

Thickness of a mildly relativistic collisional shock wave

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We consider an imperfect relativistic fluid in which a shock wave develops and we discuss its structure and thickness, taking into account the effects of viscosity and heat conduction in the form of sound absorption. The junction conditions and the nonlinear equations describing the evolution of the shock are derived with the corresponding Newtonian limit discussed in detail. As in the nonrelativistic regime, the thickness is inversely proportional to the discontinuity in the pressure. However, new terms of purely relativistic origin are also present. In particular, for a viscous polytropic gas, it is found that a purely viscous relativistic shock is thicker than its nonrelativistic counterpart, while for pure heat conduction the contrary is true.

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I. INTRODUCTION

The theory of relativistic shock waves was pioneered more than 50 years ago by Taub [1], with the related junction conditions and adiabats further discussed by Israel [2], Lichnerowicz [3], and Thorne [4]. These results were established for a relativistic perfect simple fluid, and since they do not involve any characteristic length scale, the shock front was described by a mathematical surface of zero thickness (abrupt transition). Many studies have been made extending these works to the nonlinear regime of relativistic hydrodynamics, as well as to ideal relativistic magnetohydrodynamics (e.g., see the book by Anile [5] and references therein).

In this work, we are interested in the shock wave theory for an imperfect relativistic fluid. It is well known that, for dissipative relativistic fluids, for scales smaller than the dissipation scale \mathcal{L} associated with χ/c , where χ is the viscosity or heat conduction coefficient and c is the speed of light, ordinary Navier-Stokes formulas do not apply [6]. We are concerned with fluid regimes whose characteristic lengths are larger than the dissipation scale, and therefore the classical theory for dissipative fluids may be used.

All fluids are dissipative, and a nonrelativistic shock wave propagating in a dissipative medium cannot, in general, be considered as an abrupt transition, but instead, as a region with a finite thickness. Its thickness is determined by the dissipative coefficients (i.e., viscosity and heat conductivity) [7,8]. In addition, shock waves propagating through a gas mixture that undergoes diffusion of one component show characteristics similar to those due to dissipation [9]. This fact affects not only the evolution of the wave but can also have important effects on processes that depend upon the features of the shock wave.

A pioneering study of this system was made many years ago by Koch [10], who showed that if the shock velocity is greater than a given critical value, relativistic interaction of heat transfer and momentum transfer gives rise to an increase in the velocity at the upstream end of the shock layer. The purpose of our work is to discuss several aspects of this system that were not considered in the work by Koch,

thereby providing a more complete picture of relativistic shock waves.

The main aim here is to derive the equation for the (generalized) Taub curve, as well as the general expression for the thickness of a plane shock wave in the weak relativistic regime, taking into account both the classical dissipative mechanisms (heat conduction, bulk, and shear viscosity) and the associated sound absorption process. We shall not address the important issue of diffusion in a relativistic fluid or applications of our results to astrophysical processes, both of which will be left for future work.

The paper is structured as follows. In the next section, we review briefly the Eckart formulation for a viscous and heat-conducting relativistic simple fluid. In Sec. III, we derive the junction conditions, as well as the corresponding expressions for the generalized Taub-Rankine-Hugoniot curves for a unidimensional shock wave in a nonequilibrium regime. The general expression for the thickness of the shock and its comparison with the nonrelativistic limit is derived in Sec. IV. The Newtonian limits for the remaining expressions are discussed in the Appendix. The metric signature is $(+, -, -, -)$.

**II. IMPERFECT RELATIVISTIC FLUIDS:
THE ECKART APPROACH**

The relativistic theory of imperfect fluids rests on two basic ideas. The first one is the local equilibrium hypothesis (LEH). It implies that for nonequilibrium fluids, state functions (such as entropy) depend locally on the same set of thermodynamic variables as do equilibrium fluids. In particular, the usual thermodynamic temperature and pressure concepts are maintained in the relativistic nonequilibrium regime. The second idea is the existence of a local entropy source strength (entropy variation per unit volume and unit time), which is always non-negative, as required by the second law of thermodynamics. Mathematically, the LEH is represented by the Gibbs law, whereas the entropy law takes the form of a balance equation. Using these hypothesis in the fluid equations of motion, one finds an expression for the

entropy source strength, as well as for the constitutive (phenomenological) relations. The perfect fluid equilibrium equations are recovered in the limit of a vanishing entropy production rate. However, an important point of difference in the treatment of relativistic and nonrelativistic fluids by different authors should be stressed. In contrast to the Newtonian regime, in the relativistic domain, there exists an ambiguity related to the possible choices of the macroscopic hydrodynamic four-velocity. In the so-called Eckart formulation [11,12], the four-velocity is directly related to the particle flux, while in the Landau-Lifshitz approach [7], it is defined by the energy flux. In principle, a general treatment should be able to deal with any of these ‘‘gauge’’ choices [13]. For simplicity and for the sake of a simpler comparison with previous studies, in what follows we shall adopt the Eckart formulation.

