Self-force via a Green's function decomposition

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The gravitational field in a neighborhood of a particle of small mass μ moving through curved spacetime is naturally decomposed into two parts each of which satisfies the perturbed Einstein equations through $O(\mu)$. One part is an inhomogeneous field which looks like the μ/r field tidally distorted by the local Riemann tensor. The other part is a homogeneous field that completely determines the self-force of the particle interacting with its own gravitational field, which changes the worldline at $O(\mu)$ and includes the effects of radiation reaction. Surprisingly, a local observer measuring the gravitational field in a neighborhood of a freely moving particle sees geodesic motion of the particle in a perturbed vacuum geometry and would be unaware of the existence of radiation at $O(\mu)$. In the light of all previous work this is quite an unexpected result.

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INTRODUCTION

The principle of equivalence in general relativity implies that a freely moving particle, of infinitesimal mass and size, follows a geodesic of spacetime [1]. For a small but finite mass μ, the particle perturbs the spacetime geometry at $O(\mu)$. The interaction of the particle with this perturbation of the metric is called the "self-force" which changes the worldline at $O(\mu)$ and includes the effects of radiation reaction.

A local observer measuring the gravitational field in a neighborhood of the particle, with no *a priori* knowledge of the background geometry, sees a combination of the background metric plus its $O(\mu)$ perturbation caused by the particle. Local measurements, in a neighborhood of the particle, cannot distinguish these two specific contributions. Nevertheless, we show below that the combined metric in the neighborhood of the particle can be uniquely decomposed into two distinct parts: (i) the tidally distorted μ/r piece of the perturbation of the metric, which is an inhomogeneous solution of the perturbed Einstein equations at $O(\mu)$, and (ii) the sum of the background metric and the remainder of the metric perturbation, which together are a homogeneous solution of the Einstein equations through $O(\mu)$.

The self-force is shown to result in geodesic motion of the particle, through $O(\mu)$, in the homogeneous perturbed metric, (ii) above. With only local measurements, the observer has no means of distinguishing the homogeneous perturbation from the background metric through $O(\mu)$. As the particle moves along a geodesic of this perturbed vacuum geometry, the observer sees no effect which one would be compelled to interpret as radiation reaction at $O(\mu)$. In this precisely defined fashion, we extend the understanding of the principle of equivalence in general relativity: The orbit of the particle is a geodesic through $O(\mu)$, in a locally measurable, vacuum gravitational field.

HISTORICAL BACKGROUND

In Dirac's [2] analysis of the self-force for the electromagnetic field of a particle in flat spacetime, he used the conservation of the stress-energy tensor inside a narrow world tube surrounding the particle's worldline to derive equations of motion with radiation reaction effects included. DeWitt and Brehme [3] extended this approach to allow for curvature in spacetime. Mino, Sasaki and Tanaka [4] generalized it to include the gravitational self-force as well. Quinn and Wald [5] and Quinn [6] have obtained similar results by pursuing an axiomatic approach for the gravitational, electromagnetic and scalar field self-forces.

On a formal level, these analyses provide a coherent and consistent treatment of the self-force. Analysis begins with a given worldline Γ, described by $z^a(\tau)$, which obeys the Lorentz force law through the fixed background gravitational and electromagnetic fields. The self-force gives the actual worldline an acceleration away from Γ, and it is the mass times this acceleration that we wish to determine.

Towards this end, first the retarded field is obtained in terms of the corresponding Green's function. Traditionally the Green's function has been decomposed into "direct" and "tail" parts. The resulting "direct" part of the field at a point x is determined completely by sources on the past null cone of x; in flat spacetime this is the Liénard-Wiechert potential for electromagnetism. The curvature of spacetime allows for an additional contribution from sources within the past null cone, and this is the "tail" part. The self-force on a particle is then composed of two pieces: The first piece comes from the direct part of the field and the acceleration of Γ in the background geometry; in flat spacetime this is the Abraham-Lorentz-Dirac (ALD) force. The second piece comes from the tail part of the field and is present in curved space even if Γ is a geodesic. While the decomposition of the field into the direct and tail parts has been useful for describing the self-force, neither of these parts individually is a solution of the field equation.

