PHYSICAL REVIEW D **67**, 016004 (2003)

On n-$\bar{n}$ oscillations of ultracold neutrons

Vladimir K. Ignatovich
Frank Laboratory of Neutron Physics of Joint Institute for Nuclear Research, 141980, Dubna, Moscow Region, Russia
(Received 17 November 2001; revised manuscript received 14 October 2002; published 27 January 2003)

The ultracold neutron (UCN) storage experiment for searching for n-$\bar{n}$ oscillations is discussed. The figure of merit of the UCN experiment with respect to a beam experiment is considered. The effect of neutron collisions with the walls on the production rate of the $\bar{n}$ component is analyzed.

DOI: 10.1103/PhysRevD.67.016004 PACS number(s): 11.30.Fs, 03.65.Nk, 25.40.Dn

I. INTRODUCTION

This work continues the discussion [1–10] of the merits and disadvantages of ultracold neutrons (UCN)s for the search for n-$\bar{n}$ oscillations. The question here is, how do collisions with the walls affect the transition of n to $\bar{n}$? If the walls do not affect this transition the probability of $\bar{n}$ generation by a single neutron in the storage vessel is quite large, proportional to t_s^2, where t_s is the storage time. If every collision with the wall eliminates an $\bar{n}$ component, then antineutrons can be produced only during the free flight time $t=t_f$ between two consecutive collisions, and the probability of $\bar{n}$ production in the storage vessel is proportional to $t_f^2(t_s/t_f)=t_f t_s$, where the number of free flights t_s/t_f, or the number of collisions with the walls, is introduced. However, collisions with the walls can even hamper the transition of n to $\bar{n}$. In that case the production of $\bar{n}$ by a single neutron in the storage vessel becomes even lower. We need to study how collisions with the walls affect $\bar{n}$ generation to find the most favorable conditions for a possible real experiment.

II. FIGURE OF MERIT

To have a quantitative criterion for the utility of UCNs for the searching for n-$\bar{n}$ oscillations we need to define their figure of merit compared to neutrons of higher energies. To do that, suppose we have a steady state source with the Maxwellian flux density

$$d\Phi(v)=\Phi_0\frac{v^3dv\,d\Omega}{2\pi v_T^4}\exp(-v^2/v_T^2), \quad (1)$$

where Φ_0 is the total neutron flux density, v is the neutron velocity, v_T is the thermal velocity $\sqrt{2mk_BT}$, T is the temperature, m is the neutron mass, k_B is the Boltzmann constant, and $dv\,d\Omega$ is the interval of velocities and solid angles acceptable in an experiment.

A. Production of $\bar{n}$ in a beam experiment

In a beam experiment the number of events is proportional to number of neutrons $dN_{bn}(v)$ used in the experiment, which is equal to the product of the beam density (1), the beam cross section S_b, and the total time of measurement t_m: $dN_{bn}(v)=d\Phi(v)S_bt_m$. To find the total number of produced $\bar{n}$ we must multiply $dN_{bn}(v)$ by the probability $p_1=(t_f/\tau)^2$ of $\bar{n}$ creation for every one neutron, where t_f is the flight time in the experimental device. The experimental device in a beam experiment is characterized by the area S_t of the target, where the $\bar{n}$ are registered, and by the distance L of the target from the source. The parameters S_t and L define the element of solid angle $\Omega=S_t/L^2$ in Eq. (1), and the neutron free flight time $t_f=L/v$ in the probability p_1. Thus $p_1=(L/v\tau)^2$.

Since in a beam experiment all velocities are acceptable, we integrate over v, and find that the total number of $\bar{n}$ produced in the beam is

$$N_{b\bar{n}}=t_m\Phi_0S_b\frac{S_t}{L^2}\int_0^\infty\frac{v\,dv}{2\pi v_T^4}e^{-v^2/v_T^2}\frac{L^2}{\tau^2}=t_m\frac{\Phi_0}{4\pi v_T^2\tau^2}S_bS_t. \quad (2)$$

Now we need to compare this number $N_{b\bar{n}}$ to the number $N_{\mathrm{uc}\bar{n}}$ of $\bar{n}$ produced in a UCN storage experiment.

B. Production of $\bar{n}$ in a UCN experiment

In a UCN storage experiment we have a bottle with N_n neutrons in it, stored for time t_s, which cannot be larger than the neutron decay time τ_0. The spectrum of the neutrons in the bottle is represented by the spectral density $\rho(v)$. In the case when we can neglect gravity, the spectral density is related to N_n via

$$N_n=V\int_{v^2<v_c^2}\rho(v)d^3v, \quad (3)$$

where v_c is the limiting velocity of the bottle walls, and V is its volume. In the case when the height of the bottle is higher than $z_c=v_c^2/2g$, where g is the free fall acceleration, N_n is represented by the integral

$$N_n=S\int_0^{z_c}dz\int\rho(\sqrt{v^2+2gz})d^3v, \quad (4)$$

where S is the area of the bottle bottom, and $\rho(v)$ is the neutron spectrum near the bottom [11].

The bottle is filled with neutrons by a source through a window, the dimensions of which can be characterized by the same area S_b as the cross section of the beam. If the filling process is infinitely long and losses of neutrons in the bottle are neglected, then the spectral density $\rho(v)$ in the bottle is determined from the requirement that the number of ingoing neutrons from the incident flux is equal to the number of outgoing ones:

$$d\Phi(v)\cos\theta = v\cos\theta\rho(v)d^3v, \tag{5}$$

where θ is the angle of the neutron velocity with respect to the entrance window of the bottle. Accounting for losses and for the finiteness of the filling time, if the latter is sufficiently long, will cause some corrections to these formulas, which are not essential for our estimations here.

