Nontrivial generalizations of the Schwinger pair production result

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We present new, nontrivial generalizations of the recent Tomaras-Tsamis-Woodard extension of the original Schwinger formula for charged pair production in a constant electric field. That extension generalized the Schwinger result to electric fields $E_3(x_\pm)$ dependent upon one *or* the other light-cone coordinates, x_+ or x_-, $x_\pm = x_3 \pm x_0$; the present work generalizes their result to electric fields $E_3(x_+, x_-)$ dependent upon *both* coordinates. Displayed in the form of a final, functional integral, or equivalent linkage operation, our result does not appear to be exactly calculable in the general case; and we give a simple, approximate example when $E_3(x_+, x_-)$ is a slowly varying function of its variables. We extend this result to the more general case where $\vec{E}$ can point in a varying direction, and where an arbitrary magnetic field $\vec{B}$ is present; both extensions can be cast into the form of Gaussian-weighted functional integrals over well defined factors, which are amenable to approximations depending on the nature and variations of the fields.

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A recent, nontrivial generalization of Schwinger's 1951 calculation of the probability/vol sec for e^+e^- production in a constant electric field has been given by Tomaras, Tsamis and Woodard (TTW) [1] in which the electric field may depend upon either light-cone coordinates $x_\pm = x_3 \pm x_0$, but not upon both. The form of their result is exactly the same as Schwinger's [2], which raises the question as to whether further generalizations are possible, and if so, what form they would take.

Shortly after the TTW paper appeared, a second and independent calculation of pair production in an electric field depending on either x_+ or x_- was performed [3] using functional techniques, and verifying the result of Ref. [1]. Those functional methods are here employed to attempt a further generalization to the case where the electric field depends upon both light-cone variables, $E = E_3(x_+, x_-)$. The result is more complicated than the original Schwinger form, although the important (and nontrivial) essential singularity in the region of small coupling is apparently preserved. The exact statement here requires the evaluation of an additional (and nontrivial) functional integral; however, for certain situations, such as particle production in the overlap volume of a pair of high intensity lasers [4], reasonable kinematic approximations are surely justified. If

$$\ln P_0 = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi m^2/gE}, \quad \alpha = \frac{g^2}{4\pi}$$

denotes the logarithmic density of the exact vacuum-persistence probability density of the Schwinger constant field calculation, the main result of this paper can be expressed as

$$\ln P_0' = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n\pi m^2/gE} \mathcal{M}_n$$

with $E = E(x_+, x_-)$, $\tau_n = n\pi/gE$, $J(\lambda_1, \lambda_2) = \theta(\lambda_1 - \lambda_2)\lambda_2 + \theta(\lambda_2 - \lambda_1)\lambda_1 - \lambda_1\lambda_2$, and

$$\mathcal{M}_n = (-1)^n e^{4m^2\tau_n \int\int_0^1 d\lambda_1 d\lambda_2 [\delta/\delta v_+(\lambda_1)] J(\lambda_1,\lambda_2) [\delta/\delta v_-(\lambda_2)]} \cos\left[g\tau_n \int_0^1 d\lambda\, E(x_+ - m^{-1}v_+(\lambda), x_- - m^{-1}v_-(\lambda)) \right]\Bigg|_{v_\pm \to 0} \quad (1)$$

where the "linkage operation" of Eq. (1) may be cast into a corresponding functional integral. It does not seem to be possible to evaluate Eq. (1) exactly, although kinematic approximations are certainly possible. One of these will be used below to illustrate the formula. Note that if E depends on only one variable x_+ or x_-, $\mathcal{M}_n \to 1$, and the TTW extension of the Schwinger result is obtained.

We hold to the notation and technique of Ref. [3], beginning with the exact statement of the vacuum persistence amplitude in the absence of radiative corrections

$$\langle 0|S|0\rangle = e^{L[A_{ext}]}. \tag{2}$$

The vacuum persistence probability of the present problem, here called P_0', is then given by $P_0' = \exp[2\mathrm{Re}L[A_{ext}]]$, and the probability of producing (at least) one charged pair is $P_1' = 1 - P_0'$. Note that $L[A]$ is really $L[F]$, since L is rigorously gauge invariant; but we shall continue to use the vector-potential description, beginning with the exact Fradkin representation [5]

$$L[A] = -\frac{1}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2} \exp\left[i\int_0^s ds' \sum_\mu \frac{\delta^2}{\delta v_\mu^2(s')}\right] \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot\int_0^s ds' v(s')}$$
$$\times \int d^4x e^{-ig\int_0^s ds' v_\mu(s') A_\mu(x - \int_0^{s'} v)} \mathrm{tr}(e^{g\int_0^s ds' \sigma\cdot F(x-\int_0^{s'} v)})_+ \Big|_{v_\mu\to 0} - (g\to 0). \tag{3}$$

