Classical stability of charged black branes and the Gubser-Mitra conjecture

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We have investigated the classical stability of magnetically charged black *p*-brane solutions for string theories that include the case studied by Gregory and Laflamme. It turns out that the stability behaves very differently depending on the coupling parameter between the dilaton and gauge fields. In the case of Gregory and Laflamme, it is known that the black brane instability decreases monotonically as the charge of the black branes increases and finally disappears at the extremal point. For more general cases we found that, when the coupling parameter is small, black brane solutions become stable even before reaching the extremal point. On the other hand, when the coupling parameter is large, black branes are always unstable, and moreover the instability does not continue to decrease, but starts to increase again as they approach the extremal point. However, all extremal black branes are shown to be stable even in this case. It has also been shown that the main features of the classical stability are in good agreement with the local thermodynamic behavior of the corresponding black hole system through the Gubser-Mitra conjecture. Some implications of our results are also discussed.

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I. INTRODUCTION

Black holes are very important objects for studying and understanding both general relativity and string theory. Brane configurations in string theory have been studied from various aspects, and play important roles in the context of the gauge-gravity duality [1]. From the AdS/CFT correspondence it has also been suggested that the infrared limit of nonsupersymmetric Yang-Mills theories corresponds to black hole configurations in AdS space, which can be regarded as the near horizon limit of nonextremal D-brane configurations [2]. From the study of Yang-Mills theories we can reveal properties of black holes, and it is considered that the absence of the tachyonic glueball might imply the stability of the corresponding black hole configurations [3]. On the other hand, recently the issue of black string instability and its evolution has been of great interest. It was widely believed that the evolution of the instability in a black string results in bifurcations of the horizon finally. However, Horowitz and Maeda have shown that the horizon of a black string cannot pinch off in a finite affine time [4]. Therefore an inhomogeneous black string may be the final stable configuration. Recently some authors have discussed the properties of inhomogeneous black strings and their existence [5–7].

The Schwarzschild black hole is known to be stable under linearized perturbations in general relativity [8]. However, Gregory and Laflamme discussed the stability of black branes and found that a foliation of Schwarzschild black holes (Sch.$\times S^n$) is unstable if the compactification scale (S^n radius) is larger than the order of the Schwarzschild radius—the so-called Gregory-Laflamme instability [9]. One may, roughly speaking, expect that charge prevents a black brane from being unstable because of some repulsive forces among charges as the event horizon shrinks. However, Gregory and Laflamme also showed that some charged black branes in ten-dimensional spacetime always have instability modes all the way down to the extremal limit [10]. So it is unlikely that the presence of charge can remove the instability in the black branes they considered. On the other hand, extremal black branes have been shown to be stable [11]. The stability of black branes in anti–de Sitter spacetime has also been investigated. In five-dimensional anti–de Sitter spacetime, black strings that are foliations of four-dimensional Schwarzschild black holes [12] or de Sitter–Schwarzschild black holes [13] recently have been shown to be unstable as well. However, the black string that is a foliation of four-dimensional anti–de Sitter–Schwarzschild black holes becomes stable as the horizon radius is larger than the order of the AdS_4 radius [13]. Bañados-Teitelboim-Zanelli (BTZ) black strings in *four*-dimensional spacetime turn out to be stable always [14].

In the string theory context, the black branes that Gregory and Laflamme considered [10,11] are those having a magnetic charge with respect to Neveu-Schwarz gauge fields [i.e., the case of $\bar{\alpha}=-2$ in Eq. (1) below]. Thus, for example, black *p*-branes carrying charge with respect to Ramond-Ramond gauge fields (i.e., the case of $\bar{\alpha}=0$) are not covered in their work. The properties of D-branes and their nonextremal extensions have many applications and are im-

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portant to study. The charged black brane solutions that Gregory and Laflamme considered might not be general enough to show other possible interesting stability behavior. Therefore, it will be of interest to see whether or not the stability behavior drastically changes when more general types of charged black brane solutions are considered. In fact, we find different behaviors depending on the coupling between dilaton and gauge fields. For instance, charged black branes become stable even far from the extremal point provided that this coupling is smaller than a certain critical value. In this paper, we study the classical stability of a wider class of charged black brane backgrounds which include various black *p*-branes in string theory as special cases.

Another interesting feature of black hole systems is that they can be regarded as thermodynamic systems. A black hole has entropy, temperature, and other thermodynamic quantities. The area theorem states that the total area of black holes cannot decrease, i.e., $\delta A \geqslant 0$, and it might be interpreted as stating that the total entropy cannot decrease $\delta S \geqslant 0$ in the black hole thermodynamics. The behavior of instability explained above has been commonly considered from the viewpoint of thermodynamics as a reflection that the black string is entropically less favorable compared with multiple black holes having the same mass and conserved charges. However, the classical area theorem does not necessarily require the transition of a black brane into a configuration having a larger horizon area. Thus, this global entropy argument is too naive and cannot be applied to all cases. For example, one can show that a black string in five-dimensional AdS space can be stable even if its entropy is smaller than that of five-dimensional AdS black holes [13]. As pointed out by Reall [15], the relevant consideration should be the local thermodynamic behavior.

Recently, Gubser and Mitra (GM) gave such a refinement of the entropy argument and conjectured that a black brane with a noncompact translational symmetry is classically stable if and only if it is locally thermodynamically stable (the GM conjecture) [16]. Reall argued a general correspondence between the classical instability and the presence of a negative eigenmode in the Euclidean Einstein-Hilbert action [15]. The negative eigenmode gives an imaginary contribution to the path integral and thus it is closely related to the local thermodynamic instability. In fact the local thermodynamic instability implies the existence of negative mode(s). However, there is no rigorous proof for the converse: that the existence of a negative mode indicates the thermodynamic instability. For Schwarzschild [17] and Reissner-Nordström [16] black holes in AdS space and a Schwarzschild black brane enclosed in a finite cavity [18], the equivalence between nonexistence of negative modes and local thermodynamic stability has already been examined by performing both classical and thermodynamic stability analyses explicitly.

However, it is worthwhile to study the classical stability of black branes in connection with black hole thermodynamics in a more general context. Charged black brane solutions in type II supergravity are particularly interesting. As pointed out by Reall [15], magnetically charged *p*-branes with $p \leqslant 4$ (D0, F1, D1, D2, D4) have sign changes in the specific heat as the charge increases. Although in Ref. [15] the essential feature of local thermodynamic stability for those black branes was analyzed and a sketch of the proof for the GM conjecture given in the context Reall considered, the actual analysis for the classical stability has not been performed. So it has not been examined explicitly as yet how well the classical and thermodynamic stabilities actually agree with each other. This is another main motivation of our study. By performing the classical stability analysis, we give an explicit check for the validity of the GM conjecture. As will be shown below, the agreement between our numerical results for classical perturbations and the thermodynamic stability through the GM conjecture is very good. We also show some features in the classical stability behavior that are not easily predicted by the thermodynamic behavior alone.

This paper is organized as follows. In Sec. II, we summarize the thermodynamic stability behavior of the black branes we are considering. Section III is devoted to the numerical analysis for the classical stability. We give conclusions and discussion in the last section.

II. LOCAL THERMODYNAMIC STABILITY OF BLACK *p*-BRANES

In general, it is not easy to check whether or not black string or brane background spacetimes are stable under classical linearized perturbations. Such a classical analysis involves numerical calculations in most cases. As stated above briefly, however, a simple method has been suggested recently from which one can predict the basic behavior of the classical stability. This is the so-called Gubser-Mitra conjecture [16]. In this section, before performing the classical analysis in detail, we first consider the local thermodynamic behavior of a finite segment of a black *p*-brane background spacetime.