The thermodynamic state of a relativistic simple fluid is characterized by an energy-momentum tensor $T^{\alpha\beta}$, a particle current N^α , and an entropy current S^α . The fundamental equations are expressed by the conservation laws (particles and energy momentum) and the entropy flux equation,

$$N^\mu_{;\mu} = 0, \quad T^{\mu\nu}_{;,\nu} = 0, \quad S^\mu_{;\mu} \geq 0, \quad (1)$$

where N^μ is the particle flux, $T^{\mu\nu}$ the stress tensor, and S^μ is the entropy flux (a comma denotes space-time derivatives). In the Eckart frame, the particle flux and stress tensor can be written as [12,13]

$$N^\mu = nu^\mu, \quad (2)$$

$$T^{\mu\nu} = \mu u^\mu u^\nu - p h^{\mu\nu} + \pi h^{\mu\nu} + c^{-1}(q^\mu u^\nu + q^\nu u^\mu) + \Pi^{\mu\nu}, \quad (3)$$

with the entropy flux given by

$$S^\mu = nk_B \sigma u^\mu - \frac{q^\mu}{T}. \quad (4)$$

The quantities n , ρ , p , σ , T , and k_B are the particle concentration, energy density, pressure, specific entropy (per particle), temperature, and the Boltzmann constant, respectively. The hydrodynamic 4-velocity u^μ is normalized according to $u^\mu u_\mu = 1$. The tensor

$$h^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \quad (5)$$

is the usual projector onto the local rest space of u^α . The irreversible fluxes, π , q^μ , and $\Pi^{\mu\nu}$, are defined by

$$\pi = \zeta \theta, \quad (6)$$

$$q^\mu = \kappa h^{\mu\nu} (T_{;\nu} - T a_\nu), \quad (7)$$

$$\Pi^{\mu\nu} = \eta \left(h^{\mu\alpha} u^\nu_{;\alpha} + h^{\nu\alpha} u^\mu_{;\alpha} - \frac{2}{3} \theta h^{\mu\nu} \right), \quad (8)$$

where κ , ζ , and η are the classical phenomenological coefficients (thermal conductivity, bulk, and shear viscosity) and $a_\nu = u_{\nu;\alpha} u^\alpha$ is the four-acceleration. The bulk viscosity stress, π , represents an irreversible negative pressure, and θ is the scalar of expansion (divergence of 4-velocity). The heat flux q^μ is orthogonal to the 4-velocity, i.e., $q^\mu u_\mu = 0$, whereas the shear-viscosity tensor $\Pi^{\mu\nu}$, is symmetric, trace-free, and spacelike.

For completeness, we recall that all dissipative fluxes π , q^μ , and $\Pi^{\mu\nu}$, as well as their space-time derivatives, are of first order of smallness in the equilibrium deviations. This is also true of the space-time derivatives of the reversible thermodynamic quantities n , ρ , p , σ , and T . However, the source of entropy, i.e., the divergence of the entropy flux

$$S^\mu_{;\mu} = \frac{\pi^2}{\zeta T} - \frac{q^\alpha q_\alpha}{\kappa T} + \frac{\Pi^{\alpha\beta} \Pi_{\alpha\beta}}{2 \eta T}, \quad (9)$$

is a quantity of second order of smallness. In what follows, we write the 4-velocity as $u^\mu = \gamma(1, \vec{v}/c)$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor.

III. TAUB CURVES AND ENTROPY DENSITY CHANGE

We now consider the junction conditions for a plane shock wave in a relativistic imperfect fluid and use them to derive the generalized Taub curve, as well as the associated entropy density change for weak shocks. The thickness of the shock wave due to the presence of viscosity and thermal conduction and taking into account the acoustic damping is, then, derived.

A. Junction conditions

In an ideal fluid, the relativistic junction conditions are defined by the continuity equation for the particle current N^x and the momentum and energy fluxes, i.e., the xx and $0x$ components of the energy-momentum tensor:

$$[N^x] = 0, \quad c[T^{0x}] = 0, \quad [T^{xx}] = 0. \quad (10)$$

Square brackets denote the difference between the values of any of the mentioned quantities at large distances in front of the shock and inside it. We denote side one as the (far) upstream side. Choosing the spatial component of the four-velocity along the x axis, it follows that $u^0 = \gamma$, $u^x = \gamma v^x/c$. The non-null components of the projector tensor are $h^{00} = -(u^x)^2$, $h^{0x} = -\gamma u^x$, and $h^{xx} = -\gamma^2$. For convenience, the density particle current will be expressed as $j = n \gamma v$. In this way, the conservation of the $0x$ and xx components of the energy-momentum tensor takes the form

$$n\gamma\frac{v}{c}\equiv\frac{j}{c}=\frac{j_1}{c}, \quad (11)$$

$$(w\gamma-w_1\gamma_1)=-\left(\zeta+\frac{4}{3}\eta\right)\frac{j}{c}\frac{\gamma}{n^3}\left[\frac{j}{c^2}\frac{\partial_t n}{\gamma n}+\partial_x n\right]+\frac{\kappa}{c}\left(\gamma^2+\frac{j^2}{c^2}\frac{1}{n^2}\right)\left\{\frac{1}{cn}\partial_t T+\frac{c}{j}\gamma\partial_x T-\frac{T}{cn^2}\left[\frac{j}{n}\frac{\partial_x n}{\gamma}+\partial_t n\right]\right\}, \quad (12)$$