The functional linkage operation of Eq. (3) may be recast into that of a Gaussian-weighted functional integral, since

$$e^{i\int_0^s ds[\delta^2/\delta v^2(s')]}\mathcal{F}[v]|_{v\to 0} = N\int d[v] e^{(i/4)\int_0^s ds v^2(s')}\mathcal{F}[v]$$

for arbitrary $\mathcal{F}[v]$, with the normalization N given by

$$N^{-1} = \int d[v] \exp\left[\frac{i}{4}\int_0^s ds v^2(s')\right].$$

The linkage formulation, which dispenses with normalization constants, is somewhat more convenient, and will be followed whenever possible.

To represent a given, external field in the $\hat{3}$ direction, we may choose $A_\mu = (\vec{A}_\perp, A_3, A_0) \to (0, A_3, A_0) \to (0, A_+, A_-)$ with $\vec{A}_\perp = 0$, $A_\pm(x) = \Sigma_\mu n_\pm^\mu A_\mu(x_+, x_-)$. Here, $n_\pm^\mu = (0,0,1,\mp 1)$, so that $n_\pm \cdot a = a_\pm = a_3 \pm a_0$. Note that $n_+^2 = n_-^2 = 0$, $n_+\cdot n_- = 2$, and that $p\cdot A \to p_3 A_3 - p_0 A_0 = \frac{1}{2}(p_+ A_- + p_- A_+)$. We shall choose the gauge specified by $A_- = 0$, so that $A^2 = A_\perp^2 + A_+ A_- = 0$, and $E(x_+, x_-) = -(\partial A_+/\partial x_+)(x_+, x_-)$.

As in Ref. [3], we extract the $v_\mu(s'')$ dependence inside the arguments of A_μ and $F_{\mu\nu}$, or of A_+ and E, by introducing a functional form of unity into Eq. (3), replacing $\mathcal{F}[\int_0^{s'} v_+(s''), \int_0^{s'} v_-(s'')]$ by

$$\int d[u_+] \int d[u_-] \mathcal{F}[u_+(s'), u_-(s')] \delta\left[u_+(s') - \int_0^{s'} v_+(s'')\right] \delta\left[u_-(s') - \int_0^{s'} v_-(s'')\right] \tag{4}$$

where the δ-functional notation means that when the region 0–s is broken up into many small intervals labeled by the discrete indices s_i, $i = 1, \dots, N$ (with $N\to\infty$, subsequently), $\delta[Q(s')] \to \Pi_{i=1}^N \delta(Q(s_i))$.

Note that since we are treating each $v_\pm(s')$ as a continuous (although not necessarily differentiable) function of s', the $u_\pm(s')$ introduced in Eq. (4) have, at least, a continuous first derivative.

Each δ function of Eq. (4), and the overall δ functional, may be written in terms of a standard Fourier representation; so that, in the continuous limit, Eq. (4) may be replaced by

$$N'^2 \int d[u_+] \int d[u_-] \mathcal{F}[u_+(s'), u_-(s')] \int d[\Omega_+] \int d[\Omega_-] e^{i\int_0^s ds' [u_+(s')\Omega_+(s') + u_-(s')\Omega_-(s')]}$$
$$\times e^{-i\int_0^s ds' [\Omega_+(s')\int_0^{s'} v_+(s'') + \Omega_-(s')\int_0^{s'} v_-(s'')]} \tag{5}$$

where N' is an appropriate normalization factor (which disappears from the final result). With the aid of Abel's trick, the last exponential factor of Eq. (5) may be written as

$$\exp\left\{-i\int_0^s ds' v_\mu(s')\left[n_+^\mu \int_{s'}^s ds'' \Omega_+(s'') + n_-^\mu \int_{s'}^s ds'' \Omega_-(s'')\right]\right\}$$

so that the entire v-linkage operation is immediate:

$$e^{i\int_0^s ds\,[\delta^2/\delta v^2(s')]}\, e^{i\int_0^s ds'\, v_\mu(s')[p_\mu - gA_\mu(s') - n_+^\mu \int_{s'}^s ds''\Omega_+(s'') - n_-^\mu \int_{s'}^s ds''\Omega_-(s'')]}\Big|_{v\to 0}$$
$$= \exp\left\{ -i\int_0^s ds' \left[p_\mu - gA_\mu(s') - n_+^\mu \int_{s'}^s \Omega_+ - n_-^\mu \int_{s'}^s \Omega_- \right]^2 \right\} \quad (6)$$

