Geometric dilaton gravity and smooth charged wormholes

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A particular type of coupling of the dilaton field to the metric is shown to admit a simple geometric interpretation in terms of a volume element density independent from the metric. For dimension $n=4$ two families of either magnetically or electrically charged static spherically symmetric solutions to the corresponding Maxwell-Einstein-dilaton field equations are derived. Whereas the metrics of the "magnetic" spacetimes are smooth, asymptotically flat, and have the topology of a wormhole, the "electric" metrics behave similarly as the singular and geodesically incomplete classical Reissner-Nordström metrics. At the price of losing the simple geometric interpretation, a closely related "alternative" dilaton coupling can nevertheless be defined, admitting as solutions smooth "electric" metrics.

I. INTRODUCTION

Einstein's classical theory of gravity, based on a metric, has so far passed all experimental tests concerning the motion of bodies or the deflection of light, with ever increasing precision (cf. Will [1]). But there are some disturbing limitations of a more formal nature, such as generic singularities, when not allowing "exotic matter." This is most evident in the gravitational collapse of stars and in the early phases of the Universe, notably at the "big bang singularity." In fact, it seems that we have come to live with this unsatisfactory situation as a necessary consequence of a classical description. These problems can be attributed either to Einstein's theory itself or to the inadequacy of the matter model used, or to both. In the first case, singularities seem to be a generic feature (somewhat attenuated by the expectation that they will in general be hidden behind an event horizon), unless some other "pathologies" are accepted, such as exotic matter or closed timelike lines. In the other case, for example, the apparent accelerating cosmic expansion has led to a generalization involving a "variable cosmological constant" Λ ("quintessence," "k essence"). Previously, a generalization based on a "variable gravitational constant" G has been considered, well known as Brans-Dicke theory. Also, in order for inflation to work, some specific modifications must be made. In all these generalizations, an additional real scalar field, serving the particular purpose, plays a fundamental role. Except in exceptional cases, the singularity problem remains. It is, however, generally expected that a future reconciliation of gravity and quantum theory will lead to a unique theory without the above mentioned problems. Attempts in this direction can be seen in the currently extensively studied "string-inspired cosmologies," which are based on some low energy limit of string theory and an appropriate reduction to dimension $n=4$. There a scalar field multiplying (as an exponential) the Ricci scalar and/or the metric plays a prominent role, which is interpreted physically as the dilaton field. A big advantage of these approaches is the fact that equations closely related to Einstein's are a necessary consequence.

Our aim is to show that already on a classical level, by properly including the dilaton scalar, some of the singularity problems can be avoided. Firmly based on standard differential geometry and only loosely inspired by the dilaton scalar of string theory, we formulate a theory of dilaton gravity. As criteria for the soundness of our approach, both the formulation of a minimal coupling scheme and the existence of nontrivial geodesically complete and asymptotically flat solutions of the corresponding field equations are taken.

We will proceed as follows. In Sec. II a particular form of the coupling of a scalar field ϕ (called "dilaton") to the metric tensor g_{ik} of a spacetime is proposed, which admits a straightforward and unique geometric interpretation in terms of an independent *volume element density*. A dilatonic coupling scheme is formulated in order to accomodate additional nongeometric fields. In Sec. III, the field equations corresponding to a Maxwell-Einstein-dilaton Lagrangian are derived, where also their "Einstein form" is given. In Sec. IV a family of magnetically charged static spherically symmetric solutions is derived, closely related to the well-known "string-inspired" charged black hole solutions of Garfinkle, Horowitz, and Strominger [2]. These metrics are shown to be geodesically complete (in fact, smooth) and asymptotically flat, each of them containing a *wormhole*, with no exotic matter being involved. The corresponding family of electrically charged solutions consist of singular metrics which have either an event horizon or exhibit a naked singularity. An "alternative" nongeometric coupling is shown to be, however, possible, admitting a family of smooth electrically charged solutions. In Sec. V, for the proposed geometric coupling the equivalence principle is shown to be fulfilled, in the sense that uncharged point particles still move on geodesics. In order to derive expressions for the mass and some other significant parameters, the parametrized post-Newtonian approximation is invoked. In terms of the basic parameters m, β, and γ, it is shown that the derived solutions must be considered as viable with respect to present-day astronomical empirical data. For the realm of elementary particles, it is shown that for the smooth magnetic and electric wormhole solutions, significant effects could be expected at distances

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of roughly the order of the classical electron radius. For the wormhole solutions it is shown that their mass parameter has zero as a lower bound. Also an explanation of the "repulsive" character of the dilaton involved in the wormhole solutions is given. The concluding Sec. VI formulates the main conclusions and points to some important open questions.

II. GEOMETRIC DILATON COUPLING

A. Volume geometry: Hodge duality

As is well-known (cf., e.g., Abraham, Marsden, and Ratiu [3] for the closely related concept of a *volume manifold*), a volume element density (VED) is geometrically a nondegenerate smooth n-form density volume, that is, under orientation-preserving coordinate transformations it behaves as a conventional n form, whereas under orientation reversion it gets an extra factor -1. Such a VED already allows one to invariantly express the divergence div v of a vector field v as (div v) vol$:=d(v \cdot$ vol$)$, where the dot denotes the contraction of a differential form by a vector field. This definition is well known in Hamiltonian mechanics, where it plays a major role. Also the Gauss integral theorem can already be formulated. Note that no metric has been involved so far.