Substitution of Eq. (1) into Eq. (5) gives

$$\rho(v) = (\Phi_0/2\pi v_T^4)\exp(-v^2/v_T^2). \tag{6}$$

Usually $v_c^2 \ll v_T^2$, so $\rho(v)$ can be approximated as

$$\rho(v) = \rho(v) = (\Phi_0/2\pi v_T^4)\Theta(0<v^2<v_c^2), \tag{7}$$

where $\Theta(x)$ is a step function which is equal to unity when the inequality in its argument is satisfied, and zero in the opposite case. With this density the number N_n in the bottle is

$$N_n = \begin{cases} \dfrac{2}{3}V\Phi_0\dfrac{v_c^3}{v_T^4} & \text{for Eq. (3)}, \\ \dfrac{4}{15}Sz_c\Phi_0\dfrac{v_c^3}{v_T^4} = \dfrac{2}{15}S\dfrac{v_c^2}{g}\Phi_0\dfrac{v_c^3}{v_T^4} & \text{for Eq. (4)}. \end{cases} \tag{8}$$

It is seen that according to Eq. (8) the larger is the volume V or area S, the larger is the number N_n in the bottle. However, the larger V or S, the longer is the filling time t_{in}, which should not be less than the filling time constant τ_{in}, and τ_{in} should not be larger than the neutron decay time τ_0. The filling time constant can be estimated as the emptying time constant τ_{out} of the bottle with window S_b. This constant is defined as the ratio of N_n to the number of neutrons outgoing through the window S_b per unit time. This number is

$$\dot N_n = S_b\int_{0<\theta<\pi/2} v\cos\theta\rho(v)d^3v$$

$$= S_b\frac{\Phi_0}{2\pi v_T^4}\frac{\pi v_c^4}{4}$$

$$= \Phi_0 S_b\frac{v_c^4}{8v_T^4}.$$

Thus

$$\tau_{out} = \frac{N_n}{\dot N_n} = \begin{Bmatrix} \dfrac{16V}{3S_bv_c} & \text{for Eq. (3)} \\ \dfrac{16Sv_c}{15gS_b} & \text{for Eq. (4)} \end{Bmatrix} \approx \tau_0. \tag{9}$$

It follows that $V \leq (3/16)v_c\tau_0S_b$ and $S \leq (15/16)S_bg\tau_0/v_c$.

It is seen that for $S_b \approx 0.01$ m^2 and $v_c = 5$ m/s the most appropriate bottle is a room of height z_c and floor area ≈ 16 m^2, so in the following we consider only the case (4).

The number of $\bar n$ created in a UCN storage experiment is

$$N_{\bar n}(\tau_0) = \langle p_1\rangle N_n(\tau_0), \tag{10}$$

where $\langle p_1\rangle$ is the probability of $\bar n$ creation by a single neutron. If collisions with the walls do not affect production of $\bar n$, the probability will be $(t_s/\tau)^2 = (\tau_0/\tau)^2$. The number of storage cycles performed during the measurement time t_m cannot be larger than $t_m/2t_s = t_m/2\tau_0$. Thus the total number of $\bar n$ that can be registered with 100% efficiency can be estimated as

$$N_{\bar n}(t_m) = \frac{t_m\tau_0}{\tau^2}\frac{2}{15}\Phi_0S\frac{v_c^2}{2g}\frac{v_c^3}{v_T^4},$$

and the figure of merit of the UCN compared to the beam experiment becomes

$$F_m = \frac{N_{\bar n}(t_m,\text{UCN})}{N_{\bar n}(t_m,\text{beam})} = \frac{8\pi}{15}\frac{v_c^2}{2g}(v_c\tau_0)\frac{S}{S_bS_t}\frac{v_c^2}{v_T^2}. \tag{11}$$

For $S \approx S_t = 10$ m^2, $S_b = 0.01$ m^2, $v_c = 5$ m/s, and $v_T = 2200$ m/s we obtain $F_m = 9$.

However, if every collision with the wall eliminates some $\bar n$, then $\langle p_1\rangle = \tau_0 t_f/\tau^2$, where t_f is the average flight time between two consecutive collisions with the walls. In that case F_m contains an additional small factor t_f/τ_0 which is of the order of 10^{-3} for $t_f = 1$ s.

To be more precise it is necessary to calculate the average flight time. To do that we calculate the average number of collisions per unit time per neutron with the walls in a tall cylinder of radius r in the presence of gravity. This number is defined as the ratio $\dot N_w/N_n$ of the number of neutrons striking the walls per unit time to the total number of particles in the bottle. The nominator can be represented as the sum $\dot N_w = \dot N_1 + \dot N_2$, where $\dot N_1$ is the number of neutrons striking the bottom per unit time, and $\dot N_2$ is that for the sidewalls. The first part is

$$\dot N_1 = \Phi_0 S\frac{v_c^4}{8v_T^4} = \pi r^2\Phi_0\frac{v_c^4}{8v_T^4}.$$

The second part is

$$\dot{N}_2 = 2\pi r \int_0^{z_c} dz \int_0^{v_c} v^2 dv \rho(\sqrt{v^2+2gz}) \times \int_{0<\theta<\pi/2} v \cos\theta d\Omega = \frac{\pi r \Phi_0}{v_T^4} \int_0^{z_c} dz \int_{2gz}^{v_c^2} (v^2 - 2gz) \frac{dv^2}{2} = \frac{\pi r \Phi_0}{4 v_T^4} \int_0^{z_c} dz (v_c^2 - 2gz)^2 = 2\pi r \frac{v_c^2}{2g} \frac{\Phi_0}{24} \frac{v_c^4}{v_T^4}.$$

Thus $\dot{N}_w/N_n = (15/48)(3g/v_c + v_c/r)$, which means that $t_f \approx 0.4$ s. This shows that the additional small factor is even smaller: $t_f/\tau_0 = 4\times 10^{-4}$.