where we have used the notation $A_\mu(s') \equiv A_\mu(x_+ - u_+(s'), x_- - u_-(s'))$. As in Ref. [3], the explicitly quadratic Ω_+ and Ω_- terms of the exponential factor of Eq. (6) are removed because $n_+^2 = n_-^2 = 0$ but there remains a nonzero $\Omega_+ \cdot \Omega_-$ cross term, so that Eq. (6) becomes

$$\exp\left\{ -isp^2 + igp_- \int_0^s ds' A_+(s') + 2ip_+ \int_0^s ds' s' \Omega_+(s') + 2ip_- \int_0^s ds' s' \Omega_-(s') - 2ig\int_0^s ds' \Omega_+(s') \int_0^{s'} ds'' A_+(s'') \right.$$
$$\left. - 4i\int_0^s ds_1 \int_0^s ds_2 \Omega_+(s_1) h(s_1,s_2) \Omega_-(s_2) \right\} \quad (7)$$

where $h(s_1,s_2) = \theta(s_1 - s_2)s_2 + \theta(s_2 - s_1)s_1 = \frac{1}{2}(s_1 + s_2 - |s_1 - s_2|)$ is that function of proper time which always appears when constructing representations of $G_c[A]$ [6]. In writing Eq. (7), we have used the ‘‘inverse’’ form of Abel’s manipulation.

Performing the $\int d^4p$, one requires

$$\int d^4p\, e^{-isp^2 + ip\cdot[n_- g\int_0^s A(s') + 2n_+ \int_0^s s'\Omega_+(s') + 2n_- \int_0^s s'\Omega_-(s')]}$$
$$= -i\left(\frac{\pi^2}{s^2}\right) \exp\left\{ 4i\int \int_0^s ds_1 ds_2 \Omega_+(s_1) \left(\frac{s_1 s_2}{s}\right) \Omega_-(s_2) + \frac{2i}{s}\int_0^s s'\Omega_+(s') g\int_0^s A_+(s') \right\} \quad (8)$$

where we have again used the properties : $n_+^2 = n_-^2 = 0$, $n_+ \cdot n_- = 2$. The new $\Omega_+\Omega_-$ term of Eq. (8) may be combined with the previous, like term of Eq. (7), forming the combination $\exp\{-4i\int\int_0^s \Omega_+ J \Omega_-\}$, $J(s_1,s_2) = h(s_1,s_2) - (s_1 s_2)/s$; and the functional integral over $\Omega_\pm$ may be written as

$$e^{4i\int [\delta/\delta u_+(s_1)]\, J(s_1,s_2)\, [\delta/\delta u_-(s_2)]} N'^2 \int d[\Omega_+] \int d[\Omega_-] e^{i\int_0^s ds' u_-(s')\Omega_-(s')} e^{i\int_0^s ds' \Omega_+(s')[u_+(s') + 2g(s'/s)\int_0^s ds'' A_+(s'') - 2g\int_0^{s'} ds'' A_+(s'')]}$$
$$= e^{4i\int (\delta/\delta u_+) J (\delta/\delta u_-)} \delta[u_-(s')]\, \delta\left[u_+(s') + 2g\frac{s'}{s}\int_0^s ds'' A_+(s'') - 2g\int_0^{s'} ds'' A_+(s'') \right]. \quad (9)$$

Equation (9) is multiplied by the $u_\pm$-dependent factor

$$\mathrm{tr}(e^{g\int_0^s ds' \sigma\cdot F(x_+ - u_+(s'), x_- - u_-(s'))})_+ = 4\cosh\left(g\int_0^s ds' E(s') \right) \quad (10)$$

where $E(s') \equiv E(x_+ - u_+(s'), x_- - u_-(s'))$. The $\int d[u_+] \int d[u_-]$ may be evaluated by transferring (by an integration by parts over each u variable) the differential $\delta/\delta u_\pm$ operators which act upon the δ functionals of Eq. (9) to equivalent operations upon the $u_\pm$ dependence of Eq. (10)

$$4\left|\det\frac{\delta u}{\delta f}\right| e^{4i\int (\delta/\delta u_+) J (\delta/\delta u_-)} \times \cosh\left(g\int_0^s ds' E(s') \right)\Bigg|_{u_-(s')=0,\, u_+(s')=u(s')}$$

where $|\det(\delta u/\delta f)|$ is the determinant of the transformation from the variables $u_+(s')$ to the variables $f(s')$, the latter given by the argument of the δ functional produced by the $\int d[\Omega_+]$. Here, $u(s')$ is the solution of the integral equation

$$u(s') = 2g\left[\int_0^{s'} ds'' A_+(s'') - \frac{s'}{s}\int_0^s ds'' A_+(s'') \right] \quad (11)$$

with $u_-(s') = 0$, corresponding to the other δ functional of Eq. (9).