In flat spacetime Dirac [2] decomposed the retarded electromagnetic field into two parts: (i) the "mean of the advanced and retarded fields" which is an inhomogeneous field resembling the Coulomb q/r piece of the scalar potential near the particle, and (ii) a "radiation" field, his Eqs. (11) and (13), which is a homogeneous solution of Maxwell's equations. He described the self-force as the interaction of the particle with the radiation field, a well-defined vacuum Maxwell field.

Previous descriptions of the self-force in curved spacetime [3–6] reduce to Dirac's result in the flat spacetime limit. And they provide clear, adequate expressions for the self-force. However, they do not share the physical simplicity of Dirac's analysis where the force is described entirely in terms of an identifiable, vacuum solution of the field equations. Indeed, for an electromagnetic field, the vector potential $A^{\rm self}_a$ [cf. Eq. (14) for the corresponding scalar field expression] might be said to be responsible for the self-force. But generally $A^{\rm self}_a$ does not satisfy Eq. (23) below, and the current density J^a, which would result from the application of the operator on the left hand side of Eq. (23) to $A^{\rm self}_a$, would have a nonvanishing charge distribution in the vicinity of the particle. Furthermore, $A^{\rm self}_a$ is nondifferentiable at the location of the particle with four-velocity u^a if $(R_{ab} - \frac{1}{6}g_{ab}R)u^b \neq 0$ there. Similar statements hold for scalar and gravitational fields.

In this paper we present a curved space generalization of Dirac's scheme. We find that, near the worldline Γ, even in a curved geometry, the retarded field can be decomposed into two parts such that the first, $A^{\rm S}_a$, is an inhomogeneous solution to the field equation with a point source, similar to the "mean of the advanced and retarded fields," while the second, $A^{\rm R}_a$, is a homogeneous solution which yields a complete description of the self-force. These parts are related to but distinct from the usual direct and tail parts. Scalar and gravitational fields are analyzed similarly.

Below, we first focus on the scalar field case, and then briefly describe our results for the electromagnetic and gravitational fields.

SCALAR FIELD

The force on a point particle with charge q moving in a scalar field may be deduced from (see discussion in [6])

$$F^a = q\nabla^a\psi, \tag{1}$$

where the derivative of the field is to be evaluated at the location of the particle. Usually it is implicit that ψ does not include the field from the particle itself but is composed only of the "external" field. The description of the self-force is not so straightforward. However, one can introduce a quantity $\psi^{\rm self}$, as in our Eq. (14), which when substituted into the right hand side of Eq. (1) formally yields the scalar-field self-force as described by Quinn [6]. Similar expressions for the force in terms of the derivatives of the field are given in Eq. (28) and Eq. (34) for electromagnetic and gravitational fields.

The scalar field equation

$$\nabla^2\psi \equiv \nabla^a\nabla_a\psi = -4\pi\rho \tag{2}$$

is formally solved in terms of a Green's function,

$$\nabla^2 G(x,z) = -(-g)^{-1/2}\delta^4(x-z). \tag{3}$$

The source function for a point charge moving along a worldline Γ, described by coordinates $z^a(\tau)$, is

$$\rho(x) = q\int (-g)^{-1/2}\delta^4(x-z(\tau))\, d\tau, \tag{4}$$

where τ is the proper time along the worldline of the particle with scalar charge q. The scalar field of this particle is

$$\psi(x) = 4\pi q\int G[x,z(\tau)]\, d\tau. \tag{5}$$

A symmetric scalar field Green's function is derived from the Hadamard form to be

$$G^{\rm sym}(x,z) = \frac{1}{8\pi}[u(x,z)\delta(\sigma) - v(x,z)\Theta(-\sigma)], \tag{6}$$

where $u(x,z)$ and $v(x,z)$ are biscalars, the properties of which are described by DeWitt and Brehme [3]. They are determined by a local expansion in the vicinity of Γ, and are symmetric under interchange of x and z. σ is half of the square of the distance measured along the geodesic from x to z with $\sigma<0$ for a timelike geodesic, and $\sigma=0$ on the past and future null cones of x. The expansions for the biscalars $u(x,z)$ and $v(x,z)$ are known to be convergent within a finite neighborhood of Γ if the geometry is analytic [7]. The $\Theta(-\sigma)$ guarantees that only when x and z are timelike related is there a contribution from $v(x,z)$. The terms in any Green's function containing u and v are frequently referred to as the "direct" and "tail" parts, respectively.