It makes sense to speak of a "positive" VED, and any two such VEDs differ only by a positive function λ. Assuming that we also have a nondegenerate metric g_{ik}, we can therefore always write for a general VED, vol $= |g|^{1/2} e^{\phi} dx^1 \wedge \cdots \wedge dx^n$, where we have conveniently set $\lambda = e^{\phi}$, with some scalar function ϕ.[1] This functional form ensures positivity of λ, when ϕ is continuous. Of course we could also have chosen any other smooth monotone positive function of ϕ, but this would not introduce anything new, as effectively only the "dilaton factor" e^{ϕ} matters. The scalar field ϕ thus represents an *essentially unique new geometrical degree of freedom.*

Its occurrence in the form of the factor e^{ϕ} strongly reminds one of the dilaton factor appearing in the reduced Lagrangians for the low energy limit of string theory (LELST).[2]

For a volume manifold with a nondegenerate metric, the notion of the *Hodge dual* of a differential form has to be slightly generalized. Recall that the dual $\star F$ of a plain p-form F is the result of the following construction, given with respect to some coordinate basis:

$$F_{i_1 \cdots i_p} \rightarrow F^{j_1 \cdots j_p} := g^{i_1 j_j} \cdots g^{i_p j_p} F_{i_1 \cdots i_p} \rightarrow \star F_{j_{p+1} \cdots j_n}$$

$$:= \text{vol}_{j_1 \cdots j_p j_{p+1} \cdots j_n} F^{j_1 \cdots j_p}$$

$$\equiv |g|^{1/2} e^{\phi} \varepsilon_{j_1 \cdots j_p j_{p+1} \cdots j_n} F^{j_1 \cdots j_p}, \quad (1)$$

where ε denotes the permutation symbol. The plain p-form F

is thus mapped to a $(n-p)$-form density $\star F$. As this map is one-to-one, it can be inverted to map a p-form H density to a plain $(n-p)$-form $\star^{-1}H$. Unfortunately, the nomenclature of the two different types of differential forms is far from standard[3] and so we will adhere to de Rham [4], denoting the forms we called "plain" with *even type* and the "form densities" with forms of *odd type*.

Consequently, we now define the generalized Hodge dual of any form F (even or odd) as

$$*F = \begin{cases} \star F, & F \text{ even}, \\ \star^{-1}F, & F \text{ odd}. \end{cases} \quad (2)$$

This definition makes the duality operator trivially idempotent, $*^2 F = F$.[4]

Alternatively, we could also define the Hodge duality $F \rightarrow *F$ as the *unique isomorphism* from the vector space of even (odd) p forms to the "dual" vector space of odd (even) $(n-p)$ forms, whose restriction to even forms gives

$$F \wedge *G = (F,G) \text{ vol}, \quad F,G \text{ even}. \quad (3)$$

Here the round bracket denotes the scalar product of forms based on the Riemann metric. As a consequence, we have for odd forms the corresponding relation

$$*F \wedge G = (*F,*G) \text{ vol}, \quad F,G \text{ odd}. \quad (4)$$

Based on the Hodge duality for differential forms, the operators for the *divergence* δ and the *Laplacian* Δ for differential forms can now be defined as $\delta := *d*$ and $\Delta := d\delta + \delta d$, where d denotes the operator of *exterior derivative*, which is valid for forms of any even/odd type.[5]

Why this insistence on differential forms? The main reason is that the Lagrangian scalar density is geometrically more properly understood as an odd form of maximal degree n, the energy momentum tensor being a covector-valued odd $(n-1)$ form. Of course, the electromagnetic Maxwell field, to be extensively used later together with its Hodge dual, is to be understood as an even 2 form. Also we need the divergence of a vector field and of a two form, as well as the Laplacian of a scalar field.

B. General dilaton coupling

Be it from a five-dimensional Klein-Kaluza (KK) reduction, or from a LELST compactified to $n=4$, the Lagrangians have the generic form[6]

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[1] As is common practice in the physics literature, we will denote the corresponding coefficient of the n form $dx^1 \wedge \cdots \wedge dx^n$ as "scalar density," e.g., the Lagrangian $\mathcal{L}$.

[2] Although there the equivalent factor $e^{-2\phi}$ seems to be more natural.

[3] Also the following designations are common: pseudoforms, twisted forms, Weyl tensors, and oriented tensors.

[4] Note that when distinguishing even and odd type forms, $\star^2$ does not make sense.

[5] The more precise definition of δ by de Rham introduces an extra factor -1, depending on the dimensionality of the manifold and the signature of the metric.

[6] We use the conventions of Misner, Thorne, and Wheeler [7] throughout, the squares denoting the conventional metric scalar product.

$$\mathcal{L} = |g|^{1/2} e^{\alpha\phi} [R - \beta(\nabla\phi)^2 - e^{\gamma\phi} F^2]. \tag{5}$$

The constant parameters α, β, and γ depend on the particular higher-dimensional base theory and the chosen reduction. Of course, from a LELST many more scalar fields and antisymmetric tensor fields of different degrees (''moduli'') will appear, but we keep only the dilaton scalar ϕ and a rank-2 antisymmetric tensor field F, later to be interpreted as the Maxwell field two form.[7] R denotes the omnipresent Riemann curvature scalar of general relativity. We will refer to this class of Lagrangians as Maxwell-Einstein-dilaton (MED) Lagrangians.

Often, more general Lagrangians are studied, containing free functions (cf. Esposito-Farése and Polarski [5], and references therein), but our choice is already general enough to cover the most important applications as special cases. In particular, the class of Bergmann-Wagoner Lagrangians extensively studied for about 30 years by Bronnikov *et al.* (cf. [6]) should be mentioned, having the general form $\mathcal{L} = |g|^{1/2} [f(\phi)R + h(\phi)(\nabla\phi)^2 - F^2]$ (in particular, with $h = 1$ and $f = 1 - \xi\phi^2$, where ξ is a constant parameter). Although diverse charged wormhole solutions have been obtained, they all violate some of the energy conditions. Note that the class of MED Lagrangians considered in our work essentially differs from Bronnikov's class, which completely excludes ''string-inspired'' Lagrangians.

As convenient for dimensionally reduced Lagrangians, the relativistic gravitational constant κ (as well as any factor $1/2$) is assumed to be absorbed into F^2. For example, the KK reduction leads to $\alpha = 1$, $\beta = 0$, $\gamma = 2$, whereas a typical LELST reduction has $\alpha = -2$, $\beta = -4$, $\gamma = 0$. With a scalar defined by $\Phi := e^\phi$, we can also deduce the scalar-tensor Brans-Dicke Lagrangian from this form, with parameters $\alpha = 1$, $\beta = \omega \neq 3/2$, $\gamma = -1$. Similarly, for $\beta = 3$ we get a conformal scalar coupling.

