in a D dimensional spacetime. Since spacetime is split into two, the brane can then be treated as the boundary of each half of the spacetime. Varying the Gibbons-Hawking term then gives the Israel junction conditions on the membrane [2]. These junction conditions have recently received a great deal of attention due to their use in the study of "brane worlds" (see for example [3–7]). In a brane-world scenario our universe is modeled by a 3-brane embedded in a five-dimensional "bulk" spacetime. In the simplest cases, all the standard model fields are confined to the brane, while only gravity propagates in the bulk. The brane is usually taken to be of zero thickness, and so the Israel junction conditions can be used to relate the bulk dynamics to what we, on the brane, observe.

The bulk gravitational field equations are usually assumed to be the five-dimensional Einstein equations. However in spaces of dimension greater than four it is natural to consider additional higher order curvature terms [8–11].

In general relativity, the vacuum field equations are taken to be a linear combination of the Einstein tensor and the metric. This choice is motivated by the fact that it is the most general combination of tensors which (a) is symmetric, (b) depends only on the metric and its first two derivatives, (c) is divergence free and (d) is linear in second derivatives of the metric. In fact, in four dimensions, the fourth condition is superfluous since it is implied by the other three [8]. In five dimensions the second order Lovelock tensor

$$H_{ab}=RR_{ab}-2R_{ac}R^{c}{}_{b}-2R^{cd}R_{acbd}+R_{a}{}^{cde}R_{bcde} -\frac{1}{4}g_{ab}(R^2-4R_{cd}R^{cd}+R^{cdes}R_{cdes}) \quad (1)$$

also satisfies the above conditions. Thus the most general choice of gravitational vacuum field equations in five dimensions is a linear combination of g_{ab}, G_{ab} and H_{ab}. In the absence of any experimental evidence to the contrary, all three terms should be included.

A further motivation for higher order curvature terms is that they also appear in the low energy effective field equations arising from most string theories. Since brane worlds are motivated by string theories, it is particularly natural to include the extra terms in the five-dimensional field equations.

Like the standard Einstein equations, these higher order generalizations can also be derived from an action. The tensor H_{ab} can be obtained from an action containing the Gauss-Bonnet term

$$\mathcal{L}_{\rm GB}=R^2-4R_{ab}R^{ab}+R^{abcd}R_{abcd}. \quad (2)$$

Consider the action

$$S_{\mathcal{M}}=\frac{1}{2\kappa^2}\int_{\mathcal{M}}d^Dx\sqrt{-g}\{R-2\Lambda+\alpha\mathcal{L}_{\rm GB}\} \quad (3)$$

for a D dimensional manifold $\mathcal{M}$. In a string theory context we would have $\kappa^{-2}=M_*^{D-2}$ and $\alpha\propto M_*^{-2}$, where M_* is the string mass scale.

Let us suppose, as would be the case in a co-dimension one brane world scenario, that $\mathcal{M}$ is split into two parts by a hypersurface Σ, whose two sides will be denoted $\Sigma_\pm$. Their normals, n^a, will be taken to point away from the surface and *into* the adjacent space.

Varying the action (3) with respect to the metric gives

$$\delta S_{\mathcal{M}}=\frac{1}{2\kappa^2}\int_{\mathcal{M}}d^Dx\sqrt{-g}\,\delta g^{ab}(G_{ab}+\Lambda g_{ab}+2\alpha H_{ab}) -\frac{1}{\kappa^2}\int_{\Sigma_\pm}d^{D-1}x\sqrt{-h}n_a(g^{a[c}g^{d]b}+2\alpha P^{abcd})\nabla_d\delta g_{bc} \quad (4)$$

where $h_{ab}=g_{ab}-n_an_b$ is the induced metric on Σ, and

$$P_{abcd}=R_{abcd}+2R_{b[c}g_{d]a}-2R_{a[c}g_{d]b}+Rg_{a[c}g_{d]b} \quad (5)$$

is the divergence-free part of the Riemann tensor.