The direct part of $G^{\rm sym}(x,z)$ has support only on the null cone of x, and the resultant direct part of the field for a particle moving along Γ is

$$\psi^{\rm sym}_{\rm dir}(x) = \left[\frac{qu(x,z)}{2\dot\sigma}\right]_{\tau_{\rm ret}} + \left[\frac{qu(x,z)}{2\dot\sigma}\right]_{\tau_{\rm adv}}, \tag{7}$$

where $\dot\sigma = d\sigma(x,z(\tau))/d\tau$, and $\tau_{\rm ret}$ and $\tau_{\rm adv}$ refer to the proper time of the intersection of Γ with the past and future null cones of x, respectively.

The tail part of $G^{\rm sym}(x,z)$ has support within both the past and future null cones of x, and the resultant tail part of the field for a particle moving along Γ is

$$\psi^{\rm sym}_{\rm tail}(x) = -\frac{q}{2}\left(\int_{-\infty}^{\tau_{\rm ret}} + \int_{\tau_{\rm adv}}^{\infty}\right) v(x,z)\, d\tau. \tag{8}$$

DeWitt and Brehme [3] use local expansions in the vicinity of Γ to show that [9]

$$u(x,z) = 1 + \frac{1}{12}R_{ab}\nabla^a\sigma\ \nabla^b\sigma + O(r^3), \quad x\to\Gamma; \tag{9}$$

r is the proper distance from x to Γ measured along the spatial geodesic which is orthogonal to Γ. They also show that the symmetric biscalar $v(x,z)$ is a solution of the homogeneous wave equation,

$$\nabla^2 v(x,z) = 0, \tag{10}$$

and that for x close to Γ,

$$v(x,z) = -\frac{1}{12}R(z) + O(r), \quad x \to \Gamma. \tag{11}$$

The retarded and advanced Green's functions are derived from $G^{\mathrm{sym}}(x,z)$,

$$G^{\mathrm{ret}}(x,z) = 2\Theta[\Sigma(x),z]G^{\mathrm{sym}}(x,z)$$

$$G^{\mathrm{adv}}(x,z) = 2\Theta[z,\Sigma(x)]G^{\mathrm{sym}}(x,z), \tag{12}$$

where $\Theta[\Sigma(x),z] = 1 - \Theta[z,\Sigma(x)]$ equals 1 if z is in the past of a spacelike hypersurface $\Sigma(x)$ that intersects x, and is zero otherwise. The $G^{\mathrm{rad}}(x,z)$ implicitly used by Dirac in flat spacetime is

$$G^{\mathrm{rad}}(x,z) = G^{\mathrm{ret}}(x,z) - G^{\mathrm{sym}}(x,z). \tag{13}$$

Note that the fields resulting from two different Green's functions that each obey Eq. (3) necessarily differ by a homogeneous solution of Eq. (2).

$G^{\mathrm{ret}}(x,z)$ has reasonable causal structure, and we assume for simplicity that ψ^{ret} is, in fact, the actual field resulting from the source particle.

SELF-FORCE

Careful analyses [3–6] show that contributions to the self-force result from both the source's acceleration, if Γ is not a geodesic, and from the curvature of spacetime. For these two distinct possibilities, the self-force is a consequence of the particle interacting with either the direct or the tail part of its field, respectively. By following the detailed derivations in [3–6], we see that the self-force may be considered to result, via Eq. (1), from the interaction of the particle with the quantity

$$\psi^{\mathrm{self}} = -\left[\frac{qu(x,z)}{2\dot{\sigma}}\right]_{\tau_{\mathrm{ret}}}^{\tau_{\mathrm{adv}}} - q\int_{-\infty}^{\tau_{\mathrm{ret}}} v[x,z(\tau)]\ \mathrm{d}\tau \tag{14}$$

where, unlike Dirac's radiation field, ψ^{self} is not a homogeneous solution of the field equation.