However, we should take into account that it is not a simple selection of neutrons with $v<v_c$ from the density (7) that is used for UCN accumulation in a bottle. In practice one uses a convertor with temperature $T_c \ll T$, which enhances Eq. (7) by the gain factor $G(T,T_c)$ depending on T and T_c. Thus the total number of UCNs (8) and the figure of merit (11) must be multiplied by this G factor, which in the case of solid deuterium is of the order of 10^3, and in the case of superfluid ^{4}He is estimated to be even higher.

Moreover, the $\bar{n}$ component can survive during several collisions M_{eff} with the walls [10], which additionally increases the outcome of $\bar{n}$.

With these two factors the figure of merit of the UCN experiment becomes

$$F_m = G M_{eff} \frac{t_f}{\tau_0} \left(\frac{8\pi}{15} \frac{v_c^2}{2g} (v_c \tau_0) \frac{S}{S_b S_t} \frac{v_c^2}{v_T^2} \right) = \frac{48}{15} \frac{r z_c G M_{eff}}{(3r/2 + z_c)(v_c \tau_0)} \left(\frac{8\pi}{15} \frac{v_c^2}{2g} (v_c \tau_0) \frac{S}{S_b S_t} \frac{v_c^2}{v_T^2} \right), \tag{12}$$

and for the same parameters as above it becomes of the order $4M_{eff}$, which demonstrates that the number M_{eff} is very important.

III. EFFECT OF COLLISION WITH THE WALL ON $\bar{n}$ PRODUCTION

The effect of collisions with the walls was first discussed in [4], where they were said to cause dephasing of the n and $\bar{n}$ components. However, it was not shown why dephasing spoils the rate of $\bar{n}$ production.

In [5] it was argued that the relative n-$\bar{n}$ phase is completely randomized at every collision, which means that the probability of $\bar{n}$ production is proportional to $t_f t_s$. However, in [6] it was claimed that the n-$\bar{n}$ phase shift per wall collision is a well behaved parameter, and the loss of sensitivity of UCN experiments is due only to the high absorption rate for the $\bar{n}$ component at the collisions.

Here we study once again the role of collisions with the walls. We can certify that the role is usually destructive although with some small probability the $\bar{n}$ component is even created at every collision. Below we first estimate the probability of creation, and then discuss how dephasing and losses at reflections affect the $\bar{n}$ produced during free flight between two consecutive collisions.

The neutron inside the storage vessel will be considered as a free particle without accounting for discreteness of the levels, as was suggested in [5], because the $\bar{n}$ component in the vessel is not stationary. Its storage time is of the order of 1 s, which means that every discrete level has a width comparable to or larger than the distance between the levels, as was argued in [6].

A. Estimation of $\bar{n}$ component created by a neutron at a single collision with the wall

Our approach to this problem is the same as the one used in [12,13] for the description of reflection of polarized neutrons from a magnetized mirror, when the magnetization is noncollinear to the external magnetic field. The neutron with two components n and $\bar{n}$ is a two-level system, and it can be described by a spinor [7,8] ψ, the upper component of which is n, and the lower component is $\bar{n}$. Thus $\psi = \mu\psi_n + \nu\psi_{\bar{n}}$, where μ,ν are complex numbers and $\psi_{n,\bar{n}}$ are eigenspinors of the Pauli matrix σ_z: $\sigma_z \psi_{n,\bar{n}} = \pm\psi_{n,\bar{n}}$, normalized to unity.

In general, the wave function of the particle, which we call the "Neutron," with upper case letter N, is described by a spinor $\Psi(\boldsymbol{r},t)$, which satisfies the Schrödinger equation

$$i\frac{d}{dt}\Psi = [-\Delta + U + H_z\sigma_z + H_x\sigma_x]\Psi, \tag{13}$$

where U is some interaction energy, the same for both components, H_z is some energy of opposite sign for the two components, H_x is the field, that causes the n-$\bar{n}$ transition, $\sigma_{x,z}$ are Pauli matrices, and for simplicity we use units in which $\hbar^2/2m = 1$. The energy U contains, in particular, an imaginary part $-iU''$, responsible for the free Neutron β decay.

After Neutron creation its energy is fixed, so we must look for a stationary solution of Eq. (13):

$$\Psi(\boldsymbol{r},t) = \exp(-i\omega t)\Psi(\boldsymbol{r}), \quad \Psi(\boldsymbol{r}) = \exp(i\hat{\boldsymbol{k}}\boldsymbol{r})\psi_0, \tag{14}$$

where $\hat{\boldsymbol{k}} = \boldsymbol{e}\hat{k}$, $\hat{k} \equiv k(-\boldsymbol{H\sigma}) = \sqrt{\omega - V - \boldsymbol{H\sigma}}$, $\boldsymbol{e}$ is a unit vector pointing in the direction of propagation, the vector $\boldsymbol{H}$ has components $\boldsymbol{H} = (H_x, H_z)$, $\boldsymbol{\sigma} = (\sigma_x, \sigma_z)$, and ψ_0 is a spinor, containing some fixed mixture of n and $\bar{n}$ components at the moment of collision.