We consider black *p*-brane solutions of the following action:

$$\mathrm{I}=\int d^D x\sqrt{-\overline{g}}\left[e^{-\overline{\beta}\overline{\phi}}[\overline{R}-\overline{\gamma}(\partial\overline{\phi})^2]-\frac{1}{2n!}e^{\overline{\alpha}\overline{\phi}}F_n^2\right]. \quad (1)$$

Here F_n denotes an n-form field strength. $\overline{\beta}$, $\overline{\gamma}$, and $\overline{\alpha}$ are assumed to be arbitrary constants with $\hat{\gamma}=\overline{\gamma}+(D-1)\overline{\beta}^2/(D-2)\geq 0$. However, by taking the conformal transformation[1] $\overline{g}_{MN}=e^{2\overline{\beta}\overline{\phi}/(D-2)}\hat{g}_{MN}$ and then rescaling the dilaton field $\overline{\phi}=\phi/\sqrt{2\hat{\gamma}}$, this action can be written as

$$\mathrm{I}=\int d^D x\sqrt{-\hat{g}}\left[\hat{R}-\frac{1}{2}(\partial\phi)^2-\frac{1}{2n!}e^{a\phi}F_n^2\right], \quad (2)$$

where

[1]Note that the classical stability of black branes does not change under a conformal transformation as long as the conformal factor is regular. Moreover, black hole thermodynamic quantities such as temperature [19], entropy, and specific heat are invariant under stationary conformal transformations.

$$a=\frac{\bar{\alpha}+(D-2n)\bar{\beta}/(D-2)}{\sqrt{2\hat{\gamma}}}. \tag{3}$$

Note that the sign of the parameter a is not relevant here since the action is invariant under $a\to -a$ and $\phi\to-\phi$. In the following analysis, we will consider only magnetically charged black brane solutions of this action, not electrically charged ones. This is because electrically charged solutions can be obtained from magnetically charged ones by dualizing the n-form field F. Moreover, the magnetically charged case is technically easier to treat in the perturbation analysis.[2] The metric of magnetically charged black p-brane solutions is given by [20,21]

$$\begin{aligned} d\hat{s}^2=&-\left(1+\frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{-4\tilde{d}/\Delta(D-2)}Udt^2\\ &+\left(1+\frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{4d/\Delta(D-2)}\left(\frac{dr^2}{U}+r^2d\Omega_n^2\right)\\ &+\left(1+\frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{-4\tilde{d}/\Delta(D-2)}\delta_{ij}dz^idz^j, \end{aligned}$$

$$e^{-(\Delta/2a)\phi}=1+\frac{k}{r^{\tilde{d}}}\sinh^2\mu,\quad U=1-\frac{k}{r^{\tilde{d}}},\quad \Delta=a^2+\frac{2\tilde{d}d}{D-2}. \tag{4}$$

The n-form field strength is proportional to the volume form on S^n. Here $\tilde{d}=n-1$, $d=p+1$, $D=\tilde{d}+d+2=2+n+p$, and the coordinates are $\{x^M\}=\{x^\mu,z^i\}=\{t,r,x^m,z^i\}$ with $m=1,\ldots,n$ and $i=1,\ldots,p$. Thus, the coordinates $\{z^i\}$ describe the p-dimensional spatial worldvolume, and $\{x^m\}$ the n-sphere. The spatial worldvolume is not compactified in our consideration.

Note that the metric components for the p-dimensional spatial worldvolume directions above have an overall multiplication factor depending on the r coordinate. This multiplication factor makes the spatial worldvolume nonflat, and turns out to cause some complications in the perturbation analysis. As pointed out by Reall [15], such complications can be easily avoided by performing an appropriate conformal transformation again, such as $\hat{g}_{MN}=e^{2(n-1)\phi/(D-2)a}g_{MN}$. Then the action becomes

$$\mathrm{I}=\int d^Dx\sqrt{-g}\left[e^{-\beta\phi}[R-\gamma(\partial\phi)^2]-\frac{1}{2n!}e^{\alpha\phi}F_n^2\right], \tag{5}$$

where the constants are now functions of a single parameter a as follows:

$$\beta=-\frac{n-1}{a},\quad \gamma=\frac{1}{2}-\frac{(D-1)(n-1)^2}{(D-2)a^2},$$

$$\alpha=a+\frac{(D-2n)(n-1)}{(D-2)a}. \tag{6}$$

The metric for the black p-brane background can be written as

$$ds^2=-Udt^2+V^{-1}dr^2+R^2d\Omega_n^2+\delta_{ij}dz^idz^j, \tag{7}$$

where

$$V^{-1}=\frac{[1+(k/r^{\tilde{d}})\sinh^2\mu]^{4/\Delta}}{1-k/r^{\tilde{d}}},\quad R^2=\left(1+\frac{k}{r^{\tilde{d}}}\sinh^2\mu\right)^{4/\Delta}r^2. \tag{8}$$

The mass M and magnetic charge Q per unit p-volume are given by [21]

$$M=k\left(\tilde{d}+1+\frac{4\tilde{d}}{\Delta}\sinh^2\mu\right)\quad\text{and}\quad Q=\frac{\tilde{d}k}{\sqrt{\Delta}}\sinh 2\mu, \tag{9}$$

respectively.

Note that the case of $\bar{\beta}=2$, $\bar{\gamma}=-4$, $\bar{\alpha}=-2$, and $D=10$ in Eq. (1) corresponds to the action for which the classical stability of magnetically charged black p-brane solutions has been studied by Gregory and Laflamme (GL) [10,11]. Thus, as pointed out by Reall [15], a Neveu-Schwarz 5-brane (NS5-brane) (i.e., $n=3$) is the only black brane of type II supergravity covered by their work. This case corresponds to $a=(1-n)/2$ in the conformally transformed action in Eq. (5). The F-string by dualizing the Neveu-Schwarz gauge field can also be described by the action in Eq. (1) with a different coupling to the dilaton, i.e., $\bar{\alpha}=2$ and so $a=(9-n)/2$. Hence the F-string (i.e., $n=7$) can be covered. When $\bar{\beta}=2$, $\bar{\gamma}=-4$, $\bar{\alpha}=0$, and $D=10$, it gives $a=(5-n)/2$. Solutions to Eq. (7) for this case cover Dp-branes with $p=0$, 1, 2, 4, 5, 6 of the type II supergravity carrying magnetic charge. Therefore, our study contains a broad class of black brane solutions, including most of the black branes in type II supergravity.

Now let us consider a black p-brane with unit worldvolume. Being regarded as a thermal system, it has entropy and temperature given by

$$S\sim(\cosh\mu)^{4/\Delta}r_H^n\quad\text{and}\quad T=\frac{\tilde{d}}{4\pi r_H}(\cosh\mu)^{-4/\Delta}, \tag{10}$$

respectively. Here $r_H=k^{1/\tilde{d}}$ is the horizon radius. The extremal limit of the magnetically charged black brane solution is $k\to0$ and $\mu\to\infty$ keeping the mass and charge in Eq. (9) finite (i.e., $ke^{2\mu}$ fixed). Note that in the extremal limit both temperature and entropy go to zero except for the case of

[2]Gauge perturbations can be set to zero for the sphere directions, and are decoupled from the metric and scalar perturbations for other directions.