Expression (4) contains normal derivatives of the metric variation. As with the Einstein-Hilbert action, we must add a

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boundary term in order to cancel them. For an action with a Gauss-Bonnet term (3), the appropriate term is [10]

$$S_\Sigma = -\frac{1}{\kappa^2}\int_{\Sigma_\pm} d^{D-1}x\sqrt{-h}(K+2\alpha\{J-2\hat{G}^{ab}K_{ab}\}) \quad (6)$$

where K is the trace of the extrinsic curvature, defined by $K_{ab}=h^c_a\nabla_c n_b$, and J is the trace of

$$J_{ab}=\frac{1}{3}(2KK_{ac}K^c{}_b+K_{cd}K^{cd}K_{ab}-2K_{ac}K^{cd}K_{db}-K^2K_{ab}). \quad (7)$$

Throughout this paper I will denote tensors associated with the induced metric by a caret, so $\hat{G}_{ab}$ is the $(D-1)$ dimensional Einstein tensor on Σ corresponding to h_{ab}.

Varying the action $S_\mathcal{M}+S_\Sigma$ now gives an expression which does not contain normal derivatives of δg_{ab}. If we also include a matter contribution to the action

$$S_{\rm mat}=-\int_\mathcal{M} d^Dx\sqrt{-g}\mathcal{L}_m^{(\mathcal{M})}-\int_\Sigma d^{D-1}x\sqrt{-h}\mathcal{L}_m^{(\Sigma)}, \quad (8)$$

then the variation of the total action $S=S_\mathcal{M}+S_\Sigma+S_{\rm mat}$ gives

$$G_{ab}+2\alpha H_{ab}+\Lambda g_{ab}=\kappa^2 T_{ab} \quad (9)$$

in $\mathcal{M}$, and

$$2\langle K_{ab}-Kh_{ab}\rangle+4\alpha\langle 3J_{ab}-Jh_{ab}+2\hat{P}_{acdb}K^{cd}\rangle=-\kappa^2 S_{ab} \quad (10)$$

on Σ, with $\langle X\rangle=[X(\Sigma_+)+X(\Sigma_-)]/2$ denoting the average of a quantity over the two sides $(\Sigma_\pm)$ of the hypersurface. The two energy-momentum tensors are defined by $T_{ab}=2\delta\mathcal{L}_m^{(\mathcal{M})}/\delta g^{ab}-g_{ab}\mathcal{L}_m^{(\mathcal{M})}$ and $S_{ab}=2\delta\mathcal{L}_m^{(\Sigma)}/\delta h^{ab}-h_{ab}\mathcal{L}_m^{(\Sigma)}$.

With the aid of the Gauss-Codazzi equations (A1)–(A3) below, we obtain the energy-momentum conservation equation on the hypersurface

$$D^aS_{ab}=-2\langle n^a(G_{ac}+2\alpha H_{ac})h^c{}_b\rangle=-2\kappa^2\langle n^aT_{ac}h^c{}_b\rangle, \quad (11)$$

with D^a denoting the covariant derivative corresponding to h_{ab}. This is very similar to the corresponding result in the standard brane world models [6].

Many previous papers have tried to derive the junction conditions by treating the hypersurface Σ as a δ-function contribution to $\mathcal{L}_m^{(\mathcal{M})}$, as in the original brane cosmology papers [6]. In this case there is some ambiguity as to the correct definition of H_{ab} on Σ, which has led to the suggestion that the junction conditions must depend on the thickness of the brane [12]. This would be true for a general combination of second order curvature terms, whose action would contain third (or higher) order derivatives of the metric. However, this is not true for the Gauss-Bonnet combination (2) since it has been specifically chosen not to contain such derivatives.

By allowing $n^c\partial_c g_{ab}$ to be discontinuous at the brane, and treating $(n^c\partial_c)^2 g_{ab}$ as a δ function, junction conditions which are independent of the brane thickness can be found [13–17]. However, care must be taken to regularize the δ function correctly [13,18]. This was done in Refs. [13–15], and the resulting junction conditions agree with those in this article. References [16,17] do not regularize the δ function appropriately, and obtain incorrect results.

We will now use the above results to derive the Friedmann equation for a cosmological brane world model. As has been shown by Kraus and Ida [7] (for the standard five dimensional Einstein equations), it is possible to obtain a cosmological generalization of the Randall-Sundrum [5] model by considering a static bulk spacetime, and allowing the brane to have a time dependent position in the bulk.

