Evaluating matrix elements relevant to some Lorentz violating operators

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Carlson, Carone, and Lebed have derived the Feynman rules for a consistent formulation of noncommutative QCD. The results they obtained were used to constrain the noncommutativity parameter in Lorentz violating noncommutative field theories. However, their constraint depended upon an estimate of the matrix element of the quark level operator $(\not{p}-m)$ in a nucleon. In this paper we calculate the matrix element of $(\not{p}-m)$, using a variety of confinement potential models. Our results are within an order of magnitude agreement with the estimate made by Carlson *et al.* The constraints placed on the noncommutativity parameter were very strong, and are still quite severe even if weakened by an order of magnitude.

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I. INTRODUCTION

In the recent literature, a number of ways to modify the structure of space-time which can have experimental consequences have been considered. In one of the most popular scenarios, space-time is considered to become noncommutative at short distance scales, with space-time coordinates satisfying a commutation relation of the following form [1–7]:

$$[\hat{x}^{\mu},\hat{x}^{\nu}]=i\,\theta^{\mu\nu}, \tag{1}$$

where $\hat{x}^{\mu}$ is a position four-vector promoted to an operator, and $\theta^{\mu\nu}$ is a set of c numbers antisymmetric in their indices. The most striking effects of space-time noncommutativity of the form (1) are the Lorentz violating effects appearing in field theories, which is a consequence of θ^{0i} and $\varepsilon^{ijk}\theta^{ij}$ defining preferred directions in a given Lorentz frame.

Jurčo *et al.* [3] have shown how to construct non-Abelian gauge theories in noncommutative spaces from a consistency relation. Using the same approach Carlson *et al.* [4] have derived the Feynman rules for a consistent formulation of noncommutative QCD and they have computed the most dangerous, Lorentz violating operator that is generated through radiative corrections. They have found that at the lowest order in perturbation theory, the formulation of noncommutative QCD that they have presented leads to Lorentz violating operators such as [6]

$$\theta^{\mu\nu}\overline{q}\sigma_{\mu\nu}q, \quad \theta^{\mu\nu}\overline{q}\sigma_{\mu\nu}\not{D}q, \quad \text{and} \quad \theta^{\mu\nu}D_{\mu}\overline{q}\sigma_{\nu\rho}D^{\rho}q. \tag{2}$$

In [4] the phenomenological implications of the first of these operators were studied in detail. Noting that contributions from the space-space part of $\theta^{\mu\nu}$ make $\sigma_{\mu\nu}\theta^{\mu\nu}$ act like a $\vec{\sigma}\cdot\vec{B}$ interaction with $\vec{B}$ directly related to θ^{ij}, a limit was placed on the scale of noncommutativity. One used the result of tests of Lorentz invariance in clock comparison experiments [8], which suggest that external $\vec{\sigma}\cdot\vec{B}$ like interactions are bounded at the 10^{-7} Hz level or few $\times10^{-31}$ GeV. Carlson *et al.* [4] concluded that

$$\theta\Lambda^{2}\leqslant10^{-29}, \tag{3}$$

where θ is a typical scale for elements of the matrix $\theta^{\mu\nu}$.

However, the effective Lorentz violating operator was obtained from a one loop correction to the quark propagator, and the operator proportional to $\sigma_{\mu\nu}\theta^{\mu\nu}$ also contained a factor $(\not{p}-m)$. With $\vec{B}$ constant, the evaluation of $\vec{\sigma}\cdot\vec{B}$ factors out from the evaluation of $(\not{p}-m)$, and our discussion is focused on the latter.

In [4] an *ad hoc* estimate was used for the matrix element of the operator $(\not{p}-m)$, where m is the current quark mass, in getting the limit in Eq. (3). The matrix element $\langle\not{p}-m\rangle$ was estimated to be about $M_N/3\approx300$ MeV, where M_N is the nucleon mass. However, it has been argued that the expectation value of $(\not{p}-m)$ could be much less than this naive estimate [9].

The aim of this paper is to calculate the matrix element of the operator $(\not{p}-m)$, using a variety of confinement potential models, so as to evaluate the quality of the estimate made in [4].