$\Delta = 2\tilde{d}$,[3] and the solution becomes the BPS state which is known to be stable even at the quantum level. In order for this system to be stable thermodynamically, the entropy functional given above should be a local maximum in thermodynamic phase space. Defining the Hessian of the system as

$$H = \begin{pmatrix} \dfrac{\partial^2 S}{\partial M^2} & \dfrac{\partial^2 S}{\partial M \partial Q} \\ \dfrac{\partial^2 S}{\partial M \partial Q} & \dfrac{\partial^2 S}{\partial Q^2} \end{pmatrix}, \tag{11}$$

the maximum entropy is guaranteed provided that

$$\left(\frac{\partial^2 S}{\partial M^2} \right)_Q < 0 \quad \text{and} \quad \det(H) > 0, \tag{12}$$

which become equivalent to [22]

$$C_Q = \left(\frac{\partial M}{\partial T} \right)_Q > 0 \quad \text{and} \quad \left(\frac{\partial \Phi_H}{\partial Q} \right)_T > 0. \tag{13}$$

Here Φ_H denotes the magnetic potential energy at the horizon. Using Eqs. (9), (10), we obtain explicitly

$$C_Q = -4\pi r_H^{\tilde{d}+1} (\cosh\mu)^{4/\Delta} \times \frac{2\tilde{d} + [\Delta + \tilde{d}(\Delta-2)]\cosh 2\mu}{2\tilde{d} + (\Delta - 2\tilde{d})\cosh 2\mu}, \tag{14}$$

$$\left(\frac{\partial \Phi_H}{\partial Q} \right)_T = \frac{r_H^{-\tilde{d}} \Delta \cosh 2\mu}{2\tilde{d} + (\Delta - 2\tilde{d})\cosh 2\mu}. \tag{15}$$

Since the term $\Delta + \tilde{d}(\Delta-2)$ is positive definite, one can find that C_Q and $(\partial \Phi_H / \partial Q)_T$ always have opposite signs in the above. Consequently, the two conditions in Eq. (13) cannot be satisfied simultaneously. In other words, the black brane system in consideration cannot be thermodynamically stable for processes in which both the mass M and charge Q vary. Therefore, according to the GM conjecture, this may indicate that the black brane background is always unstable under classical perturbations. As mentioned by Reall [15], however, the charge of the black brane cannot fluctuate since there is no matter field carrying charge in our theory. Thus, only thermodynamic processes through which the charge does not change are relevant. Accordingly, the local thermodynamic stability of a black brane is determined solely by the sign of the specific heat C_Q above.

The specific heat in Eq. (14) is always negative if $\Delta - 2\tilde{d} \geq 0$, or, equivalently, $|a| \geq a_{\rm cr}$ where the critical value of a depends on the form n of the gauge field or the dimension p of the spatial worldvolume as

$$a_{\rm cr} = \frac{n-1}{\sqrt{(D-2)/2}} = \frac{D-3-p}{\sqrt{(D-2)/2}}. \tag{16}$$

For $|a| < a_{\rm cr}$, however, the specific heat is negative if $0 \leq \mu < \mu_{\rm cr}$, but positive if $\mu_{\rm cr} < \mu < \infty$. At $\mu = \mu_{\rm cr}$, it diverges, i.e., $C_Q \to \mp\infty$ as $\mu \to \mu_{\rm cr} \mp 0$. Here the critical value of the parameter μ is

$$\sinh^2 \mu_{\rm cr} = \frac{2(D-3-p)(p+1) + (D-2)a^2}{2[2(D-3-p)^2 - (D-2)a^2]}. \tag{17}$$

Based on the GM conjecture, therefore, it can be expected under classical linearized perturbations that a black p-brane background is always unstable provided $|a| \geq a_{\rm cr}$, *independent of its charge*. For $|a| < a_{\rm cr}$, however, it becomes stable for $\mu > \mu_{\rm cr}$. For instance, the black p-branes considered by Gregory and Laflamme [10,11] have $a = (1-n)/2$ as explained above. This value is exactly the same as the critical value for $D = 10$ in Eq. (16), i.e., $a^2 = a_{\rm cr}^2$, implying negative specific heat always. Thus it is expected that all black branes considered by them, including the NS5-brane, will be unstable, and this behavior is shown by them. For the F-string, we have $a = 1 < a_{\rm cr} = 3$. For Dp-brane solutions of type II string theory, we have $a = (p-3)/2$ and $a_{\rm cr} = (7-p)/2$. Therefore, one can easily see that the specific heat changes sign for magnetically charged D0-, F1-, D1-, D2-, and D4-branes, but not for D5- and D6-branes as mentioned by Reall [15]. The case of D3-branes will not be considered in this paper since the action for the self-dual gauge field is not known. Moreover, even if we simply use the action in Eq. (1) and impose the self-duality condition, we encounter some problems in analyzing linearized perturbation equations. Finally, we would like to point out that, except for cases of $|a| = a_{\rm cr}$, the specific heat goes to zero in the extremal limit, satisfying a formulation of the ordinary thermodynamic third law.

III. CLASSICAL PERTURBATION ANALYSIS

So far, we have analyzed the local thermodynamic stability of black p-brane systems which gives some hints at the stability of such background spacetimes through the GM conjecture. Now let us perform the classical stability analysis explicitly. We consider small metric perturbations about black p-brane background spacetimes and see whether or not there exists any mode that is regular spatially, but grows exponentially in time.

Since the background black p-brane spacetime in Eq. (7) is independent of the time coordinate t and spatial worldvolume coordinates z^i, one can assume that

$$\delta g_{MN} = h_{MN}(x^\mu, z^i) = e^{\Omega t + i m_i z^i} H_{MN}(r, x^m),$$

$$\delta\phi = e^{\Omega t + i m_i z^i} f(r, x^m),$$

$$\delta F = e^{\Omega t + i m_i z^i} \delta F(r, x^m) \tag{18}$$

[3]For the case of $\Delta = 2\tilde{d}$, interestingly the temperature does not vanish although the entropy still goes to zero in the extremal limit.

for unstable mode solutions. Here $\Omega > 0$ and the Kaluza-Klein (KK) mass m_i is a continuous real number. As explained in Refs. [14,15], one can further use the diffeomorphism symmetry so that the scalar and vector parts of the metric fluctuations h_{MN} are set to zero for *massive* modes from the viewpoint of the Kaluza-Klein reduction [23]

$$H_{\mu i} = H_{ij} = 0. \tag{19}$$

Since in general non-s-wave fluctuations are expected to be more stable than s-wave ones, we consider only spherically symmetric perturbations.[4] Thus, as shown in Refs. [8], the following gauge choice can be made:

$$H_{tm} = H_{rm} = 0, \quad H^m_n = K(r)\,\delta^m_n \tag{20}$$

for such perturbations. The other components H_{tt}, H_{tr}, H_{rr}, and the dilaton and field strength perturbations are also independent of the angular coordinates x^m. That is, they are functions of the radial coordinate r only.

For these forms of perturbation above, one can easily show that linearized equations for gauge fluctuations δF are decoupled, i.e., $\nabla_N(e^{\alpha\phi}\delta F^{NP_1\cdots P_{n-1}}) = 0$. Moreover, as in Ref. [10], it can be shown that for background spacetimes in Eq. (7) there exists no unstable mode solution for the spherically symmetric fluctuations. One can first show that the fluctuations proportional to the volume form on S^n should vanish if $\Omega \neq 0$. Thus only the remaining components $\delta F_{ti_1\cdots i_{n-1}}$, $\delta F_{ri_1\cdots i_{n-1}}$, $\delta F_{tri_1\cdots i_{n-2}}$, and $\delta F_{i_1\cdots i_n}$ of such fluctuations could be nonzero. Here $i_1, \ldots, i_{n-1}$ are the spatial worldvolume coordinates $\mathbf{z}$. By using the Bianchi identity $\nabla_{[N}\delta F_{P_1\cdots P_n]} = 0$ for $(NP_1\cdots P_n) = (tri_1\cdots i_{n-1})$ and the perturbation equations for $(P_1\cdots P_{n-1}) = (ri_1\cdots i_{n-2}), (ii_1\cdots i_{n-2})$ above, one obtains coupled equations for $\mathbf{f}_t \equiv \Sigma_i m_i \delta F_{tii_1\cdots i_{n-2}}$, $\mathbf{f}_r \equiv \Sigma_i m_i \delta F_{rii_1\cdots i_{n-2}}$, and $\delta F_{tri_1\cdots i_{n-2}}$ which can easily be decoupled as

$$V\mathbf{f}_t'' + (\cdots)\mathbf{f}_t' - \left(m^2 + \frac{\Omega^2}{U}\right)\mathbf{f}_t = 0, \tag{21}$$

where $m^2 = \Sigma_i m_i^2$. Since both U and V are positive outside the horizon, one can see that the only regular solution of this equation is $\mathbf{f}_t = 0$, giving $\mathbf{f}_r = \delta F_{tri_1\cdots i_{n-2}} = 0$ correspondingly. Similarly, one also finds that there are no nonvanishing regular solutions for other components of the fluctuations. Therefore we set $\delta F = 0$ in the following analysis.