$$\frac{j^2}{c^2}\left(\frac{w}{n}-\frac{w_1}{n_1}\right)+(p-p_1)=-\left(\zeta+\frac{4}{3}\eta\right)\frac{j}{c}\frac{\gamma^2}{n^2}\left[\frac{j}{c^2}\frac{\partial_t n}{\gamma n}+\partial_x n\right]+2\frac{\kappa}{c}\gamma\frac{j^2}{c^2}\frac{1}{n}\left\{\frac{1}{cn}\partial_t T+\frac{c}{j}\gamma\partial_x T-\frac{T}{cn^2}\left[\frac{j}{n}\frac{\partial_x n}{\gamma}+\partial_t n\right]\right\}, \quad (13)$$

where we have introduced the specific enthalpy (per particle),

$$w=\frac{\mu+p}{n}, \quad (14)$$

and used $u_{,0}^0=u^x\partial_t u^x/c\gamma$, $u_{,x}^0=u^x\partial_x u^x/\gamma$. It has been assumed that at large distances from the shock, the flux is uniform, i.e., all gradients vanish.

B. Generalized Taub curve

In order to obtain the expression for the change in the entropy across the shock, we follow a procedure similar to that adopted by Thorne [4]. First we multiply Eq. (13) by $(w/n+w_1/n_1)$ and then combine the result with $j^2=n_1^2(u_1^x)^2c^2=n^2(u^x)^2c^2$, obtaining

$$w^2u^{x2}-w_1^2u_1^{x2}+(p-p_1)\left(\frac{w}{n}+\frac{w_1}{n_1}\right)=-\left(\zeta+\frac{4}{3}\eta\right)\gamma^2\left(\frac{w}{n}+\frac{w_1}{n_1}\right)\frac{j}{cn^2}\left[\frac{j}{nc^2}\frac{\partial_t n}{\gamma}+\partial_x n\right]+2\frac{\kappa}{c}\gamma\frac{j^2}{c^2}\frac{1}{n}\left(\frac{w}{n}+\frac{w_1}{n_1}\right)\left\{\frac{1}{n}\frac{1}{c}\partial_t T+\frac{c}{j}\gamma\partial_x T-\frac{T}{cn^2}\left[\frac{j}{n}\frac{\partial_x n}{\gamma}+\partial_t n\right]\right\}. \quad (15)$$

Multiplying Eq. (12) by $(w\gamma+w_1\gamma_1)$, we get

$$(w^2\gamma^2-w_1^2\gamma_1^2)=-\left(\zeta+\frac{4}{3}\eta\right)(w\gamma+w_1\gamma_1)\frac{j}{c}\frac{\gamma}{n^3}\left[\frac{j}{c^2}\frac{\partial_t n}{\gamma n}+\partial_x n\right]+\frac{\kappa}{c}\left(\gamma^2+\frac{j^2}{c^2}\frac{1}{n^2}\right)(w\gamma+w_1\gamma_1)\left\{\frac{1}{cn}\partial_t T+\frac{c}{j}\gamma\partial_x T-\frac{T}{cn^2}\left[\frac{j}{n}\frac{\partial_x n}{\gamma}+\partial_t n\right]\right\}. \quad (16)$$

Finally, subtracting Eq. (16) from Eq. (15) and using $\gamma^2=1+(u^x)^2$, we obtain

$$w^2-w_1^2=(p-p_1)\left(\frac{w}{n}+\frac{w_1}{n_1}\right)+\left(\zeta+\frac{4}{3}\eta\right)\gamma^2\gamma_1w_1\frac{1}{n^2}\frac{j}{c}\left[\frac{1}{\gamma_1n_1}-\frac{1}{\gamma n}\right]\left[\frac{j}{c^2}\frac{\partial_t n}{\gamma n}+\partial_x n\right]+ \kappa\left[w\gamma+w_1\gamma_1-2\frac{j^2}{c^2}w_1\gamma_1\frac{\gamma}{n}\left(\frac{1}{\gamma_1n_1}-\frac{1}{\gamma n}\right)\right]\left\{\frac{\gamma}{j}\partial_x T+\frac{1}{nc^2}\partial_t T-\frac{T}{c^2n^2}\left[\frac{j}{n}\frac{\partial_x n}{\gamma}+\partial_t n\right]\right\}. \quad (17)$$

Equation (17) together with the definition of j/c [Eq. (11)] are the generalized Taub junction conditions for a plane shock wave in an imperfect relativistic simple fluid.