The bulk metric can be written in the form

$$ds^2=-h(r)dT^2+\frac{dr^2}{h(r)}+r^2\Omega_{ij}dx^idx^j \quad (12)$$

where Ω_{ij} is the three dimensional metric of space with constant curvature $k=-1,0,1$. For $\mathcal{L}_m^{(\mathcal{M})}=0$, the field equations are solved by [11]

$$h=k+\frac{r^2}{4\alpha}\left(1-\sqrt{1+\frac{4}{3}\alpha\Lambda+8\alpha\frac{\mu}{r^4}}\right) \quad (13)$$

with μ being an arbitrary constant. In the $\alpha\to 0$ limit, μ is equal to the black hole mass.

We define the position of the brane as $r=a(\tau)$ and $T=t(\tau)$, which is parametrized by the proper time on the brane τ. The induced metric is

$$ds^2=-d\tau^2+a(\tau)^2\Omega_{ij}dx^idx^j, \quad (14)$$

the tangent vector of the brane is $u^a=(\dot{T},0,0,0,\dot{r})$ and $n_a=(-\dot{r},0,0,0,\dot{T})$. Normalization of n^a implies

$$-h^2\dot{T}^2+\dot{r}^2=-h. \quad (15)$$

We take the brane matter to be a perfect fluid, so $S_{ab}=(\rho+p)u_au_b+ph_{ab}$. The (uu) component of Eq. (10) is then

$$\left(1+\frac{8}{3}\alpha H^2+4\alpha\frac{k}{r^2}\right)\frac{\langle h\dot{T}\rangle}{r}-\frac{4}{3}\alpha\frac{\langle h^2\dot{T}^3\rangle}{r^3}=-\frac{\rho}{6}. \quad (16)$$

For simplicity let us assume Z_2 symmetry across the brane. Squaring Eq. (16) and simplifying with Eq. (15) gives a cubic equation for H^2. This has the real solution

$$H^2=-\frac{k}{a^2}+\frac{c_++c_--2}{8\alpha} \quad (17)$$

where

$$c_\pm=\left(\sqrt{\left(1+\frac{4}{3}\alpha\Lambda+8\alpha\frac{\mu}{a^4}\right)^{3/2}+\frac{\alpha\rho^2}{2}}\pm\rho\sqrt{\frac{\alpha}{2}}\right)^{2/3}. \quad (18)$$

This Friedmann equation agrees with Ref. [13], but not Ref. [16,17]. The only difference between these two conflicting results is one factor of 3, but surprisingly this gives a substantially different Friedmann equation. For $\rho=\text{const}$, Eq. (17) also agrees with Ref. [14]. Reference [19] uses different boundary terms [20] to obtain a different Friedmann equation. However the terms used do not give a consistent action, and so the result is incorrect.

As in the usual brane cosmology [6], the standard Friedmann equation can be recovered at late time (large a) by splitting ρ into a cosmological constant and a matter part [13].

The boundary terms (6) are easily generalized to actions where other fields couple to the curvature tensors. Variation of the action

$$S=\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^D x\sqrt{-g}[\Phi(x^\mu)R-2\Lambda+\alpha\Psi(x^\mu)\mathcal{L}_{\text{GB}}] -\frac{1}{\kappa^2}\int_{\Sigma_\pm} d^{D-1}x\sqrt{-h}[\Phi(x^\mu)K+2\alpha\Psi(x^\mu) \times\{J-2\hat{G}^{ab}K_{ab}\}]+S_{\text{mat}} \tag{19}$$

produces the field equations

$$\Phi G_{ab}-\nabla_a\nabla_b\Phi+g_{ab}\nabla^2\Phi+\Lambda g_{ab}+2\alpha\Psi H_{ab} +4\alpha P_{eacb}\nabla^e\nabla^c\Psi=\kappa^2 T_{ab} \tag{20}$$

and the boundary conditions

$$\langle\Phi(K_{ab}-Kh_{ab})-h_{ab}n^e\partial_e\Phi\rangle+2\alpha\langle 3\Psi J_{ab}-\Psi J h_{ab} +2\Psi\hat{P}_{acdb}K^{cd}\rangle+2\alpha\langle\{2\hat{G}_{ab}+2K_{ea}K^e_b-2KK_{ab} +h_{ab}[K^2-K_{cd}K^{cd}]\}n^e\partial_e\Psi\rangle+8\alpha\langle(K_{a[c}h_{b]d} +K_{c[a}h_{d]b}-Kh_{a[c}h_{b]d})D^cD^d\Psi\rangle=-\frac{1}{2}\kappa^2 S_{ab}. \tag{21}$$

All the problematic boundary terms, like those appearing in Eq. (4), cancel out. In a string theory context Φ and Ψ would typically be functions of the dilaton or moduli fields.