As shown in Ref. [15], now the dilaton perturbation equation is given by

$$\nabla^2 f - 2\beta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu f - H^{\mu\nu}\nabla_\mu\nabla_\nu\phi + \beta H^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \nabla_\mu\phi\nabla_\nu\left(H^{\mu\nu} - \frac{1}{2}H^\rho_\rho g^{\mu\nu}\right) + \frac{a}{2(n-1)!}e^{(\alpha+\beta)\phi}\left[H^{\mu\nu}F_{\mu\rho_1\cdots\rho_{n-1}}F_\nu{}^{\rho_1\cdots\rho_{n-1}} - \frac{\alpha+\beta}{n}F^2 f\right] = m^2 f. \tag{22}$$

The linearized equations for the metric perturbations $H_{\mu\nu}$ are given by

$$\nabla^2 H_{\mu\nu} - 2\nabla_{(\mu}\nabla^\rho H_{\nu)\rho} + \nabla_\mu\nabla_\nu H^\rho_\rho - 2R_{\rho(\mu}H^\rho_{\nu)} + 2R_{\mu\rho\nu\sigma}H^{\rho\sigma} + \beta(2\nabla_{(\mu}H^\rho_{\nu)} - \nabla^\rho H_{\mu\nu})\nabla_\rho\phi - 2\beta\nabla_\mu\nabla_\nu f + 4(\gamma+\beta^2) \times\nabla_{(\mu}\phi\nabla_{\nu)}f - \frac{1}{(n-1)!}e^{(\alpha+\beta)\phi}[(n-1)H^{\rho\sigma}F_{\mu\rho\lambda_1\cdots\lambda_{n-2}}F_{\nu\sigma}{}^{\lambda_1\cdots\lambda_{n-2}} - (\alpha+\beta)fF_{\mu\lambda_1\cdots\lambda_{n-1}}F_\nu{}^{\lambda_1\cdots\lambda_{n-1}}] = m^2 H_{\mu\nu}. \tag{23}$$

The $\{\mu i\}$ and $\{ij\}$ components are

$$\nabla_\nu H^\nu_\mu - \beta H^\nu_\mu\nabla_\nu\phi - 2(\gamma+\beta^2)f\nabla_\mu\phi = 0, \tag{24}$$

$$H^\rho_\rho - 2\beta f = 0, \tag{25}$$

respectively. Here the covariant derivative and curvature tensors in the above equations are constructed using the metric $g_{\mu\nu}$ only. In other words, perturbation equations on D-dimensional black p-brane backgrounds become coupled second order differential equations for dilaton and gravitational fields with mass m on $(D-p)$-dimensional black hole backgrounds, which is expected from the viewpoint of Kaluza-Klein dimensional reduction.

As pointed out in Ref. [15] already, notice also that the dilaton perturbation equation, Eq. (22), is actually not independent, but can be obtained from the trace of Eq. (23) and Eqs. (24), (25). In Eq. (25), one finds that the dilaton fluctuation f can be computed once the metric fluctuations $H_{\mu\nu}$ are obtained. Thus, we need to consider only the metric perturbation variables, which are H_{tt}, H_{tr}, H_{rr}, and K. For the stability analysis, however, it suffices to consider the behavior of the so-called threshold modes, i.e., KK massive modes having $\Omega = 0$. This is because infinitely heavy massive modes cannot be expected to give instability. There should be a maximum value m^* for the instability to exist. For unstable modes in the case of black brane systems, if they exist, it is known that the corresponding "imaginary" fre-

[4]We will give an argument below that consideration of s-wave perturbations only might be enough to show stability.

quency Ω usually starts to increase as the KK mass increases, reaches some maximum value, but finally decreases to zero at the so-called threshold mass m^*. Thus, as long as the set of equations in Eqs. (23)–(25) allows a threshold mode solution with nonvanishing mass, the instability exists. When $\Omega=0$, there is the additional merit that a further gauge choice is possible to set $H_{tr}=0$ as in Ref. [15]. Therefore, one finally has

$$(H^{\mu}_{\nu})=\mathrm{diag}(\varphi(r),\psi(r),\chi(r),\ldots,\chi(r)). \tag{26}$$

That is, the perturbed metric for threshold modes can be expressed in the following form:

$$ds^2=-U(1+\varphi e^{im_iz^i})dt^2+V^{-1}(1+\psi e^{im_iz^i})dr^2 + R^2(1+\chi e^{im_iz^i})d\Omega_n^2+d\mathbf{z}^2. \tag{27}$$

Now Eqs. (23)–(25) reduce to

$$\varphi''+\left(\frac{U'}{2U}+\frac{V'}{2V}+n\frac{R'}{R}-\beta\phi'\right)\varphi'+\left[\frac{U''}{U}-\frac{U'}{U}\left(\frac{U'}{2U}-\frac{V'}{2V}-n\frac{R'}{R}+\beta\phi'\right)-\frac{m^2}{V}\right]\varphi-\frac{U'}{U}\psi' -\left[\frac{U''}{U}-\frac{U'}{U}\left(\frac{U'}{2U}-\frac{V'}{2V}-n\frac{R'}{R}+\beta\phi'\right)\right]\psi=0, \tag{28}$$

$$\psi''+\left(\frac{U'}{2U}+\frac{V'}{2V}+n\frac{R'}{R}-\frac{2\gamma+3\beta^2}{\beta}\phi'\right)\psi'+\frac{m^2}{V}\psi-\left(\frac{U'}{U}+2\frac{\gamma+\beta^2}{\beta}\phi'\right)\varphi'-2n\left(\frac{R'}{R}+\frac{\gamma+\beta^2}{\beta}\phi'\right)\chi'=0, \tag{29}$$

$$\psi'+\left(\frac{U'}{2U}+n\frac{R'}{R}-\frac{\gamma+2\beta^2}{\beta}\phi'\right)\psi-\left(\frac{U'}{2U}+\frac{\gamma+\beta^2}{\beta}\phi'\right)\varphi-n\left(\frac{R'}{R}+\frac{\gamma+\beta^2}{\beta}\phi'\right)\chi=0. \tag{30}$$

Using Eq. (30), one can easily see that χ can be evaluated from φ and ψ, and that Eqs. (28), (29) become two second order coupled equations for φ and ψ only as follows:

$$r(r^{\tilde{d}}-k)\varphi''+[(\tilde{d}+1)r^{\tilde{d}}-k]\varphi'-m^2r^{\tilde{d}+1-4\tilde{d}/\Delta}(r^{\tilde{d}}+k\sinh^2\mu)^{4/\Delta}\varphi-\tilde{d}k\psi'=0, \tag{31}$$

$$r^2(r^{\tilde{d}}-k)^2(r^{\tilde{d}}+k\sinh^2\mu)\psi''+r(r^{\tilde{d}}-k)^2\left[2\tilde{d}\left(r^{\tilde{d}}-\frac{2}{\Delta}k\sinh^2\mu\right)-(\tilde{d}-3)(r^{\tilde{d}}+k\sinh^2\mu)\right]\psi'-\left(m^2r^{\tilde{d}+2-4\tilde{d}/\Delta}(r^{\tilde{d}}-k)\times(r^{\tilde{d}}+k\sinh^2\mu)^{1+4/\Delta}+\tilde{d}k\left\{W+\frac{2}{\Delta}\left[2\tilde{d}^2+(\tilde{d}+3)\left(a^2-\frac{2\tilde{d}^2}{D-2}\right)\right]\sinh^2\mu(r^{\tilde{d}}-k)^2\right\}\right)\psi+\tilde{d}kW\varphi=0. \tag{32}$$

Here the functions U, V, and R are substituted explicitly, and

$$W=\tilde{d}(r^{\tilde{d}}-k)(r^{\tilde{d}}+k\sinh^2\mu)-\frac{2}{\Delta}\left(a^2-\frac{2\tilde{d}^2}{D-2}\right)\times\sinh^2\mu(r^{\tilde{d}}-k)^2+\tilde{d}k\cosh^2\mu\ \ r^{\tilde{d}}. \tag{33}$$

The question of whether the black p-branes are stable or unstable under linearized perturbations now becomes whether or not there exists some Kaluza-Klein mass parameter m for which the above coupled equations allow any regular solution outside the event horizon.

