Baryon charge radii and quadrupole moments in the $1/N_c$ expansion: The 3-flavor case

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We develop a straightforward method to compute charge radii and quadrupole moments for baryons both with and without strangeness, when the number of QCD color charges is N_c. The minimal assumption of the single-photon exchange ansatz implies that only two operators are required to describe these baryon observables. Our results are presented so that SU(3) flavor and isospin symmetry breaking can be introduced according to any desired specification, although we also present results obtained from two patterns suggested by the quark model with gluon exchange interactions. The method also permits to extract a number of model-independent relations; a sample is $r_\Lambda^2/r_n^2 = 3/(N_c+3)$, independent of SU(3) symmetry breaking.

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I. INTRODUCTION

Between the static properties of hadrons, e.g., masses, electromagnetic moments, matter and charge radii, and low-energy dynamical properties such as scattering lengths and decay rates, a wide variety of accurately measured data lies outside the perturbative range of QCD. While techniques such as lattice gauge theory, QCD sum rules, and a wide variety of models have been developed to study this extremely interesting energy regime, the only known perturbative approach for strongly interacting Yang-Mills theories that holds at all energies is the $1/N_c$ expansion [1], where N_c is the number of color charges.

The baryon sector is particularly amenable to this expansion, since baryons with N_c colors contain N_c valence quarks. Then, each additional quark participating in an interaction (specified by an operator with known transformation properties under spin and flavor) brings in a factor of the QCD coupling constant $\alpha_s \propto 1/N_c$ due to the requirement of one or more gluons to connect this quark to the interaction. An operator in which n quarks interact is called an n-body operator; its suppression in powers of $1/N_c$ tends to increase as n increases. This "operator method" has recently been used to study a wide variety of baryon observables (see Ref. [2] for a recent list of references). The strength of the $1/N_c$ expansion is that additional gluons do not spoil this counting, and the only other powers of N_c that need be taken into account arise from the combinatorics of the quarks. It should also be noted that the glue and sea quark pairs in the baryon, not just the valence quarks, are subsumed by the operator method [3].

This work extends the results of two of our recent papers on baryon observables in the $1/N_c$ expansion, regarding charge radii [3] and quadrupole moments [2], to baryons with nonzero strangeness. These two types of observables are studied here in a single paper because the method of calculation is, as seen below, very similar in the two cases. As mentioned in the previous works, a treatment of the strange sector was not undertaken at the time of their writing because the group theory in the 3-flavor sector is more involved and warrants a separate treatment. The operator method calculation specific to the case $N_c = 3$, which appears very similar to the general QCD parametrization [4] was carried out for baryon quadrupole moments in [5]; in that case the calculation can readily be carried out for states with or without strangeness since it is relatively simple to perform calculations using the full baryon spin-flavor wave functions with only 3 quarks.

Here we demonstrate that the solution of the full 3-flavor N_c-quark problem can nonetheless be handled entirely using SU(2) Clebsch-Gordan (CG) coefficients, ultimately because the strange states are related to nonstrange states through the SU(2) U-and V-spin subgroups of SU(3) [6]. In the case of the ground-state baryons, i.e., those belonging to the large-N_c generalization of the SU(6) **56**-plet, the total symmetry of the wave function under simultaneous exchange of spin and flavor indices simplifies this procedure considerably.

In this paper we focus only on those aspects of the problem unique to the strange sector; the reader is directed to Refs. [2] and [3] for a more thorough discussion of details of the $1/N_c$ operator method, as well as outlook for the experimental measurement of baryon quadrupole moments. The remainder of this paper is organized as follows: In Sec. II we briefly discuss the experimental situation regarding measurement of baryon charge radii. Section III presents the details of the 3-flavor calculation; the most important feature of the analysis of [2] and [3] survives, namely, that only one operator appears at the 2-body level and one at the 3-body level, leading to a large number of constraints between the observables. Section IV presents a sample of the plethora of results obtained from this calculation, e.g., by including frequently used patterns of SU(3) flavor symmetry breaking. Section V summarizes. The tables and Appendixes contain the extensive results of the calculation in forms designed to be most useful to interested researchers.

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II. CHARGE RADII: PROSPECTS FOR MEASUREMENT

The mean-square charge radius r_B^2 of a baryon B is defined through the elastic Sachs charge form factor $F(q^2)$, a function of the photon 4-momentum transfer q^2 often denoted by $G_E(q^2)$, by the relation

$$r_B^2 \equiv -6 \frac{1}{Q} \left. \frac{dF(q^2)}{dq^2} \right|_{q^2=0}, \qquad (2.1)$$

while the total charge $Q = F(0)$ is omitted if $Q=0$.

The operator giving rise to charge radii is a scalar that does not change electric charge or strangeness. Thus, it connects only states with the same total J, I_3, and Y. However, the operator can connect states of different isospin I since the electromagnetic interaction does not conserve this quantum number. Each baryon state thus has a charge radius, and in addition there exists a $r^2_{\Sigma^0\Lambda}$ transition charge radius.