Let us see what happens at the collision with the wall. The potentials U' and $\boldsymbol{H}'$ inside the matter may be different from U and $\boldsymbol{H}$ outside it. Thus, if we suppose that the wall occupies a semi-infinite half space $x>0$, the stationary Schrödinger equation for the neutron becomes

$$[-\Delta-\omega+(U+\boldsymbol{H}\boldsymbol{\sigma})\Theta(x<0)+(U'+\boldsymbol{H}'\boldsymbol{\sigma})\Theta(x>0)]\Psi(\boldsymbol{r})=0, \qquad (15)$$

where $\Theta(x)$ is a step function equal to 1 or 0, when the inequality in its argument is or is not satisfied, respectively. We use the stationary equation, because we are interested in elastic reflection from the wall. All inelastic processes that lead to inelastic scattering and to $n,\bar{n}$ losses can be included in Eq. (15) via the imaginary parts of U' and H_z'. In the following we omit U in Eq. (15), because we can incorporate it into ω.

The solution of Eq. (15) can be represented in the form

$$\Psi(\boldsymbol{r})=\exp(i\boldsymbol{k}_{\|}r_{\|})\psi(x),$$

where $\boldsymbol{k}_{\|}$ are parallel to the wall components of the neutron wave vector. Substitution into Eq. (15) reduces it to a one-dimensional equation

$$[-d^2/dx^2-k^2+\boldsymbol{H}\boldsymbol{\sigma}\Theta(x<0)+(U'+\boldsymbol{H}'\boldsymbol{\sigma})\Theta(x>0)]\psi(x)=0, \qquad (16)$$

where $k=\sqrt{\omega-k_{\|}^2}$ is normal to the wall component of the neutron wave vector in the absence of external fields.

The solution of Eq. (16), which contains the incident wave

$$\psi_0(\boldsymbol{r})=\Theta(x<0)\exp(i\hat{k}x)\xi_0 \qquad (17)$$

at $x<0$, contains also the reflected and refracted ones with reflection ρ and refraction τ matrix amplitudes found by matching the three waves at the interface. This matching gives [12]

$$\hat{\rho}=[k(-\boldsymbol{\sigma}\boldsymbol{H})+k'(-\boldsymbol{\sigma}\boldsymbol{H}')]^{-1}[k(-\boldsymbol{\sigma}\boldsymbol{H})-k'(-\boldsymbol{\sigma}\boldsymbol{H}')], \qquad (18)$$

where $k(-\boldsymbol{\sigma}\boldsymbol{H})=\sqrt{k^2-\boldsymbol{\sigma}\boldsymbol{H}}$, $k'(-\sigma\boldsymbol{H}')=\sqrt{k^2-U'-\boldsymbol{\sigma}\boldsymbol{H}'}$. To find the amplitude of $\bar{n}$ creation at a single collision with the wall we need to find the matrix element $\langle\bar{n}|\hat{\rho}|n\rangle$. To calculate it we use the following relations valid for an arbitrary function $f(x)$ and arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$f(\boldsymbol{\sigma}\boldsymbol{a})=\frac{1}{2}[f(a)+f(-a)]+\frac{1}{2}\frac{\boldsymbol{\sigma}\boldsymbol{a}}{a}[f(a)-f(-a)],$$

$$f(\boldsymbol{\sigma}\boldsymbol{a})f(-\boldsymbol{\sigma}\boldsymbol{a})=f(a)f(-a), \qquad (18')$$

$$\frac{1}{f(\boldsymbol{\sigma}\boldsymbol{a})}=\frac{f(-\boldsymbol{\sigma}\boldsymbol{a})}{f(a)f(-a)}, \quad (\boldsymbol{\sigma}\boldsymbol{a})(\boldsymbol{\sigma}\boldsymbol{b})=(\boldsymbol{a}\boldsymbol{b})+i[\boldsymbol{a}\boldsymbol{b}]\boldsymbol{\sigma}. \qquad (18'')$$

Using these rules we transform expression (18) to the following:

$$\hat{\rho}=\frac{k(H)k(-H)+k'(H')k'(-H')-\frac{\boldsymbol{\sigma}\boldsymbol{H}}{2H}k_-k_+'+\frac{\boldsymbol{\sigma}\boldsymbol{H}'}{2H'}k_+k_-'+i\frac{\boldsymbol{\sigma}[\boldsymbol{H}\boldsymbol{H}']}{2HH'}k_-k_-'}{k(H)k(-H)+k'(H')k'(-H')+\frac{1}{2}k_+k_+'+\frac{\boldsymbol{H}\boldsymbol{H}'}{2HH'}k_-k_-'}, \qquad (19)$$

where $k(\pm H)=\sqrt{k^2\pm H}$, $k'(\pm H')=\sqrt{k^2\pm H'}$, $k_\pm=k(H)+k(-H)$, and $k_\pm'=k'(H')+k'(-H')$. The transitions n-$\bar{n}$ are provided by the matrices σ_x and σ_y, so the amplitude of this transition is

$$\langle\bar{n}|\rho|n\rangle=\frac{-\frac{H_x}{2H}k_-k_+'+\frac{H_x'}{2H'}k_+k_-'+\frac{H_xH_z'-H_zH_x'}{2HH'}k_-k_-'}{k(H)k(-H)+k'(H')k'(-H')+\frac{1}{2}k_+k_+'+\frac{\boldsymbol{H}\boldsymbol{H}'}{2HH'}k_-k_-'}. \qquad (20)$$

