Linear response, validity of semiclassical gravity, and the stability of flat space

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A quantitative test for the validity of the semiclassical approximation in gravity is given. The criterion proposed is that solutions to the semiclassical Einstein equations should be stable to linearized perturbations, in the sense that no gauge invariant perturbation should become unbounded in time. A self-consistent linear response analysis of these perturbations, based upon an invariant effective action principle, necessarily involves metric fluctuations about the mean semiclassical geometry, and brings in the two-point correlation function of the quantum energy-momentum tensor in a natural way. This linear response equation contains no state dependent divergences and requires no new renormalization counterterms beyond those required in the leading order semiclassical approximation. The general linear response criterion is applied to the specific example of a scalar field with arbitrary mass and curvature coupling in the vacuum state of Minkowski spacetime. The spectral representation of the vacuum polarization function is computed in n dimensional Minkowski spacetime, and used to show that the flat space solution to the semiclassical Einstein equations for $n=4$ is stable to all perturbations on distance scales much larger than the Planck length.

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I. INTRODUCTION

There are many well known difficulties that arise when attempting to combine quantum field theory and general relativity into a full quantum theory of gravity. Almost certainly, a consistent quantum theory at the Planck scale requires a fundamentally different set of principles from those of classical general relativity, in which even the concept of spacetime itself is likely to be radically altered. Yet, over a very wide range of distance scales, from that of the electroweak interactions (10^{-16} cm) to cosmology (10^{27} cm), the basic framework of a spacetime metric theory obeying general coordinate invariance is assumed to be valid, and receives phenomenological support both from the successes of flat space quantum field theory at the lower end of this distance scale, and classical general relativity at its upper end. Hence, whatever the full quantum theory of gravity entails, it should reduce to an effective low energy field theory on this very broad range of some 43 orders of magnitude of distance [1,2].

To the extent that quantum effects are relevant at all in gravitational phenomena within this range of scales, one would expect to be able to apply *semi*classical techniques to the low energy effective theory of gravity. In the semiclassical approximation to gravity the spacetime metric g_{ab} is treated as a classical c-number field and its quantum fluctuations are neglected, although quantum fluctuations of the other fields are taken into account. The semiclassical approach has been discussed and studied for some time now, and a considerable body of results has been obtained [3]. Yet a definitive answer to the question of what is the limit of validity of this approach has remained somewhat unclear.

It is our purpose in this paper to propose a well-defined *quantitative* criterion for the validity of the semiclassical approximation to gravity, within the semiclassical formalism itself, namely that solutions to the semiclassical Einstein equations should be stable against linearized perturbations of the geometry. This criterion may be formulated within the framework of linear response theory [4–6].

It is important to distinguish what we mean in this paper by the semiclassical approximation from the ordinary loop expansion, which is sometimes also called semiclassical. In the ordinary loop expansion of the effective action, $\hbar$ is the formal (loop expansion) parameter. As a result both the matter and gravitational quantum fluctuations are treated on exactly the same footing, and the back-reaction of these fluctuations on the metric (being first order in the expansion parameter $\hbar$) is neglected. If one does attempt to include such effects in some modified loop expansion, the technical issues involved in defining a one-loop effective action for gravitons that respects both linearized gauge and background field coordinate invariance must be faced. These are difficult enough to have impeded progress in the standard loop expansion in gravity [7,8]. An unambiguous definition of the corresponding conserved and gauge invariant energy-momentum tensor for gravitons in an arbitrary curved space-

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time has not yet been given [9]. Apart from such technical difficulties it should be clear that a simple loop expansion is ill-suited physically to many applications that have been and are likely to be of interest in semiclassical gravity, such as particle creation in the early universe, or black hole radiance, where the quantum effects of matter significantly affect the background geometry after some period of time, but (it is usually assumed) the quantum fluctuations of the geometry itself can be neglected. Whenever quantum effects of matter are expected significantly to affect the classical geometry, the standard loop expansion, which treats these effects as order $\hbar$ and small, must certainly break down.

The semiclassical approximation to gravity we discuss in this paper treats the matter fields as quantum but the spacetime metric as classical, and allows for the consistent back-reaction of the quantum matter on the classical geometry. This asymmetric treatment can be justified formally by replicating the number of matter fields N times and taking the large N limit of the quantum effective action for the matter fields in an arbitrary background metric g_{ab} [10]. Then, the semiclassical equations for the metric are derived by varying the effective action, with local gravitational terms included. Since no assumption of the weakness or perturbative nature of the metric is assumed, the large N expansion is able to address problems in which gravitational effects on the matter are strong, and the matter fields can have a large cumulative effect on the classical geometry in turn. The absence of quantum gravitational effects in the lowest order large N approximation also means that the technical obstacles arising from the quantum fluctuations of the geometry are avoided. General coordinate invariance is assured, provided only that the matter effective action is regularized and renormalized in a manner which respects that invariance [7]. In that case the quantum expectation value of the matter energy-momentum tensor $\langle T_{ab}\rangle$ is necessarily conserved.

Assuming that the classical energy-momentum tensor for the matter field(s) vanishes (an assumption that may be easily relaxed if necessary), the unrenormalized semiclassical back-reaction equations take the form [11]

$$G_{ab}+\Lambda g_{ab}=8\pi G_{_N}\langle T_{ab}\rangle. \tag{1.1}$$

Here G_{ab} is the Einstein tensor, Λ is the cosmological constant (which may be taken to be zero in some applications), $G_{_N}$ is Newton's constant, and $\langle T_{ab}\rangle$ is the expectation value of the energy-momentum tensor operator of the quantized matter field(s). Among the technical issues that must be confronted is the renormalization of the expectation value of T_{ab}, a quartically divergent composite operator in $n=4$ spacetime dimensions. The renormalization of its expectation value requires the introduction of fourth order counterterms in the effective action, that modify the geometric terms on the left hand side of Eq. (1.1) [3].

Once a renormalized semiclassical theory has been defined, one possible route to investigating its validity is to compare calculations in a theory of quantum gravity with similar semiclassical calculations. Since a well-defined, full quantum theory is lacking, this has been done only in some simplified models of quantum gravity. Large quantum gravity effects were found in three-dimensional models by Ashtekar [12] and Beetle [13]. In four dimensions, Ford has considered the case of graviton production in a linearized theory of quantum gravity on a flat space background, and compared the results with the production of gravitational waves in semiclassical gravity [14]. He found that they were comparable when the renormalized energy-momentum (connected) correlation function,

$$\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}\equiv\langle T_{ab}(x)T_{cd}(x')\rangle-\langle T_{ab}(x)\rangle\langle T_{cd}(x')\rangle \tag{1.2}$$

satisfied the condition

$$\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}\ll\langle T_{ab}(x)\rangle\langle T_{cd}(x')\rangle. \tag{1.3}$$

The limits of validity of the semiclassical approximation have also been studied without making reference to a specific model of quantum gravity. Kuo and Ford [15] proposed that a measure of how strongly the semiclassical approximation is violated can be given by how large the quantity,

$$\Delta_{abcd}(x,x')\equiv\left|\frac{\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}}{\langle T_{ab}(x)T_{cd}(x')\rangle}\right| \tag{1.4}$$

is, where it is assumed that the expectation values in this expression are suitably renormalized. It is important to note that Eq. (1.4) is coordinate dependent, since both the numerator and denominator are *tensor* quantities. The situation is complicated further by the regularization and renormalization issues that arise in defining the quantities appearing in this expression. Using normal ordering, Kuo and Ford [15] computed the quantity

$$\Delta(x)\equiv\left|\frac{\langle T_{00}(x)T_{00}(x)\rangle_{\rm con}}{\langle T_{00}(x)T_{00}(x)\rangle}\right| \tag{1.5}$$

for a free scalar field in flat space for several states including the Casimir vacuum. They found that it vanishes in a coherent state, whereas in many other cases, including the Casimir vacuum, it is of order unity.

Wu and Ford [16] computed the radial flux component of Eq. (1.4), in the cases of a moving mirror in 2-dimensions and an evaporating black hole far from the event horizon in both 2 and 4 dimensions. They found that it was of order unity over time scales comparable to the black hole mass, but that it averages to zero over much larger times. In a normal ordering prescription they found state dependent divergent terms. They also showed that in the simple case of radiation exerting a force on a mirror, the quantum fluctuations in the radiation pressure are due to a state dependent cross term in the energy-momentum tensor correlation function [17,18].

Phillips and Hu [19] used zeta function regularization to compute $\Delta(x)$ with the denominator replaced by the quantity $\langle T_{00}(x)\rangle^2$, for a free scalar field in some curved spacetimes having Euclidean sections. They also computed $\Delta(x)$ for a scalar field in flat space in the Minkowski vacuum state, using both point splitting and a smearing operator to remove the divergences [20]. For the flat space calculation they found that $\Delta(x)$ depends on the direction the points are split, but that it is of order unity regardless of how the points are split. They used their results to criticize the Kuo-Ford conjecture and to suggest that the criteria for the validity of the semiclassical approximation should depend on the scale at which the system is being probed.

Wu and Ford [18] addressed the Kuo-Ford conjecture and the above mentioned criticism of it by Phillips and Hu. They stated that the conjecture is incomplete because it does not address the effect of divergent state dependent terms. They suggested that any criterion for the validity of the semiclassical approximation should be a nonlocal one that involves integrals over the world lines of test particles. They also argued that the question of whether the semiclassical approximation is valid depends on the specifics of a given situation, including the scales being probed and the choice of initial quantum state.

Although it is somewhat unclear what the dimensionless small parameter is that controls the inequality (1.3), Ford's initial work and these subsequent discussions draw attention to the importance of the higher point correlation functions of the energy-momentum tensor. It is quite clear, at least in qualitative terms, that if the higher point connected correlation functions of T_{ab} are large (in an appropriate sense to be determined), it cannot be correct to neglect them completely, as the semiclassical equations (1.1) certainly do.

Another context in which the quantity $\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}$ plays a role is stochastic semiclassical gravity [1,21–25]. In this case the probability distribution function for the quantum noise is obtained from the symmetric part of this correlation function [21]. A dissipation kernel has also been shown to be related to the antisymmetric part of this correlation function [26]. Stochastic semiclassical gravity is an interesting attempt to go beyond the semiclassical approximation. However, for the purposes of the present work, we do not make any stochastic assumptions and determine to investigate the validity of the semiclassical equations within the large N approximation itself.

The energy-momentum correlation function $\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}$ has been directly computed for a scalar field in a two dimensional spacetime with a moving boundary [27], for scalar fields and the Maxwell field in Minkowski spacetime [28,29], and for a massless minimally coupled scalar field in de Sitter spacetime, in the case that the points are spacelike separated and geodesically connected [30]. It has also been computed indirectly through the nonlocal kernel appearing in the deviation of $\langle T_{ab}\rangle$ from flat space [31–33], from a Robertson-Walker spacetime [34], and from a general conformally flat spacetime [35]. The noise and dissipation kernels in stochastic semiclassical gravity are related to the energy-momentum tensor correlation function [26]. These quantities have been computed exactly or approximately for scalar fields of various types in several situations including Minkowski spacetime [36,37], hot flat space [26], the far field limit of a black hole in equilibrium with a thermal field [38], Robertson-Walker spacetimes [21,22,39,40], Bianchi type I spacetimes [23], and a weakly curved spacetime using a covariant expansion in powers of the curvature [41].