The first expression in Eq. (14) is finite and differentiable in the coincidence limit, $x \to \Gamma$. When substituted into the right hand side of Eq. (1) this expression provides the curved space generalization of the ALD force, and local expansions of $u(x,z)$ and $\dot{\sigma}(x,z)$ in [3–6] give the resultant force in terms of the acceleration of Γ and components of the Riemann tensor.

The integral in Eq. (14) comes from the tail part of the Green's function. Its derivative results, in part, from the implicit dependence of τ_{ret} upon x. Quinn [6] computes this contribution to the derivative to be

$$-qv[x,z(\tau_{\mathrm{ret}})]\nabla_a \tau_{\mathrm{ret}} = q[v\dot{\sigma}^{-1}\nabla_a \sigma]_{\tau_{\mathrm{ret}}}$$

$$= -\frac{qR(x)}{12r}(x_a - z_a) + O(r), \quad x \to \Gamma, \tag{15}$$

the spatial part of which is not defined when x is on Γ. In the usual self-force analysis, one first averages $\nabla_a \psi^{\mathrm{self}}$ over a small, spatial two-sphere surrounding the particle, thus removing the spatial part of Eq. (15). Then one takes the limit of this average as the radius of the two-sphere goes to zero, thereby obtaining a well defined contribution to the self-force [3–6].

HOMOGENEOUS FIELD

We now provide an alternative expression for the field responsible for the self-force.

Given one Green's function which solves Eq. (3), a second may be generated by adding to the first any biscalar which is a homogeneous solution of Eq. (3). $v(x,z)$ is just such a biscalar and is also symmetric, $v(x,z) = v(z,x)$ [3]. Thus, a new symmetric Green's function is

$$G^{\mathrm{S}}(x,z) \equiv G^{\mathrm{sym}}(x,z) + \frac{1}{8\pi}v(x,z) = \frac{1}{8\pi}[u(x,z)\delta(\sigma) + v(x,z)\Theta(\sigma)], \tag{16}$$

which has no support within the null cone. We use $G^{\mathrm{S}}(x,z)$ only in a local neighborhood of the particle, and do not depend upon any knowledge of its global existence. $G^{\mathrm{S}}(x,z)$ does have support on the null cone of x, just as $G^{\mathrm{sym}}(x,z)$ does, and also outside the null cone, at spacelike separated points. The use of $G^{\mathrm{S}}(x,z)$ is thus not complicated by the need for knowledge of the entire past history of the source and is amenable to local analysis. The corresponding field

$$\psi^{\mathrm{S}}(x) = \left[\frac{qu(x,z)}{2\dot{\sigma}}\right]_{\tau_{\mathrm{ret}}} + \left[\frac{qu(x,z)}{2\dot{\sigma}}\right]_{\tau_{\mathrm{adv}}} + \frac{q}{2}\int_{\tau_{\mathrm{ret}}}^{\tau_{\mathrm{adv}}} v(x,z)\ \mathrm{d}\tau \tag{17}$$

is an inhomogeneous solution of Eq. (2) just as ψ^{ret} is. In the pioneering spirit of Dirac, it is natural to define

$$G^{\mathrm{R}}(x,z) \equiv G^{\mathrm{ret}}(x,z) - G^{\mathrm{S}}(x,z) \tag{18}$$