C. Weak shock wave: Entropy density change

We now consider the weak shock case, i.e., that for which all discontinuities are small. This means that differences, such as $V-V_1$, $p-p_1$, etc., between the values in front of

the transition layer and inside it are small. Thus differentiation with respect to x or ct increases the order of smallness by 1, i.e., dV/dx is a quantity of second order of smallness. From Eq. (17), we see that the term involving the viscosity coefficients is of third order, while for heat conduction, the

terms proportional to $(W\gamma + w_1 \gamma_1)$ are of second order. The number particle conservation law can be written as

$$\frac{u^\alpha n_{,\alpha}}{n} = -\theta, \quad (18)$$

and considering that to lowest order, the temperature gradient satisfies [13]

$$\frac{u^\alpha T_{,\alpha}}{T} = -\left(\frac{\partial p}{\partial \mu}\right)_n \theta, \quad (19)$$

the enthalpy density change, given by Eq. (17), can be expressed as

$$w^2 - w_1^2 = (p - p_1)(w/n + w_1/n) + \kappa(w\gamma + w_1 \gamma_1) \left\{ \frac{\gamma}{j} \partial_x T - \frac{j}{\gamma c^2 n^2} \partial_x T + \frac{T}{\gamma n c} \left[1 - \left(\frac{\partial p}{\partial \mu}\right)_n \right] \theta \right\}. \quad (20)$$

Note that the term proportional to T is proportional to c^{-2} through the dependence of θ on the four-velocity and time derivative [cf. Eq. (18)].

In the dissipationless regime, the resulting expression for the Taub adiabat is formally very similar to the Newtonian expression, as can be seen in Refs. [7] and [4]. For an imperfect relativistic fluid, the change in entropy in the transition layer is also of second order in the pressure, just as it is in the nonrelativistic case. However, as one may see from Eq. (20), three new purely relativistic terms come into play. In the Appendix, we show that Eq. (20) yields the Newtonian expression previously found in the literature (e.g., Ref. [7]).

To find the expression for the difference in the entropy density values far upstream and in the transition layer, we follow a standard procedure [4] and develop w/n around its upstream value in powers of $(p - p_1)$. We write the first law of thermodynamics as $dw = dp/n + Tds$, where s is the entropy per particle, and then we multiply by w , using the development of w/n . Keeping the zeroth order in wT in the second term and integrating, we get

$$w^2 - w_1^2 = 2w_1 T_1 (s - s_1) + 2 \frac{w_1}{n_1} (p - p_1) + \left[\frac{\partial}{\partial p} \left(\frac{w}{n} \right) \right]_{s,1} \times (p - p_1)^2 + \frac{1}{3} \left[\frac{\partial^2}{\partial p^2} \left(\frac{w}{n} \right) \right]_{s,1} (p - p_1)^3. \quad (21)$$

As the derivatives of T and n are already of second order, we consider $(w\gamma + w_1 \gamma_1) \approx 2 w_1 \gamma_1$ in Eq. (20). With this approximation and comparing with Eq. (21), we obtain the entropy density change,

$$s - s_1 \approx \frac{\kappa}{T_1} \gamma_1 \left\{ \frac{\gamma}{j} \partial_x T - \frac{j}{\gamma c^2 n} \partial_x T + \frac{T}{\gamma n c} \left[1 - \left(\frac{\partial p}{\partial \mu}\right)_n \right] \theta \right\}. \quad (22)$$

Therefore, as in the nonrelativistic case, the entropy density change is proportional to the heat conduction coefficient. The nonrelativistic limit of this expression is trivial and coincides with the known expression [7].

IV. SHOCK WAVE THICKNESS

Relativistic or nonrelativistic shocks are described by an evolving nonlinear wave. On the other hand, waves propagating in a viscous, heat-conducting medium are damped. This fact can be phenomenologically described by an extra imaginary term in the dispersion relationship for the wave, i.e., by writing $\omega \approx v_s k - i c L k^2$, where ω is the frequency, k is the wave number, v_s is the sound speed, and L is the absorption length (see Refs. [7,14]). The equation we are seeking must be of the form [7]

$$\left(\frac{\partial}{\partial t} - v_s \frac{\partial}{\partial x} \right) f - v_s \alpha_p f \frac{\partial}{\partial x} f = c L \frac{\partial^2}{\partial x^2} f, \quad (23)$$

where f is a suitable function that describes the wave profile. To obtain this equation, we shall follow a two-step procedure: we first find the nonlinear term (in the next subsection) and then proceed to find the quasiacoustic damping contribution (in the subsequent subsection).

A. Nonlinear term in shock waves

In order to find the nonlinear contribution, we need only to consider equations for an ideal fluid. We consider now the local reference frame, in which the medium is at rest (co-moving frame), and let δv be a unidimensional velocity perturbation. We have (cf. Ref. [12])

$$\frac{\partial}{\partial t} \mu + \delta v \frac{\partial}{\partial x} \mu + (\mu + p) \left[\frac{1}{c^2} \delta v \frac{\partial}{\partial t} \delta v + \frac{\partial}{\partial x} \delta v \right] = 0, \quad (24)$$

$$\frac{(\mu + p)}{c^2} \left[\frac{\partial \delta v}{\partial t} + \delta v \frac{\partial}{\partial x} \delta v \right] + \left[\frac{\delta v}{c^2} \frac{\partial p}{\partial t} + \frac{\partial}{\partial x} p \right] = 0. \quad (25)$$