However, all these MED Lagrangians can be transformed *a modulo* a trivial divergence by a Weyl conformal transformation $g'_{ik} = g_{ik} e^{\alpha\phi}$ into a so-called *Einstein frame*, characterized by $\alpha' = 0$. For dimension $n = 4$, this results in $\beta' = \beta + 3/2\,\alpha^2$, $\gamma' = \gamma + \alpha$. In such a frame, formally the conventional Einstein Lagrangian is obtained, with a massless Klein-Gordon (KG) field ϕ and a Maxwell field F gravitationally coupled with an effective coupling ''constant'' $e^{(\alpha+\gamma)\phi}$. The scalar field ϕ is ghost-free, i.e., *nonexotic*, if and only if $\beta' \geq 0$.

More generally, for any dimension n, a dilaton-based conformal transformation of the metric, $g'_{ik} = g_{ik} e^{\lambda\phi}$ (λ a constant parameter), leads to the following transformation behavior of the parameters of the MED Lagrangian density:

$$\alpha \to \alpha' = \alpha - \lambda,$$

$$\beta \to \beta' = \beta + (n-1)(n-2)/4\,(\alpha^2 - \alpha'^2), \tag{6}$$

$$\gamma \to \gamma' = \gamma + \lambda.$$

Although the causal structure is not altered as long as ϕ is continuous,[8] some basic metric-based relations (e.g., length and ''straightness'') are not. For the physical interpretation therefore some conformal frame has to be taken as the fundamental one, characterized by a particular form of the Lagrangian density. This will depend on the particular coupling to other fields, in particular, to point masses. However for ''string-inspired'' dilaton theories most authors agree that an Einstein frame should be taken, primarily justified by the availability of the familiar interpretatory apparatus of the classical Einstein gravity.[9]

C. Geometric dilatonic coupling and minimal coupling scheme

Let us now introduce a particularly simple coupling, characterized by the parameters $\alpha = 1$, $\beta = 0$, $\gamma = 0$. Evidently, the dilaton enters the corresponding MED Lagrangian in a mode which can be interpreted geometrically in terms of a general metric-based volume element density, as previously described. The Lagrangian simply becomes $\mathcal{L} = |g|^{1/2} e^\phi (R - F^2)$. Let us call such a coupling a *geometric dilaton coupling* (GDC) and the corresponding conformal frame a *G frame* (*G* for ''geometric'').

The particularly simple form of the GDC suggests the following *minimal geometric dilatonic coupling scheme* (MGDCS): assuming we have already a Lagrangian density without dilaton and satisfying the prerequisites of general covariance, $\mathcal{L}_0 = |g|^{1/2} L$, we get a GDC by just correcting any occurence of the Riemann volume element density $|g|^{1/2}$ by the dilaton factor e^ϕ: $\mathcal{L} := \mathcal{L}_0 e^\phi = |g|^{1/2} e^\phi L$, even when it explicitly occurs inside L.

Taking the standard Maxwell-Einstein Lagrangian as the prototypical example, we get as a result of the MGDCS the GDC Lagrangian:

$$\text{MGCDS:}\quad \mathcal{L} = |g|^{1/2}(R - 1/2\,F^2) \mapsto \mathcal{L}$$
$$= |g|^{1/2} e^\phi (R - 1/2\,F^2). \tag{7}$$

Similarly, a massive point particle could also be incorporated, to give $\mathcal{L} = |g|^{1/2} e^\phi R - m(\dot{x}^2)^{1/2} \delta_T$, where the Dirac delta distribution is supported by the world-line T of the particle, given by $T: \tau \mapsto x^i(\tau)$. Note that the mass term does not acquire a dilaton factor, as it does not contain the factor $|\det g|^{1/2}$.

III. GDC FIELD EQUATIONS

The field equations derived from the geometrically coupled Maxwell-Einstein-dilaton Lagrangian $\mathcal{L} = |g|^{1/2} e^\phi (R - 1/2\,F^2)$ are, up to a factor $|g|^{1/2} e^\phi$,

$$G_{ik} - \sigma^2 \Theta'_{ik} - \sigma \Theta''_{ik} = M_{ik}, \tag{8}$$

$$R = 1/2\,F^2, \tag{9}$$

$$0 = \operatorname{div} F, \tag{10}$$

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[7]Sometimes ''potential'' terms $V(\phi)$ also appear or are introduced ''by hand.''

[8]Which is, however, not in general the case.

[9]Compare, e.g., Gasperini and Veneziano [8].

where

$$M_{ik} := F_{ir}F_{ks}\,g^{rs} - 1/4\,F^2\,g_{ik}, \tag{11}$$

$$\Theta''_{ik} := \nabla_i\nabla_k\phi - \nabla^2\phi\,g_{ik}, \tag{12}$$

$$\Theta'_{ik} := \nabla_i\phi\nabla_k\phi - (\nabla\phi)^2\,g_{ik}, \tag{13}$$

and

$$\sigma := (n-2)/2. \tag{14}$$

Here div F denotes the dilaton-generalized divergence of a 2 form, which in a coordinate base can also be written as

$$(\mathrm{div}\,F)^i := |g|^{-1/2}\,e^{-\phi}\,\partial_j(|g|^{1/2}\,e^{\phi}\,F^{ij}) \equiv \nabla_j F^{ij} + \phi_j F^{ij}. \tag{15}$$

Also we have assumed that F has a gauge potential, $F = dA$, with respect to which the corresponding variation is performed. Combining Eq. (9) with the trace of Eq. (8) and assuming $n \geq 2$, we get an explicit *dilaton equation* for $\Phi := e^{\sigma\phi}$,

$$\nabla^2\Phi - 1/(n-1)\,R\,\Phi = 0. \tag{16}$$

Remarks

(i) Neither the dilaton scalar ϕ nor the dilaton factor e^{ϕ} do explicitly appear in the primary field equations (8)–(10), except through their derivatives, this invisibility of the "effective gravitational coupling constant" e^{ϕ} underlines the geometric character of the theory.