Before going further, several points should be mentioned here. One may wonder if metric fluctuations alone with a frozen dilaton perturbation could produce instability. When $f=0$, Eq. (25) gives $-n\chi=\varphi+\psi$, and so from Eq. (30) one has an additional first order equation for φ and ψ. This raises the question of whether or not this equation is consistent with Eqs. (31), (32). It turns out that the inconsistency is proportional to $\sinh^2\mu$. We find therefore that metric fluctuations alone without dilaton perturbation cannot produce linearized instability when black branes are charged, i.e., $\mu\neq0$. It is possible only for uncharged black branes, i.e., $\mu=0$. This property could also be expected by applying the same argument as in the work of Gregory and Laflamme [10]. It is shown in Ref. [10] that for small charge cases $f=f_0+\vartheta(\sinh^2\mu)$ where f_0 is independent of μ. Now it can easily be shown in Eq. (22) that $f_0=0$.

Secondly, Eqs. (31), (32) are invariant under rescalings of $m\to\alpha m$, $r_H\to r_H/\alpha$, $\mu\to\mu$, and $r\to r/\alpha$ for an arbitrary constant α. This scaling symmetry can be easily seen by defining dimensionless variables such as $\rho=r/r_H$ and $\bar{m}=mr_H$, or by observing that such rescalings with $t\to t/\alpha$ and $z^i\to z^i/\alpha$ simply result in an overall constant rescaling of the metric under which field equations are not affected. This symmetry implies that the threshold mass must be inversely proportional to the horizon radius, i.e., $m^{*}\sim 1/r_H$. Accordingly, it suffices to study instability modes just for a single value of r_H.

Since the perturbation equations above are coupled second order and linear in ψ and φ, there are four linearly independent mode solutions in general. The asymptotic solutions of Eqs. (31), (32) at spatial infinity (i.e., $r\sim\infty$) are given by

$$\varphi(r)\simeq e^{\pm mr}u_{\pm}(r) \simeq e^{\pm mr}\left[r^{-(\tilde{d}+1)/2}\mp\frac{(\tilde{d}-1)(\tilde{d}+1)}{8m}r^{-(\tilde{d}+3)/2}+\cdots\right], \tag{34}$$

$$\psi(r)\simeq e^{\pm mr}v_{\pm}(r) \simeq e^{\pm mr}\left[r^{-(\tilde{d}+3)/2}\mp\frac{(\tilde{d}+1)(\tilde{d}+5)}{8m}r^{-(\tilde{d}+3)/2}+\cdots\right] \tag{35}$$

up to overall arbitrary constants. By finding asymptotic solutions in the vicinity of the event horizon as well, i.e., $\rho=1+\varepsilon$, one can have four sets of mode solutions whose asymptotic behaviors are given by

$$\psi_I\sim\begin{cases}\varepsilon+\vartheta(\varepsilon^2),\\ A_I e^{-\bar{m}\rho}v_{-}(\rho)+B_I e^{\bar{m}\rho}v_{+}(\rho),\end{cases}$$

$$\psi_{II}\sim\begin{cases}1+\vartheta(\varepsilon^2),\\ A_{II} e^{-\bar{m}\rho}v_{-}(\rho)+B_{II} e^{\bar{m}\rho}v_{+}(\rho),\end{cases} \tag{36}$$

$$\psi_{III}\sim\begin{cases}\varepsilon\ln\varepsilon+\vartheta(\varepsilon^2\ln\varepsilon),\\ A_{III} e^{-\bar{m}\rho}v_{-}(\rho)+B_{III} e^{\bar{m}\rho}v_{+}(\rho),\end{cases}$$

$$\psi_{IV}\sim\begin{cases}\varepsilon^{-1}-\dfrac{\bar{m}^2}{\tilde{d}}(\cosh\mu)^{8/\Delta}\ln\varepsilon+\vartheta(\varepsilon^2),\\ A_{IV} e^{-\bar{m}\rho}v_{-}(\rho)+B_{IV} e^{\bar{m}\rho}v_{+}(\rho),\end{cases} \tag{37}$$

and correspondingly

$$\varphi_I\sim\begin{cases}\varepsilon+\vartheta(\varepsilon^2),\\ \bar{A}_I e^{-\bar{m}\rho}u_{-}(\rho)+\bar{B}_I e^{\bar{m}\rho}u_{+}(\rho),\end{cases}$$

$$\varphi_{II}\sim\begin{cases}1+\dfrac{\bar{m}^2}{\tilde{d}}(\cosh\mu)^{8/\Delta}\varepsilon+\vartheta(\varepsilon^2),\\ \bar{A}_{II} e^{-\bar{m}\rho}u_{-}(\rho)+\bar{B}_{II} e^{\bar{m}\rho}u_{+}(\rho),\end{cases} \tag{38}$$

$$\varphi_{III}\sim\begin{cases}\varepsilon\ln\varepsilon-\varepsilon+\vartheta(\varepsilon^2\ln\varepsilon),\\ \bar{A}_{III} e^{-\bar{m}\rho}u_{-}(\rho)+\bar{B}_{III} e^{\bar{m}\rho}u_{+}(\rho),\end{cases}$$

$$\varphi_{IV}\sim\begin{cases}-\varepsilon^{-1}-\dfrac{\bar{m}^2}{\tilde{d}}(\cosh\mu)^{8/\Delta}\ln\varepsilon+\vartheta(\varepsilon^2),\\ \bar{A}_{IV} e^{-\bar{m}\rho}u_{-}(\rho)+\bar{B}_{IV} e^{\bar{m}\rho}u_{+}(\rho),\end{cases} \tag{39}$$

where the upper lines are for $\rho\to 1$ and the lower lines for $\rho\to\infty$. All the solutions are simply linear superpositions of these modes. Note that, for the mode IV, $\varphi_{IV}\neq\psi_{IV}$ as one approaches the horizon whereas for the other three modes $\varphi=\psi$.

Now let us turn to the question of boundary conditions for solutions ψ and φ. Since they are linearized perturbations, they should be "small" at any positions outside the event horizon. Thus, at spatial infinity we require that both ψ and φ should be decaying, that is, exponentially decreasing. In the vicinity of the horizon, the metric itself is regular in Kruskal coordinates if, and only if, the perturbation is bounded and $\varphi=\psi$ at the horizon [15]. All modes except for the mode IV above satisfy these conditions. However, it is very important to notice that this is not enough for the regularity of linearized perturbations. Note that the second derivative of the mode III is proportional to $\ln\varepsilon$. Accordingly, some curvature quantity, for instance, the perturbation of the Ricci scalar curvature associated with this mode, becomes singular at the horizon. In other words, the mode solution III produces a curvature singularity even if the metric perturbation itself is regular at the horizon. The mode solutions I and II do not produce any curvature singularity at the horizon. By requiring regular curvature at the horizon in addition, therefore, the boundary conditions we impose for regular perturbations are that both ψ and φ are linear combinations of modes I and II only near the horizon and that they should decay at the spatial infinity.