Expanding μ and p , we have

$$\mu = \mu_0 + \frac{c^2}{v_s^2} \delta p + \frac{1}{v_s} \left(\frac{\partial^2 \mu}{\partial p^2} \right)_s \delta p^2, \quad (26)$$

$$p = p_0 + \delta p, \quad (27)$$

where μ_0 and p_0 are the background values. For a wave propagating to the left, we can write $\delta v = -(c^2/v_s) \delta p / (\mu + p)$ and using $\partial/\partial t = v_s \partial/\partial x$ in the second-order terms, Eq. (25) can be written as

$$\frac{1}{c^2} (\mu_0 + p_0) \frac{\partial \delta v}{\partial t} + \frac{\partial}{\partial x} \delta p = \frac{2}{(\mu_0 + p_0)} \delta p \frac{\partial}{\partial x} \delta p \quad (28)$$

and Eq. (24) as

$$\begin{aligned} & \frac{c^2}{v_s^2} \frac{\partial}{\partial t} \delta p + (\mu_0 + p_0) \frac{\partial}{\partial x} \delta v \\ &= v_s \left\{ \frac{2}{(\mu_0 + p_0)} \frac{c^4}{v_s^4} - \left(\frac{\partial^2 \mu}{\partial p^2} \right)_s \right\} \delta p \frac{\partial}{\partial x} \delta p. \end{aligned} \quad (29)$$

Deriving Eq. (28) with respect to x and Eq. (29) with respect to t and subtracting the resulting expressions, we get

$$\begin{aligned} & \left(\frac{1}{v_s} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left(\frac{1}{v_s} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \delta p \\ &= \left(\frac{v_s}{c^2} \alpha_1 \frac{\partial}{\partial t} - \alpha_2 \frac{\partial}{\partial x} \right) \left[\delta p \frac{\partial}{\partial x} \delta p \right], \end{aligned} \quad (30)$$

where

$$\alpha_1 = \left\{ \frac{2}{(\mu_0 + p_0)} \frac{c^4}{v_s^4} - \left(\frac{\partial^2 \mu}{\partial p^2} \right)_s \right\}, \quad (31)$$

$$\alpha_2 = \frac{2}{(\mu_0 + p_0)}. \quad (32)$$

Substituting $\partial/\partial t = v_s \partial/\partial x$ and eliminating $\partial/\partial x$ in both terms, we get

$$\left(\frac{1}{v_s} \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \delta p - \alpha_p \delta p \frac{\partial}{\partial x} \delta p = 0, \quad (33)$$

where we define

$$\begin{aligned} \alpha_p &\equiv \frac{1}{2} \left(\frac{v_s^2}{c^2} \alpha_1 - \alpha_2 \right) \\ &= \frac{1}{2} \left[\frac{2}{(\mu + p)} \frac{c^2}{v_s^2} - \frac{v_s^2}{c^2} \left(\frac{\partial^2 \mu}{\partial p^2} \right)_s - \frac{2}{(\mu + p)} \right]. \end{aligned} \quad (34)$$

This equation has the same form as does the nonrelativistic one (see Ref. [7]), with $(\mu + p)$ replacing the rest mass density, the relativistic energy density derived twice with respect to the pressure, and the ratio of the sound speed to the light speed appearing explicitly. The last term in the square brackets is a purely relativistic correction.

B. Dissipative term of the shock wave equation

The acoustic relativistic damping length required by the complete nonlinear equation of a shock wave was derived in another context by Weinberg [14]. We refer the interested reader to this work in order to see details of the derivation. Here, we just quote the final expression,

$$\begin{aligned} cL &= \frac{1}{2(\mu + p)} \left\{ c^2 \left(\zeta + \frac{4}{3} \eta \right) + \kappa(\mu + p) \left(\frac{1}{C_v} - \frac{1}{C_p} \right) \right. \\ &\quad \left. + \kappa T \left[\frac{v_s^2}{c^2} - \frac{2}{C_v} \left(\frac{\partial p}{\partial T} \right)_n \right] \right\}, \end{aligned} \quad (35)$$

observing that Weinberg's expression is recovered when $C_v(C_p) = nc_v(nc_p)$ and $c = 1$. The above expression is the same as the nonrelativistic result (see Ref. [7]), with $(\mu + p)$ replacing the density of the rest mass and a relativistic correction proportional to κT appears explicitly.

C. Solving for the thickness

The complete equation of the evolution of a shock wave is obtained by adding to Eq. (33) a term proportional to the second derivative with respect to x , which takes into account the dissipation. The final equation is then

$$\left(\frac{\partial}{\partial t} - v_s \frac{\partial}{\partial x} \right) \delta p - v_s \alpha_p \delta p \frac{\partial}{\partial x} \delta p = cL \frac{\partial^2 \delta p}{\partial x^2}. \quad (36)$$

Following the usual analysis [7], we assume that δp has the following dependence:

$$\delta p = \delta p(\xi), \quad \xi = x + v_w t, \quad (37)$$

where v_w is the velocity of the wave. With this solution, Eq. (36) becomes

$$\frac{d}{d\xi} \left[(v_w - v_s) \delta p - \frac{1}{2} v_s \alpha_p \delta p^2 - cL \frac{d}{d\xi} \delta p \right] = 0. \quad (38)$$

The solution to Eq. (36) is then [7]

$$p = \frac{1}{2}(p_1 + p_2) + \frac{1}{2}(p_2 - p_1) \tanh \frac{(p_2 - p_1)(x + v_w t)}{4(c/v_s)(L/\alpha_p)}, \quad (39)$$

where p_1 is the pressure far upstream and p_2 is the pressure far downstream. In the reference frame where the shock is at rest, we have for the pressure variation

$$p - \frac{1}{2}(p_1 + p_2) = \frac{1}{2}(p_2 - p_1) \tanh \left(\frac{x}{\Delta} \right), \quad (40)$$

where we have defined the ‘‘thickness’’ of the shock by

$$\Delta = \frac{4cL}{v_s \alpha_p (p_2 - p_1)}. \quad (41)$$

We see that this expression is identical in form to the nonrelativistic one and proportional to the inverse of the pressure difference. The relativistic corrections are contained in the factors α_p and cL .