(ii) The dilaton scalar does *not* couple to the trace (which here vanishes for $n=4$) of the energy momentum tensor as in the Brans-Dicke theory, but to the independent scalar F^2.

(iii) The tensor Θ' has almost the form of the energy momentum tensor for a massless Klein-Gordon field, except for a factor -1 instead of $-1/2$.

(iv) The dilaton equation is purely geometric, as it does not contain the Maxwell field; for dimension $n=6$ it reduces to the *conformal wave equation*[10] $\nabla^2\Phi - 1/5\,R\,\Phi = 0$.

Transformed to an E frame (and asssuming $n \geq 2$), these field equations can, however, be reduced to the more familiar looking locally equivalent forms[11]

$$G_{ik} = (n-1)/2\,\sigma\,\Theta_{ik} + \lambda\,M_{ik}, \tag{17}$$

$$\sigma\,\nabla^2\phi = 1/2(n-1)\,\lambda\,F^2, \tag{18}$$

$$\nabla_i(\lambda\,F^{ij}) = 0. \tag{19}$$

Here $\lambda := e^{\phi}$, and Θ is the Klein-Gordon energy momentum tensor,

$$\Theta_{ik} := \nabla_i\phi\nabla_k\phi - 1/2\,(\nabla\phi)^2\,g_{ik}. \tag{20}$$

Unfortunately, in Eq. (17) the KG term Θ does not couple with the same factor λ as the constant factor of the M term, thus making a conventional interpretation in terms of a "variable effective gravitational coupling constant" $\lambda := e^{\phi}$ somewhat problematic.

The merit of the E frame formulation lies, however, in the fact that for this most frequently used frame, some solutions to slightly more general couplings are already known. We will in the following take advantage of this.

IV. STATIC SPHERICALLY SYMMETRIC SOLUTIONS

A. Smooth GDC solutions of magnetic type

In the following, we will deal exclusively with the case $n=4$ ($\sigma=1$). The MED field equations in the E frame then reduce to

$$G_{ik} = 3/2\,\Theta_{ik} + \lambda\,M_{ik}, \tag{21}$$

$$\nabla^2\phi = 1/6\,\lambda\,F^2, \tag{22}$$

$$\nabla_i(\lambda\,F^{ij}) = 0. \tag{23}$$

We want to obtain static spherically symmetric (SSS) solutions to the GDC field equations. This is most easily done staying in the E frame, where SSS solutions to slightly more general equations, depending on an extra parameter a, have been found by Garfinkle, Horowitz, and Strominger [2] (GHS).[12] They start from the following Lagrangian density:

$$\mathcal{L} = |g|^{1/2}[R - 2(\nabla\phi)^2 - 1/2\,e^{-2a\phi}\,F^2], \tag{24}$$

where we trivially rescaled their F with a factor $1/\sqrt{2}$ in order to have a closer correspondence with the classical Maxwell field. Our G-frame Lagrangian can evidently be conformally mapped to the GHS Lagrangian with the particular choice of the parameter $a = \pm 1/\sqrt{3}$.

Their general "magnetic-type" solution can then be written as[13]

$$ds^2 = -\lambda^2 dt^2 + \lambda^{-2} dr^2 + R^2 d\Omega^2, \tag{25}$$

$$\lambda^2 := X_+ X_-^{(1-a^2)/(1+a^2)}, \tag{26}$$

$$R^2 := r^2 X_-^{2a^2/(1+a^2)}, \tag{27}$$

$$e^{-2a\phi} = X_-^{2a^2/(1+a^2)}, \tag{28}$$

$$F = F_m := q\sin\theta\,d\theta\wedge d\varphi, \tag{29}$$

where

$$X_+ := 1 - r_+/r, \quad X_- := 1 - r_-/r. \tag{30}$$

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[10]Note that the R factor is dimension-dependent (cf. Wald [9], Appendix D).

[11]For notational reasons, we do not differentiate between the G frame g_{ik} and the E frame $g'_{ik} := e^{\phi}g_{ik}$.

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[12]In the context of a broader framework, equivalent solutions have been found earlier by Gibbons and Maeda [10] (see also Horowitz [11], relating these solutions to the GHS solution).

[13]In accordance with the notation of GHS, λ and R here refer only to Eq. (25).

The parameters q, r_+, r_- are, however, restricted by[14]

$$q^2 = 2 r_+ r_- /(1+a^2). \tag{31}$$

Mapping the $a^2 = 1/3$ GHS solution back to the G frame, we then get the following "magnetic" family of solutions:

$$ds^2 = -X_+ dt^2 + (X_+ X_-)^{-1} dr^2 + r^2 d\Omega^2, \tag{32}$$

$$e^\phi = X_-^{1/2}, \quad F = F_m := q \sin\theta\, d\theta \wedge d\varphi, \tag{33}$$

where

$$q^2 := 3/2\, r_+ r_-. \tag{34}$$

However, in these coordinates the metric is *not* regular at $r = r_-$, and the dilaton scalar does not even exist for $r < r_-$. Introducing a new coordinate ρ (solution of $d\rho/dr = \varepsilon X_-^{-1/2}$) by means of

$$\rho(r) = \varepsilon r_- (\xi^{1/2}(\xi-1)^{1/2} + \ln(\xi^{1/2} + (\xi-1)^{1/2})), \quad \xi := r/r_-, \tag{35}$$

removes these drawbacks. The parameter $\varepsilon = \pm 1$ characterizes the two ρ branches joined at $\rho = 0$ and mapping to the single r range $r \geq r_- > 0$. The inverse function $r(\rho)$ is smooth in ρ. In these new cordinates (which now properly include the locus $r = r_-$, respectively, $\rho = 0$), the resulting metric

$$ds^2 = -X_+ dt^2 + X_+^{-1} d\rho^2 + r^2(\rho) d\Omega^2 \tag{36}$$

is nondegenerate[15] and can be shown to be *smooth* in a neighborhood of $\rho = 0$ which does not include $\rho = \rho(r_+)$.[16] Also, e^ϕ becomes smooth there, and the expression for the Maxwell field F_m remains unchanged (and smooth).