Although technical problems such as renormalization and coordinate invariance complicate matters, this body of previous work suggests that the correlation function $\langle T_{ab}(x)T_{cd}(x')\rangle_{\rm con}$ should play an important role in determining the validity of the semiclassical approximation. However, the proper context for incorporating and making use of the information contained in this correlation function in a well-defined (i.e., finite), quantitative framework, that respects general coordinate invariance, has remained somewhat unclear.

The criterion we propose in this paper, that solutions to the semiclassical Einstein equations should be stable against linearized perturbations of the geometry, provides just such a framework. According to standard linear response theory [4,5,42], the linearized equations for the perturbed metric depend on the retarded two-point correlation function of the energy-momentum tensor evaluated in the semiclassical background metric g_{ab} [6]. In this case, the correlation function can be computed using the closed time path (CTP) effective action [43]. The result is a retarded correlation function that involves the commutator of two energy-momentum tensor operators. Hence the perturbations are manifestly causal. Moreover, the UV divergences found in the unrenormalized linear response equations are exactly those required to renormalize the semiclassical theory itself. This ensures that no state-dependent divergences occur. Finally, gauge transformations of the linearized metric fluctuations, h_{ab}, are easily handled within the linear response framework, so that ambiguities related to quantities such as Eq. (1.4) do not arise. Thus, standard linear response theory provides a well-defined test of the validity of the semiclassical approximation to gravity, which directly involves $\langle T_{ab}(x)T_{cd}(x')\rangle$ and its renormalization, in a manner that is in complete accordance with the physical principles of general covariance and causality.

Since this criterion for the validity of the semiclassical approximation lies strictly within the context of that approximation itself, one avoids problems such as gauge invariance of the energy-momentum tensor for gravitons, that inevitably appear if one tries to go beyond the semiclassical approximation and include quantum effects due to the gravitational field. Although these effects certainly are not contained in the semiclassical Einstein equations (1.1), it is possible to study the properties of linearized gravitational fluctuations about the self-consistent solution of Eq. (1.1), simply by taking one higher variation of the effective action that leads to that equation. This second variation involves the two-point correlation function (1.2), evaluated in the self-consistent background geometry.

To understand qualitatively the role of the two-point correlation function in the validity of the semiclassical approximation, it is helpful to consider the physical analogy between semiclassical gravity and semiclassical electromagnetism. The connected correlation function (1.2) measures the gravitational vacuum polarization, which contributes to the proper self-energy of the linearized graviton fluctuations around the background metric, just as the current two-point correlation function, $\langle j^a(x)j^b(x')\rangle_{\rm con}$, measures the electromagnetic vacuum polarization which contributes to the proper self-energy of the photon [44,45]. Hence, if these polarization effects are significant, the semiclassical approximation has certainly broken down, at least in the form specified by Eq. (1.1), where all fluctuations of the metric have been ignored. In quantum electromagnetism (QED) we know exactly how to take these fluctuation effects into account, namely by scattering and interaction Feynman diagrams involving the photon propagator. These processes are important not only in scattering between a few particles at high energies, but also in low energy processes in hot or dense plasmas [5]. Analogous statements should be applicable to gravity. Thus, if the linear response validity criterion is not satisfied, there will be no avoiding the technical difficulties and physical consequences of treating the fluctuations of the gravitational field itself, even if we seek to understand only the *infrared* behavior of a semiclassical approximation to the effective theory of gravity, far below the Planck energy scale.

As a particular illustration of the validity criterion, we apply it to the example of a scalar field with arbitrary mass and curvature coupling in the vacuum state of Minkowski spacetime. We express the retarded correlation function of the linear response analysis in flat space in terms of a Källén-Lehmann spectral representation [45]. The positivity of the spectral representation is sufficient to demonstrate that there are no unstable modes of the linearized semiclassical equations around flat space at distance scales far larger than the Planck scale, and hence, that flat spacetime is completely infrared stable in semiclassical gravity. The semiclassical stability of Minkowski spacetime has been investigated previously by several authors [32,33,46], and instabilities have been found which involve strictly Planck scale variations of the metric fluctuations in space and/or time, which arise from the terms fourth order in derivatives of the metric that are needed to renormalize Eqs. (1.1). Their existence clearly precludes the validity of the semiclassical large N approximation at Planck time or distance scales. Prescriptions for explicitly reducing the order of the equations, which eliminates these Planck scale solutions, have been proposed [47,48] and discussed in some detail [49]. Whether or not these prescriptions are accepted in the general case, it is quite clear *a priori* that the semiclassical approximation (1.1) can be viewed at best only as the low energy effective field theory limit of a more complete quantum theory [1,2], and that no reliable results can be obtained from this approximation in the Planckian regime. However, the flat space example treated in some detail in this paper shows explicitly that the semiclassical approximation does give mathematically meaningful and physically sensible results, when properly restricted to its range of validity at space and distance scales very much larger than the Planck scale.

The organization of the paper is as follows. In the next section the properties of the large N semiclassical approximation in gravity and its renormalization within the covariant effective action framework are reviewed. In Sec. III the linear response theory for the semiclassical back-reaction equations is described. The form of the two-point correlation function for the energy-momentum tensor that appears in the linear response equations is given, and its properties and renormalization are discussed. Then our proposal for a necessary condition for the validity of the semiclassical approximation is presented. In Sec. IV the use of our criterion is illustrated for the case of a scalar field with arbitrary mass and curvature coupling in the vacuum state in Minkowski spacetime. The linear response analysis implies that flat spacetime is stable under small fluctuations at large wavelengths. Our results are discussed further in Sec. V. Some additional applications of our criterion to the study of quantum effects in cosmological and black hole spacetimes are suggested. There are two Appendixes. The first deals with the general decomposition of tensors and polarization operators in Minkowski spacetime. The second contains the technical details of the computation of the retarded correlation response function for a scalar field in Minkowski spacetime.

II. SEMICLASSICAL GRAVITY AND RENORMALIZATION

The most direct route to the semiclassical equations (1.1) is via the effective action method in the large N limit. We consider the specific example of N noninteracting scalar fields. Generalizations to interacting fields and fields of other spin are straightforward, but as they are not required to expose the main elements of the stability criterion, we treat only this simplest case in detail. We begin by reviewing the effective action formulation of the semiclassical Eqs. (1.1) without regard to boundary conditions or the state of the field. Thus, the equations in this section are valid for both the $\langle out|in\rangle$ and $\langle in|in\rangle$ formalisms. We postpone to the next section the introduction of the CTP method which selects real and causal $\langle in|in\rangle$ expectation values. It is this latter form that must be used for the linear response analysis.

The classical action for one scalar field (of arbitrary mass and curvature coupling) is

$$S_{\rm m}[\Phi,g]=-\frac{1}{2}\int d^4x\,\sqrt{-g}\,[(\nabla_a\Phi)g^{ab}(\nabla_b\Phi)+m^2\Phi^2+\xi R\Phi^2], \tag{2.1}$$

where ∇_a denotes the covariant derivative for the metric g_{ab}, ξ is the dimensionless curvature coupling, and R is the scalar curvature. The path integral over the free scalar field Φ is Gaussian and may be computed formally by inspection, with the result

$$\int [\mathcal{D}\Phi]\exp\left(\frac{i}{\hbar}S_{\rm m}[\Phi,g]\right)=\exp\left(-\frac{1}{2}\mathrm{Tr}\ln G^{-1}[g]\right)\equiv\exp\left(\frac{i}{\hbar}S^{(1)}_{\rm eff}[g]\right), \tag{2.2}$$

where

$$G^{-1}[g]\equiv-\Box+m^2+\xi R, \tag{2.3}$$

is the inverse propagator of the scalar field in the background metric g_{ab}, and the (generally nonlocal) functional

$$S^{(1)}_{\rm eff}[g]=\frac{i\hbar}{2}{\rm Tr}\ln G^{-1}[g], \tag{2.4}$$

may be regarded as the effective action due to the quantum effects of the scalar field in this metric. It contains an explicit factor of $\hbar$. No assumption about the smallness of the metric deviations from flat spacetime or any other preferred spacetime has been made.

The expectation value of the energy-momentum tensor of the quantum matter field in this background can be formally obtained by the variation

$$\langle T_{ab}(x)\rangle=-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}S^{(1)}_{\rm eff}[g]$$
$$=-i\hbar\mathcal{D}_{ab}\,G[g](x,x')|_{x'=x}, \tag{2.5}$$

where $\mathcal{D}_{ab}G[g](x,x')$ in the coincident limit is

$$\mathcal{D}_{ab}G[g](x,x)\equiv\mathcal{D}_{ab}G[g](x,x')|_{x'=x}=\left[\frac{1}{4}(2\nabla_a\nabla_b-g_{ab}\Box)-\frac{1}{2}m^2g_{ab}+\xi(g_{ab}\Box-\nabla_a\nabla_b+G_{ab})\right]G[g](x,x)$$
$$+\left(-\delta_a^c\delta_b^d+\frac{1}{2}g_{ab}g^{cd}\right)\nabla_c\nabla_dG[g](x,x')|_{x'=x}. \tag{2.6}$$

By Noether's theorem, this (unrenormalized) expectation value of T_{ab} is covariantly conserved, provided that the effective action $S^{(1)}_{\rm eff}[g]$ is invariant under general coordinate transformations. However, $\langle T_{ab}\rangle$ is divergent because of the singular nature of the limit $x'\to x$ in Eq. (2.5), which requires a careful UV regularization and subtraction procedure consistent with coordinate invariance, before a finite renormalized value for its expectation value can be defined [3].

In physical terms the UV regularization and renormalization procedure mean that the theory is not strictly defined at arbitrarily short time and distance scales. The lack of information about the physics at those arbitrarily small scales may be absorbed into a finite number of parameters in the effective low energy theory at larger scales. Since the effective Lagrangian and energy-momentum tensor have canonical scale dimension n (in n spacetime dimensions), the number of parameters is given by the number of local coordinate invariant scalars up to dimension n. In $n=4$ dimensions, these are the parameters of the Einstein-Hilbert action plus the coefficients of the two independent fourth order invariants R^2 and $C_{abcd}C^{abcd}$, where R is the scalar curvature and C_{abcd} is the Weyl tensor, respectively. Thus, in order to renormalize the theory we require the total low energy effective gravitational action,

$$S_{\rm eff}[g]=S^{(1)}_{\rm eff}[g]+\frac{1}{16\pi G_N}\int d^4x\sqrt{-g}(R-2\Lambda)$$
$$-\frac{1}{2}\int d^4x\sqrt{-g}(\alpha C_{abcd}C^{abcd}+\beta R^2), \tag{2.7}$$

with arbitrary dimensionless constants α and β. Renormalization means that G_N, Λ, α, and β are at first bare parameters, which may be chosen to depend on the UV cutoff (introduced to regulate the divergences in the one-loop term $S^{(1)}_{\rm eff}[g]$) in such a way as to cancel those divergences and render the total action, $S_{\rm eff}[g]$, independent of the cutoff. Hence the four parameters of the local geometric terms (up to fourth order derivatives of the metric which are *a priori* independent of $\hbar$) must be considered as parameters of the same order as the corresponding divergent terms in $S^{(1)}_{\rm eff}[g]$, which from Eq. (2.4) is first order in $\hbar$. Formally, this may be justified by considering N identical copies of the matter field, so that $S^{(1)}_{\rm eff}[g]$ is replaced by $NS^{(1)}_{\rm eff}[g]$ and G_N^{-1}, Λ/G_N, α, and β are rescaled by a factor of N. In this way all the terms in Eq. (2.7) are now of the same order in N as $N\to\infty$.