D. Analysis of the thickness

In this subsection, we shall estimate the effect of the relativistic corrections to see if they increase or decrease the

shock thickness. We shall examine them in the weak relativistic limit. We then write (see the Appendix)

$$cL = \Lambda_{\text{NR}} - \lambda = \Lambda_{\text{NR}} \left[1 - \frac{\lambda}{\Lambda_{\text{NR}}} \right] \quad (42)$$

and

$$\alpha_p = v_s^2 \tilde{\alpha}_{\text{NR}} - \eta = v_s^2 \tilde{\alpha}_{\text{NR}} \left[1 - \frac{\eta}{v_s^2 \tilde{\alpha}_{\text{NR}}} \right], \quad (43)$$

with Λ_{NR} , λ , $\tilde{\alpha}_{\text{NR}}$, and η given in the Appendix. The nonrelativistic expression for the shock thickness is [7] $\delta = 4a/\tilde{\alpha}_{\text{NR}}(p_2 - p_1)$, where $a = \Lambda_{\text{NR}}/v_s^3$. We must evaluate

$$\frac{\Delta}{\delta} = \frac{cL \tilde{\alpha}_{\text{NR}}}{v_s \alpha_p a} \approx 1 + \frac{\eta}{v_s^2 \tilde{\alpha}_{\text{NR}}} - \frac{\lambda}{\Lambda_{\text{NR}}}, \quad (44)$$

where the semiequality holds for the weak relativistic case. Using the expressions in the Appendix, we find

$$\begin{aligned} \frac{\Delta}{\delta} = 1 + \frac{2v_s^2}{\rho c^2} & \frac{\left\{ 1 + \frac{(\varepsilon + p)}{\rho v_s^2} + \frac{v_s^2}{2} \rho \left(\frac{\partial^2 \varepsilon}{\partial p^2} \right)_s \right\}}{\left[\frac{2}{\rho} - v_s^4 \left(\frac{\partial^2 \rho}{\partial p^2} \right)_s \right]} \\ & - \frac{1}{\rho c^2} \frac{\left[\left(\zeta + \frac{4}{3} \eta \right) (\varepsilon + p) + \frac{\kappa T}{c_v} \left(\frac{\partial p}{\partial T} \right)_n \right]}{\left[\left(\zeta + \frac{4}{3} \eta \right) + \frac{\kappa}{2} \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right]}. \end{aligned} \quad (45)$$

For a more direct comparison of the relativistic thickness with the standard Newtonian result, we consider a polytropic gas and evaluate the above expression in two special cases: with viscosity alone and with thermal conduction alone.

Polytropic gas

In a classical polytropic gas, the energy density and enthalpy density are given by $\varepsilon = c_v T = p/(\Gamma - 1)$ and $w = c_p T = \Gamma p/(\Gamma - 1)$, where $\Gamma = c_p/c_v = \text{const}$, respectively. Hence $(\partial^2 \varepsilon/\partial p^2)_s = 0$ and $(\partial p/\partial T)_n = c_v(\Gamma - 1)$ and $1/c_v - 1/c_p = (\Gamma - 1)^2 T/\Gamma p$. Replacing these formulas in the classical expressions for the internal energy ε in Eq. (45), we get

$$\begin{aligned} \frac{\Delta}{\delta} = 1 + \frac{v_s^2}{c^2} \frac{2\gamma}{(\Gamma^2 - 1)} \\ & - \frac{1}{\rho c^2} \frac{\left[\left(\zeta + \frac{4}{3} \eta \right) \frac{\Gamma p}{(\Gamma - 1)} + \kappa T (\Gamma - 1) \right]}{\left[\left(\zeta + \frac{4}{3} \eta \right) + \frac{\kappa}{2} \frac{(\Gamma - 1)^2 T}{\Gamma p} \right]}. \end{aligned} \quad (46)$$

a. Only viscosity. If thermal conduction is absent, we find

$$\frac{\Delta}{\delta} = 1 + \frac{v_s^2}{c^2} \frac{1}{(\Gamma + 1)}. \quad (47)$$

We see that, in this case, the relativistic shock is thicker than the nonrelativistic one, with the increment proportional to the sound speed.

b. Only heat conduction. If viscosity is absent, we obtain

$$\frac{\Delta}{\delta} = 1 - 2 \frac{v_s^2}{c^2} \frac{1}{(\Gamma^2 - 1)}. \quad (48)$$

In this case, the relativistic shock is thinner than its nonrelativistic counterpart and the correction is again proportional to the sound speed.