As the metric is symmetric under $\rho \rightarrow -\rho$, and there is now always a two sphere with minimal area $A = 4\pi r_-^2$ [corresponding to a "radius" $r(0) = r_-$], the geometric interpretation is that of a *wormhole* with throat at $\rho = 0$. This notion of wormhole, based on a *local reflection symmetry*, is, however, different and more general than the usual one, which only allows wormholes with *timelike* throats (e.g., Visser [12], compare also Hayward [13]).

If $r_+ \geq r_-$ the locus $r = r_+$ can be shown to be a *regular event horizon* (even in the "degenerate" case $r_+ = r_-$), and the metric can be smoothly extended through it by standard procedures (e.g., by using Eddington-Finkelstein coordinates). For $r_+ < r_-$ there is no black hole and the complete metric can be expressed in the single coordinate chart given above. The wormhole topology (connecting universes with their own asymptotic regions) is common to all metrics of this family and can in this sense be considered as *generic* among the class of SSS solutions.

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[14]Despite the suggestive notation, it is *not* required that $r_+ \geq r_-$.

[15]Except for the usual easily removable degeneracy at the z axis.

[16]Note the close similarity of the form of this metric with Schwarzschild's, to which it reduces for $r_- = 0$.
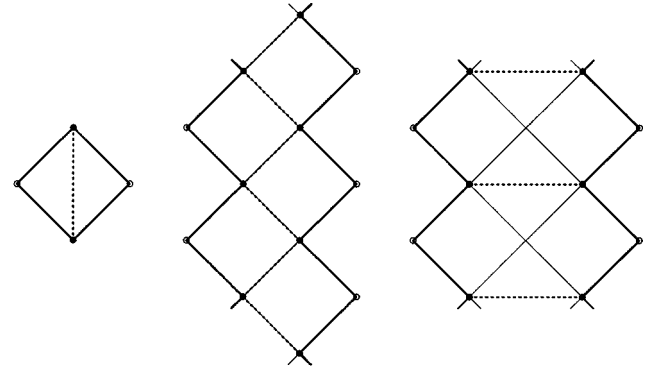
Type I: $r_- > r_+ \geq 0$ Type II: $r_- = r_+ > 0$ Type III: $r_+ > r_- > 0$
($q^2 > 6 m^2 > 0$, and   ($q^2 = 6 m^2 > 0$)   ($0 < q^2 < 6 m^2$)
$q = m = 0, r_- > 0$)

FIG. 1. Carter-Penrose diagrams for the extended wormhole metrics. Thick lines: null infinity, thin lines: event horizon, dashed lines: wormhole throat, circles: $i^\pm, i^0$.

The Carter-Penrose (CP) diagrams corresponding to these extensions fall into the three distinct types I: $r_- > r_+ \geq 0$, II: $r_- = r_+ > 0$, III: $r_+ > r_- > 0$, and are shown in Fig. 1 (where we also used the more physical characterization by means of charge q versus mass m, to be justified later). Depending on the type of the solution, the throat of the wormhole is *timelike* for type I, *null* for type II (coinciding with the event horizon), and *spacelike* for type III.

B. Gauge potential for the magnetic GDC solution

Although the question of an appropriate gauge potential is most often ignored (being trivial in the "electric" case), we will now exhibit a smooth potential in the sense of a $U(1)$-gauge theory. The existence of such a potential makes the smooth SSS solution complete. Consider the $u(1)$-valued (i.e., purely imaginary)

$$\tilde{A}_\pm := -in/2 (\cos\vartheta \mp 1) d\varphi, \tag{37}$$

where the upper sign refers to the upper hemisphere $\vartheta \neq \pi$ and the lower sign to the lower hemisphere $\vartheta \neq 0$. Evidently, $\tilde{F} := d\tilde{A} = in/2 \sin\vartheta\, d\vartheta \wedge d\varphi$. The transition function for the potential in the overlap of the two hemispheres is given by $S := e^{in\varphi}$: $A_+ = A_- + S^{-1} dS$. For consistency $n \in \mathbf{N}$ must hold (cf. Göckeler and Schücker [14] for more details). Reverting to the corresponding real field, $iF := \tilde{F}$, this amounts to $q = n/2$. Taking properly into account the terms appearing in the "gauge derivative" for an electrically charged particle in the field of a magnetic monopole, $\nabla = \partial + ie/\hbar A$, we obtain *Dirac's quantization condition*: $pq/\hbar = n/2$. For $n = 1$ and $p = e$ the minimal magnetic charge is $g = 1/2 e/\hbar \approx 68.5 e$, giving the factor $(g/e)^2 \approx 4.7 \times 10^3$ needed in Sec. V.

C. Singular GDC solutions of electric type

As already shown by Garfinkle, Horowitz, and Strominger [2], from a magnetically charged solution (g, ϕ, F) of their E frame field equations, an electrically

charged one can formally be obtained by taking $(g, -\phi, *F)$, where $*F$ is the (generalized) Hodge dual of F.[17] But then, the transformation back to the G frame inevitably leads to a metric immanently degenerate at $r = r_-$

$$ds^2 = X_-(-X_+ dt^2 + X_+^{-1} d\rho^2 + r^2(\rho) d\Omega^2), \quad (38)$$

which is the image of the degenerate conformal mapping with factor X_- of the smooth "magnetic" metric considered before.[18] Dilaton factor and Maxwell field are given by

$$e^\phi = X_-^{-1/2}, \quad F = F_e := p/r^2 X_-^{1/2} d\rho \wedge dt, \quad (39)$$

where $p^2 = 3/2\, r_+ r_-$. A gauge potential for F is

$$A_e := \varepsilon p/r_- X_-\, dt. \quad (40)$$

Note that no charge quantization is involved and that F vanishes at the singularity of the metric, $\rho = 0$ ($r = r_-$), in fact, both F and its potential are smooth there.