This formal rescaling by N is carried out at the level of the generating functional of connected p-point vertices, $S_{\rm eff}[g]$ (which are the *inverse* of p-point Green's functions), rather than the Green's functions themselves. Therefore, it has the net effect of resumming the quantum effects contained in the one-loop diagrams of the matter field(s) to all orders in the metric g_{ab}. The large N expansion and its relationship to the standard loop expansion have been extensively studied in both Φ^4 theory and electrodynamics (both scalar and spinor QED) in flat space [50]. The QED case is most analogous to the present discussion with the classical vector potential A_μ replaced by the metric g_{ab}. The large N approximation (2.7) is also invariant under changes in the ultraviolet renormalization scale (by definition of the UV cutoff dependence of the local counterterms which cancel against those of the matter action), and is equivalent to the UV renormalization group (RG) improved one-loop approximation.

It is the large N, RG improved one-loop approximation that is necessary to derive the renormalized semiclassical equations (1.1) with back-reaction, for only in such a resummed loop expansion can the one-loop quantum effects of $\langle T_{ab}\rangle$ influence the nominally classical background metric g_{ab}. As mentioned in the previous section, in the ordinary (unimproved) loop expansion the quantum fluctuations of the matter can make at most small corrections to the background metric. The large N approximation also preserves the covariance properties of the theory, since it can be derived from an invariant action functional (2.7). The divergences in $\langle T_{ab}\rangle$ are in one-to-one correspondence with the local counterterms in the action $S_{\rm eff}[g]$, whose variations with respect to g_{ab} produce, in addition to the terms in the classical Einstein equations, the fourth order tensors,

$$
\begin{aligned}
{}^{(1)}H_{ab} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}\int \mathrm{d}^4x\sqrt{-g}R^2\\
&= 2g_{ab}\Box R - 2\nabla_a\nabla_b R + 2RR_{ab} - \frac{g_{ab}}{2}R^2, \qquad (2.8a)
\end{aligned}
$$

$$
\begin{aligned}
{}^{(C)}H_{ab} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}\int \mathrm{d}^4x\sqrt{-g}C_{abcd}C^{abcd}\\
&= 4\nabla^c\nabla^d C_{acbd} + 2R^{cd}C_{acbd}. \qquad (2.8b)
\end{aligned}
$$

Hence the variation of the effective action (2.7) gives the equations of motion for the spacetime metric for zero expectation value of the free scalar field Φ:

$$
-\alpha\ {}^{(C)}H_{ab} - \beta\ {}^{(1)}H_{ab} + \frac{1}{8\pi G_N}(G_{ab}+\Lambda g_{ab}) = \langle T_{ab}\rangle_R, \qquad (2.9)
$$

where $\langle T_{ab}\rangle_R$ is the renormalized expectation value of the energy-momentum tensor of the scalar field, and all the parameters are now understood to take finite renormalized values. In order for the renormalized parameters to be defined unambiguously, we require that any terms of precisely the form of the local geometric tensors on the left hand side of Eq. (2.9), specified at an arbitrary but fixed renormalization scale μ, are removed from the expectation value on the right side of Eq. (2.9) by an explicit subtraction procedure at that scale μ. A concrete example of this subtraction procedure in flat spacetime is given in Sec. IV.

It is worth emphasizing that the UV renormalization of the energy-momentum tensor and the covariant form of the equations of motion (2.9) are justified by formal appeal to an underlying covariant action principle (2.7), whose variation they are. Although particular regularization and renormalization procedures, such as noncovariant point splitting or adiabatic subtraction, may break explicit covariance, the result must be of the form (2.9), with a covariantly conserved $\langle T_{ab}\rangle_R$, or the procedure does not correspond to the addition of local counterterms up to dimension $n=4$ in the effective action, as required by the general principles of renormalization theory. Thus, the renormalization of the effective action (2.7) suffices in principle to renormalize the equations of motion (2.9) and *all* of its higher variations, a fact we make use of in the next section.

The large N approximation is equivalent to a Gaussian path integration for the quantum matter fields, in which the spacetime metric and gravitational degrees of freedom have been treated as c numbers, coupled only to the expectation value of the energy-momentum tensor through Eq. (1.1). Since the energy-momentum tensor expectation value can be expressed as a coincidence limit of local derivatives of the one-loop matter Green's function $G[g](x,x)$ in the background metric g_{ab} through Eq. (2.5), it requires solving the differential equation $G^{-1}[g]\circ G[g]=1$, or more explicitly

$$
(-\Box + m^2 + \xi R)G[g](x,x') = \frac{\delta^4(x,x')}{\sqrt{-g}}, \qquad (2.10)
$$

concurrently with the semiclassical back-reaction equation (2.9). It is the exact solution of this equation, without any perturbative re-expansion of $G[g]$, and the resulting self-consistent solution of Eq. (2.9) for the metric g_{ab}, that constitutes the principal nonperturbative RG improved feature of the large N limit.

The equations of motion (2.9), which are the original Eqs. (1.1) modified by the additional terms required by the UV renormalization of $\langle T_{ab}\rangle$, are fourth order in derivatives of the metric. This feature, which is not present in QED (but is a general feature of effective field theories that are characterized by derivative expansions), has been the source of much discussion in the literature [32,33,46–49]. As is well known from the general theory of differential equations, if the order of the equations is changed by adding higher derivative terms, the solutions of the modified equations fall into two classes, viz., those that approach the solutions of the lower order equations as the new parameters $\alpha,\beta\to 0$, and those which become singular in that limit. The latter class of solutions are not present in the lower order theory and correspond physically to solutions which vary on Planck length and time scales (in order for the higher derivative terms to be of the same order as the lower derivative Einstein terms). There is clearly no experimental basis for taking these solutions seriously (since they would predict that even empty flat space is unstable to arbitrarily short length and time scale perturbations) [32,33,46,49]. Instead, the modern framework of effective field theories suggests that we should regard the Planck scale as the physical UV cutoff which defines the extreme limit of possible validity of semiclassical gravity, and that we should confine our attention to only those predictions of the theory which involve length scales ℓ much greater than the Planck length $\ell_{\rm Pl}$. In this regime, the effects of the higher order local terms in Eq. (2.9) are suppressed by at least two powers of $\ell_{\rm Pl}/\ell$, provided the solutions remain regular in the limit of vanishing renormalized coefficients α and β of the higher order terms. We are interested in this paper primarily in defining a validity and stability criterion of the semiclassical approximation at length scales $\ell \gg \ell_{\rm Pl}$, and only comment briefly on the Planck scale solutions again in the Discussion.

III. CTP, LINEAR RESPONSE AND THE STABILITY CRITERION

In this section we present our criterion for the validity of the semiclassical approximation which relies on a linear response analysis. This analysis makes use of $\langle in|in\rangle$ expectation values which can be realized using the CTP formalism [43]. We begin by reviewing a few details of this formalism that are needed to derive the causal linear response equation.

The desired $\langle in|in\rangle$ expectation values are obtained by integrating the path integral (2.2) along a contour from the initial time up to a late time in the future, and then backwards to the initial time. This results in a doubling of the field variables with a new CTP index (denoted by capital roman letters $A,B,C,\ldots$), running over the values $A=1,2$, that specify the forward or backward part of the contour, respectively. After all manipulations are performed the resulting expressions are evaluated by equating field variables on the two contours.

In the CTP formalism the leading order effective action for the matter fields is formally identical to Eq. (2.4) with the replacement,

$$G^{-1}[g]\rightarrow(\mathcal{G}^{-1}[g])_{AB}\equiv c_{AB}G^{-1}[g^A], \tag{3.1}$$

with $\mathcal{G}[g]$ a 2×2 matrix in the CTP indices and $c_{AB}=\mathrm{diag}(1,-1)$ the CTP metric. Thus, $\mathcal{G}^{-1}_{11}[g]$ depends only on fields of type 1, while $\mathcal{G}^{-1}_{22}[g]$ depends only on fields of type 2. The signs in the CTP metric, c_{AB}, keep track of the direction of the time contour, positively directed forward in time for field variables of the first type, and negatively directed backward in time for field variables of the second type. The corresponding CTP effective action will be denoted by $\mathcal{S}^{CTP}_{\mathrm{eff}}$. Performing the variation of $\mathcal{S}^{CTP}_{\mathrm{eff}}$ with respect to the first CTP component of the metric variable, g^1_{ab}, is formally identical to that of $S^{(1)}_{\mathrm{eff}}[g]$ in Eq. (2.5), and gives $\langle in|T_{ab}(x)|in\rangle$, the unrenormalized diagonal matrix element of the energy-momentum tensor, which is real for Hermitian T_{ab}.

The linear response equation can be obtained by expanding the CTP effective action, in a functional Taylor series to one higher order, around a given semiclassical geometry g_{ab} that solves Eq. (2.9). Writing

$$g_{ab}\rightarrow g_{ab}+h_{ab}, \tag{3.2}$$

one finds that to second order in h_{ab} the CTP effective action is

$$\begin{aligned}\mathcal{S}^{CTP}_{\mathrm{eff}}[g+h]=\mathcal{S}^{CTP}_{\mathrm{eff}}[g]+\int \mathrm{d}^4x\,\frac{\delta\mathcal{S}^{CTP}_{\mathrm{eff}}[g]}{\delta g^A_{ab}(x)}\,h^A_{ab}(x)\\+\frac{1}{2}\int \mathrm{d}^4x\int \mathrm{d}^4x'\frac{\delta^2\mathcal{S}^{CTP}_{\mathrm{eff}}[g]}{\delta g^A_{ab}(x)\,\delta g^B_{cd}(x')}\\\times h^A_{ab}(x)h^B_{cd}(x')+\cdots,\end{aligned} \tag{3.3}$$

where the first variation vanishes by Eq. (2.9). Varying with respect to h^1_{ab} and then setting $h^1=h^2=h$ and $g^1=g^2=g$, gives the linear response equation which is equivalent to the first variation of the semiclassical Einstein equations, namely

$$\begin{aligned}\delta\left[-\alpha^{(C)}H_{ab}-\beta\,{}^{(1)}H_{ab}+\frac{1}{8\pi G_N}(G_{ab}+\Lambda g_{ab})\right]=\delta\langle T_{ab}\rangle\\=\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)+\frac{1}{2}\int \mathrm{d}^4x'\sqrt{-g(x')}\,\Pi^{(\mathrm{ret})cd}_{ab}(x,x')h_{cd}(x'),\end{aligned} \tag{3.4}$$

where

$$\Pi^{(\mathrm{ret})cd}_{ab}(x,x')=\Pi^{11cd}_{ab}(x,x')+\Pi^{12cd}_{ab}(x,x'), \tag{3.5}$$

is the nonlocal connected, retarded polarization tensor and $M_{ab}{}^{cd}$ is the purely local part of the variation of $\langle in|T_{ab}(x)|in\rangle$ at x. We follow here the notation of Ref. [6], except for an opposite sign convention in the definition of the energy-momentum tensor in Eq. (2.2) of that work.