[cf. Eq. (13)]. Remarkably, like $G^{\mathrm{ret}}(x,z)$, $G^{\mathrm{R}}(x,z)$ has no support inside the future null cone. Corresponding to $G^{\mathrm{R}}(x,z)$, we construct

$$\psi^{\mathrm{R}} = \psi^{\mathrm{ret}} - \psi^{\mathrm{S}} = -\left[\frac{qu(x,z)}{2\dot{\sigma}}\right]_{\tau_{\mathrm{ret}}}^{\tau_{\mathrm{adv}}} - q \times \left(\int_{-\infty}^{\tau_{\mathrm{ret}}} + \frac{1}{2}\int_{\tau_{\mathrm{ret}}}^{\tau_{\mathrm{adv}}}\right) v(x,z)\ \mathrm{d}\tau \tag{19}$$

analogous to Dirac's radiation field. By construction $\psi^{\rm R}$ is a homogeneous solution of Eq. (2) and has the consequent property that it is smooth in the coincidence limit, $x\to\Gamma$. We note its relation to $\psi^{\rm self}$

$$\psi^{\rm R}=\psi^{\rm self}-\frac{1}{2}q\int_{\tau_{\rm ret}}^{\tau_{\rm adv}}v(x,z)\ \ {\rm d}\tau. \tag{20}$$

Our most significant technical result is, in fact, that the self-force is determined by the particle's interaction with $\psi^{\rm R}$, since the integral term in Eq. (20) gives no contribution to a self-force. For a field point x which is near Γ, the integrand may be expanded using Eq. (11). The dominant part of the integral from $\tau_{\rm ret}$ to $\tau_{\rm adv}$ then brings in a factor of $\tau_{\rm adv}-\tau_{\rm ret}=2r+O(r^2)$ times $v(x,x)$, and the integral term of Eq. (20) is

$$-qrv(x,x)+O(r^2)=\frac{1}{12}qrR(x)+O(r^2),\quad x\to\Gamma. \tag{21}$$

The derivative of $\frac{1}{12}qrR(x)$ gives an outward pointing, spatial unit vector near Γ; this exactly cancels the troublesome part of $\nabla_a\psi^{\rm self}$ in Eq. (15) which is thus absent from $\nabla_a\psi^{\rm R}$. The derivative of the remainder term $O(r^2)$ is zero in the limit that x approaches Γ and gives no contribution to the self-force. Thus, the self-force may be seen to be due exclusively to the interaction of the particle with $\psi^{\rm R}$ via Eq. (1). We regard this approach as preferable, because $\psi^{\rm R}$ is differentiable at the location of the particle, so that averaging is no longer required in computing the self-force. Even more importantly, $\psi^{\rm R}$ is a homogeneous solution of Eq. (2).

ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

The analysis for the scalar field is easily generalizable to both electromagnetic and gravitational fields by the addition of extra indices to $u(x,z)$ and $v(x,z)$ to create corresponding bivectors and bitensors, and by the introduction of $\bar{g}^{ab'}(x,z)$, which is the bivector of parallel displacement [3]. The primed indices below refer to the source point z, the unprimed indices to the field point x as above. The definitions and relationships for the various Green's functions follow the same pattern as above and are not repeated below.

For the electromagnetic field, the Lorentz gauge requires

$$\nabla_aA^a=0, \tag{22}$$

so that Maxwell's equations become

$$\nabla^2A^a-R^a{}_bA^b=-4\pi J^a. \tag{23}$$

DeWitt and Brehme [3] show that

$$u_{ab'}(x,z)=\bar{g}_{ab'}(x,z)u(x,z), \tag{24}$$

and that (as $x\to\Gamma$)

$$v_{ab'}(x,z)=\frac{1}{2}\bar{g}_a{}^{c'}\left(R_{b'c'}-\frac{1}{6}g_{b'c'}R\right)+O(r). \tag{25}$$

If $(R_{ab}-\frac{1}{6}g_{ab}R)u^b\neq0$ at the particle, then $A_a^{\rm self}$ is nondifferentiable there.