If $r_+ > r_-$ and $r_+ > 0$, the locus $r = r_+$ is a regular event horizon, hiding the spacelike singularity. For the "degenerate" case $r_+ = r_-$, there is still a horizon, but the singularity becomes timelike. For $r_+ < r_-$, the singularity is timelike and even naked. The corresponding CP diagrams agree with those of the Reissner-Nordström family of solutions, except for the case $r_+ > r_-$, where the diagram agrees with Schwarzschild's, which has a spacelike singularity.

D. Alternative dilaton coupling

A closer look at the general GHS solution reveals that only the choice $a^2 = 1/3$ allows one to remove the offending common X_- factor from the SSS metric by an appropriate conformal transformation. This again is given by the same factor $e^{-2a\varphi}$, as from the G frame to the E frame, resulting in the "alternative" dilaton coupling (ADC) Lagrangian

$$\mathcal{L} = |g|^{1/2}(e^{-\phi} R - 1/2\, e^\phi F^2). \quad (41)$$

Evidently, it *cannot* directly be interpreted in terms of a volume manifold and corresponding coupling scheme. The field equations corresponding to this alternative Lagrangian are then

$$G_{ik} - \Theta'_{ik} + \Theta''_{ik} = e^{2\phi} M_{ik}, \quad (42)$$

$$R = -1/2\, e^{2\phi} F^2, \quad (43)$$

$$0 = \text{div}\ (e^{2\phi} F), \quad (44)$$

with the corresponding dilaton equation derived from them,

$$\Box \phi = -1/6\, e^{2\phi} F^2. \quad (45)$$

Here the divergence and the Laplacian are defined based on a volume element density $|g|^{1/2} e^{-\phi}$. Except for the manifest appearance of dilaton factors the essential change is a sign reversal in the dilaton equation.

This shows that smooth electric wormhole solutions are possible when sacrificing the geometrical interpretation. Their metrics agree with those of the smooth GDC wormholes. The coupling is again ghost-free and the material part still obeys the energy conditions. Among the class of MED Lagrangians considererd, it is the only Lagrangian with smooth SSS solutions.

An "alternative" coupling scheme, involving arbitrary nongeometrical fields, could, however, be formulated tentatively as follows: (i) apply the standard GDC scheme, (ii) denote the map from the G frame to the E frame by $\mathcal{E}$, and (iii) define the "alternate" A frame (and corresponding Lagrangian) as the image of the G frame under the map $\mathcal{E}^2$.

When including point masses, such a coupling scheme would lead to nongeodesic behavior for their trajectories (see Sec. V). This could be considered as a drawback. But the main objection against this coupling is of course that it has been deliberately constructed so as to possess smooth electrically charged SSS solutions, and also its lack of any direct geometric interpretation.

E. Comparison with other SSS solutions

As already noted by Garfinkle, Horowitz, and Strominger, *all* the nontrivial E-frame metrics of the GHS family of solutions are either geodesically incomplete and/or singular, with the exception of the "cornucopion" metric, which is an extreme solution for $a = 1$ interpreted in the string frame.

To my knowledge all other static spherically symmetric solutions directly or indirectly related to Maxwell-Einstein-dilaton gravity violate some of the energy conditions, and must be considered as classically "unphysical."[19] Therefore we will not dicuss them here.

Unfortunately, general enough existence or no-go theorems do not yet exist, save for particular couplings and the corresponding conformal frames. For example, for the closely related vacuum Brans-Dicke theory, it has been shown by Nandi, Bhattacharjee, Alam, and Evans [15] that while in the Jordan frame there do exist wormhole-solutions for the (unphysical) range $-3/2 < \omega < -4/3$ of the BD parameter, which are, however, plagued by "badly diseased" naked singularities, in the E frame there do not exist such solutions at all, unless energy-violating regions are deliberately introduced.

However, we must admit that while our GDC/ADC wormhole solutions are smooth as regards metric, volume element density (i.e., dilaton factor e^ϕ, respectively $e^{-\phi}$), and gauge potentials, they are not, when instead considering the dilaton scalar ϕ itself, which diverges to $-\infty$ (respectively $+\infty$) at the throat of the wormhole. Although this poses no problem for the smoothness of the corresponding

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[17]A manifestation of the "weak/strong coupling duality" of string theory.

[18]Note that X_- is non-negative, considered as a function of ρ.

[19]In fact, Bronnikov's wormhole solutions turn out to be highly unstable (cf. [6]). This is also the case for the recently found "ghostly" massless wormhole solution of Armendáriz-Picón [16], as discussed by Shinkai and Hayward [17].

TABLE I. PPN parameters for charged static spherically symmetric metrics.

	$2m$	β	γ	γ asymptotics
GDC/magnetic				
ADC/electric	r_+	$1 + 1/6\,\epsilon^2$	$1 + 1/6\,\epsilon^2$	$O(\epsilon^2)$
GDC/electric	$r_+ + r_-$	$1 + 1/3\,\epsilon^2$	$(1+x)^{-1}$	n. a.
GHS/E frame	r_+	$1 + 1/4\,\epsilon^2$	1	$O(\epsilon^0)$
GHS/S frame	$r_+ - r_-$	1	$(1+x)/(1-x)$	$O(\epsilon^1)$
Reissner-Nordström	$r_+ + r_-$	$1 - 1/2\,\epsilon^2$	1	$O(\epsilon^0)$

dilaton factors (which just smoothly vanish there), the geometric interpretation in terms of a corresponding nondegenerate volume element density cannot anymore be strictly maintained, as it is not manifestly positive. This must in fact be considered a flaw, albeit not a serious one, as the physical interpretation does not suffer.

V. INTERPRETATION

A. GDC/ADC and Einstein's equivalence principle

Let us emphasize that the GDC theory presented here is still a *metric* theory, in the sense that massive point sources move on timelike *geodesics* of the metric, when expressed in a G frame. This can be most easily seen by noting that the volume integral over the GDC Lagrangian for a point particle effectively splits into the volume integral over the "geometric" part of the Lagrangian plus a conventional line integral $m\int(\dot{x}^2)^{1/2}d\tau$, which by variation of the arc length then leads to the standard geodesic equation.[20]

As the alternative coupling scheme seems to be too artificial to be really believed, we introduce the point particle term "by hand" into the A frame Lagrangian. Therefore Einstein's equivalence principle (EEP) is satisfied both for the GDC and (trivially) ADC formulations.