E. Entropy density change

With the expression for the pressure given by Eq. (40), we can express the entropy change as a function of the pressure discontinuity. We begin by writing explicitly the derivatives in the expression for θ , namely Eq. (18) and replacing the time derivative with $\partial/\partial t = v_s \partial/\partial x$. Using $dT/dx = (\partial T/\partial p)_s dp/dx + (\partial T/\partial s)_p ds/dx \approx (\partial T/\partial p)_s dp/dx$ and $dn/dx = (\partial n/\partial p)_s dp/dx + (\partial n/\partial s)_p ds/dx \approx (\partial n/\partial p)_s dp/dx$ in Eq. (22) and evaluating dp/dx from Eq. (40), we obtain the following expression for the entropy density change in a reference system in which the shock is at rest:

$$\begin{aligned} s - s_1 \approx \frac{\kappa}{T_1} \gamma_1 & \left\{ \left[\frac{1}{j} - \frac{j}{\gamma c^2 n} \right] \left(\frac{\partial T}{\partial p} \right)_s - \frac{T(v_s + v)}{\gamma n^2 c^2} \left[1 - \left(\frac{\partial p}{\partial \mu} \right)_n \right] \right. \\ & \left. \times \left(\frac{\partial n}{\partial p} \right)_s \right\} \frac{v_s \alpha_p}{8cL} \frac{(p_2 - p_1)^2}{\cosh^2(x/\Delta)}, \end{aligned} \quad (49)$$

where the factor c in cL does not add an extra power in the speed of light (see the Appendix). In the nonrelativistic case [7], the entropy reaches a maximum inside the shock [8] and is of second order in the pressure discontinuity.

V. CONCLUSION

In this paper, we have extended previous studies of shock waves done in the nonrelativistic domain to the weak relativistic case. Considering dissipative relativistic fluids in the range of validity of the Navier-Stokes-Fourier theory [6], we have obtained expressions for the entropy density change and the shock thickness that coincide in form with nonrelativistic ones. In each of the factors in the equations, purely relativistic corrections appear explicitly. We studied the expression for the shock thickness for a polytropic gas and analyzed the effect of corrections in two important limits defined by the presence of viscosity or heat conduction. When only heat conduction is taken into account, the relativistic shock is thinner than for the nonrelativistic case. This result can be understood by observing that heat conducting fluids can develop ‘‘thermal discontinuities’’ [7,8], i.e., they allow for discontinuities in the velocity, pressure, and density of the fluid flow, while the temperature remains constant. On

the other hand, when only viscosity is present, the shock thickness is larger than for its nonrelativistic counterpart and, hence, the tendency to erase singularities is stronger in the relativistic limit than in the Newtonian one. This difference in the effect of the relativistic corrections can also be understood as follows. Viscosity provides the mechanism to convert a portion of kinetic energy of the gas flowing into the discontinuity into heat. This conversion is equivalent to the transformation of the energy of ordered motion of gas molecules into the energy of random motion by the dissipation of molecular motion. In this respect, heat conduction has an indirect effect on the conversion process since it only participates in the transfer of the energy of random motion of the molecules from one point to another, but it does not directly affect the ordered motion. The corresponding relativistic corrections seem to amplify these effects.

A comment on the entropy change is in order. When the preshock gas has a low temperature, we are in the strictly Newtonian limit and, in this sense, the fact that the entropy density change reduces to its Newtonian analog is equivalent to requiring that the theory has the correct low-speed limit. This fact contains no new information. But when the preshock fluid has relativistic internal speeds, the shock weakness does not imply a Newtonian propagation velocity of the shock and, hence, this case is not covered by the Newtonian treatment. In this sense, the result that we have obtained, that the entropy change still reduces to the Newtonian expression, is new and potentially interesting.

Finally, it should be mentioned that although first-order theories are successful in revealing the physics underlying a large class of phenomena, they present some experimental and theoretical drawbacks. In its classical version, the linear constitutive equations (6)–(8) are not adequate at high frequencies or short wavelengths, as is manifested in experiments on ultrasound propagation in rarefied gases and on neutron scattering in liquids [15]. In addition, they also allow for the propagation of perturbations with arbitrarily high speeds, which although unsatisfactory on classical grounds, is completely unacceptable from a relativistic point of view. Furthermore, they do not have a well-posed Cauchy problem and their equilibrium states are not stable. Several authors have formulated relativistic second-order theories which circumvent these deficiencies [16–19]. In a forthcoming paper, we intend to extend our considerations to this class of theories.

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APPENDIX

In this appendix, we obtain the nonrelativistic limits of the magnitudes discussed in this paper. We begin with the non-

relativistic limit of $\gamma^2(\mu+p)$, neglecting the pressure since $nm c^2 \gg p$ in the nonrelativistic limit. Thus, $\gamma^2(\mu+p) \rightarrow \mu = \gamma^2 n m c^2 + \gamma^2 \epsilon$, where ϵ is the internal energy density, i.e., the energy associated with internal degrees of freedom. In the limit of small velocities, $nm \rightarrow \rho/\gamma$, where ρ is the mass density and, therefore, $\gamma^2 n m c^2 \rightarrow \gamma \rho c^2 = (1-v^2/c^2)^{-1/2} \rho c^2 \approx \rho c^2 + (1/2)\rho v^2$. Taking $\gamma=1$ in the expression for the internal energy density, we obtain the desired limit: $\gamma^2(\mu+p) \rightarrow \rho c^2 + (1/2)\rho v^2 + \rho \epsilon$, where ϵ is the internal energy per particle.