At the time of this writing, only the p, n, and Σ^- charge radii have been measured. A charge radius is the first nontrivial moment of a Coulomb monopole ($C0$) transition amplitude, and thus charge radii of hadrons are typically measured through their coupling to Coulomb photons in elastic electron scattering.

In the case of the proton, a dedicated measurement of the elastic electron-proton cross section at very low momentum transfer [7] led to the proton root mean-square charge radius $r_p \equiv \sqrt{r_p^2} = 0.862(12)$ fm [$r_p^2 = 0.743(21)$ fm^2], a value that is considerably larger than the famous dipole result of $r_p = 0.80$ fm obtained by Hofstadter *et al.* [8]. A dispersion-theoretical analysis of electron scattering data available through 1995 [9] gives the result $r_p^2 = 0.717(15)$ fm^2. However, measurements using the hydrogen Lamb shift as an alternative source of information on the proton size tend to produce larger values. For example, Ref. [10] extracts a value of $r_p = 0.890(14)$ fm [$r_p^2 = 0.792(25)$ fm^2], while Ref. [11] extracts a value of $r_p^2 = 0.780(25)$ fm^2 from the experiment of Ref. [12]. A reanalysis [13] that includes Coulomb and recoil corrections for $A > 1$ targets produces a result that accommodates both types of data:

$$r_p^2 = 0.779(25) \ \ \text{fm}^2, \qquad (2.2)$$

which we use in our analysis. The other available experimental values are

$$r_n^2 = -0.113(3)(4) \ \ \text{fm}^2 \ \ [14], \qquad (2.3)$$

and the very recently published result

$$r_{\Sigma^-}^2 = 0.61(12)(9) \ \ \text{fm}^2 \ \ [15]. \qquad (2.4)$$

The measurement of $r^2_{\Sigma^-}$ in particular suggests the possibility of measuring the charge radii of other long-lived strange baryons, the Σ^+, Λ, Ξ^-, Ξ^0, and Ω^-. In such cases the impediments are experimental in nature, particularly the problem of producing a beam of baryons of sufficient quality that electron scattering events can be separated from background events. But nonetheless one can anticipate such difficulties being overcome in the future.

For the spin-3/2 resonances and Σ^0, however, the short lifetimes mean that charge radii can only be observed in off-shell processes. For example, the Δ^{++} charge radius could be extracted in principle through the process $\pi^+ p \to \pi^+ p e^- e^+$, but in this case one would need to use model dependence in separating resonant from continuum $\pi^+ p$ scattering, as well as isolate the source of the virtual photon $\gamma^* \to e^+ e^-$ as coming solely from the Δ. Processes with real photons, which for example can be used to measure magnetic moments, cannot be utilized to probe the Coulomb transitions from which charge radii are extracted. The situation is very similar to that described in Ref. [2] for the baryon quadrupole moments. Therefore, our results divide into two categories: Predictions for observables that may be measured in the near-to-medium future, for which the quality of our results can be checked, and those that can only be predicted but not measured any time soon.

III. DETAILS OF THE CALCULATION

A. Constructing the states

Naively, one would expect that SU(3) CG coefficients are required to construct a state with strangeness. This potential complication can be avoided by noting that the strange states are related to the nonstrange states via U-and V-spin SU(2) subgroups of SU(3). In an arbitrary SU(3) representation, one method of calculating matrix elements is to start with the nonstrange states and apply the lowering operators U_- and V_-, imposing along the way orthogonality in isospin among otherwise degenerate states such as Σ^0 and Λ, which is precisely the same approach as used to obtain flavor eigenstates.

However, this method does not exploit the full symmetry available to the $J^P = 1/2^+$ and $3/2^+$ ground-state baryon SU(3) multiplets (We do not call them "octet" and "decuplet," since the corresponding multiplets for arbitrary N_c are much larger [16].) These flavor multiplets belong to the large-N_c analogue of the SU(6) **56**-plet, which is completely symmetric under simultaneous exchange of spin and flavor indices. That is to say, the *I*-, *U*-, and *V*-spin symmetries are correlated with the SU(2) spin symmetry in a particularly convenient way, as we now explore.

Let N_α denote the number of valence quarks of type α in the baryon; then $N_c = N_u + N_d + N_s$ and $I_3 = (N_u - N_d)/2$. Since the spin-flavor wave function is completely symmetric, the spin wave function of quarks of type α within the baryon must be completely symmetric, which means that the N_α spin-1/2 quarks of type α carry the maximum possible spin value, $S_\alpha = N_\alpha/2$. This information alone allows to analyze the nonstrange states. In the strange baryon case, note that each physical baryon state is still specified by its isospin I, which uniquely determines the symmetry of the flavor wave function carried by just u and d quarks. Owing to the complete symmetry under spin $\times$ flavor, the spin wave function carried by the u and d quarks together must possess exactly the same symmetry properties as the corresponding flavor wave function, and hence, the spin $\mathbf{S}_{ud} \equiv \mathbf{S}_u + \mathbf{S}_d$ and isospin

I have the same eigenvalue, $S_{ud}=I$.