By exactly the same arguments as in [3], the only possible solution to Eq. (11) is $u(s') = u'(s') = 0$, which means that, as in [3],

$$\left|\det\frac{\delta u}{\delta f}\right| = \frac{gsE(x_+,x_-)}{\sinh(gsE(x_+,x_-))} \tag{12}$$

with the difference that this E can depend upon both x_+ and x_-.

The sign of E entering into Eqs. (10) and (12) is irrelevant, and we shall suppose it is positive. Our expression for $L[A]$ can therefore be put into a form which closely resembles that of Schwinger, and of Ref. [3]

$$L[A] = \frac{i}{8\pi^2}\int d^4x \int_0^\infty \frac{ds}{s^2} e^{-ism^2} \frac{gE}{\sinh(gsE)} e^{4ism^2 \int\int d\lambda_1 d\lambda_2 [\delta/\delta v_+(\lambda_1)] J(\lambda_1,\lambda_2) [\delta/\delta v_-(\lambda_2)]}$$
$$\times \cosh\left[gs\int_0^1 d\lambda E(x_+ - m^{-1}v_+(\lambda), x_- - m^{-1}v_-(\lambda))\right]_{v_\pm\to 0} - (g\to 0). \tag{13}$$

In writing Eq. (13), we have observed that, in the interval 0–s, any continuous function $u_\pm(s')$ may be given by a Fourier series representation as a function of (s'/s), and we therefore replace $u_\pm(s')$ by $m^{-1}v_\pm(s'/s) \equiv m^{-1}v_\pm(\lambda)$, where $\lambda = s'/s$, and the factor m^{-1} is inserted for dimensional reasons (and will cancel away when the functional derivatives $\delta/\delta v_\pm$ are taken). However, because

$$\frac{\delta u_\pm(s_1)}{\delta u_\pm(s_2)} = \delta(s_1 - s_2) = \frac{1}{s}\delta(\lambda_1 - \lambda_2),$$

$$\frac{\delta v_\pm(s_1)}{\delta v_\pm(s_2)} = \delta(\lambda_1 - \lambda_2)$$

one must adopt the relation $\delta/\delta u_\pm(s') = (m/s)[\delta/\delta v_\pm(\lambda')]$. After extracting an overall factor of s, $J(\lambda_1,\lambda_2)$ is understood to be given by

$$J(\lambda_1,\lambda_2) = \theta(\lambda_1-\lambda_2)\lambda_2 + \theta(\lambda_2-\lambda_1)\lambda_1 - \lambda_1\lambda_2.$$

Extracting the Re$L[A]$ from an expression involving an operator such as that of Eq. (13) requires some further thought, and we therefore imagine that the linkage operator of Eq. (13) is expanded in powers of J, so that the rotation of contour $s \to -i\tau$ is permitted; here, that expansion yields powers of $4m^2\tau\int\int_0^1[\delta/\delta v_+(\lambda_1)]J(\lambda_1,\lambda_2)[\delta/\delta v_-(\lambda_2)]$, which generate real results because E and all of its derivative are real. Under this contour rotation, one again finds that the contributions to Re$L[A]$ arise from zeroes of the denominator $\sin(g\tau E)$, occurring for $\tau \to \tau_n - i\varepsilon$, $\tau_n = n\pi/gE$; and we then sum up all J-dependent terms, so that

$$2\text{Re}L[A] = -\frac{\alpha}{\pi^2}\int d^4x E^2 \sum_{n=1}^\infty \frac{1}{n^2} e^{-n\pi m^2/gE}\mathcal{M}_n(x_+,x_-)$$

with $\mathcal{M}_n$ defined in Eq. (1). This seems to be as far as the exact analysis can be performed.