If we consider a conformally flat spacetime of the form

$$ds^2=-e^{2A(y)}(dT^2+dx^i dx_i)+dy^2 \tag{22}$$

and take Φ, Ψ and $\mathcal{L}_m^{(\Sigma)}=\lambda/\kappa^2$ to be functions of y only, the generalized Israel junction conditions reduce to

$$2\langle 3\Phi A'+\Phi'-4\alpha A'^2(A'\Psi+3\Psi')\rangle=-\lambda. \tag{23}$$

This agrees with other results in the literature [15].

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APPENDIX

To prove Eq. (10), we first use the Gauss-Codazzi equations

$$R_{pqrs}h^p{}_a h^q{}_b h^r{}_c h^s{}_d=\hat{R}_{abcd}+K_{bc}K_{ad}-K_{ac}K_{bd}, \tag{A1}$$

$$n^a R_{aqrs}h^q{}_b h^r{}_c h^s{}_d=D_dK_{bc}-D_cK_{bd}, \tag{A2}$$

$$\hat{R}_{bd}=R^a{}_{qcp}h^c{}_a h^q{}_b h^p{}_d+KK_{bd}-K_{bc}K^c_d \tag{A3}$$

and contractions of them to expand n^aP_{abcd} in terms of K_{ab}, n^a and quantities associated with the induced metric:

$$n^aP_{abcd}=2D_{[d}K_{c]b}+2D_eK^e_{[d}h_{c]b}+2h_{b[d}D_{c]}K+2\hat{G}_{b[c}n_{d]} +2(K^e{}_b-Kh^e{}_b)K_{e[c}n_{d]}+(K^2-K_{ae}K^{ae})h_{b[c}n_{d]}. \tag{A4}$$

The covariant derivative D is defined by $D_aX_{bc\cdots}=h^m{}_a h^p{}_b h^q{}_c\cdots\nabla_m X_{pq\cdots}$.

To find the variation of S_Σ [Eq. (6)] with respect to g_{ab}, we first note that the normalization of n_a implies $\delta n_a=\frac{1}{2}n_an^cn^d\delta g_{cd}$. Thus, after a little algebra, and using the definitions of K_{ab} and D_a, we obtain

$$h^c{}_a h^d{}_b\delta K_{cd}=n^e\nabla_{[e}\delta g_{p]q}h^p{}_{(a}h^q{}_{b)}-\frac{1}{2}\delta g^{cd}K_{c(a}h_{b)d} -\frac{1}{2}D_{(a}(h^c{}_{b)}\delta g_{ce}n^e) \tag{A5}$$

and, from Eq. (A3),

$$h^c{}_a h^d{}_b\delta\hat{R}_{cd}=D_{(a}D^e(h^c{}_{b)}h^d{}_e\delta g_{cd})-\frac{1}{2}D^eD_e(h^c{}_a h^d{}_b\delta g_{cd}) -\frac{1}{2}D_aD_b(h^{cd}\delta g_{cd}). \tag{A6}$$

Now, with the aid of integration by parts and the relation

$$Y^{abe}\nabla_e\delta g_{ab}=D_e(Y^{abe}\delta g_{ab})+2n^c\delta g_{cd}Y^{(dp)e}K_{pe} +\delta g^{ab}D^eY_{abe}, \tag{A7}$$

which holds for any tensor Y^{abe} that is orthogonal to the normal n^c, the variation of the total action S can be reduced to Eq. (10). The more general case (21) can be dealt with in the same way.

The energy conservation equation (11) can be proved by using $D^a\hat{P}_{acdb}=0$ and the Gauss-Codazzi equations (A1)–(A3) to express the divergence of Eq. (10) in terms of bulk tensors.

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