For various values of the parameters in Eqs. (31), (32), we have searched for regular solutions satisfying the boundary conditions described above by using MATHEMATICA. In more detail, we start from a regular solution at the horizon in the following form:

$$\psi=C\psi_I+E\psi_{II},\qquad \varphi=C\varphi_I+E\varphi_{II}. \tag{40}$$

At spatial infinity, they will become

$$\psi\sim(CA_I+EA_{II})e^{-\bar{m}\rho}v_{-}(\rho)+(CB_I+EB_{II})e^{\bar{m}\rho}v_{+}(\rho), \tag{41}$$

$$\varphi\sim(C\bar{A}_I+E\bar{A}_{II})e^{-\bar{m}\rho}u_{-}(\rho)+(C\bar{B}_I+E\bar{B}_{II})e^{\bar{m}\rho}u_{+}(\rho). \tag{42}$$

By solving Eqs. (31), (32) numerically, one can check if there exists any combination of constants C, E, and $\bar{m}$ such that the coefficients of the exponentially growing parts for both ψ and φ at spatial infinity vanish in Eqs. (41), (42), i.e.,

$$CB_I+EB_{II}=0,\qquad C\bar{B}_I+E\bar{B}_{II}=0. \tag{43}$$

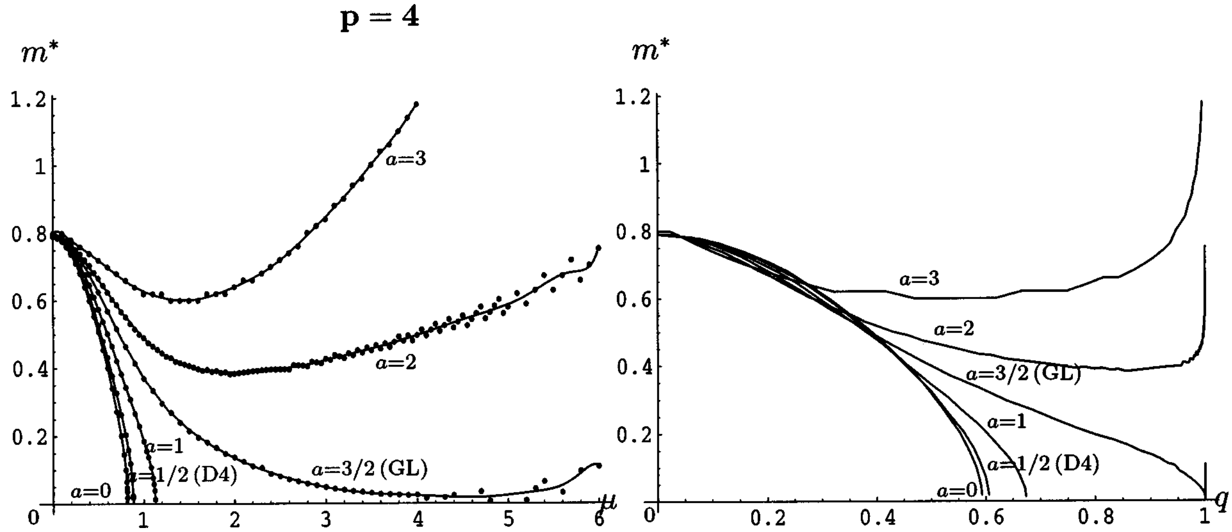

FIG. 1. Behavior of threshold masses for black four-branes in $D=10$ at various values of a with fixed mass density $M=2^5$. $m^* \simeq 1.581/r_H \simeq 0.791$ with $r_H=2$ at $\mu=0$. $\mu_{\rm cr}\simeq 0.818$ (0.8184), 0.881 (0.8814), and 1.125 (1.1254) for the cases of $a=0$, 1/2, and 1, respectively. Here the critical values obtained from the GM conjecture are noted in parentheses. On the right-hand side the same data are plotted in terms of the nonextremality parameter q. $\mu=6$ corresponds to $q\simeq 0.99996$, 0.99995, 0.99991 for $a=3/2$, 2, 3.

However, we have used a different but equivalent method that turned out to be more stable and efficient in actual numerical search. For instance, one needs to check only for varying $\bar{m}$, as will be shown below. Defining

$$P(\bar{m},\mu,a,\tilde{d},D)=B_I\bar{B}_{II}-B_{II}\bar{B}_I, \tag{44}$$

the existence of nontrivial constants C and E satisfying Eq. (43) is guaranteed only if $P=0$. Thus, having given the initial data as in Eqs. (36), (38) near the horizon, we solve the coupled equations Eqs. (31), (32) numerically, and evaluate the quantity $P(\bar{m},\mu,a,\tilde{d},D)$ by using numerical values for $\psi_{I,II}$ and $\varphi_{I,II}$ at sufficiently large ρ. For given other parameters of the black p-branes, we vary the Kaluza-Klein mass $\bar{m}$ only and search for the value $\bar{m}^*$ at which $P=0$. This can be achieved by finding an $\bar{m}^*$ around which the function P above changes its sign. If there exists such an $\bar{m}^*$, this mode is indeed the threshold unstable mode.[5]

Once such $\bar{m}^*=\bar{m}^*(\mu,a,\tilde{d},D)$ is obtained numerically, the threshold mass $m^*=\bar{m}^*/r_H=\bar{m}^*/k^{1/\tilde{d}}$ can be evaluated by fixing r_H or k. Recall that a black p-brane in D dimensions is characterized by two physical quantities Q and M, or equivalently r_H and μ, and one coupling parameter a. We would like to see how the stability behaves as the charge Q of a black brane increases when its mass density M is kept the same.[6] We have chosen M such that the threshold masses at $\mu=0$ for the GL cases reproduce those in Gregory and Laflamme's work [10,11], i.e., $r_0=r_H=2$. The nonextremality parameter can be defined as

$$q\equiv\frac{Q}{Q_{\max}}=\frac{2}{\sqrt{\Delta}}\frac{Q}{M}=\frac{\sinh 2\mu}{(\tilde{d}+1)\Delta/2\tilde{d}+2\sinh^2\mu}. \tag{45}$$

Here $Q_{\max}$ is the maximum charge density allowed for a given mass density M. Since $(\tilde{d}+1)\Delta/2\tilde{d}\geqslant 1$, q is a monotonic function of μ and $q=1$ is the extremal case. Notice that as μ or q increases the horizon radius $r_H=k^{1/\tilde{d}}$ decreases according to Eq. (9) and becomes zero for the extremal brane when M is fixed. For black branes having different mass density M', one can get

$$m^*(\mu;M')=\left(\frac{M}{M'}\right)^{1/\tilde{d}}m^*(\mu;M). \tag{46}$$

The behaviors of some of the threshold masses we have obtained are illustrated in Fig. 1. There we considered black four-branes in $D=10$ and the mass density $M=2^5$. Our numerical results show that threshold masses at $\mu=0$ are nonvanishing for any values of a, implying that all uncharged black four-branes in Eq. (7) are unstable under linearized perturbations associated with Kaluza-Klein massive modes with $0<m<m^*$ as expected. In other words, instability occurs for small perturbations whose wavelengths in spatial worldvolume directions are in the range of λ_* $(=2\pi/m^*)$ $<\lambda<\infty$. The value of the threshold mass at $\mu=0$ can be seen to be independent of a, as it should be, since the a dependence of Eqs. (31), (32) disappears as $\mu\to 0$. Numerically we find $m^*\simeq 0.791$ at $\mu=0$, which agrees with the numerical result obtained by Gregory and Laflamme [10].

[5]In the actual numerical calculations, we used the sign change of $\psi_{II}/\psi_I-\varphi_{II}/\varphi_I$ for $\rho\gg 1$, which becomes equivalent when $B_{I,II},\bar{B}_{I,II}\neq 0$.

[6]In the Gregory-Laflamme work, they fix the size of the outer horizon r_0 in the string frame and vary the inner horizon size r_i. Here $r_0^{\tilde{d}}=r_H^{\tilde{d}}\cosh^2\mu$ and $r_i^{\tilde{d}}=r_H^{\tilde{d}}\sinh^2\mu$.