Without approximation, the linkage operation of Eq. (13) may be converted to a pair of functional integrals,

$$e^{iab\int(\delta/\delta v_+)J(\delta/\delta v_-)}\mathcal{F}[v_+,v_-]|_{v_\pm\to 0}$$
$$= \det[J]\int d[\chi_+]\int d[\chi_-] e^{i\int \chi_+ J \chi_-}\mathcal{F}\left[a\int J\chi_+, b\int J\chi_-\right] \tag{14}$$

and mean field or stationary phase approximations derived for the right-hand side of Eq. (14). Perhaps the simplest approximation of all occurs when $E(x_+,x_-) = E_+(x_+) + E_-(x_-)$, a form suggested by recent estimates of pair production in the overlap region of two, crossed, high intensity lasers [4]. If all the derivatives of E of order higher than the second are neglected,

$$\int_0^1 d\lambda E_\pm(x_\pm - m^{-1}v_\pm(\lambda))$$
$$\simeq E_\pm(x_\pm) - m^{-1}\int_0^1 d\lambda v_\pm(\lambda)\frac{\partial E_\pm(x_\pm)}{\partial x_\pm}$$
$$+ \frac{m^{-2}}{2}\int_0^1 d\lambda v_\pm^2(\lambda)\frac{\partial^2 E_\pm(x_\pm)}{\partial x_\pm^2} + \cdots$$

which approximation assumes that the fractional variations of the electric field are very small over distances on the order of the Compton wavelength.

Then, it is easy to see that the linkage operation yields

$$\mathcal{M}_n = \cos\left[\frac{a^2}{2}Q(\alpha_+^2\gamma_- + \alpha_-^2\gamma_+)\right]\exp\left[-4a^2\alpha_+\alpha_- R\right.$$
$$\left. - \frac{1}{2}\text{Tr}\ln(1 + a^2\gamma_+\gamma_- I)\right] \tag{15}$$

where $a = 4\pi n/gE$, $\alpha_\pm = n\pi(\partial/\partial x_\pm)\ln(E/m^2)$, $\gamma_\pm = (n\pi/E)(\partial^2/\partial x_\pm^2)E$, $E = E_+ + E_-$ and

$$Q = \int\int_0^1 d\lambda_1 d\lambda_2 \left\langle \lambda_1 \left| I\left(\frac{1}{1 + a^2\gamma_+\gamma_- I}\right)\right|\lambda_2\right\rangle$$

$$R=\int\int_0^1 d\lambda_1 d\lambda_2\langle\lambda_1|J(1+a^2\gamma_+\gamma_- I)^{-1}|\lambda_2\rangle$$

with

$$I(\lambda_1,\lambda_2)=\langle\lambda_1|I|\lambda_2\rangle=\int_0^1 d\lambda J(\lambda_1,\lambda)J(\lambda,\lambda_2).$$

Note that, formally, for large n, $R\sim n^{-4}$ while $a^2x_+x_-\sim n^4$, so that the exponential terms of Eq. (15) cannot grow rapidly with n. One therefore infers that the essential singularity structure of the original Schwinger result is preserved.

For an electric field E_3, which depends upon $x_\perp$, one can retain the condition $A_-=A_\perp=0$, but require A_+ to depend on $x_\perp$, as well as x_+ and x_-. One then introduces variables $u_\perp(s')$ and $\Omega_\perp(s')$, and following the above analysis finds

$$L[A]=\frac{i}{32\pi^2}\int_0^\infty\frac{ds}{s^3}e^{-ism^2}\int d^4x\int d[u_+]\int d[u_-]\int d[u_\perp]\mathrm{tr}(e^{g\int_0^s\sigma\cdot F})_+$$
$$\times e^{4i\int(\delta/\delta u_+)J(\delta/\delta u_-)+i\int(\delta/\delta u_\perp)J(\delta/\delta u_\perp)}\delta\left[u_+(s')+2g\frac{s'}{s}\int_0^s A_+(s'')-2g\int_0^{s'}A_+(s'')\right]\delta[u_-(s')]\delta[u_\perp(s')]. \quad (16)$$

An integration by part on the $u_\pm$, $u_\perp$ dependence then allows the linkage operator to act upon the $\mathrm{tr}(e^{g\int_0^s\sigma\cdot F})_+$ term only; and one sees that the Jacobian of the transformation between the $u_\pm(s')$ and $f(s')$ variables is exactly the same as in Eq. (12), except that here $E=E_3(x_+,x_-,x_\perp)$. If we restrict $\vec{E}$ to lie in the 3 direction, then again, $\mathrm{tr}(e^{g\int_0^s\sigma\cdot F})_+\to 4\cosh(\int_0^s ds' gE(s'))$ and the limits $u_\pm=u_\perp=0$ are to be taken after the linkage operator of Eq. (16) acts upon the $u_\pm$, $u_\perp$ dependence inside $4\cosh(\int_0^s ds' gE(s'))$, with $E=E_3(x_+-u_+(s'),\ x_-u_-(s'),\ x_\perp-u_\perp(s'))$.