To demonstrate that Eq. (3.5) is indeed the retarded polarization tensor, we carry out the variation of the (unrenormalized) CTP effective action for the scalar matter field explicitly, so that

$$\begin{aligned}\delta\langle in|T_{ab}(x)|in\rangle&=-i\hbar\int d^4x'\,\frac{\delta\mathcal{D}_{ab}}{\delta g^B_{cd}(x')}\,\mathcal{G}[g](x,x)h^B_{cd}(x')-i\hbar\int d^4x'\mathcal{D}_{ab}\,\frac{\delta\mathcal{G}[g](x,x)}{\delta g^B_{cd}(x')}\,h^B_{cd}(x')\\&=\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)+i\hbar\int \mathrm{d}^4x'\sqrt{-g'}\{c_{11}\mathcal{D}_{ab}\mathcal{G}_{1A}[g](x,x')c_{AB}(-\mathcal{D}^{cd})\mathcal{G}_{B1}[g](x',x)\}h_{cd}(x')\\&=\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)-i\hbar\int \mathrm{d}^4x'\sqrt{-g'}\{\mathcal{D}_{ab}\mathcal{G}_{11}[g](x,x')\mathcal{D}^{cd}\mathcal{G}_{11}[g](x',x)\\&\quad-\mathcal{D}_{ab}\mathcal{G}_{12}[g](x,x')\mathcal{D}^{cd}\mathcal{G}_{21}[g](x',x)\}h_{cd}(x').\end{aligned} \tag{3.6}$$

The minus sign in $(-\mathcal{D}^{cd})$ enters because the variation with respect to g_{cd} is opposite in sign from the variation with respect to g^{ab} used to define $\mathcal{D}_{ab}$ in Eqs. (2.5) and (2.6).

The definitions of the various components of the CTP matrix Green's function of the scalar field are [43,50,51]

$$\mathcal{G}_{12}[g](x,x')=i\langle in|\Phi(x')\Phi(x)|in\rangle\equiv G_<(x,x'),$$

$$\begin{aligned}\mathcal{G}_{21}[g](x,x') &= i\langle in|\Phi(x)\Phi(x')|in\rangle \equiv G_>(x,x') = G_<(x',x),\\ \mathcal{G}_{11}[g](x,x') &= i\langle in|T[\Phi(x)\Phi(x')]|in\rangle \equiv \theta(t,t')G_>(x,x') + \theta(t',t)G_<(x,x'),\\ \mathcal{G}_{22}[g](x,x') &= \mathcal{G}^*_{11}[g](x,x') = -\theta(t',t)G_>(x,x') - \theta(t,t')G_<(x,x').\end{aligned} \tag{3.7}$$

Hence suppressing momentarily the spacetime indices in the last line of Eq. (3.6), the CTP structure of that expression is

$$\begin{aligned}&\theta(t,t')[G_>(x,x')]^2 + \theta(t',t)[G_<(x,x')]^2 - G_<(x,x')G_>(x',x)\\ &= \theta(t,t')\{[G_>(x,x')]^2 - [G_<(x,x')]^2\} = -\frac{1}{2}\theta(t,t')\langle in|[\Phi^2(x),\Phi^2(x')]|in\rangle,\end{aligned} \tag{3.8}$$

where we have used the definitions (3.7) and the properties of the Heaviside step function $\theta(t,t')$ for unequal arguments, thus ignoring possible ambiguities at the coincident points $x=x'$. Restoring the spacetime indices, we find that the nonlocal term of the variation of $\langle T_{ab}\rangle_R$ in Eq. (3.4) can be written formally as

$$\frac{1}{2}\int \mathrm{d}^4x' \sqrt{-g'}\,\Pi^{(\mathrm{ret})cd}_{ab}(x,x')h_{cd}(x') = \frac{i\hbar}{2}\int \mathrm{d}^4x' \sqrt{-g'}\,\theta(t,t')\langle in|[T_{ab}(x),T^{cd}(x')]|in\rangle h_{cd}(x'), \tag{3.9}$$

which is real and causal.

This derivation is still formal because of the singular behavior of the retarded polarization operator at coincident points $x=x'$. This singular behavior is related to the short distance behavior of the formal expressions and their renormalization. The singular behavior of commutators of physical currents and their various time ordered products has been recognized for some time [44], and has been discussed in the gravitational context in Ref. [6]. The proper covariant definition of the singular functions requires combining the retarded commutator with the first local (contact) term, $\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)$ in Eq. (3.6), in such a way that the divergences in the sum of the two quantities can be renormalized via the usual counterterms, namely exactly the same counterterms at the level of the effective action which are necessary to renormalize the semiclassical Eqs. (2.9) themselves. Alternatively, one may calculate the time asymmetric part of the response function, which is free of singularities in the limit $x\to x'$, and *define* the renormalized time symmetric part of the full response function (including the local contact terms) by a covariant regularization and renormalization procedure, which gives unique answers up to finite redefinitions of the coefficients α and β in the fourth order renormalized effective action. It is this latter procedure which we carry out explicitly by means of a dispersion integral, after Fourier transforming Eq. (3.9) in the flat space example provided in the next section.

The linearized fluctuation $h_{ab}(x)$ obeys an integro-differential equation (3.4) in which the integral depends only on the past of x, due to the causal boundary conditions, and which involves the two-point correlation function of the matter energy-momentum tensor. According to the general principles of linear response analysis, this retarded correlation function is evaluated in the background geometry of the leading order solution of the semiclassical equations (2.9).

The polarization operator, $\Pi^{(\mathrm{ret})cd}_{ab}(x,x')$, is determined by the second variation of the same effective action that determines the energy-momentum tensor, and it also obeys the same covariant conservation law,

$$\nabla^a\Pi^{(\mathrm{ret})cd}_{ab}(x,x') = \nabla'_c\Pi^{(\mathrm{ret})cd}_{ab}(x,x') = 0, \quad \text{for} \quad x\neq x'. \tag{3.10}$$

Equations (3.4) are covariant in form and therefore are nonunique up to linearized coordinate (gauge) transformations

$$\delta g_{ab} \to \delta g_{ab} + \nabla_a X_b + \nabla_b X_a, \tag{3.11}$$

for any vector field X_a. Singular gauge transformations in the initial data for δg_{ab} are certainly not allowed, and some care is required to decide whether time dependent linearized gauge transformations which grow in time without bound are allowed or not. Since the action principle is fundamental to the present approach, any transformation of the form (3.11), for which the action (2.7) is not invariant (due to boundary or surface terms), is not a true invariance and should be excluded from the set of allowable gauge transformations of the linear response equations (3.4).

We now state our stability criterion for the semiclassical approximation. A necessary condition for the validity of the large N semiclassical equations of motion (2.9) is that the linear response equations (3.4) should have no solutions with finite non-singular initial data for which any linearized gauge invariant scalar quantity grows without bound. Such a quantity must be constructed only from the linearized metric perturbation h_{ab} and its derivatives, and it must be invariant under allowed gauge transformations of the kind described by Eq. (3.11).

The existence of any solutions to the linear response equations with unbounded growth in time, that cannot be removed by an allowed linearized gauge transformation (3.11), implies that the influence of the growing gravitational fluctuations on the semiclassical background geometry are large, and must be taken into account in the evolution of the background itself. That is to say, if the gravitational fluctuations around the background grow, even if they were initially small, then the leading order semiclassical equations (2.9), which neglect these fluctuations, must eventually break down.

IV. STABILITY OF FLAT SPACETIME

Flat spacetime is a solution of the semiclassical Einstein equations for vanishing expectation value of T_{ab} and cosmological term, with the quantum matter field in its Lorentz invariant vacuum ground state. This is the simplest solution of the semiclassical Eqs. (2.9) to which we can apply our validity criterion, and for which the polarization operator, $\Pi_{ab}^{(\mathrm{ret})cd}$, can be evaluated in closed form. In addition to illustrating the application of the criterion to a well-defined specific case, the analysis of the normal modes which solve the linear response Eq. (3.4), will permit us to reach a definitive conclusion on the stability of Minkowski spacetime to quantum perturbations on distance and time scales far larger than the Planck scale.

The linear response equation (3.4) around a Minkowski background, η_{ab}, can be decomposed into scalar, vector and tensor components according to the decomposition and projection operators defined in Appendix A. The variations of the local tensors appearing on the left hand side of Eq. (3.4) are given by Eqs. (A16). Thus, Eq. (3.4) around flat space may be written in the form

$$\left[\alpha\Box^2-\frac{1}{16\pi G_N}\Box\right]P_{ab}^{(T)cd}h_{cd}(x)+\left[6\beta\Box^2+\frac{1}{8\pi G_N}\Box\right]P_{ab}^{(S)cd}h_{cd}(x)=\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)+\frac{1}{2}\int d^4x'\,\Pi_{ab}^{(\mathrm{ret})cd}(x,x')h_{cd}(x'). \tag{4.1}$$

The nonlocal vacuum polarization tensor [right hand side of Eq. (4.1)] can be decomposed into exactly the same two scalar and tensor projections (see both Appendixes A and B), $P_{ab}^{(T)cd}\Pi^{(T)(\mathrm{ret})}+P_{ab}^{(S)cd}\Pi^{(S)(\mathrm{ret})}$, and a Källén-Lehmann spectral representation [5,4,45] given for the Fourier transform of each of these two gauge invariant scalar functions,

$$\Pi^{(i)(\mathrm{ret})}(k^0,\vec{k})=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\sigma^{(i)}(\omega,\vec{k})}{\omega-k^0-i\epsilon},\quad i=T,S. \tag{4.2}$$

This is a form of Cauchy's theorem for

$$\mathrm{Im}[\Pi^{(i)(\mathrm{ret})}(k^0,\vec{k})]=\frac{1}{2}\sigma^{(i)}(k^0,\vec{k})\equiv\pi\rho^{(i)}(s)\,\mathrm{sgn}(k^0),$$

$$s=(k^0)^2-|\vec{k}|^2. \tag{4.3}$$

In Fourier space the nonlocal real convolution in Eq. (4.1) becomes a simple multiplication with $h_{cd}(k)$. Its real part is even and its imaginary part odd under time reversal, which is taken into account by the $\mathrm{sgn}(k^0)$ function in Eq. (4.3), and which is a consequence of the causal, retarded boundary conditions of the CTP formalism.

Since the purely local term, $M_{ab}{}^{cd}$, is time reversal invariant, it does not contribute to the imaginary part of the dispersion relation (4.2) of $\Pi^{(i)(ret)}(k^0,\vec{k})$ which is finite and well defined. The proper definition of the local term is connected with the renormalization procedure needed to reconstruct the real part of the Fourier transform of Eq. (4.1) from its imaginary part, and in fact, the dispersion integral in Eq. (4.2) does *not* exist due to the large k behavior, viz. k^4, of the Lorentz invariant spectral functions $\rho^{(i)}(s=-k^2)$. This divergent behavior of the unrenormalized dispersion integral (4.2) as $k^2\to\infty$, is nothing but the ambiguities of the coincident limit $x\to x'$ in the retarded polarization function in a different guise. The divergent terms are proportional to $\delta^4(x,x')$ and up to four derivatives thereof, which by Lorentz invariance must be of exactly the same form as the local terms on the left hand side of the linear response Eq. (3.4). Thus, these divergences, as well as the local term $M_{ab}{}^{cd}$, can be handled by the same renormalization procedure needed to define the expectation value of the dimension 4 operator T_{ab} in Eq. (1.1), namely by subtraction of the allowed covariant counterterms up to dimension 4. In the flat space dispersion integral (4.2), this is easily accomplished by subtracting the first three terms in its Taylor series expansion around $k^2=0$, and *defining* the renormalized real part of the retarded correlation function by

$$\begin{aligned}\mathrm{Re}[\Pi^{(i)(\mathrm{ret})}(k^2)|_{\mathrm{R}}]&\equiv\mathrm{Re}\left\{\Pi^{(i)(\mathrm{ret})}(k^2)-\Pi^{(i)(\mathrm{ret})}(0)-k^2\left[\left.\frac{\partial\Pi^{(i)(\mathrm{ret})}}{\partial k^2}\right|_{k^2=0}\right]-\frac{(k^2)^2}{2}\left[\left.\frac{\partial^2\Pi^{(i)(\mathrm{ret})}}{\partial(k^2)^2}\right|_{k^2=0}\right]\right\}\\&=-(k^2)^3\mathcal{P}\int_0^\infty\frac{ds}{s^3}\,\frac{\rho^{(i)}(s)}{s+k^2},\end{aligned} \tag{4.4}$$

where $\mathcal{P}$ denotes the principal part prescription for the integral when k is timelike $(-k^2=s)$. The subtractions do not affect the time odd imaginary part of the retarded polarization function. The integral over s in the real part is now well-defined and UV finite, and may even be computed in terms of elementary functions, in the case of a scalar field of arbitrary mass $m>0$ and curvature coupling ξ. The details of this calculation are given in Appendix B. The three subtractions in Eq. (4.4) correspond physically to renormalizing the coefficients of the cosmological constant (Λ), Newton's constant $(1/G_N)$, and the coefficients of the fourth order terms (α and β). These are of order $(k^2)^0$, k^2 and $(k^2)^2$ respectively. The renormalized values of these parameters at $k^2=0$ are what appear then on the left hand side of Eq. (2.9). The singular local term, $M_{ab}{}^{cd}$ in Eq. (4.1) is effectively removed by these subtractions in flat space as well, so that the entire linear response equation becomes well-defined and covariant. Indeed had we performed the renormalization at the level of the effective action directly then it would be clear that no local term ambiguities appear in the renormalized equations.