However, for both the standard GHS and the generalized GHS solution the validity of the EEP is undecided, as an equivalent to a coupling scheme to external sources has not been formulated. In the following we will therefore simply *assume* the EEP to hold also for the corresponding string-inspired theory. This will allow us in the following to interpret the parametrized post-Newtonian (PPN) parameter m for *all* solutions considered as the mass of the gravitational source.

B. PPN viability and experimental verifiability

However, the compatibility with the EEP is not sufficient for the viability of a generalized theory of gravity, as it does not refer to any particular solution. A framework to check just this (in particular, SSS solutions) is well known under the name of parametrized post-Newtonian (PPN) approximation (cf. Will [18]). We will give in tabular form only the resulting most important parameters mass m, β, and γ, and compare them on the one hand with the corresponding pa-

rameters for the $\alpha^2 = 1$ GHS solutions (parametrized by (M, Q, ϕ_0),[21] where $M := r_+/2$, $q^2 := r_+ r_-$,[22] $q := Qe^{-\phi_0}$), both in the E frame and in the S frame, and on the other hand with the PPN parameters of the Reissner-Nordström metric, also given in the same form[23]

$$ds^2 = -\lambda^2 dt^2 + \lambda^{-2}dr^2 + r^2 d\Omega^2,$$

$$\text{where} \quad \lambda^2 := (1 - r_+/r)(1 - r_-/r). \tag{46}$$

In this PPN approximation, the metric has the following "isotropic" form:

$$ds^2 = -[1 - 2m/r + 2(m/r)^2\,\beta]dt^2 + (1 + 2m/r\,\gamma)$$
$$\times (dr^2 + r^2 d\Omega^2). \tag{47}$$

Setting $\epsilon := q/m$, $x := r_-/r_+$ and assuming $m \neq 0$, $r_+ \neq 0$, we can collect the results in the Table I, where also the asymptotic behavior of γ is given for $\epsilon \to \infty$. As for the electric GDC solution ϵ is limited by $\epsilon^2 \leq 3/2$, in this case an asymptotic expression does not make sense. The metrics for the GDC and ADC wormholes agree, so they share a common table entry.

Before proceeding, let us observe that the S-frame GHS "cornucopion" solution, being degenerate ($r_+ = r_-$), has *vanishing* PPN mass m and so does not fit into the table (β and γ undefined). This applies also to both the zero-mass magnetic GDC and electric ADC solutions, where $r_+ = 0$.

For heavenly bodies, where most empirical data comes from, we have $|\epsilon| = |q/m| \ll 1$. Assuming $r_+ \approx m$, we also have $x := r_-/r_+ \equiv 2/3\,(q/r_+)^2 \approx \epsilon^2 (m/r_+)^2 \ll 1$, so all these metrics are *equally viable*, having $\beta \approx \gamma \approx 1$.

However, applied to charged elementary particles (where up to now no PPN data seems to exist), the corresponding γ show significant differences in their dependence on the "specific charge" ratio q/m. This is most pronounced for the smooth SSS metric, where $\gamma \sim \epsilon^2$. For the smooth electric metric, taking the charge and mass of the electron, we get the factor $\epsilon^2 \approx 10^{40}$, which is essentially the smallest of Dirac's big numbers.

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[20]This can also be justified with a "pure dust" model of matter, based on relativistic thermodynamics (paper in preparation).

[21]M is not necessarily to be identified with the PPN mass.

[22]Note that we have to rescale F with $1/\sqrt{2}$ in order to have common conventions.

[23]Note that this form of the metric is only possible if $m^2 \geq q^2$, i.e., $\epsilon^2 \leq 1$.

This would lead for the electric ADC metric to significant effects near the wormhole throat $r \approx r_- \approx q^2/m$, which for the electron is of the order of the *classical electron radius* $r_{El} \approx 3 \times 10^{-13}$ cm. For the magnetic GDC metric, this estimate does change less than one order of magnitude, when taking $q = e/2\alpha$ (corresponding the Dirac's charge quantization) and assuming the mass of the proton, $m \approx 1836 m_e$, giving a throat "radius" of about 8×10^{-13} cm. The energies corresponding to these distances (a few MeV) would be well within the reach of present-day experimental technology.

In comparison, for the *S*-frame GHS metric deviations would appear only at distances (and corresponding energies) of the order of $r \approx e$, which is even one order of magnitude smaller than the Planck length $r_{Pl} \approx 2 \times 10^{-33}$ cm.

Of course, magnetic monopoles have not been observed (not even by their characteristic electromagnetic signature). However, for the possible dilatonic effects of electrically charged particles the situation looks more favorable. A realistic estimate of the effects involved would have to take into account a (not yet existing) quantum field theory adapted to curved backgrounds with nontrivial topology.

C. Zero-mass solutions; boundedness of mass

From a formal geometric viewpoint, the classical Reissner-Nordström family of metrics are equally meaningful for zero and even negative values of their mass parameters; only the character of their everpresent singularities at $r = 0$ change to the worse. From a physical standpoint, one would like to be able to exclude such negative-mass solutions. In the classical Einstein theory for isolated systems, and assuming suitable energy conditions, this has been achieved only fairly recently (see references given in Wald [9]).

The boundedness of the mass, $m \geq 0$, is, however, guaranteed by the families of charged SSS solutions described above, when insisting on *smoothness*.