For arbitrary $\vec{E}$ and $\vec{B}$ fields, one must include at the very beginning $A_\perp$ $(x_+,\ x_-,\ x_\perp)\neq 0$, although it is still convenient to retain the gauge condition $A_-=0$. Now there will appear in this final generalization of Eq. (16) the extra factor

$$\mathcal{F}[u]=\exp\left[-ig\int_0^s ds'' A_\perp^2(s'')+ig/s\left(\int_0^s ds'' A_+(s'')\right)^2\right]$$

where, following the notation of Eq. (6), $A_\perp(s')=A_\perp(x_+-u_+(s'),x_--u_-(s'),x_\perp-u_\perp(s'))$, and $\mathcal{F}[u]$ multiplies the trace of the now more complicated ordered exponential (OE) $(e^{g\int_0^s\sigma\cdot F})_+$. Following techniques developed for unitary OEs [7], the present OE can be approximately evaluated analytically when $\vec{E}$ and $\vec{B}$ directions change in ways that can be characterized as slowly varying (''adiabatic'') or rapidly fluctuating (''stochastic''); and the same linkage operator as in Eq. (16) is then to act upon the product $\mathcal{F}[u](e^{g\int_0^s\sigma\cdot F})_+$. Note that there will appear another Jacobian, involved in the variable change from $u_\perp(s')$ to $f_\perp(s')$, as in Eq. (12), and both Jacobians will simultaneously involve $\vec{E}$ and $\vec{B}$. Alternatively, if one wishes to avoid the direct evaluation of these Jacobians, one can rewrite this generalization of Eq. (16) in terms of functional integrals, and attempt to use mean field or stationary phase methods for their approximate evaluation. This extension of Eq. (16) provides a functional representation of $L[A]$ appropriate for the most general choice of $\vec{E}$ and $\vec{B}$ fields. When the latter are constants, and in particular for $\vec{B}=0$, it reduces immediately to Schwinger's 1951 solution, while it generates a definition of $L[A]$ at any stage inbetween.

Finally, a more symmetric formulation, explicitly depending only on the $F_{\mu\nu}$ may be obtained by starting from formula

$$L[A]=-\frac{1}{2}\int_0^\infty\frac{ds}{s}e^{-ism^2}\int d^4x\int\frac{d^4p}{(2\pi)^4}e^{i\int_0^s(\delta^2/\delta v^2)}e^{ip\cdot\int_0^s v}N'\int d[\Omega]\int d[u]e^{i\int_0^s u_\mu\Omega_\mu}e^{-i\int_0^s\Omega_\mu\int_0^{s'}v_\mu}$$
$$\times e^{-ig\int_0^s ds' v_\mu(s')u_\nu(s')\int_0^1\lambda d\lambda F_{\mu\nu}(x-\lambda u(s'))}\mathrm{tr}(e^{g\int_0^s\sigma\cdot F(x-u(s'))})_+-(g\to 0) \quad (17)$$

where every factor of $\int_0^{s'}ds'' v_\mu(s'')$ has been replaced by $u_\mu(s')$. Performing the linkage operation, $\int d^4p$ and $\int d[\Omega]$, then leads to the result

$$L[A]=\frac{i}{32\pi^2}\int_0^\infty\frac{ds}{s^3}e^{-ism^2}\int d^4x\prod_{\mu=1}^{4}\int d[u]\Delta[u_\mu]e^{i\int(\delta/\delta u_\mu)J(\delta/\delta u_\mu)}\mathrm{tr}(e^{g\int_0^s\sigma\cdot F})_+e^Q \tag{18}$$

$$\Delta[u_\mu]=\delta\left[u_\mu(s')-2g\int_0^s ds''\left(\theta(s'-s)-\frac{s'}{s}\right)\int_0^1\lambda d\lambda F_{\mu\nu}(x-\lambda u(s''))u_\nu(s'')\right]$$

$$Q=-ig^2\int_0^s ds'\int_0^1\lambda d\lambda\int_0^1\lambda' d\lambda'\int_0^s ds'' u_\nu(s')u_\sigma(s'')F_{\mu\nu}(x-\lambda u(s'))F_{\mu\sigma}(x-\lambda u(s''))\left[\delta(s'-s'')-\frac{1}{s}\right].$$

The only solutions for each $u_\mu(s')$ allowed by the product of the four δ functional of Eq. (18) are $u_\mu(s')\equiv 0$, although the Jacobians obtained from each variable change $u_\mu(s')\to f_\mu(s')$ will each be unity only for the Schwinger case of constant $F_{\mu\nu}$; in this latter situation, the $\exp[i\int(\delta/\delta u_\mu)J(\delta/\delta u_\mu)]$ operation may be recast into that of soluble Gaussian functional integrals over the $u_\mu(s')$, leading back to Schwinger's original result.