The sample of potentials included four different confining potentials, two of them purely Lorentz scalar and two of them equal mixtures of scalar and vector. The first scalar potential is a baglike potential

$$V(r)=\begin{cases}V_0 & \text{if } r\geqslant R,\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

We also consider the one dimensional case for the nicety of the analytical result,

$$V(z)=\begin{cases}V_0 & \text{if } z\leqslant-a/2 \text{ or } z\geqslant a/2,\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

The $V_0\to\infty$ limit gives, of course, the MIT bag model [10,11] if one does not consider the bag energy. We will consider models of vector + scalar confinement next, using in one case a linear spatial potential and in the other case a harmonic one:

$$V(r)=\frac{1}{2}(1+\gamma^0)(V_0+\lambda r) \tag{6}$$

or

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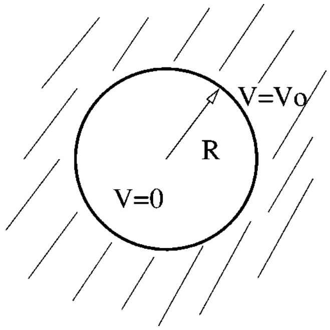

FIG. 1. 3D scalar central confinement.

$$V(r)=\frac{1}{2}(1+\gamma^0)Cr^2. \tag{7}$$

Finally we shall consider a purely scalar harmonic potential:

$$V(r)=Cr^2. \tag{8}$$

In the following sections it will be assumed that the current quark mass of 5–10 MeV can be neglected compared to the quark eigenenergy of several hundred MeV.

II. SCALAR SQUARE-WELL POTENTIAL

For any given potential *V*, from the Dirac equation we have that

$$(\not{p}-m)\psi=V\psi, \tag{9}$$

and therefore

$$\langle \not{p}-m\rangle=\langle V\rangle. \tag{10}$$

In the three dimensional case, for the central potential $V(r)$ presented in Eq. (4), the solutions of the Dirac equation for the ground state, with $m=0$, in two regions (I)$\mathbf{r}<\mathbf{R}$, and (II)$\mathbf{r}>\mathbf{R}$ (Fig. 1) have the following form

$$\psi_I(r)=N_I\begin{pmatrix} j_0(Er) \\ i\boldsymbol{\sigma}\cdot\hat{r}j_1(Er)\end{pmatrix}\chi^{(s)}, \tag{11}$$

$$\psi_{II}(r)=N_{II}\begin{pmatrix} h_0^{(1)}(ik_0r) \\ -\boldsymbol{\sigma}\cdot\hat{r}\sqrt{\dfrac{V_0-E}{V_0+E}}h_1^{(1)}(ik_0r)\end{pmatrix}\chi^{(s)}, \tag{12}$$

where $k_0=\sqrt{V_0^2-E^2}$, j_0,j_1 are spherical Bessel functions, and $h_0^{(1)},h_1^{(1)}$ are spherical Hankel functions of the first kind. The ground state energy can be found from the energy eigenvalue equation

$$j_1(ER)=j_0(ER)\left[\frac{1+k_0R}{(V_0+E)R}\right], \tag{13}$$

while for $V_0\to\infty$ the eigenvalue equation is $j_1(ER)=j_0(ER)$, as is familiar from the MIT bag model [10,11].

However, we know there are long range forces between baryons. If one wants to accommodate long range forces in this type of model, then one has to allow quarks to penetrate the walls of the potential well with some finite probability. Therefore the height of the potential, V_0, should be finite. A reasonable choice for V_0 and R can be obtained by fitting the model parameters to obtain reasonable values, for example, for the mean square of the charge radius of the nucleon $\langle r^2\rangle$ and for the axial vector coupling constant g_A. We get a good fit by choosing $R=1.12$ fm and $V_0=3$ GeV for which we find $\langle r^2\rangle=0.64$ fm^2 and $g_A=1.15$, as compared to experimental values of 0.76 fm^2 and 1.27, respectively [12]. Solving Eq. (13) for this choice of parameters for the ground state energy of a quark we find $E=348$ MeV.

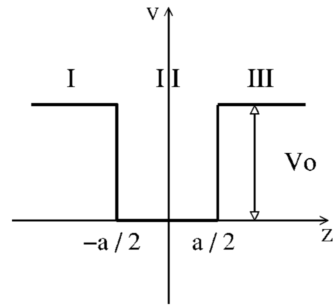

FIG. 2. One dimensional scalar square-well confinement.

Using the solutions given in Eqs. (11) and (12), we find

$$\langle \not{p}-m\rangle=21 \text{ MeV}. \tag{14}$$

Exploration of the integrals appearing in $\langle V(r)\rangle$ shows that $\langle \not{p}-m\rangle\to 0$ as $1/V_0$, when $V_0\to\infty$.