This expression can be simplified, if we suppose that $H_x'\approx H_x\ll H_z\ll H_z'$, which means that the transition rates in vacuum and matter are the same, and the energy difference for n and $\bar{n}$ states in matter is considerably higher than in vacuum. If the energy difference H in vacuum is considerably lower than the neutron energy k^2, then we can approximate $k(H)\approx k(-H)\approx k$ and $k_-\approx H/k$. As a result the relation (20) is reduced to

$$\langle\bar{n}|\rho|n\rangle=\frac{H_x}{H'}\frac{k^2k_-'-H'k'(-H')}{k[k+k'(H')][k+k'(-H')]}. \qquad (21)$$

We see that the amplitude is of the order of $H_x/k^2\ll H_xt_f/\hbar$, so in the following we can completely ignore it.

B. The number of $\bar{n}$ created in an ideal spherical bottle without gravity

To estimate the role of phases and absorption in $\bar{n}$ generation it is sufficient to consider the simplest case of a spherical bottle of radius R with ideal walls and without gravity. This means that the reflection from the walls is always specular, and the flight paths between consecutive collisions with the walls for a given angle of incidence are equal.

In the following we neglect the decrease with time of the n state because of neutron decay, transitions to the $\bar{n}$ state, and losses at every collision with the walls. We consider the neutron during the storage time t_s as a particle with the n state normalized to unity, and its wave function before the first collision with the wall is equal to ξ_n without any phase. After that the wave function of the n state acquires phases appearing at every collision and accumulated during free propagations between collisions. Thus, before the mth collision with the wall the wave function is $(e_n\rho_n)^{m-1}\xi_n$, where e_n is the phase factor $e_n=\exp(ik_nl_f)$ accumulated along the flight path l_f between collisions, $k_n=\sqrt{k^2-H_z}\approx k-H_z/2k$, $k=\sqrt{E}$, and ρ_n is the reflection amplitude, which we approximate by $\exp(i\chi_n)$ with real phase χ_n appearing at every collision with the walls. All the parameters l_f,χ_n depend on the Neutron energy E and the angle of incidence θ on the wall.

The particle in the n state creates an $\bar{n}$ state and we calculate its amplitude. Following the notations of [7] let us denote the amplitude of the $\bar{n}$ state before the mth collision with the wall by β_{m-1}, and find β_m. This amplitude consists of two parts: $\beta_m=\beta_m'+\beta_m''$, where $\beta_m'=e_{\bar{n}}\rho_{\bar{n}}\beta_{m-1}$ is related to reflection of the $\bar{n}$ component with the reflection amplitude $\rho_{\bar{n}}$ and propagation in free space between the mth and $(m+1)$st collisions described by the phase factor $e_{\bar{n}}=\exp(ik_{\bar{n}}l_f)$ with $k_{\bar{n}}=\sqrt{k^2+H_z}\approx k+H_z/2k$. Back transition from $\bar{n}$ to n is neglected.

The second part is created by the n component, which before the mth collision was $(e_n\rho_n)^{m-1}\xi_n$. After mth reflection and propagation to the $(m+1)$st collision, the wave function of this n component becomes $\psi=(e_n\rho_n)^{m-1}\exp(i\hat{k}l_f)\rho_n\xi_n$, where $\hat{k}_n=\sqrt{k^2-2\boldsymbol{H\sigma}}\approx k-\boldsymbol{H\sigma}/k$. Since

$$\exp(-i\boldsymbol{H\sigma}l_f/k)=\cos(Hl_f/k)-i\frac{\boldsymbol{H\sigma}}{H}\sin(Hl_f/k),$$

where $H=\sqrt{H_x^2+H_z^2}\approx H_z$, the product $\exp(i\hat{k}l_f)\xi_n$ contains an $\bar{n}$ component with amplitude $e\gamma$, where

$$e=\exp(ikl_f),\quad \gamma=\frac{H_x}{H}\sin(Hl_f/k)=t_f/\tau.$$

Thus

$$\beta_m''=(e_n\rho_n)^{m-1}\alpha,\quad \alpha=\rho_ne\gamma. \tag{22}$$

Now we can put down the recurrence relation for β_m:

$$\beta_m=e_{\bar{n}}\rho_{\bar{n}}\beta_{m-1}+(e_n\rho_n)^{m-1}\alpha. \tag{23}$$

Let us denote $\beta_m=(e_n\rho_n)^{m-1}\alpha x_m$; then the recurrence relation (25) is reduced to

$$x_m=qx_{m-1}+1,\quad q=\frac{\rho_{\bar{n}}}{\rho_n}\frac{e_{\bar{n}}}{e}=\rho e^{i\phi},$$

$$\rho=|\rho_{\bar{n}}|,\quad \phi=H_z\frac{l_f}{k}+\chi_{\bar{n}}-\chi_n, \tag{24}$$

with $x_1=1$. The recurrence relation (24) has solution $x_m=(1-q^m)/(1-q)$, which means that

$$\beta_m=(e_n\rho_n)^{m-1}\frac{1-q^m}{1-q}\alpha. \tag{25}$$

The parameter ρ in Eq. (24) is less than unity, $\rho^2=|\rho_{\bar{n}}|^2=1-\mu$, because of absorption and scattering of the $\bar{n}$ component. We suppose that absorption is the main part of μ and neglect scattering. Absorption μ means registration of $\bar{n}$ with probability μ.