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APPENDIX

We present here an independent calculation of the determinant of Eq. (12), defined by the transformation from the functional coordinates $u_+(s')$ to $f(s')$, where

$$f(s')=u_+(s')+2g\left[\frac{s'}{s}\int_0^s ds'' A_+(s'')-\int_0^{s'}ds'' A_+(s'')\right] \tag{A1}$$

and $A_+(s')=A_+(x_+-u_+(s'),x_--u_-(s'))$. Because of the delta functionals of Eq. (9), every $u_-(s')=0$, while the only possible solution of $f(s')=0$ corresponds to $u_+(s')=0$; this latter point has been discussed in Ref. [3], but for completeness is reproduced below.

The inverse determinant, constructed from the quantity

$$\frac{\delta f(s_1)}{\delta u_+(s_2)}=\delta(s_1-s_2)+2g\left[\frac{s_1}{s}-\theta(s_1-s_2)\right]E(x_+,x_-)$$
$$\equiv\langle s_1|(1+M)|s_2\rangle \tag{A2}$$

is therefore to be evaluated at $u_\pm(s')=0$, as stated. Reference [3] displays that determinant as the solution of a relevant differential equation; here, one follows a direct evaluation, by introducing the operators

$$\mathcal{S}=\kappa\int_0^s ds'\int_0^s ds''\left(\frac{s'}{s}\right)|s'\rangle\langle s''|,$$

$$\Theta=\kappa\int_0^s ds'\int_0^s ds''\theta(s'-s'')|s'\rangle\langle s''|$$

where the unit, operator in this space is given by

$$1\mathrm{I}=\int_0^s ds'|s'\rangle\langle s'|,\quad\text{so that }\ \langle s_1|s_2\rangle=\delta(s_1-s_2).$$

With $\kappa=2gE$, one has $\langle s_1|\mathcal{S}|s_2\rangle=\kappa(s_1/s)$, and $\langle s_1|\Theta|s_2\rangle=\kappa\theta(s_1-s_2)$, so that $M=\mathcal{S}-\Theta$.

The quantity $\det[\delta f/\delta u_+]$ can be written as $\exp[\mathrm{tr}\ln(1+M)]$, and a direct evaluation of the latter's exponential factor proceeds as follows. With $\mathrm{tr}\ln(1+M)=\mathrm{tr}\ln(1+\mathcal{S}-\Theta)=\mathrm{tr}\ln(1-\Theta)+\mathrm{tr}\ln(1+\mathcal{S}[1/(1-\Theta)])$, it is easy to see that

$$\left\langle s_1\left|\frac{1}{1-\Theta}\right|s_2\right\rangle=\delta(s_1-s_2)+\kappa\theta(s_{12})$$
$$\times[1+\kappa s_{12}+(\kappa s_{12})^2/2!+\cdots]$$
$$=\langle s_1|1|s_2\rangle+e^{\kappa s_1}\langle s_1|\Theta|s_2\rangle e^{-\kappa s_2},$$

$$s_{12}=s_1-s_2$$

so that

$$\left\langle s_1\left|\mathcal{S}\frac{1}{1-\Theta}\right|s_2\right\rangle=\frac{\kappa s_1}{s}\left[1+\int_{s_2}^s ds'\kappa e^{\kappa(s'-s_2)}\right]$$
$$=\kappa\left(\frac{s_1}{s}\right)e^{\kappa(s-s_2)}.$$

Similarly,

$$\left\langle s_1\left|\left(\mathcal{S}\frac{1}{1-\Theta}\right)^2\right|s_2\right\rangle=Q\left\langle s_1\left|\mathcal{S}\frac{1}{1-\Theta}\right|s_2\right\rangle,$$

$$Q=\frac{1}{\kappa s}[e^{\kappa s}-1-\kappa s]$$

so that

$$\left\langle s_1\left|\left(\mathcal{S}\frac{1}{1-\Theta}\right)^n\right|s_2\right\rangle=Q^{n-1}\kappa\left(\frac{s_1}{s}\right)e^{\kappa(s-s_2)}$$

and hence

$$\left\langle s_1\left|\ln\left(1+\mathcal{S}\frac{1}{1-\Theta}\right)\right|s_2\right\rangle=\kappa\left(\frac{s_1}{s}\right)e^{\kappa(s-s_2)}\frac{1}{Q}\ln(1+Q)$$

or

$$\mathrm{trln}\left(1+\mathcal{S}\frac{1}{1-\Theta}\right)=\frac{1}{Q}\ln(1+Q)\int_0^s ds_1\kappa\left(\frac{s_1}{s}\right)e^{\kappa(s-s_1)}$$
$$=\ln(1+Q).$$