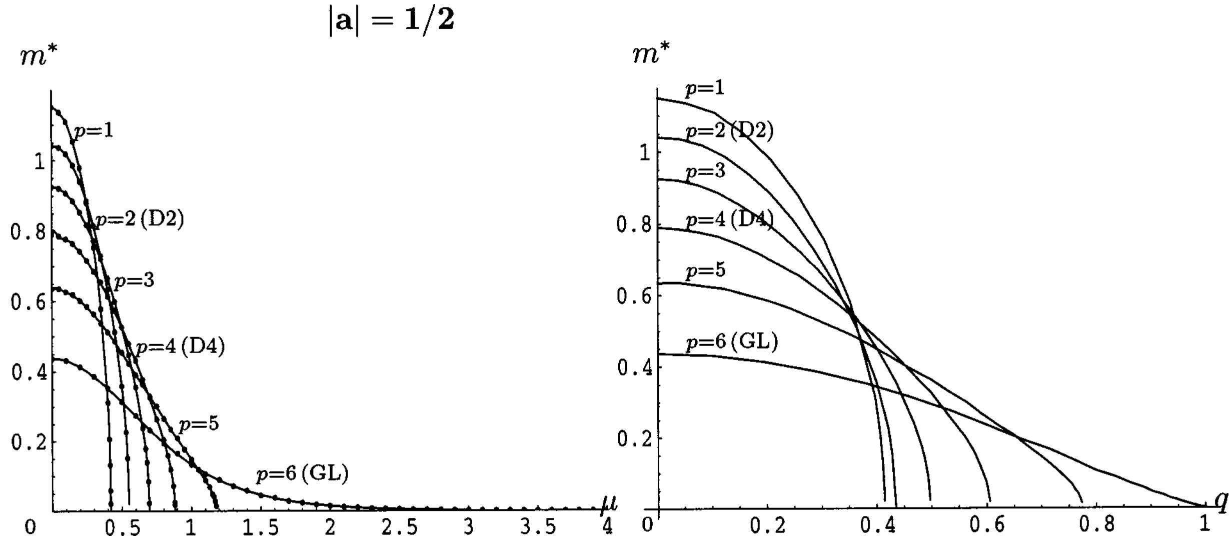

FIG. 2. Behavior of threshold masses for black p-branes with fixed M in $D=10$ in the theory of $|a|=1/2$. At $\mu=0$, $r_H=2$ and $m^*=\overline{m}^*/r_H \simeq 1.153$, 1.044, 0.925, 0.791, 0.635, 0.440 for $p=1,\ldots,6$. Critical values for transition points are $q_{\rm cr}\simeq 0.413$, 0.435, 0.497, 0.606, 0.773, >0.9993 for $p=1,\ldots,6$.

When black branes become charged, the background dilaton field becomes nontrivial (e.g., $\phi\neq 0$) and could play some important role. As can be seen in Fig. 1, the stability behaves very differently depending on the coupling parameter a between the dilaton and gauge fields in the Einstein frame Eq. (2). We summarize its behavior in three types, i.e., $a<3/2$, $a=3/2$, and $a>3/2$.

In the work by Gregory and Laflamme [10,11], it was shown that the black brane instability decreases monotonically as a black brane approaches to its extremal point.[7] It is also noticed that numerical instability starts to occur near the extremal point, but the extremal brane was shown to be stable by analyzing its stability separately [11]. The case for which numerical study was performed explicitly by Gregory and Laflamme corresponds to the GL case in Fig. 2, i.e., $a=-1/2$ and $p=6$. Our result confirms theirs up to $\mu\simeq 4$ or $q\simeq 0.9993$. When $\mu\gtrsim 4$, numerical instability starts to occur as already observed in Refs. [10,11]. For the GL case in Fig. 1 (i.e., $a=3/2$ and $p=4$), the stability behavior is similar at least up to $\mu\simeq 4$ or $q\simeq 0.99799$. Our data for $\mu\gtrsim 4$ are not reliable due to numerical instability.

For the cases of $a=0$, 1/2, and 1, we find that threshold masses become zero at certain finite values of $\mu_{\rm cr}\simeq 0.818$, 0.881, and 1.125, respectively. Thus the Gregory-Laflamme instability disappears as $\mu\to\mu_{\rm cr}$ for such cases and so the corresponding black four-branes become in fact *stable*, at least under spherically symmetric linearized perturbations, for $\mu_{\rm cr}<\mu\leqslant 6$, presumably all the way down to the extremal point (i.e., $\mu\sim\infty$). The same data are plotted in terms of the nonextremality parameter q on the right-hand side. Corresponding critical values for transitions in the stability are $q_{\rm cr}\simeq 0.593$, 0.606, 0.674 for $a=0$, 1/2, 1, respectively. This critical value $q_{\rm cr}$ seems to increase monotonically from a minimum for $a=0$ to unity for the GL case. Therefore one sees that there indeed exist some charged black branes which are stable even far from the extremality.

For the cases of $a=2$ and 3, the threshold masses decrease as black four-branes get charged as before, but interestingly they start to increase again at around $\mu\simeq 2$ (i.e., $q\simeq 0.9972$) and $\mu\simeq 1.5$ (i.e., $q\simeq 0.9950$), respectively. This turning point becomes smaller as a increases. Thus it shows that the instability is *not* always reduced as a black brane gains more charge. Such stability behavior in the presence of charge has never been expected in the literature as far as we know. Moreover, the threshold mass seems to diverge as the black brane approaches the extremal point as can be seen in Fig. 1 up to $q\simeq 0.99995$, 0.99991 for $a=2$, 3. As shown below, however, extremal black branes are stable in these cases as well.

Now let us check the stability of extremal black branes. These cases are expected to be stable since they correspond to BPS ground states in string theory. Our numerical analysis cannot be directly used to show its stability since numerical instability occurs as the extremality is being approached. Instead, we treat the extremal case separately. As one can show explicitly, the equations for linear perturbations in the extremal case are obtained simply by putting $k=0$ and $\mu=\infty$ with $ke^{2\mu}=4c$ fixed in Eqs. (31), (32):

$$\varphi''+\frac{\tilde{d}+1}{r}\varphi'-m^2\left(1+\frac{c}{r^{\tilde{d}}}\right)^{4/\Delta}\varphi=0, \tag{47}$$

[7]By decreasing instability, we mean that the threshold mass is decreasing. Thus, the instability actually shrinks in the parameter range of Ω and m. Consequently, smaller values of Ω imply that instability modes grow less rapidly in time. So in this sense we may say that a black brane having larger charge is less unstable than one having smaller charge in the case considered by Gregory and Laflamme.

$$(r^{\tilde{d}}+c)\psi''+\frac{1}{r}\left[(\tilde{d}+3)r^{\tilde{d}}-4c\left(\frac{\tilde{d}}{\Delta}+\frac{\tilde{d}-3}{4}\right)\right]\psi' -\left\{m^2(r^{\tilde{d}}+c)\left(1+\frac{c}{r^{\tilde{d}}}\right)^{4/\Delta}+\frac{2c\tilde{d}}{\Delta r^2}\left[2\tilde{d}^2+(\tilde{d}+2)\times\left(a^2-\frac{2\tilde{d}^2}{D-2}\right)\right]\right\}\psi-\frac{2c\tilde{d}}{\Delta r^2}\left(a^2-\frac{2\tilde{d}^2}{D-2}\right)\varphi=0. \tag{48}$$

The boundary conditions are simply the regularity of φ and ψ, which are different from those of nonextremal cases, and the asymptotic solutions near the event horizon are also different and cannot be obtained simply by taking the extremal limit of those in Eqs. (36)–(39). This difference comes because we cannot ignore the terms including k at the near horizon as long as k is nonvanishing. Now it is straightforward to see that the only regular solution of Eq. (47) is $\varphi=0$ since the coefficient of φ is negative definite. Similarly, the only regular solution for ψ is zero. Therefore all extremal black branes considered in this paper are stable at least under s-wave fluctuations, although the threshold mass seems to diverge for $a>3/2$ as the extremality is being approached.