It may be of some pedagogic value to give the equivalent result for the 1D case (Fig. 2). The wave function for $|z|<a/2$ is just the free solution of the Dirac equation, and the solutions for $|z|>a/2$ are obtained from the free solution by the substitution $E\to E-V_0$. We obtain

$$\langle \not{p}-m\rangle=2V_0\int_{a/2}^{\infty}\bar{\psi}\psi dz=\frac{E}{1+a\sqrt{V_0^2-E^2}}. \tag{15}$$

One can note immediately that when the height of the potential $V_0\to\infty$ then $\langle \not{p}-m\rangle\to 0$, unless $a\to 0$. For the choice of parameters made above, we obtain

$$\langle \not{p}-m\rangle=14 \text{ MeV}, \tag{16}$$

where for the ground state energy E we have used a value of 260 MeV, from the energy eigenvalue equation.

III. SCALAR + VECTOR LINEAR CONFINEMENT

Let us consider now the confinement problem of a spin 1/2 particle in a confining potential of the form

$$V(r)=\frac{1}{2}(1+\gamma^0)(V_0+\lambda r). \tag{17}$$

This linear potential model for quark confinement was used in [13] to calculate several properties of low lying baryons. In [13] the authors assumed nonzero quark masses. The straightforward modification of the wave functions for the case of vanishing current quark masses yields the following solution for the lowest energy eigenstate of the Dirac equation for the potential (17):

$$\Psi(r)=N\begin{pmatrix}\Phi(r)\\ \boldsymbol{\sigma}\cdot\mathbf{p}/E\Phi(r)\end{pmatrix}\chi^{(s)}, \tag{18}$$

$$\Phi(r)=\sqrt{\frac{K}{4\pi\,\mathrm{Ai}'^2(a1)}}\frac{1}{r}\mathrm{Ai}(Kr+a_1), \tag{19}$$

where $K=(\lambda E)^{1/3}$. The energy eigenvalue E and the normalization constant N are given in Eq. (20):

$$E=V_0-\frac{\lambda a_1}{K},\quad N^2=\frac{3E}{4E-V_0}. \tag{20}$$

In [13] an analytic expression was obtained for the mean square charge radii of the baryons and in [14] Ferreira obtained an analytic expression for the magnetic moment of the proton. We modified those expressions for the zero current quark mass case and used them together with the energy eigenvalue equation (20) to fit our model parameters V_0 and λ. We choose $V_0=-626$ MeV and $\lambda=0.98$ GeV/fm to fit $\langle r^2\rangle$ exactly and give the value of μ_p, closest to the data obtaining

$$E=420\ \text{MeV},\quad \langle r^2\rangle=0.76\ \text{fm}^2,\quad \text{and}\quad \mu_p=2.44\ \text{n.m.} \tag{21}$$

For the above mentioned values of the model parameters we find that

$$\langle \not{p}-m\rangle=27\ \text{MeV}. \tag{22}$$

IV. SCALAR + VECTOR HARMONIC CONFINEMENT

Consider now a potential of the form

$$V(r)=\frac{1}{2}(1+\gamma^0)Cr^2. \tag{23}$$

The solution of the Dirac equation with this potential is given in [15]. The authors of [15] write the lowest energy state Dirac spinor as

$$\psi(r)=\frac{1}{\sqrt{4\pi}}\begin{pmatrix}if(r)/r\\ \boldsymbol{\sigma}\cdot\hat{r}g(r)/r\end{pmatrix}\chi^{(s)}, \tag{24}$$

where $\chi^{(s)}$ is a Pauli spinor, with the normalization $\int\psi^\dagger\psi d^3r=\int_0^\infty(f^2+g^2)dr=1$. Then the upper and lower components of the solution are

$$f(r)=N\left(\frac{r}{r_0}\right)e^{-r^2/2r_0^2},$$

$$g(r)=-\frac{N}{\sqrt{3}}\left(\frac{r}{r_0}\right)^2e^{-r^2/2r_0^2}, \tag{25}$$

$$N=\sqrt{8/(3r_0\sqrt{\pi})},\quad r_0^2E_0^2=3,\quad C=\frac{1}{9}E_0^3,$$

where E_0 is the ground state eigenenergy and r_0 is a state dependent scale parameter.

Now we can calculate the matrix element of interest,

$$\langle \not{p}-m\rangle=\int\bar{\psi}\frac{1}{2}(1+\gamma^0)Cr^2\psi d^3r$$
$$=\int_0^\infty f(r)^2Cr^2dr=\left(\frac{C}{3}\right)^{1/3}=\frac{E_0}{3}. \tag{26}$$