Further,

$$\langle s_1|\ln(1-\Theta)|s_2\rangle=-\langle s_1|(\Theta+\tfrac{1}{2}\Theta^2+\tfrac{1}{3}\Theta^3+\cdots)|s_2\rangle$$
$$=-\kappa\theta(s_{12})\left[1+\frac{\kappa}{2!}s_{12}+\frac{\kappa^2}{3!}(s_{12})^2+\cdots\right]$$

so that the only contribution to this quantity's trace, with $s_1=s_2$, and $\theta(0)=1/2$, comes from its leading term, $\mathrm{trln}(1-\Theta)=-\kappa s/2$. With $1+Q=(1/\kappa s)[e^{\kappa s}-1]$, one finds

$$\exp[\mathrm{trln}(1+M)]=\frac{2}{\kappa s}\sinh\left(\frac{\kappa s}{2}\right) \tag{A3}$$

and the inverse of this factor produces exactly the required determinant of Eq. (12).

Finally, and for completeness and clarity, we present the argument that the only solution of the restrictions $f(s')=0$ for all $0\leqslant s'\leqslant s$ corresponds to $u_+(s')=0$. Since it is clear from Eq. (9) that all $u_-(s')=0$, and all $A_-=0$, we suppress the $+$ subscripts, and consider the solutions of

$$u(s')=2g\int_0^{s'}ds''A(s'')-2g\frac{s'}{s}\int_0^s ds''A(s'') \tag{A4}$$

where $A(s')=A(x_+-u(s'),x_-)$, $E(s')=E(x_+-u(s'),x_-)=-\partial A/\partial x_+$.

Since $u(s')$ was defined as $\int_0^{s'}ds''v(s'')$, where $v(s'')$ corresponds to a velocity at proper time s'' (see [6]), we assume on physical grounds (and even for virtual particles) that v is a continuous function of its argument, although no restriction is placed on its derivatives. Hence $u(s')$ is a continuous function with (at least) a continuous first derivative.

Evaluating $u(s')$ at $s'=0$ and at $s'=s$, shows that $u(0)=u(s)=0$. Further, $u'(s')$ is given by

$$u'(s')=2gA(s')-2g\frac{1}{s}\int_0^s ds''A(s'') \tag{A5}$$

and therefore

$$u'(0)=u'(s)=2g\left[A(x_+,x_-)-\frac{1}{s}\int_0^s ds''A(x_+-u(s''),x_-)\right] \tag{A6}$$

which may or may not be zero. Let us suppose that the right-hand side (RHS) of Eq. (A6) is positive, and it then follows that $u(s')$ must vanish at (at least) one point, say s_1, $0\leqslant s_1\leqslant s$; and that at s_1, $u'(s_1)$ must be negative. But from Eq. (A5) one sees that, if $u(s_1)=0$, then $u'(s_1)=u'(0)=u'(s)$, and has the wrong sign. Therefore, there must be at least two other points, one in the interval between 0 and s_1, and the other between s_1 and s, where u will vanish with a negative slope. But, again, from Eq. (A5), whenever u vanishes its slope is the same, here assumed positive. The argument can be repeated indefinitely, and the only possible conclusion is that $u'(s)=0$, for all points s' in the interval between 0 and s. Then, from Eq. (A5), it follows that $u'(s')=0$. This is the reason why both $\det[\delta u_+/\delta f]$ and the functional-operation factor in the equation following Eq. (10) are evaluated at $u_\pm(s')=0$; and for the latter factor, this limit is taken after the functional derivatives are performed.

One final remark, to insure clarity. The replacement, just after Eq. (16), of the trace of the ordered exponential (OE), $\mathrm{tr}(e^{g\int_0^s ds'\sigma\cdot F})_+$, by $4\cosh(\int_0^s ds' gE(s'))$, when $\vec{E}$ points in one direction (e.g., the 3 direction) only, is easy to see using the old-fashioned representation

$$\gamma_\mu=i\alpha_3\gamma_4,\quad \alpha_3=\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

where σ_3 is the Pauli matrix; in this way, $\sigma\cdot F\to\alpha_3E_3$. Since α_3 commutes with itself, the commutator $[\sigma\cdot F(s_1),\sigma\cdot F(s_2)]$ vanishes for any points s_1 and s_2, and the OE may be replaced by an ordinary exponential. Then, a simple expansion in powers of g produces for the trace the result $4\cosh(\int_0^s ds' gE(s'))$.

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