Since the projections onto scalar and tensor modes are linearly independent (in fact, orthogonal), the coefficients of the two projection operators must satisfy the linear response relation separately. Transferring the polarization part to the left hand side of Eq. (4.1) and taking account of the renormalization just described, yields two independent dispersion formula, namely,

$$k^2\left[2\alpha k^2+\frac{1}{8\pi G_N}+(k^2)^2\int_0^\infty\frac{ds}{s^3}\,\frac{\rho^{(T)}(s)}{(s+k^2-i\epsilon\,\mathrm{sgn}(k^0))}\right]=0, \tag{4.5a}$$

$$k^2\left[12\beta k^2-\frac{1}{4\pi G_N}+(k^2)^2\int_0^\infty\frac{ds}{s^3}\,\frac{\rho^{(S)}(s)}{(s+k^2-i\epsilon\,\mathrm{sgn}(k^0))}\right]=0. \tag{4.5b}$$

The two spectral functions $\rho^{(i)}(s)$ are calculated explicitly for the free scalar field with arbitrary mass and curvature coupling in Appendix B. For this case, the spectral functions have support only when $s>4m^2$, which corresponds to the two particle threshold for timelike gravitational fluctuations. However, some conclusions can be drawn from the two dispersion relations above using only the fact that both spectral functions are positive for both transverse tensor and scalar gravitational perturbations of flat space.

Let us examine first the tensor dispersion relation. It clearly is always satisfied by $k^2=0$. This solution corresponds to the physical, transverse linearized gravitational waves propagating in a flat space background. The coefficient of k^2 at $k^2=0$ is unchanged from the classical value by the quantum parameter α and vacuum polarization corrections. Therefore, these linearized gravitational waves carry the same energy density in the semiclassical approximation as they do in the classical Einstein theory.

Next we may examine the interior of the brackets to determine if there are any other solutions to the tensor linear response equations. Solutions with $k^2=-(k^0)^2+|\vec{k}|^2>0$ correspond to unstable modes with imaginary frequencies, since we can always consider these modes in a frame where $\vec{k}=0$. When $k^2>0$ the $-i\epsilon$ prescription is not needed, and ϵ may be set to zero. Thus, by making use only of the positivity of $\rho^{(T)}$, we observe that the bracket is strictly positive for $k^2>0$, provided

$$16\pi\alpha G_N k^2+1>0. \tag{4.6}$$

If $\alpha\geqslant 0$ this is always satisfied, and indeed this constraint on α is required by positivity of the energy density $2^{(C)}H_{00}$ corresponding to the fourth order $C_{abcd}C^{abcd}$ term in the action. This demonstrates that there are no unstable transverse tensor perturbations of flat spacetime for $\alpha\geqslant 0$. Since this conclusion relies only on the positivity of the spectral function $\rho^{(T)}(s)$, it requires only causality, a bounded Hamiltonian, and a well-defined positive Hilbert space norm for the quantum matter theory. Hence it is valid much more generally than for the specific scalar field example.

Going further, we may inquire as to the existence of additional stable solutions characterized by propagating tensor wave modes with timelike k ($k^2<0$). The bracket in Eq. (4.5a) vanishes if

$$16\pi\alpha G_N k^2+1+8\pi G_N k^2 F^{(T)}=0, \tag{4.7}$$

where

$$F^{(T)}\equiv k^2\int_0^\infty\frac{ds}{s^3}\,\frac{\rho^{(T)}(s)}{[s+k^2-i\epsilon\,\mathrm{sgn}(k^0)]}, \tag{4.8}$$

is a dimensionless function of k^2/m^2 (and the sign of k^0), given explicitly for the case of a scalar field by Eq. (B44). It is clear that if this function remains bounded for all k^2/m^2, the equality (4.7) can never be satisfied for $16\pi G_N|k^2|\ll 1$, since both the third (polarization) term and first (α) term can never be of order unity. In fact, from the explicit form of $F^{(T)}$ for a scalar field, given by Eq. (B44), we find

$$\mathrm{Re}\left[F^{(T)}\left(\frac{k^2}{m^2}\right)\right]\to\frac{1}{960\pi^2}\ln\left(\frac{|k^2|}{m^2}\right)\quad\text{as}\quad |k^2|\to\infty, \tag{4.9}$$

so that the function does grow without bound, but only logarithmically. Hence the relation (4.7) cannot be satisfied except at k^2 approaching G_N^{-1}, provided $m^2>0$. If $\alpha<0$ then the preceding analyses for $k^2>0$ and $k^2<0$ interchange roles, with the conclusion unchanged. Thus, there are no tensor mode solutions of the linear response Eq. (3.4) on length scales much larger than the Planck length for a massive field theory around flat space, other than the usual linearized gravitational waves of the classical theory. On physical grounds one must expect this result to hold for any quantum matter field polarization tensor of finite mass obeying the same general properties of our scalar field example.

The logarithmic divergence in the response function $F^{(T)}$ when $k^2\to\infty$ is a consequence of the large s (UV) behavior of the spectral function proportional to s^2, and is generic, with only the value of the finite coefficient of the logarithm dependent on the matter content. However the appearance of m^2 in the lower limit of the logarithm is a result of our definition of the renormalized α parameter at $k^2=0$, which allows no other scale to appear in the logarithm. This definition is no longer tenable in the zero mass limit. Instead, one may renormalize the parameter at an arbitrary nonzero value of $k^2=\mu^2$, related to our previous definition by

$$\alpha_R(\mu^2)=\alpha+\frac{1}{1920\pi^2}\ln\left(\frac{\mu^2}{m^2}\right). \tag{4.10}$$

Then, μ^2 replaces m^2 in the logarithm of Eq. (4.9), and the $m^2\to 0$ limit may be taken safely by maintaining $\alpha_R(\mu^2)$ finite. The conclusions about the absence of solutions to the transverse linear response equation (4.5a) at length scales much greater than the Planck length remain unchanged.

Turning now to the scalar component of the linear response equation (4.5b), we note the opposite sign in the coefficient of the $1/G_N$ term, which is the well-known negative metric sign of the conformal factor in the Einstein-Hilbert action. From several different analyses [52,53] it is known that there are no physical wavelike scalar excitations of flat space in either the classical or semiclassical theory. Physically the reason is that the conformal factor is constrained by the diffeomorphism invariance. This implies that the expansion of the gravitational action to second order in the metric perturbations about flat space, $h^{(S)ab}k^2h^{(S)}_{ab}$ should be treated as proportional to $|\tilde{\chi}|^2$, where $\tilde{\chi}$ is the Fourier transform of a new scalar field variable with no kinetic term in the Einstein-Hilbert action. This redefinition may be understood as required also in the covariant path integral treatment of the linearized gravitational fluctuations around flat space [53]. The net effect of either the covariant or canonical analysis is to remove the overall $k^2=0$ solution from the scalar sector, as it is not dynamically allowed by the constraints.

Finally, the analysis of the expression within the brackets of Eq. (4.5b) shows that there are no scalar mode solutions (stable or otherwise) with $8\pi G_N|k^2|\ll 1$, for exactly the same reason as in the tensor case, notwithstanding the sign change in the Einstein term. In the scalar case the explicit form of the response function (B45) yields

$$\mathrm{Re}\left[F^{(S)}\left(\frac{k^2}{m^2}\right)\right]\to\frac{1}{96\pi^2}(1-6\xi)^2\ln\left(\frac{|k^2|}{m^2}\right)\quad\text{as}\quad|k^2|\to\infty,\tag{4.11}$$

which shows that the linear response equation cannot be satisfied for either sign of k^2 unless $G_N|k^2|$ becomes of order unity. This result was first obtained in Ref. [53] for the case of a massive, but conformally coupled field with $\xi=1/6$. In the conformal case, the large s behavior of the scalar spectral function $\rho^{(S)}$ is much less severe, as is clear from its explicit form given in Eq. (B38), and only two subtractions suffice. This corresponds to only a finite renormalization of the βR^2 term in the effective action. Ignoring the β term completely, the only twice subtracted dispersion formula gives then

$$1+4\pi G_N k^2\int_0^\infty\frac{ds}{s^2}\,\frac{\rho^{(S)}(s)}{[s+k^2-i\epsilon\,\mathrm{sgn}(k^0)]}=0,\tag{4.12}$$

in the scalar sector. This form of the linear response equation was used in Ref. [53] to demonstrate the stability of flat space to scalar (conformal) fluctuations in the infrared limit, i.e., on wavelengths far larger than the Planck length. Indeed, substituting the explicit form of $\rho^{(S)}$, Eq. (B37b) with $\xi=1/6$, into Eq. (4.12), shows that it is identical with Eq. (4.13) of [53] with $2m^4\rho(s)/3$ of that reference equal to $\rho^{(S)}(s)$ here.

When $\xi\neq 1/6$ this argument cannot be used since the scalar spectral function behaves as s^2 for large s and the twice subtracted dispersion integral in Eq. (4.12) diverges. However, analysis of the fully subtracted response function behavior in Eq. (4.11) above leads to the same result. As in the tensor case the logarithmic growth with $|k^2|$ is generic, with only the coefficient of the logarithm depending on the matter content, possibly vanishing in some special cases such as $\xi=1/6$. Also as in the tensor case the lower limit of the logarithm can be made finite in the $m^2\to 0$ limit by redefining the renormalized β coefficient at a finite $\mu^2\neq 0$ analogously to Eq. (4.10), namely,

$$\beta_R(\mu^2)=\beta+\frac{1}{1152\pi^2}(1-6\xi)^2\ln\left(\frac{\mu^2}{m^2}\right).\tag{4.13}$$

One obtains then the same conclusion as for the tensor case, namely that there are no new solutions of the linear response equations, stable or unstable, far from the Planck regime, despite the opposite sign of the classical Einstein term in the scalar sector. There is an unstable solution in the scalar sector at

$$k^2\simeq\frac{1}{48\pi G_N\beta_R(k^2)},\tag{4.14}$$

which signals the breakdown of the semiclassical approximation to flat space in the Planckian regime.