The total number $N_{\bar{n}}$ of $\bar{n}$ per single Neutron in a storage experiment is equal to the sum $N_{\bar{n}}=N_{\bar{n}}'+N_{\bar{n}}''$ of the number of $\bar{n}$ registered in $M+1$ collisions with the walls during storage before emptying the vessel, $N_{\bar{n}}'$, and of accumulated neutrons, that are registered after emptying the vessel, $N_{\bar{n}}''$:

$$N_{\bar{n}}=N_{\bar{n}}'+N_{\bar{n}}''=\mu\sum_{m=2}^{M+1}|\beta_{m-1}|^2+|\beta_{M+1}|^2$$
$$=\mu p\sum_{m=2}^{M+1}\left|\frac{1-q^{m-1}}{1-q}\right|^2+p\left|\frac{1-q^M}{1-q}\right|^2, \tag{26}$$

where $p=|\gamma|^2=t_f^2/\tau^2$ and $M+1\approx M=t_s/t_f=t_sk/l_f$. From Eq. (26) it follows that, if $q=\rho$, i.e., $\phi=0$, then in the limit $\mu\to 0$ or $\rho\to 1$, the first part $N_{\bar{n}}'$ becomes 0, because nothing is registered at the walls, and the second $N_{\bar{n}}''$, [the last term in Eq. (26)] becomes $M^2p=t_s^2/\tau^2$, which means the coherent accumulation of $\bar{n}$ during storage. This shows that we can call ϕ the "decoherence phase" although this phase is coherently added at every collision with the wall.

In the general case, when $\rho<1$ and $\phi\neq 0$ the number of $\bar{n}$ [Eq. (26)] after summation is

$$N_{\bar{n}}=\frac{\mu p}{|1-q|^2}\left[M-2\,\mathrm{Re}\left(q\frac{1-q^M}{1-q}\right)+|q|^2\frac{1-|q|^{2M}}{1-|q|^2}\right]$$
$$+p\left|\frac{1-q^M}{1-q}\right|^2, \tag{27}$$

where $\mathrm{Re}(x)$ denotes the real part of x. If $\rho^M\ll 1$, i.e., $1/M\ll\mu\ll 1$, we can neglect q^M and reduce Eq. (27) to the form

$$N_{\bar n}=\frac{\mu p}{|1-q|^2}\left[M+\frac{1}{\mu}+\frac{-2\,\mathrm{Re}(q)+3|q|^2}{1-|q|^2}\right]$$
$$=\frac{4p}{16\sin^2(\phi/2)+\mu^2}[M\mu+2+4\sin^2(\phi/2)]$$
$$=\frac{t_s t_f}{\tau^2}M_{eff}, \tag{28}$$

where we introduced the effective number of collisions

$$M_{eff}=\frac{4\mu}{16\sin^2(\phi/2)+\mu^2}. \tag{29}$$

We should take into account that for $\phi=0$ the denominator in Eq. (29) is μ^2. Thus $M_{eff}<4/\mu$. If $\mu=0.1$ we can have $M_{eff}=40$. In that case the UCN experiment, according to the figure of merit (12), for $G=1000$ becomes 160 times more effective than the beam experiment.

However, if ϕ is not small, then $M_{eff}\approx\mu/4$, which means that the UCN experiment is more or less effective only if $\mu\approx 1$, i.e., the $\bar n$ component is completely absorbed at every collision with the wall.

As was pointed out in [7], it is possible to manipulate the decoherence phase ϕ by changing the external field H_z. The phase ϕ consists of two parts. The first one $\phi_1=Hl_f/k=(H/k^2)kl_f$ is related to free flight, and the second one

$$\phi_2=\chi_{\bar n}-\chi_n$$
$$=-2\left[\arccos\left(\frac{k}{\sqrt{u_{\bar n}}}\right)-\arccos\left(\frac{k}{\sqrt{u_n}}\right)\right]$$
$$=-2\arccos\left(\frac{k^2+\sqrt{u_n-k^2}\sqrt{u_{\bar n}-k^2}}{\sqrt{u_n}\sqrt{u_{\bar n}}}\right) \tag{30}$$

is related to the difference of the reflection phases. We can show that these two parts can compensate each other. To do that we represent the phase ϕ in its full form, where $l_f=2R\cos\theta$, and k in Eq. (30) is replaced by its normal component $k\cos\theta$, θ being the incidence angle. For simplicity we introduce dimensionless variables $y^2=k^2/u_{\bar n}$ and $x=y\cos(\theta)$. In these variables the phase ϕ becomes

$$\phi(x,y)=a\frac{x}{y^2}-2\arccos(s[x^2+\sqrt{1/s^2-x^2}\sqrt{1-x^2}]), \tag{31}$$

where $a=2HR/\sqrt{u_{\bar n}}$ and $s=\sqrt{u_{\bar n}/u_n}$ are dimensionless parameters. If R and s are given, say, $R=1$ m and $s^2=0.9$ (according to [14] this is possible), then we can choose the external field H to get $\phi<\mu$ in a sufficiently wide range of x and y.

In Fig. 1 the phase ϕ is represented in a wide range of x for $y=1$ and $a=0.15$, which corresponds to H of the order of 10^{-5} G.