Similar to the scalar field, $A_a^{\rm S}$ is an inhomogeneous solution of Eq. (23). That it also satisfies the gauge condition (22) follows from an argument similar to that after Eq. (3.37) in [3]. We have

$$\nabla^aA_a^{\rm S}=\int\nabla_{a'}(G^{\rm S}J^{a'})\sqrt{-g}\ \ {\rm d}^4x', \tag{26}$$

assuming that J^a is conserved, where $G^{\rm S}$ is the scalar Green's function of Eq. (16). That $G^{\rm S}$ has no support within the past or future null cone implies that the integral is zero when written as a boundary integral.

With the definition

$$A_a^{\rm R}\equiv A_a^{\rm ret}-A_a^{\rm S}, \tag{27}$$

$A_a^{\rm R}$ gives a homogeneous electromagnetic field. The electromagnetic self-force becomes

$$F^a=qg^{ac}(\nabla_cA_b^{\rm R}-\nabla_bA_c^{\rm R})\dot{z}^b. \tag{28}$$

This combines with the Lorentz force law from the background to determine the actual worldline of the particle.

For the gravitational field, the harmonic gauge requires

$$\nabla_a\overline{h}^{ab}=0, \tag{29}$$

and, with $R_{ab}=0$, the perturbed Einstein equations are

$$\nabla^2\overline{h}_{ab}+2R_a{}^c{}_b{}^d\overline{h}_{cd}=-16\pi T_{ab}, \tag{30}$$

where $\overline{h}_{ab}=h_{ab}-\frac{1}{2}g_{ab}h^c{}_c$ is the trace reversed version of the metric perturbation h_{ab}. Mino, Sasaki and Tanaka [4] show that

$$u_{abc'd'}(x,z)=2\bar{g}_{ac'}(x,z)\bar{g}_{bd'}(x,z)u(x,z), \tag{31}$$

and that (as $x\to\Gamma$)

$$v_{abc'd'}(x,z)=-\bar{g}_a{}^{e'}\bar{g}_b{}^{f'}R_{c'e'd'f'}(z)+O(r). \tag{32}$$

If $R_{cadb}u^cu^d\neq0$ at the particle, then $\overline{h}_{ab}^{\rm self}$ is nondifferentiable there.

Similar to the electromagnetic field case, $\overline{h}_{ab}^{\rm S}$ is an inhomogeneous solution of Eq. (30) satisfying the gauge condition (29), if T_{ab} is conserved. With the definition

$$\overline{h}_{ab}^{\rm R}\equiv\overline{h}_{ab}^{\rm ret}-\overline{h}_{ab}^{\rm S}, \tag{33}$$

$\overline{h}_{ab}^{\rm R}$ is differentiable on Γ, and

$$F^a=-\mu(g^{ab}+\dot{z}^a\dot{z}^b)\dot{z}^c\dot{z}^d\left(\nabla_ch_{db}^{\rm R}-\frac{1}{2}\nabla_bh_{cd}^{\rm R}\right). \tag{34}$$

Subject to this force, the particle moves along a worldline which is actually a geodesic for a metric composed of the background geometry complemented by $h_{ab}^{\rm R}$. While geodesic motion has been demonstrated in the past [4,5], only in our case is the reference metric, $g_{ab}+h^{\rm R}_{ab}$, a vacuum solution of the Einstein equations through $O(\mu)$.

DISCUSSION

For the clearest demonstration of the impact of our analysis, we consider the free motion of a particle of small mass μ in the purely gravitational case. With no *a priori* knowledge of the background geometry, an observer makes local measurements of the metric within a neighborhood of the worldline of the particle. That field has two distinct contributions. The first is the background metric combined with $h^{\rm R}_{ab}$—this combination appears as an ''external,'' homogeneous field; no local measurement distinguishes $h^{\rm R}_{ab}$ from the background. The second comes from $h^{\rm S}_{ab}$; for free motion the observer knows this contribution to be simply the μ/r part of the metric with its tidal distortion from the ''external,'' homogeneous field [8]. As a consequence of Eq. (34), the local observer naturally sees that the worldline of the particle is a geodesic in the combined, ''external'' homogeneous field which he measures. Making only local measurements near the worldline, the observer sees no radiation, no local source for the ''external'' field and also no effect which he would be compelled to describe as radiation reaction.

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