For black p-branes with different p ($\neq 4$), the basic stability behavior is essentially the same as above. Only the values of threshold masses and critical μ change slightly. Figure 2 shows our results for various p-branes in the theory of $|a|=1/2$. The mass density is chosen to be $M=(8-p)2^{7-p}$, giving $r_H=2$ at $\mu=0$. If results for other values of a are added, one can see similar patterns as in Fig. 1 for each p-brane. Black p-branes with $p=1,\dots,5$ becomes stable if $\mu\geqslant\mu_{\rm cr}$ whereas the instability of the six-brane persists all the way down to at least $\mu=4$ (i.e., $q=0.9993$).

Now it will be very interesting to see how well the classical stability behavior of black branes described above agrees with that predicted by the local thermodynamic behavior through the GM conjecture. First of all, let us consider critical values $\mu_{\rm cr}$ beyond which black branes become stable classically from being unstable.

p	1	2	3	4	5	6
$\mu_{\rm cr}$ (num.)	0.418	0.549	0.695	0.881	1.178	>4
$\mu_{\rm cr}$ (GM)	0.4186	0.5493	0.6954	0.8814	1.1791	∞

(49)

The table above shows the results for the case of $|a|=1/2$ in Fig. 2. One can see that they are in very good agreement. Other cases we have checked are marked with black dots in Fig. 3. The solid lines are obtained by using Eq. (17) in the GM conjecture. All critical values of μ obtained numerically in our classical perturbation analysis agreed well with those in the GM conjecture. Based on such good agreement, this diagram shows how the classical stability under small perturbations will behave in general. That is, for a black brane with given dimension p for the spatial worldvolume, this brane will be unstable (i.e., $\mu_{\rm cr}=\infty$) under small perturbations if $|a|\geqslant a_{\rm cr}=(D-3-p)/\sqrt{(D-2)/2}$. If $|a|<a_{\rm cr}$, the brane is still unstable for $\mu<\mu_{\rm cr}(a,p,D)$, but becomes stable for $\mu\geqslant\mu_{\rm cr}$. As can be seen in Fig. 3 for $D=10$, the critical values $\mu_{\rm cr}$ increase monotonically as $|a|$ or p increases. Critical values for various black brane solutions of type II supergravity are marked in Fig. 3 explicitly.

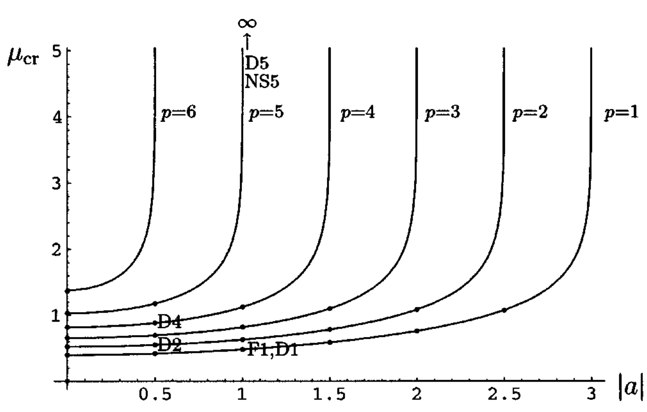

FIG. 3. Critical values of the parameter μ for various black p-branes in $D=10$ at which the threshold mass vanishes, $m^*=0$. The solid lines are obtained from the Gubser-Mitra conjecture and the black dots from our numerical results for several values of a.

IV. DISCUSSION

To conclude, we have investigated the stability of magnetically charged black brane solutions for the low energy string theory in Eq. (1) under small perturbations. It turns out that all uncharged black branes in our consideration are unstable under linearized perturbations. When the black brane becomes charged, however, the stability behavior depends on how strongly the n-form field couples to the dilaton field. In more detail, our results seem to show that, when the coupling is weak enough in the theory in Eq. (2) (e.g., $|a|<a_{\rm cr}$), black branes become stable as they get charged enough (e.g., $\mu\geqslant\mu_{\rm cr}$), even before they reach the extremal point. When the coupling is strong enough (e.g., $|a|\geqslant a_{\rm cr}$), however, black brane solutions of this theory are always unstable. Moreover, the instability starts to increase again as the charge is larger than a certain value and seems to diverge near the extremality. For example, F1, D1, D2, and D4 black branes of type II supergravity could be stable classically for large charge, whereas NS5 and D5 black brane solutions are always un-

stable all the way down to the extremal point. The case of $a=a_{\rm cr}$ is the boundary between these two categories, which is the case Gregory and Laflamme studied, and so the instability monotonically decreases to zero at the extremal point. All extremal black branes are shown to be stable separately. It has also been shown that our results for the classical stability agree well with the qualitative behavior predicted by the local thermodynamic stability through the Gubser-Mitra conjecture. That is, the critical values for $a_{\rm cr}$ and $\mu_{\rm cr}$ obtained numerically in our classical perturbation analysis agree very well with those in Eqs. (16), (17) for the sign change of the specific heat for the black brane regarded as a thermal body.

In the classical perturbation analysis above, we considered only spherically symmetric perturbations. Thus, even if it turned out that there exists no instability mode for a black brane solution with $\mu>\mu_{\rm cr}$ in the analysis above, this does not necessarily mean that such a black brane is stable under small perturbations. There might exist some instability mode when we considered all non-s-wave perturbations as well. However, we give some evidence that the only possible instability mode comes from s-wave fluctuations as follows. Although it was in a different context, one can find in Ref. [24] that the higher angular momentum fluctuations for Schwarzschild black brane backgrounds do not produce unstable modes. As $\mu\to 0$, since the black brane solutions considered in the present paper become Schwarzschild black branes and the metric perturbation equations become completely decoupled from others, one can easily see that the s-mode instability is the only instability for *uncharged* black branes in this paper. Now one can apply the same argument as in Ref. [11]. When charge is added, we observed that there is a stabilizing influence for the s-wave perturbations if $|a| \leqslant a_{\rm cr}$. Therefore, it is not expected that higher angular momentum modes will exhibit instability in *charged* black branes since they do not give it even for the uncharged case.

The spatial worldvolume is assumed to be noncompact in the description above. Consequently, the KK mass spectrum is continuous since there are translational symmetries in spatial worldvolume directions. What will happen to the stability behavior if the worldvolume is compactified with a scale L? In this case, the KK mass becomes discrete and has a minimum mass, e.g., $m_{\rm min}=2\pi/L$. Thus, KK masses smaller than the minimum (i.e., $0\leqslant m<m_{\rm min}$) are not allowed due to the compactification. Accordingly, even if the threshold masses obtained by solving the perturbation equations for black branes are nonvanishing, such black branes are actually stable provided that $m^*<m_{\rm min}$, or equivalently that the compactification scale is small enough (i.e., $L<2\pi/m^*$) [9,10,25].

Although the critical values for the transition points are in good agreement between the classical stability analysis and the local thermodynamic stability through the GM conjecture, there are some other aspects that might be in disagreement. For instance, one can see in cases of $|a|\leqslant a_{\rm cr}$ that the magnitude of the specific heat increases whereas the black brane instability decreases as the charge increases with fixed mass density. Moreover, the specific heat is divergent at the transition point (i.e., $\mu=\mu_{\rm cr}$). In the classical perturbation analysis, however, there does not seem to exist any singular behavior around $\mu=\mu_{\rm cr}$. For given μ in Fig. 1, we found that the threshold mass increases as a increases. One might wonder if the magnitude of the specific heat also increases. One can see that it actually decreases monotonically, even for a fixed Q/M instead of a fixed μ. Maybe these discrepancies in some details of the stability behavior simply indicate that one needs to find some appropriately modified thermodynamic quantity, other than the specific heat, in order to compare other details between classical and thermodynamic stability analyses for black branes.

Finally, it would be very interesting to understand why the instability starts to increase again as the extremality is being approached for black branes in theories of $a>a_{\rm cr}$. Further work is required.

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