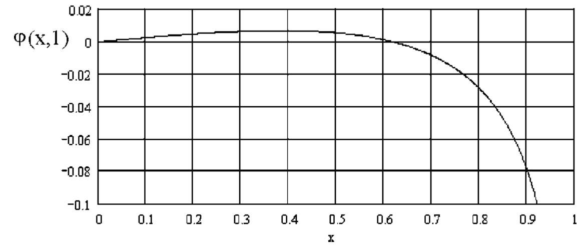

FIG. 1. $\phi(x,1)$ for $a=0.15$ and $s^2=0.9$.

To get some information about the dependence of M_{eff} on the energy k^2 we can average Eq. (29) over the angle θ:

$$\langle M_{eff}(y)\rangle=\int_0^y\frac{2x\,dx}{y^2}\frac{4\mu}{16\sin^2[\phi(x,y)/2]+\mu^2}. \tag{32}$$

The result is presented in Fig. 2 for the same parameters as in Fig. 1, and for $\mu=0.1$.

It is seen that in a sufficiently wide energy range the effective number of collisions is larger than 10, which according to Eq. (12) means high efficiency of the UCN experiment.

C. Nonideal vessel

It is clear that with the same parameters as those found for the ideal bottle we have good conditions for a storage experiment even in a nonideal bottle with rough wall surfaces and with the gravity field included. Indeed, we can easily represent the total $\bar n$ component β_{m-1} before the mth collision, if just before the first collision with the wall the neutron is in the state ξ_n:

$$\beta_{m-1}=\sum_{j=2}^{m}\alpha(j)\prod_{i=1}^{j-1}e_n(i)\rho_n(i)\prod_{i=j}^{m}e_{\bar n}(i)\rho_{\bar n}(i)$$
$$=\prod_{i=1}^{m}e_n(i)\rho_n(i)\sum_{j=2}^{m}\alpha(j)\prod_{i=j}^{m}q(i),$$

where

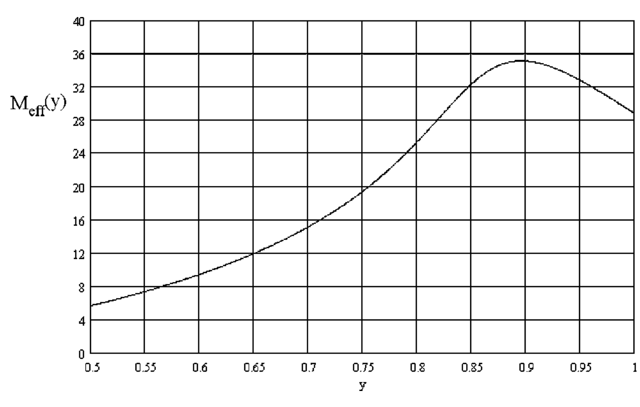

FIG. 2. $M_{eff}(y)$ for $a=0.15$, $s^2=0.9$, and $\mu=0.1$.

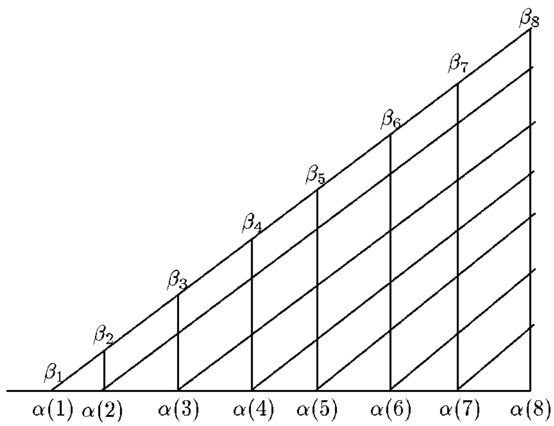

FIG. 3. Diagram of $\bar{n}$ accumulation in storage vessel.

$$\alpha(i)=\rho_n(i)e(i)\gamma(i), \quad \gamma(i)=\frac{H_x}{H}\sin\left(H\frac{l_i}{k}\right),$$

$$q(i)=\rho(i)\exp(i\phi_i),$$

$$e_n(i)=\exp(ik_n l_i), \quad e_{\bar{n}}(i)=\exp(ik_{\bar{n}} l_i), \quad e(i)=\exp(ikl_i),$$

l_i is the free flight path in the flight before the ith collision with the wall, $\rho_{n,\bar{n}}(i)$ are reflection amplitudes with angles at the $(i-1)$st collision of

$$\rho_n(i)=\exp(i\chi_n[i]), \quad \rho_{\bar{n}}(i)=\rho(i)\exp(i\chi_{\bar{n}}[i]),$$

$$\rho(i)=|\rho_{\bar{n}}(i)|, \quad \phi_i=H\frac{l_i}{k}+\chi_{\bar{n}}(i)-\chi_n(i),$$

and we set $e_n(1)\rho_n(1)=1$ and $\rho_{\bar{n}}(2)e_{\bar{n}}(2)=1$.

All that can be illustrated by Fig. 3, where the horizontal line represent the Neutron in the state ξ_n. The points on it represent collision moments, and the segment after the ith point represents the phase factor $e_n(i)\rho_n(i)$. The vertices $\alpha(i)$ represent the $\bar{n}$ component created by the Neutron after the ith collision with the wall. The inclined lines represent the history of the $\bar{n}$ component: the points on them are the collisions, which correspond to multiplication by $\rho_{\bar{n}}$, and the segments after them are phase factors $e_{\bar{n}}(i)$.