1. Nonrelativistic limit of the Taub curve

At the limit $c \rightarrow \infty$ in Eq. (20), the terms in the square brackets can be neglected. Thus,

$$w^2 - w_1^2 = (p - p_1) \left(\frac{w}{n} + \frac{w_1}{n_1} \right) + \frac{\kappa}{j} (w + w_1) \partial_x T, \quad (\text{A1})$$

or

$$w^2 - (p - p_1) \frac{w}{n} - \frac{\kappa}{j} w \partial_x T = w_1^2 + (p - p_1) \frac{w_1}{n_1} + \frac{\kappa}{j} w_1 \partial_x T. \quad (\text{A2})$$

Taking the square root,

$$\begin{aligned} w \left[1 - (p - p_1) \frac{1}{wn} - \frac{\kappa}{jw} \partial_x T \right]^{1/2} \\ = w_1 \left[1 + (p - p_1) \frac{1}{w_1 n_1} + \frac{\kappa}{jw_1} \partial_x T \right]^{1/2}. \end{aligned} \quad (\text{A3})$$

Assuming that $(p - p_1)$ and $\partial_x T$ are small, we have

$$\begin{aligned} w \left(1 - \frac{1}{2} (p - p_1) \frac{1}{wn} - \frac{1}{2} \frac{\kappa}{jw} \partial_x T \right) \\ = w_1 \left(1 + \frac{1}{2} (p - p_1) \frac{1}{w_1 n_1} + \frac{1}{2} \frac{\kappa}{jw_1} \partial_x T \right) \end{aligned} \quad (\text{A4})$$

or, rearranging terms,

$$w - w_1 = \frac{1}{2} (p - p_1) \left(\frac{1}{n} + \frac{1}{n_1} \right) + \frac{\kappa}{j} \partial_x T, \quad (\text{A5})$$

which is the standard nonrelativistic expression for the Taub curve [7].

2. Nonrelativistic limit of the shock thickness

The nonrelativistic limit of the shock thickness is derived from Eqs. (34) and (35), the expressions for nonrelativistic α_p and L , respectively. Using the expression for $\gamma^2(\mu+p)$, derived in the introduction to the Appendix, we rewrite Eq. (34) for α_p as

$$\alpha_p = \frac{v_s^2}{2} \left\{ \frac{2}{\rho(1 + \varepsilon/\rho c^2 + p/\rho c^2)} \frac{1}{v_s^4} - \left[\frac{\partial^2}{\partial p^2} \left(\rho + \frac{\varepsilon}{c^2} \right) \right]_s - \frac{2}{v_s^2(\rho c^2 + \varepsilon + p)} \right\}. \quad (\text{A6})$$

In the weak relativistic limit, we obtain

$$\alpha_p = \alpha_{\text{NR}} - \eta, \quad (\text{A7})$$

with

$$\alpha_{\text{NR}} = \frac{v_s^2}{2} \left[\frac{2}{\rho v_s^4} - \left(\frac{\partial^2 \rho}{\partial p^2} \right)_s \right] = v_s^2 \tilde{\alpha}_{\text{NR}}, \quad (\text{A8})$$

$$\eta = \frac{1}{\rho c^2} \left\{ 1 + \frac{(\varepsilon + p)}{\rho v_s^2} + \frac{v_s^2}{2} \rho \left(\frac{\partial^2 \varepsilon}{\partial p^2} \right)_s \right\}. \quad (\text{A9})$$

It is convenient to express the dispersion relationship as $k = \gamma\omega/v_s + i\gamma^2 cL\omega^2/v_s^3$, where L is defined in Eq. (35). Using $(\mu + p)/c^2 = \rho + (\varepsilon + p)/c^2$, in the weak relativistic limit, we have

$$cL = \Lambda_{\text{NR}} - \lambda = \Lambda_{\text{NR}} \left[1 - \frac{\lambda}{\Lambda_{\text{NR}}} \right], \quad (\text{A10})$$

where

$$\Lambda_{\text{NR}} = \frac{1}{2\rho} \left[\left(\xi + \frac{4}{3} \eta \right) + \frac{\kappa}{2} \left(\frac{1}{c_v} - \frac{1}{c_p} \right) \right], \quad (\text{A11})$$

$$\lambda = \frac{1}{2\rho^2 c^2} \left[\left(\xi + \frac{4}{3} \eta \right) (\varepsilon + p) + \frac{\kappa T}{2} \frac{2}{c_v} \left(\frac{\partial p}{\partial T} \right)_n \right], \quad (\text{A12})$$

with $c_v = C_v/\rho$, the specific heat per unit mass.

Using Eqs. (A7)–(A12) in Eq. (41) and taking the limit $c \rightarrow \infty$, we obtain the standard α_p expression for the thickness of a nonrelativistic shock [7],

$$\delta = \frac{8aV^2}{(\partial^2 V/\partial p^2)_s (p_2 - p_1)}. \quad (\text{A13})$$

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