V. DISCUSSION

We have presented a criterion for the validity of the semiclassical approximation for gravity that involves solving the linear response equation (3.4), to determine the stability of solutions to the semiclassical equations (2.9). If, for a given state and background geometry that solves the semiclassical equations, one or more solutions to the linear response equations experience unbounded growth in a gauge invariant sense, then the semiclassical approximation is not valid for that particular geometry, at least not for that particular state. Clearly this is a necessary, though perhaps not a sufficient condition for the validity of the approximation.

As discussed in the Introduction, various methods have been suggested previously to test the validity of the semiclassical approximation by making use of the two-point correlation function for the energy-momentum tensor. The linear response criterion provides a natural and well-defined way for this two-point correlation function to enter into the determination of the validity of the semiclassical approximation. Further, linear response involves quantities that lie entirely within the semiclassical approximation itself, since the polarization tensor is computed on the semiclassical background geometry. The large N method, augmented by the causal CTP formulation of the effective action, provides a well-defined framework for applying the validity criterion, which is equivalent to a stability criterion for the semiclassical solution.

In the covariant effective action formulation, it is clear that the UV renormalization counterterms are the same as those needed to define the semiclassical approximation itself and that there are no state-dependent divergences. Although the matter energy-momentum tensor correlator by itself suffers from possible ambiguities at coincident points, these are removed by a proper covariant regularization and renormalization procedure, which ties these divergences in the correlator at $x=x'$ to counterterms in the purely local gravitational effective action. The resulting combination of *all* terms in Eq. (3.4) then becomes well-defined.

We have illustrated the use of the stability criterion with the simple example of a quantized scalar field with arbitrary mass and curvature coupling in the vacuum state of Minkowski spacetime. In this case it is possible to carry out the analysis to completion and show that flat space is stable in the infrared limit. There are no solutions to the linear response equations (3.4) around flat space, except the usual transverse, traceless gravitational wave excitations of the classical Einstein theory, provided we restrict ourselves to solutions with $4\pi G_N|k^2|\ll O(1)$. The exact finite number of order unity on the right hand side of this inequality determines the values of k^2 for which new Planckian solutions and instabilities will appear. Its value depends on both the matter theory and the values of the renormalized coefficients α and β of the fourth order terms. The existence of such growing modes, which violate the validity criterion proposed here, informs us not that flat spacetime is unstable, but only that the quantum fluctuations of the geometry should be included in some consistent way at short scales. It is then the semiclassical notion of flat space as a pseudo-Riemannian manifold endowed with a smooth metric down to arbitrarily short length scales that is breaking down at the Planck scale. The semiclassical approximation which does not incorporate the effects of these quantum fluctuations on the mean geometry is certainly not valid in the Planck regime. Because of the unstable tachyon mode (4.14) in the scalar sector, the semiclassical approximation contains the signal of its own breakdown at such short scales according to our validity criterion, but otherwise leads to a completely satisfactory stability of flat space for all perturbations obeying $4\pi G_N|k^2|\ll 1$.

That empty flat space with quantum matter in its vacuum ground state should be stable, and quantum gravitational effects negligible excluding at the Planck scale, is hardly surprising. It does mean that the predictions of the semiclassical approximation at least are not in complete disagreement with observations in this case. In addition to providing an explicit example of how to handle the energy-momentum tensor correlation function by standard renormalization methods to obtain well-defined answers, working out this case in detail also provides an important clue as to how the validity criterion may fail to be satisfied in more interesting cases. What is required is simply that the polarization tensor of the matter fluctuations become singular, i.e., large and unbounded, in some region. Only in this way can the natural suppression of $8\pi G_N|k^2|\ll 1$ in flat space be overcome. A nontrivial example, where new modes may be expected, is a finite temperature quantum matter field in an Einstein–de Sitter model [54]. It would be interesting to apply the validity criterion proposed here to a consistent solution of the semiclassical Einstein equations possessing thermal matter.

Further important and interesting examples to which the criterion may be applied are solutions with event horizons, such as Schwarzschild and de Sitter spacetimes, as well as more general cosmological solutions of the semiclassical equations. If the linearized solutions of Eq. (3.4) show any growing modes (due to a singular behavior of the polarization tensor in such cases), then one would be led to the conclusion that inclusion of gravitational fluctuations beyond the leading order semiclassical approximation would be required. A number of different arguments lead to the conclusion that de Sitter spacetime is not the stable ground state of a quantum theory of gravity with a cosmological term [55]. In fact, the two-point correlation function of the energy-momentum tensor for a scalar field was estimated in [6], and argued to contribute to a gauge invariant growing mode on the horizon time scale. This proposition could be tested by a detailed calculation of the two-point correlation function of the energy-momentum tensor and the solutions of the linear response equations (3.4) in de Sitter space.

A second important application of the criterion is to black hole spacetimes. Ever since the discovery of black hole radiance, it has been recognized that the quantum behavior of black holes is qualitatively different from the classical analogs at *long* times, since semiclassical black holes decay at late times, while classical black holes are stable. In the Hartle-Hawking state [56] one can construct a static solution to the semiclassical equations (2.9) that is quite close to the classical one near the horizon [57–59]. On thermodynamic grounds this state is expected to be unstable [60]. However, the stability of this self-consistent solution has not been investigated in a dynamical approach. The validity criterion proposed in this paper provides a clear dynamical principle for the stability or instability of the self-consistent solutions in both the black hole and de Sitter cases.

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APPENDIX A: TENSOR DECOMPOSITION AND SPECTRAL REPRESENTATION IN FLAT SPACE

An arbitrary metric perturbation, h_{ab}, around n dimensional flat space, η_{ab}, can be decomposed in the following way:

$$h_{ab}=h^{\perp}_{ab}+\partial_a v^{\perp}_b+\partial_b v^{\perp}_a+\left(\partial_a\partial_b-\frac{1}{n}\eta_{ab}\Box\right)w+\frac{\eta_{ab}}{n}h, \tag{A1}$$

where $h^{\perp}_{ab}$ is transverse and traceless with respect to the flat metric η_{ab}

$$\partial^a h^{\perp}_{ab}=0=\partial_b h^{\perp}_{ab}, \quad \text{and} \quad \eta^{ab}h^{\perp}_{ab}=0, \tag{A2}$$

and $v^{\perp}_a$ is transverse

$$\partial^a v^{\perp}_a=0. \tag{A3}$$

By taking partial derivatives and traces and using these defining properties, the various terms in the decomposition can be isolated successively, viz.

$$h=\eta^{ab}h_{ab}\,, \tag{A4}$$

$$w=-\frac{1}{n-1}\Box^{-1}(\eta^{ab}-n\Box^{-1}\partial^a\partial^b)h_{ab}\,, \tag{A5}$$

$$v_a^{\perp}=\Box^{-1}(\delta_a^c-\Box^{-1}\partial_a\partial^c)\partial^d h_{cd}\,, \tag{A6}$$

$$\begin{aligned}h_{ab}^{\perp}=\bigg[&\delta_a^c\delta_b^d-\frac{1}{n-1}\eta_{ab}\eta^{cd}+\frac{2}{n-1}\Box^{-2}\partial_a\partial_b\partial^c\partial^d\\&+\Box^{-1}\bigg(-\delta_a^c\partial_b\partial^d-\delta_b^c\partial_a\partial^d+\frac{1}{n-1}\eta_{ab}\partial^c\partial^d\\&+\frac{1}{n-1}\eta^{cd}\partial_a\partial_b\bigg)\bigg]h_{cd}\,,\end{aligned} \tag{A7}$$

where $\Box^{-1}$ denotes the propagator inverse of $\Box\equiv\eta^{ab}\partial_a\partial_b$.

Under an infinitesimal coordinate (gauge) transformation,

$$h_{ab}\to h_{ab}+\partial_a X_b+\partial_b X_a\,, \tag{A8}$$

the change in h_{ab} can be absorbed into a redefinition of the various components of the decomposition according to

$$\begin{aligned}h&\to h+2\Box Y,\\w&\to w+2Y,\\v_a^{\perp}&\to v_a^{\perp}+X_a^{\perp}\,,\\h_{ab}^{\perp}&\to h_{ab}^{\perp}\,,\end{aligned} \tag{A9}$$

where X_a has been decomposed into its transverse and longitudinal parts, as

$$X_a=X_a^{\perp}+\partial_a Y,\quad \text{with}\quad \partial^a X_a^{\perp}=0. \tag{A10}$$

From these transformations we observe that the transverse, traceless tensor $h_{ab}^{\perp}$ and the linear combination of scalars

$$h-\Box w=\frac{n}{n-1}(\eta^{cd}-\Box^{-1}\partial^c\partial^d)h_{cd}\,, \tag{A11}$$

are invariant under infinitesimal coordinate transformations. Hence we may define the projections onto the scalar (spin-0) and transverse, traceless tensor (spin-2), gauge invariant terms in the general decomposition of the symmetric tensor perturbation h_{ab} by

$$\begin{aligned}h_{ab}^{(S)}&\equiv\frac{1}{n-1}(\eta_{ab}-\Box^{-1}\partial_a\partial_b)(\eta^{cd}-\Box^{-1}\partial^c\partial^d)h_{cd}\\&=\frac{1}{n}(\eta_{ab}-\Box^{-1}\partial_a\partial_b)(h-\Box w)\equiv P_{ab}^{(S)cd}h_{cd}\,,\end{aligned} \tag{A12a}$$

$$h_{ab}^{(T)}\equiv h_{ab}^{\perp}\equiv P_{ab}^{(T)cd}h_{cd}\,. \tag{A12b}$$

The remaining terms in the decomposition contain all the gauge dependence. We denote the vector perturbation [containing both transverse (spin-1) and longitudinal (spin-0) components] by

$$h_{ab}^{(V)}=\Box^{-1}(\delta_a^c\partial_b+\delta_b^c\partial_a-\Box^{-1}\partial_a\partial_b\partial^c)\partial^d h_{cd}\equiv P_{ab}^{(V)cd}h_{cd}\,. \tag{A13}$$

Thus, the general symmetric tensor metric perturbation can be written as the sum of three projected components

$$h_{ab}=h_{ab}^{(S)}+h_{ab}^{(V)}+h_{ab}^{(T)}=\sum_{i=S,V,T}P_{ab}^{(i)cd}h_{cd}\,. \tag{A14}$$

The three projectors are orthonormal, i.e.,

$$P_{ab}^{(i)ef}P_{ef}^{(j)cd}=\delta^{ij}P_{ab}^{(i)cd}, \tag{A15}$$

and complete, and define a unique decomposition (modulo the $[n(n+1)]/2$ conformal Killing vectors in flat spacetime).

Because they are conserved tensors derived from invariant action functionals, all the local tensors on the left hand side of the linear response Eq. (3.4) must be expressible in terms of only the scalar and tensor components of the metric fluctuations. Indeed, by explicit computation in $n=4$ dimensions,

$$\delta\ {}^{(C)}H_{ab}=-\Box^2 h_{ab}^{(T)}=-\Box^2 P_{ab}^{(T)cd}h_{cd}\,, \tag{A16a}$$

$$\delta\ {}^{(1)}H_{ab}=-6\Box^2 h_{ab}^{(S)}=-6\Box^2 P_{ab}^{(S)cd}h_{cd}\,, \tag{A16b}$$

$$\begin{aligned}\delta G_{ab}&=\Box\left(-\frac{1}{2}h_{ab}^{(T)}+h_{ab}^{(S)}\right)\\&=-\frac{1}{2}\Box P_{ab}^{(T)cd}h_{cd}+\Box P_{ab}^{(S)cd}h_{cd}\,,\end{aligned} \tag{A16c}$$

where $h_{ab}\equiv\delta g_{ab}$ is the metric perturbation (variation).