The set of points on the vertical line at the ith collision represent the coherent sum β_{m-1} of all the $\bar{n}$ components surviving to this point.

The fraction of $\bar{n}$ registered during storage is

$$N'_{\bar{n}}=\left|\sum_{m=2}^{M}\mu(m)\sum_{j=2}^{m}\alpha(j)\prod_{i=j}^{m}q(i)\right|^2,$$

where $\mu(m)$ is the absorption probability at the mth collision. The fraction of $\bar{n}$ created during storage and registered with 100% efficiency after emptying the vessel is

$$N''_{\bar{n}}=\left|\sum_{j=2}^{M}\alpha(j)\prod_{i=j}^{M}q(i)\right|^2.$$

Thus

$$N_{\bar{n}}=\left|\sum_{m=2}^{M}\mu(m)\sum_{j=2}^{m}\alpha(j)\prod_{i=j}^{m}q(i)\right|^2+\left|\sum_{j=2}^{M}\alpha(j)\prod_{i=j}^{M}q(i)\right|^2, \tag{33}$$

and it is easy to check that Eq. (33) is reduced to Eq. (27) when all $m(m)$ and $q(i)$ are equal.

The flight paths l_i and angles at reflections are random, and therefore $q(i)$ are also random variables; however, for small μ and small $H\approx 10^{-5}$ G, all the phases $\phi(i)$ are small, and we can replace $q(i)$, $\mu(i)$ with $\langle q\rangle+\delta q$, $\mu(i)=\langle\mu\rangle+\delta\mu$, where $\langle x\rangle$ is an average value of x and δx is a random variable with zero average. Calculations with average $\langle\mu\rangle$ and $\langle q\rangle$ will give the same result as for the ideal spherical bottle, and the corrections related to $\delta q\,\delta\mu$, even for large dispersions $\langle(\delta q)^2\rangle\approx\langle q\rangle^2$, and $\langle(\delta\mu)^2\rangle=\langle\mu\rangle^2$ will not spoil the result essentially.

IV. CONCLUSION

In this paper we derived a figure of merit for an UCN experiment compared to a beam experiment (12), and investigated the requirements to get high efficiency of the UCN experiment.

We investigated also once again the effect of UCN collisions with the wall on $\bar{n}$ production. We found that collisions can even produce the $\bar{n}$ component; however, the probability of such production [Eq. (21)] is negligible.

We considered also the effect of absorption (the coefficient μ) and of the phase difference ϕ of the n and $\bar{n}$ component propagations and reflection from the walls. We showed that the effective number of flights (29) M_{eff} between collisions, during which the $\bar{n}$ component is accumulated, can be large, $M_{eff}\propto 1/\mu$, if $\phi\leqslant\mu$, but becomes small, $M_{eff}\propto\mu<1$, when $\phi\gg\mu$.

We have also shown that with an external magnetic field we can control the phase difference ϕ and reduce it to get high M_{eff} in a wide range of UCN spectra inside the storage bottle, as is shown in Fig. 2.

Our considerations confirm and give additional support to the results of [6,7].

ACKNOWLEDGMENTS

The author is grateful to Yu. Kamyshkov and A. Serebrov for interest in this work.

[1] R. Golub and J. M. Pendlebury, Rep. Prog. Phys. **42**, 439 (1979).

[2] H. Yoshiki, KEK Report No. 180-10, 1981.

[3] K. G. Chetyrkin, M. V. Kazarnovsky, V. A. Kuzmin, and M. E. Shaposhnikov, Phys. Lett. **99B**, 358 (1981).

[4] M. V. Kazarnovsky, V. A. Kuzmin, and M. E. Shaposhnikov "n-$\overline{n}$ Oscillations: On a Possibility of Observation with Ultracold Neutrons" Report No. IYaI-P-0223, Moscow, 1981; Pis'ma Zh. Eksp. Teor. Fiz. **34**, 49 (1981).

[5] S. Marsh and K. W. McVoy, Phys. Rev. D **28**, 2793 (1983).

[6] R. Golub and H. Yoshiki, Nucl. Phys. **A501**, 869 (1989).

[7] R. Golub, H. Yoshiki, and R. Gaehler, Nucl. Instrum. Methods Phys. Res. A **284**, 16 (1989).

[8] A. Bottino, V. de Alfaro, and C. Giunti, Z. Phys. C **47**, 31 (1990).

[9] S. K. Lamoreaux, R. Golub, and J. M. Pendlebury, Europhys. Lett. **14**, 503 (1991).

[10] H. Yoshiki and R. Golub, Nucl. Phys. **A536**, 648 (1992).

[11] V. K. Ignatovich, *The Physics of Ultracold Neutrons (UCN)* (Clarendon Press, Oxford, 1990).

[12] V. K. Ignatovich, Pis'ma Zh. Eksp. Teor. Fiz. **28**, 311 (1978).

[13] D. A. Korneev, V. I. Bodnarchuk, and V. K. Ignatovich, Pis'ma Zh. Eksp. Teor. Fiz. **63**, 900 (1996); in *Proceedings of the International Symposium on Advances in Neutron Optics and Related Research Facilities, Kumatori, 1996* [J. Phys. Soc. Jpn. **65**, 7 (1996)].

[14] Ye. S. Golubeva and L. A. Kondratyuk, Nucl. Phys. B (Proc. Suppl.) **56A**, 103 (1997).