First, the particular solutions with $r_+ = 0$, $r_- > 0$ have smooth metrics with *vanishing mass*, $m = 0$. As the corresponding charge q necessarily vanishes, they are *vacuum solutions* of GDC/ADC gravity.[24]

Now, let us assume negative mass m, i.e., $r_+ < 0$. Then we must necessarily also have $r_- < 0$ if the solution is to be charged. But then the transformation $r \mapsto \rho(r)$ to regular coordinates fails to produce a metric locally regular at $\rho(0)$, regularity can also not be achieved by any other map. Remain the uncharged possible solutions with $m < 0$. This means necessarily $r_- = 0$, but this is exactly the negative-mass Schwarzschild metric, which is well known to have a naked singularity.[25] Therefore as claimed, for the family of smooth SSS solutions there must be the *lower mass bound* $m \geq 0$, in order that the metric remains smooth. The nonflat

smooth SSS metrics can thus for $m \neq 0$ be characterized by the two physically meaningful parameters m, q, where $m > 0$, $q \neq 0$, whereas the massless solutions ($m = 0$) are uncharged ($q = 0$) and are uniquely characterized by their "scalar charge" $r_- \geq 0$.

Incidentally, the conserved total energy residing in the Maxwell field can be straightforwardly calculated, giving $E = 3m$ for both types I and II and $E = q^2/m < 3m$ for type III, indicating that there is a *gravitational binding energy* $\Delta_g E \leq 2m$, saturated for type I and II. Note that there is consistency in the sense that the vanishing of the mass m corresponds to the vanishing both of the charge q and the Maxwell field F.

D. Energy conditions; repulsion

Due to fairly general theorems, in the context of classical gravity (e.g., Friedman, Schleich and Witt [19], cf. also Visser's monography [12]), wormhole metrics like the smooth SSS metrics would necessarily somewhere exhibit "exotic matter," in the form of regions with negative energy density of the material source.[26] The Maxwell stress tensor M_{ik} evidently obeys automatically even the dominant energy condition (DEC) and was used as the only material source in our system of field equations; moreover it was coupled in the "orthodox" way (up to a dilaton factor in the case of the electric ADC solution) on the right-hand side of the field equation. $G - \Theta' - \Theta'' = M$, respectively $G - \Theta' + \Theta'' = e^{2\phi} M$. Therefore it can be said that our solutions do *not* contain any exotic matter, their material sources obeying the DEC. This is also evident in the *E* frame, where the KG stress tensor (adding to the Maxwell stress tensor multiplied by the dilaton factor) obeys the DEC. In fact, the energy tensor for our dilaton metrics does even obey the strong energy condition (SEC), which plays a prominent role in the singularity theorems of classical gravity.

So how does it come that there seems to be a "repulsive force" holding open the throat of the wormhole? This can be seen by invoking the Raychaudhuri identity for geodesics. First note that this identity cannot be applied directly in the *G* frame, as the field equations involve extra geometric terms besides the Einstein tensor. Mapping geodesics from the *G* frame or from the *A* frame to the *E* frame (where Einstein's equations formally hold) will result in additional nongeodesic terms: $\dot{\vec{u}} = 0 \rightarrow \dot{\vec{u}} = 1/2\,[(u \cdot d\phi)\,\vec{u} + g^{-1} d\phi]$. Such terms $\sim d\phi$ are also known from Nordström's scalar theory of gravitation (a precursor of Einstein's metric theory).[27] These additional terms also prevent the Raychaudhuri identity from being applied. For the smooth SSS metrics (and $r > 2m$) they have a repulsive effect, $g^{-1} d\phi = 1/2\, r_-/r^2\,(X_+/X_-)^{1/2}\,n_\rho\,(n_\rho := X_+^{1/2} \partial_\rho)$, which for type I

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[24]Closely related uncharged massless wormhole solutions have been investigated very recently by Armendáriz-Picón [16], although for a "ghostly" KG/dilaton scalar.

[25]The positive-mass Schwarzschild metrics would also be excluded by smoothness.

[26]The few known "nonexotic" wormhole solutions of the classical Maxwell-Einstein theory (e.g., Schein and Aichelburg [20]) in fact break some of the standard assumptions, like having closed timelike lines.

[27]For a short history of scalar-tensor theories of gravitation, see Brans [21].

and type III becomes unbounded at the former locus of the throat, where the metric is now singular. Therefore when interpreted in the E frame, test particles seem to be always effectively *repelled* by the object "sitting" at $\rho = 0$. In contrast, being driven by $e^{-\phi}$, the singular electric SSS GDC solution would always appear to attract the test particle.

VI. CONCLUSIONS

With respect to the criteria mentioned in the Introduction, we have been able to show that among the class of Maxwell-Einstein-dilaton Lagrangians there exist two essentially different couplings allowing for well-behaved static spherically symmetric solutions. However, only the GDC Lagrangian admits a simple coupling scheme. Moreover, it has an immediate geometric interpretation in terms of a volume manifold.

When it comes to making a choice between alternative Lagrangian-based theories of dilaton gravity, of course it depends on the weight one is willing to give to the existence of well-behaved (i.e., smooth) solutions and to the generality of MED Lagrangians. In the context of classical gravity, the "regularizing" nature of the GDC/ADC dilaton could be welcomed as a "new degree of freedom" to tame some of the inherent divergences. The main obstacle is, however, still the "magnetic" nature of the geometric GDC solutions, whereas for the "electric" ADC solutions, it is their lack of geometric interpretation and the nonuniqueness of a corresponding coupling scheme. There are no indications that more "sophisticated" Lagrangians could resolve this dilemma.

The stability of the smooth wormhole solutions has not been touched in our work, and constitutes the major open issue. However, in view of the fulfillment of all the energy conditions, the prospects seem to be promising.

It would be also be highly desirable to have general *stationary* spherically symmetric GDC/ADC solutions. Some basic questions in this context: are there still smooth charged wormhole solutions? Do nontrivial geometric vacuum solutions again exist? Does the "magnetic-electric dilemma" still persist? Can stability be proven to hold in a general sense?

A satisfactory answer to these questions would of course challenge the role of classical relativity and the corresponding (nondilatonic) black holes.

Applications to the different setting of cosmology would be particularly interesting: there is the possibility that the dilaton could again act repulsively, thus contributing to the observed accelerated cosmic expansion.

The diverse global spacetime models derivable from the basic extensions described here could also serve as "sandboxes" to develop and test some other fundamental theories in situations where the two-dimensional formulations are too limited to be realistic.

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[28]For a recent review of 2D dilaton gravity, see Grumiller, Kummer, and Vassilevich [23].

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