The scalar and tensor projectors onto the space of gauge invariant metric perturbations can be written in momentum space in the compact forms,

$$P_{ab}^{(S)cd}(k)=\frac{1}{n-1}\theta_{ab}\theta^{cd}, \tag{A17a}$$

$$P_{ab}^{(T)cd}(k)=\frac{1}{2}(\theta_a^c\theta_b^d+\theta_a^d\theta_b^c)-\frac{1}{n-1}\theta_{ab}\theta^{cd}, \tag{A17b}$$

where we have introduced the tensor θ_{ab},

$$\theta_{ab}\equiv\eta_{ab}-\frac{k_a k_b}{k^2}, \tag{A18}$$

which obeys $k^a\theta_{ab}=k^b\theta_{ab}=0$. Therefore, the scalar and tensor projectors are also transverse:

$$k^b P_{ab}^{(S)cd}(k) = k^b P_{ab}^{(T)cd}(k) = 0. \tag{A19}$$

Because of this property the correlation function of two conserved energy-momentum tensors in momentum space

$$\Pi_{ab}^{>\,cd}(k) = i \int \frac{d^n x}{(2\pi)^n} e^{ik\cdot(x-x')} \langle T_{ab}(x) T^{cd}(x')\rangle, \tag{A20}$$

may be expanded in terms of the gauge invariant scalar and tensor projectors only. By conservation of T_{ab} this correlator must be transverse, with zero projection onto the vector subspace, i.e. [61],

$$\Pi_{ab}^{>cd}(k) = P_{ab}^{(S)cd}(k)\Pi^{(S)}(k) + P_{ab}^{(T)cd}\Pi^{(T)}(k), \tag{A21}$$

in terms of two scalar functions of momentum k.

The most convenient way of expressing the retarded correlation function in Fourier space is first to introduce the spectral function representation [45] in the spin-2 (T) and spin-0 (S) sectors, in terms of the Euclidean four-momentum $(k_4, \vec{k})$

$$\Pi_E^{(i)}(ik_4, \vec{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sigma^{(i)}(\omega, \vec{k})}{\omega - ik_4}. \tag{A22}$$

The retarded correlator is given then by the analytic continuation, $ik_4 \to k^0 + i\epsilon$, i.e.,

$$\Pi^{(i)(\mathrm{ret})}(k^0, \vec{k}) = \Pi_E^{(i)}(ik_4 = k^0 + i\epsilon, \vec{k}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sigma^{(i)}(\omega, \vec{k})}{\omega - k^0 - i\epsilon}, \tag{A23}$$

which corresponds to Eq. (4.2) of the text. Moreover, since $\sigma^{(i)}(\omega, \vec{k})$ is an odd function of ω (by the reality of $\Pi_E^{(i)}$), which otherwise depends only on the Lorentz invariant combination $s \equiv \omega^2 - \vec{k}\cdot\vec{k}$, we can define Lorentz invariant positive spectral functions $\rho^{(i)}(s)$ by

$$\sigma^{(i)}(\omega, \vec{k}) = 2\pi \mathrm{sgn}(\omega) \rho^{(i)}(s), \tag{A24}$$

and obtain the dispersion formula,

$$\Pi_{ab}^{(\mathrm{ret})cd}(k) = P_{ab}^{(S)cd}(k) \int_0^{\infty} \frac{ds\, \rho^{(S)}(s)}{s + k^2 - i\epsilon \mathrm{sgn}(k^0)} + P_{ab}^{(T)cd}(k) \int_0^{\infty} \frac{ds\, \rho^{(T)}(s)}{s + k^2 - i\epsilon \mathrm{sgn}(k^0)}, \tag{A25}$$

which follows by substituting Eq. (A24) into Eq. (4.2), dividing the integration range over ω into positive and negative ω, renaming the integration variable, and using $ds = 2\omega d\omega$. Thus, the spectral functions $\rho^{(i)}(s)$ of the two independent scalar functions given in Eq. (A21) can be obtained by computing the simple correlator in Euclidean momentum and evaluating the imaginary part, after performing the specific analytic continuation (4.2), namely

$$\mathrm{Im}[\Pi_E^{(i)}(ik_4 = k^0 + i\epsilon)] = \pi \mathrm{sgn}(k^0) \rho^{(i)}(s = -k^2), \tag{A26}$$

which is also obtained by continuing the Euclidean

$$(k_4)^2 + \vec{k}\cdot\vec{k} \to k^2 - i\epsilon \mathrm{sgn}(k^0), \tag{A27}$$

to the Lorentzian $k^2 = \eta_{ab}k^a k^b = -(k^0)^2 + \vec{k}\cdot\vec{k}$. The usefulness of this representation is that the imaginary part of the correlator is given by spectral functions which have simple positivity properties and which are completely free of ultraviolet divergences. These appear only when the real part of the correlator is constructed by the fully covariant dispersion integrals over s in Eq. (A25), and may be handled by standard methods that make clear their relation to covariant local countertems in the effective action. The covariant renormalization of these dispersion integrals by explicit subtractions in flat space is described in Sec. IV.

APPENDIX B: GRAVITATIONAL VACUUM POLARIZATION TENSOR IN FLAT SPACE

The classical energy-momentum tensor for a scalar field in n dimensional Minkowski spacetime is given by

$$T_{ab}|_{\mathrm{flat}} = (1-2\xi)\nabla_a\Phi\nabla_b\Phi + \left(2\xi - \frac{1}{2}\right)\eta_{ab}\nabla_c\Phi\nabla^c\Phi - 2\xi\Phi\nabla_a\nabla_b\Phi + 2\xi\eta_{ab}\Phi\nabla_c\nabla^c\Phi - \frac{1}{2}\eta_{ab}m^2\Phi^2, \tag{B1}$$

which can be rewritten as

$$T_{ab}|_{\mathrm{flat}} = \left[\frac{1}{4}(2\nabla_a\nabla_b - \eta_{ab}\nabla_c\nabla^c) - \frac{1}{2}m^2\eta_{ab} + \xi(\eta_{ab}\nabla_c\nabla^c - \nabla_a\nabla_b)\right]\Phi^2 + (-\delta_a^c\delta_b^d + \tfrac{1}{2}\eta_{ab}\eta^{cd})\Phi(\nabla_c\nabla_d\Phi). \tag{B2}$$

When this is substituted into the Fourier transform of the energy-momentum tensor two-point connected correlation function in Euclidean space and the two possible Wick contractions of $\langle\Phi^2(x)\Phi^2(x')\rangle_{\mathrm{con}}$ are taken into account, we obtain

$$\Pi_{ab}{}^{cd}|_E(k) = \int d^n x \langle T_{ab}(x) T^{cd}(x')\rangle\Big|_E e^{ik\cdot(x-x')} = 2\mathcal{D}_{ab}^{(1)}(k)\mathcal{D}^{(1)cd}(k)H(k) + 2\mathcal{D}_{ab}^{(1)}(k)\mathcal{D}_{c'd'}^{(2)cd}I^{c'd'}(k)$$
$$+ 2\mathcal{D}_{ab}^{(2)a'b'}\mathcal{D}^{(1)cd}(k)I_{a'b'}(k) + 2\mathcal{D}_{ab}^{(2)a'b'}\mathcal{D}_{c'd'}^{(2)cd}J_{a'b'}{}^{c'd'}(k) + 2\mathcal{D}_{ab}^{(2)a'b'}\mathcal{D}_{c'd'}^{(2)cd}K_{a'b'}{}^{c'd'}(k). \tag{B3}$$

Here $k=(k_4,\vec{k})$ is the Euclidean momentum, and we have introduced the following tensors:

$$\mathcal{D}^{(1)}_{ab}(k)=\xi(k_ak_b-\eta_{ab}k^2)+\frac{1}{4}(\eta_{ab}k^2-2k_ak_b)+\frac{1}{2}m^2\eta_{ab}, \tag{B4}$$

$$\mathcal{D}^{(2)cd}_{ab}=-\delta^c_a\delta^d_b+\frac{1}{2}\eta_{ab}\eta^{cd}, \tag{B5}$$

and the notation for the following integrals:

$$H(k)=\int\frac{d^np}{(2\pi)^n}\frac{1}{p^2+m^2}\frac{1}{(p+k)^2+m^2}, \tag{B6}$$

$$I^{cd}(k)=\int\frac{d^np}{(2\pi)^n}\frac{p^cp^d}{p^2+m^2}\frac{1}{(p+k)^2+m^2}, \tag{B7}$$

$$J_{ab}{}^{cd}(k)=\int\frac{d^np}{(2\pi)^n}\frac{p_ap_bp^cp^d}{p^2+m^2}\frac{1}{(p+k)^2+m^2}, \tag{B8}$$

$$K_{ab}{}^{cd}(k)=\int\frac{d^np}{(2\pi)^n}\frac{p_ap_b}{p^2+m^2}\frac{(p^c+k^c)(p^d+k^d)}{(p+k)^2+m^2}. \tag{B9}$$

We regularize all integrals by means of dimensional regularization in n dimensions [62,63]. By introducing a Feynman parameter x, the previous integrals are evaluated to yield [63]

$$H(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}, \tag{B10}$$

$$I^{cd}(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\left\{\frac{\eta^{cd}}{2-n}[m^2+k^2x(1-x)]+x^2k^ck^d\right\}, \tag{B11}$$

$$\begin{aligned}J_{ab}{}^{cd}(k)=&\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\left\{\frac{1}{n(n-2)}(\tau^{(1)cd}_{ab}+\tau^{(2)cd}_{ab})[m^2+k^2x(1-x)]^2\right.\\&\left.-\frac{x^2}{(n-2)}(\tau^{(3)cd}_{ab}+\tau^{(4)cd}_{ab})[m^2+k^2x(1-x)]+x^4\tau^{(5)cd}_{ab}\right\},\end{aligned} \tag{B12}$$

$$\begin{aligned}K_{ab}{}^{cd}(k)=&\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\left\{\frac{1}{n(n-2)}(\tau^{(1)cd}_{ab}+\tau^{(2)cd}_{ab})[m^2+k^2x(1-x)]^2\right.\\&\left.-\frac{x^2}{(n-2)}\tau^{(3)cd}_{ab}[m^2+k^2x(1-x)]+\frac{x(1-x)}{(n-2)}\tau^{(4)cd}_{ab}[m^2+k^2x(1-x)]+x^2(1-x)^2\tau^{(5)cd}_{ab}\right\}.\end{aligned} \tag{B13}$$