So, we can see that, in the case of scalar + vector confinement of equal strengths, $\langle \not{p}-m\rangle$ is determined only by the spin independent part of the Dirac spinor and is equal to one-third of the ground state energy. In [16] it was also shown that for three massless quarks in their lowest $1s$ orbit, with energy eigenvalues E_0 for each quark, the center-of-mass energy obtained with the potential (23) is just E_0; hence the nucleon mass in this model is $M_N=2E_0$ (instead of $M_N=3E_0$, as in nonrelativistic and nonrecoil models). Therefore, $E_0=540$ MeV and

$$\langle \not{p}-m\rangle=180\ \text{MeV}. \tag{27}$$

V. PURE SCALAR HARMONIC POTENTIAL

Tegen [15] considered scalar + vector harmonic confinement in calculating the weak neutron decay constant g_A/g_V and found too small a value for g_A/g_V, compared to experiment. In [16] and [17], a pure scalar harmonic potential $V(r)=Cr^2$ was studied numerically, and yielded more satisfactory results for g_A and for the rms charge radius. We find that

$$\langle \not{p}-m\rangle=C\int_0^\infty r^2[f(r)^2-g(r)^2]dr, \tag{28}$$

where $f(r)$ and $g(r)$ are defined as in Eq. (24).

We have fitted the numerical solution presented graphically in [16] with $C=830$ MeV/fm^2 to calculate our integral of interest (28). The fitted wave functions are presented in Fig. 3, and, as a benchmark for evaluation of the quality of the fit, we have calculated $\langle r^2\rangle$ and g_A and obtained values of 0.61 fm and 1.26, respectively, as compared to $\langle r^2\rangle=0.64$ fm and $g_A=1.26$ found in [16].

Thus we obtained, without any additional tuning, the following result:

$$\langle \not{p}-m\rangle=160\ \text{MeV}. \tag{29}$$

VI. SUMMARY

In this paper we have calculated, for the ground state of the quark in a nucleon, the matrix element of the operator $(\not{p}-m)$, using a variety of confinement potential models, under the assumption that the constituent quarks obey the Dirac equation. The motivation has been to solidify the esti-

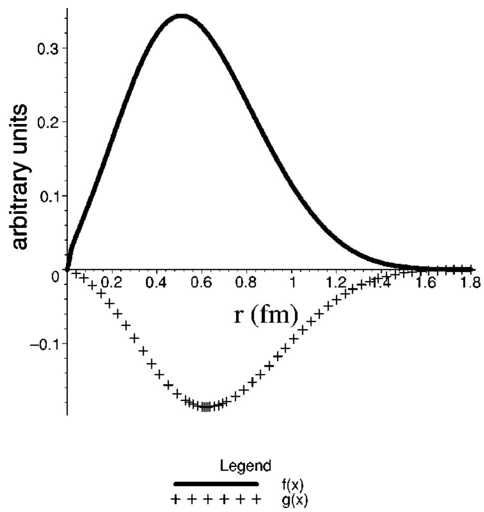

FIG. 3. Fit to the numerical solution of the Dirac equation for a pure scalar harmonic confinement.

mates of the noncommutativity parameter of canonical (Lorentz violating) noncomutative QCD, where some leading order Lorentz violating effects are proportional to factors of $\langle \not{p} - m \rangle$.

Interestingly, we found the following results:

$$\langle \not{p} - m \rangle = \begin{cases} 21\ \text{MeV} & \text{for } 3\ D \text{scalar central potential,} \\ 27\ \text{MeV} & \text{for scalar+vector linear potential,} \\ 180\ \text{MeV} & \text{for scalar+vector harmonic potential,} \\ 160\ \text{MeV} & \text{for pure scalar harmonic potential.} \end{cases} \tag{30}$$

We note that, in the case of scalar central confinement as considered in Sec. II, $\langle \not{p} - m \rangle$ vanishes as $1/V_0$ when $V_0 \to \infty$, but it is different from zero in general. We note also that the value of $\langle \not{p} - m \rangle$ obtained for the scalar + vector linear confinement model is close to that obtained for a scalar 3D potential well.

We have also shown that, in the case of scalar + vector harmonic confinement of equal strengths, $\langle \not{p} - m \rangle$ is determined only by the spin independent part of the Dirac spinor and is equal to one-third of the ground state energy.

For pure scalar harmonic confinement of the form $V(r) = Cr^2$, $\langle \not{p} - m \rangle$ was obtained using a fit to the numerical solution of the Dirac equation presented graphically in [16], and appears to have a value pretty close to that obtained for the scalar + vector harmonic confinement model.

Results obtained in this paper are within an order of magnitude agreement with the estimate made by Carlson *et al.* [4]. The results obtained in [4] were used there to constrain the noncommutativity parameter in Lorentz violating noncommutative field theories. The constraints were very strong, and are still quite severe even if weakened by an order of magnitude. These results may be taken as a motivation to look for space-time noncomutativity in Lorentz-covariant ways [18–21].

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