These expressions are given in terms of the following five basis tensors:

$$\tau^{(1)cd}_{ab}(k)=\eta_{ab}\eta^{cd}, \tag{B14}$$

$$\tau^{(2)cd}_{ab}(k)=\delta^c_a\delta^d_b+\delta^d_a\delta^c_b, \tag{B15}$$

$$\tau^{(3)cd}_{ab}(k)=\eta_{ab}k^ck^d+\eta^{cd}k_ak_b, \tag{B16}$$

$$\tau^{(4)cd}_{ab}(k)=\delta^c_ak_bk^d+\delta^d_ak_bk^c+\delta^c_bk_ak^d+\delta^d_bk_ak^c, \tag{B17}$$

$$\tau^{(5)cd}_{ab}(k)=k_ak_bk^ck^d. \tag{B18}$$

In order to obtain the Euclidean polarization tensor we need to compute the corresponding tensor products specified in Eq. (B3). Once we do this we can write the polarization operator (in n dimensions and for $\xi=0$) as

$$\Pi_{ab}{}^{cd}|_E(k,\xi=0)=\sum_{j=1}^{5}F_j(k)\tau^{(j)cd}_{ab}(k), \tag{B19}$$

where

$$F_1(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\left[\frac{n^2-2n-4}{2n(n-2)}[m^2+k^2x(1-x)]^2+\left(\frac{xk^2}{n-2}(1-nx)+\frac{k^2}{2}-m^2\right)[m^2+k^2x(1-x)]+\frac{k^4x^2}{4}[(1-x)^2+x^2]+k^2x^2\left(-\frac{k^2}{2}+m^2\right)+\frac{1}{2}\left(-\frac{k^2}{2}+m^2\right)^2\right], \quad (B20)$$

$$F_2(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\frac{2}{n(n-2)}[m^2+k^2x(1-x)]^2, \quad (B21)$$

$$F_3(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\left[\left(\frac{(-2x+(n+2)x^2)}{(n-2)}-\frac{1}{2}\right)[m^2+k^2x(1-x)]-\frac{k^2}{4}+\frac{m^2}{2}-\frac{k^2}{2}x^2[(1-x)^2+x^2-2]-x^2m^2\right], \quad (B22)$$

$$F_4(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\frac{x(1-2x)}{(n-2)}[m^2+k^2x(1-x)], \quad (B23)$$

$$F_5(k)=\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\left[\frac{1}{2}+x^2((1-x)^2+x^2-2)\right]. \quad (B24)$$

The ξ dependent part of the polarization operator can be written as

$$\Pi_{ab}{}^{cd}|_E(k,\xi)=\xi\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}\times\left[k^2(\theta_{ab}\eta^{cd}+\theta^{cd}\eta_{ab})\left(-[m^2+k^2x(1-x)]-\frac{k^2}{2}+m^2+k^2x^2\right)+(\theta_{ab}k^ck^d+k_ak_b\theta^{cd})k^2\times(1-2x^2)\right]+2\xi^2k^4\theta_{ab}\theta^{cd}\times\frac{\pi^{n/2}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 dx[m^2+k^2x(1-x)]^{(n/2)-2}. \quad (B25)$$

In order to obtain the retarded polarization operator we need to analytically continue the Euclidean momentum to a Lorentzian momentum. This analytic continuation is defined by

$$k_4^2+\vec{k}^2\to\eta_{ab}k^ak^b-i\epsilon\,\mathrm{sgn}(k^0), \quad (B26)$$

with $\epsilon\to 0^+$ (see Appendix A). Notice that the continuation depends on the sign of k^0. We also take the limit $n\to 4$ and write $n=4-\delta$, with $\delta\to 0^+$. The real part of the polarization tensor has a pole at $n=4$, but its imaginary part comes only from the logarithmic branch cut of the function, $[m^2+k^2x(1-x)]^{-\delta/2}$ expanded around $\delta=0$,

$$\mathrm{Im}\{[m^2+k^2x(1-x)]^{-\delta/2}\}=\frac{\pi\delta}{2}\mathrm{sgn}(k^0)\,\theta[-m^2-\eta_{ab}k^ak^bx(1-x)]+\mathcal{O}(\delta^2), \quad (B27)$$

so that the pole in $\Gamma(\delta/2)\to 2/\delta$ at $\delta=0$ is canceled. Thus the imaginary part is finite in the limit $n\to 4$. Because $x(1-x)\leqslant 1/4$ in the interval $[0,1]$, the step function condition is satisfied only if $(s/4)-m^2>0$, where $s=-k^2=-\eta_{ab}k^ak^b$, and $x\in[x_-,x_+]$, with $x_\pm$ the real roots of the quadratic polynomial $m^2-x(1-x)s=0$. In particular

$$x_\pm=\frac{1}{2}(1\pm r),\quad \text{with}\quad r\equiv\sqrt{1-\frac{4m^2}{s}}. \quad (B28)$$

Therefore,

$$\mathrm{Im}\left\{\lim_{\delta\to 0^+}\int_0^1 dx\left[\Gamma\left(\frac{\delta}{2}\right)[m^2+k^2x(1-x)]^{-\delta/2}\right]\cdots\right\}=\int_{x_-}^{x_+}dx\,\pi\,\mathrm{sgn}(k^0)\,\theta(s-4m^2)\cdots, \quad (B29)$$

where the ellipsis denotes any function of x to be integrated.

Thus, the x integrals all become simple powers of x, and for the case $n=4$, we obtain

$$\mathrm{Im}[F_1(s)]=\mathrm{sgn}(k^0)\frac{\theta(s-4m^2)}{16\pi}\frac{r}{15}\left(2m^4+4m^2s+\frac{3}{4}s^2\right), \quad (B30)$$

$$\mathrm{Im}[F_2(s)]=\mathrm{sgn}(k^0)\frac{\theta(s-4m^2)}{16\pi}\frac{r}{15}\left(2m^4-m^2s+\frac{1}{8}s^2\right), \quad (B31)$$

$$\mathrm{Im}[F_3(s)]=\mathrm{sgn}(k^0)\frac{\theta(s-4m^2)}{16\pi}\frac{r}{15s}\left(2m^4+4m^2s+\frac{3}{4}s^2\right), \quad (B32)$$

$$\mathrm{Im}[F_4(s)]=\mathrm{sgn}(k^0)\frac{\theta(s-4m^2)}{16\pi}\frac{r}{15s}\left(2m^4-m^2s+\frac{1}{8}s^2\right), \quad (B33)$$

$$\mathrm{Im}[F_5(s)]=\mathrm{sgn}(k^0)\,\frac{\theta(s-4m^2)}{16\pi}\,\frac{r}{15s^2}(6m^4+2m^2s+s^2). \tag{B34}$$

These functions are not linearly independent, as the covariant conservation of the energy-momentum tensor implies that

$$F_1(s)-sF_3(s)=0, \tag{B35a}$$

$$F_2(s)-sF_4(s)=0, \tag{B35b}$$

$$F_3(s)+2F_4(s)-sF_5(s)=0. \tag{B35c}$$

In fact the combinations, $2F_2$ and $3F_1+2F_2$, yield the tensor and scalar spectral functions of Eq. (A25), respectively.

Likewise, after computing the x integrals, the ξ dependent polarization tensor becomes

$$\mathrm{Im}[\Pi_{ab}^{cd}(s,\xi)]=\mathrm{sgn}(k^0)\frac{\theta(s-4m^2)}{24\pi}\sqrt{1-\frac{4m^2}{s}}\theta_{ab}\theta^{cd} \times[-\xi s(s+2m^2)+3\xi^2s^2], \tag{B36}$$

which is explicitly transverse and proportional to the scalar projector $P_{ab}^{(S)cd}$. Combining this ξ dependent contribution with the previous ξ independent part, and recalling Eqs. (A17) and (A26), we may now identify the two independent tensor and scalar spectral functions,

$$\rho^{(\mathrm{T})}(s)=\frac{\theta(s-4m^2)}{60\pi^2}\sqrt{1-\frac{4m^2}{s}}\left(\frac{s}{4}-m^2\right)^2\geqslant 0, \tag{B37a}$$

$$\rho^{(\mathrm{S})}(s)=\frac{\theta(s-4m^2)}{24\pi^2}\sqrt{1-\frac{4m^2}{s}}\left[m^2+\frac{(1-6\xi)s}{2}\right]^2\geqslant 0. \tag{B37b}$$

Both spectral functions are positive, as they must be, and agree with results (for $\xi=0$) reported in [28], and (for arbitrary m and ξ) reported in [29].

In the case that the curvature coupling takes its conformal value, $\xi=1/6$, the scalar spectral function does not have terms proportional to s^2 or to m^2s, and becomes

$$\rho^{(\mathrm{S})}|_{\xi=1/6}(s)=\theta(s-4m^2)\frac{m^4}{24\pi^2}\sqrt{1-\frac{4m^2}{s}}, \tag{B38}$$

which agrees with [53], after account is taken of a relative factor of $2m^4/3$ in the definition of the spectral function $\rho^{(S)}(s)$ here, relative to $\rho(s)$ of that work.

Finally the integrals appearing in the Källén-Lehmann representations (A25) are all of the form,

$$I_{n,l}\equiv k^2\int_{4m^2}^{\infty}\frac{ds}{s^{l+1}(s+k^2)}\left(1-\frac{4m^2}{s}\right)^{n+(1/2)} \tag{B39}$$

for $k^2>0$ and n and l integers. By making the change of variables $s=4m^2/(1-u^2)$, all integrals of this kind may be reduced to linear combinations of

$$I_n(z)\equiv I_{n,l=0}=2\int_0^1 du\,\frac{u^{2n+2}}{z^2-u^2}, \tag{B40}$$

where

$$z\equiv\sqrt{1+\frac{4m^2}{k^2}}, \tag{B41}$$

and the $I_n(z)$ functions obey the recursion formula,

$$I_n(z)=-\frac{2}{2n+1}+z^2I_{n-1}(z), \tag{B42}$$

with

$$\begin{aligned}I_0(z)&=-2+z\ln\left(\frac{z+1}{z-1}\right)\quad\text{for}\quad z>1\\&\equiv-2+f\left(\frac{k^2}{m^2}\right)\quad\text{for}\quad k^2>0.\end{aligned} \tag{B43}$$

Using these relations, the response function for the tensor fluctuations can be written as

$$\begin{aligned}F^{(T)}\left(\frac{k^2}{m^2}\right)&\equiv k^2\int_{4m^2}^{\infty}\frac{ds}{s^3(s+k^2)}\;\rho^{(T)}(s)\\&=\frac{1}{960\pi^2}\left[-\frac{2}{5}-\frac{2}{3}z^2+z^4I_0(z)\right]\\&=\frac{1}{960\pi^2}\left[-\frac{46}{15}-\frac{56}{3}\frac{m^2}{k^2}-\frac{32m^4}{(k^2)^2}\right.\\&\quad\left.+\left(1+\frac{4m^2}{k^2}\right)^2f\left(\frac{k^2}{m^2}\right)\right],\end{aligned} \tag{B44}$$

while the corresponding response function for the scalar fluctuations is

$$\begin{aligned}F^{(S)}\left(\frac{k^2}{m^2}\right)&\equiv k^2\int_{4m^2}^{\infty}\frac{ds}{s^3(s+k^2)}\rho^{(S)}(s)\\&=\frac{1}{96\pi^2}\left\{\frac{1}{15}+\frac{2}{3}(1-6\xi)-\frac{2}{3}\frac{m^2}{k^2}\right.\\&\quad\left.+\left[(1-6\xi)-\frac{2m^2}{k^2}\right]^2\left[-2+f\left(\frac{k^2}{m^2}\right)\right]\right\}.\end{aligned} \tag{B45}$$

As $k^2\to 0$, $z\to\infty$, and the function I_0 (or f) is analytic at $z^{-1}=0$. However, as k^2 changes sign, z^{-1} becomes pure imaginary and

$$f\left(\frac{k^2}{m^2}\right)=2\left(\frac{4m^2}{s}-1\right)^{1/2}\tan^{-1}\left[\left(\frac{4m^2}{s}-1\right)^{-1/2}\right]$$
$$\text{for}\quad 0<s=-k^2\leqslant 4m^2, \tag{B46}$$

which remains real in this range. Finally, when $s=-k^2>4m^2$, f develops an imaginary part, viz.

$$f\left(\frac{k^2}{m^2}\right)=z\,\ln\left(\frac{1+z}{1-z}\right)-i\pi z\,\mathrm{sgn}(k^0)$$
$$\text{for}\quad z=\sqrt{1-\frac{4m^2}{s}},$$
$$s=-k^2\geqslant 4m^2,\quad 0\leqslant z<1. \tag{B47}$$

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