But now one need only combine the state of ud combined spin $S_{ud}=I$ and isospin quantum numbers I,I_3 with the symmetrized strange quarks carrying spin S_s to obtain the complete state with spin eigenvalues J,J_3, where $\mathbf{J}=\mathbf{S}_{ud}+\mathbf{S}_s$. To be precise,

$$|JJ_3;\ II_3\ \ (S_uS_dS_s)\rangle = \sum_{S_{ud}^z,\ S_s^z} \left(\begin{array}{cc|c} I & S_s & S \\ S_{ud}^z & S_s^z & S_z \end{array} \right) \sum_{S_u^z,\ S_d^z} \left(\begin{array}{cc|c} S_u & S_d & I \\ S_u^z & S_d^z & S_{ud}^z \end{array} \right) \times |S_uS_u^z\rangle |S_dS_s^z\rangle |S_sS_s^z\rangle, \tag{3.1}$$

where the parentheses denote CG coefficients. Now, in order to compute the matrix elements of any particular operator, one need only sandwich it between a bra and ket of the form of Eq. (3.1) and use the Wigner-Eckart theorem.

B. The operator basis

In the analysis of both charge radii and quadrupole moments we use the single-photon exchange assumption, i.e., that the photon probing these baryon observables couples to only one quark line within the baryon. Although physically rather mild, this assumption drastically reduces the number of distinct operators that need to be considered. (However, note that the full basis of operators can certainly be used, as shown for the charge radii [3] or for the magnetic and quadrupole moments [17].) Using indices i,j,k to indicate quarks within the baryon, the most general operator expansion for the charge radius operator out to the 3-body level—all that is necessary since 4-,5-, etc. body operators acting upon physical baryons are linearly dependent on those at lower order—reads

$$-6\frac{dF(q^2)}{dq^2}\bigg|_{q^2=0} = A \sum_i^{N_c} a_i + \frac{B}{N_c}\sum_{i\neq j}^{N_c} b_i c_j \ \ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + \frac{C}{N_c^2}\sum_{i\neq j\neq k} d_i e_j f_k \ \ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, \tag{3.2}$$

where the coefficients a,b,c,d,e,f depend only on quark flavor; to obtain the results of Ref. [3], set $a_i=b_i=f_i=Q_i$, where Q_i is the charge of the ith quark, and $c_i=d_i=e_i=1$. The corresponding expansion for the quadrupole moment operator reads

$$\mathcal{Q} = \frac{B'}{N_c}\sum_{i\neq j} b_i' c_j' (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j) + \frac{C'}{N_c^2}\sum_{i\neq j\neq k} d_i' e_j' f_k' \ (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j), \tag{3.3}$$

where the coefficients b',c',d',e',f' again depend only on quark flavor, and the results of Ref. [2] may be obtained by setting $b_i'=f_i'=Q_i$, $c_i'=d_i'=e_i'=1$. Note that most models use a universal form of SU(3) symmetry breaking for all operators, so that typically the corresponding primed and unprimed coefficients are equal. Values used for Q_i are discussed in Sec. IV below. Note that 3-body operators with 3 Pauli matrices are absent [2] due to time-reversal symmetry. The use of primed and unprimed coefficients with the same labels echoes the similar form of the operators in the two cases; indeed, in a number of models the primed and unprimed coefficients are related [18].

One can obtain the most general charge radius and quadrupole moment expressions by computing the 12 fundamental matrix elements

$$\langle \alpha\beta\rangle^{(0)} \equiv \langle \boldsymbol{S}_\alpha\cdot\boldsymbol{S}_\beta\rangle,$$

$$\langle \alpha\beta\rangle^{(2)} \equiv \langle 3S_\alpha^z S_\beta^z - \boldsymbol{S}_\alpha\cdot\boldsymbol{S}_\beta\rangle, \tag{3.4}$$

where (α,β) are the 6 distinct pairs (u,u), (u,d), (u,s), (d,d), (d,s), and (s,s) (note the symmetry under $\alpha\leftrightarrow\beta$). Then, charge radii or quadrupole moments for baryons in the presence of arbitrary SU(3) or isospin symmetry breaking can be computed simply by taking an appropriate linear combination of these primitive matrix elements.

In particular, while the indices i,j,k indicate each of the N_c quarks within the baryon, let the indices α,β,γ indicate collectively the quarks of a particular flavor $u,d,$ or s. The coefficients for the most general operators that appear for the charge radii or quadrupole moments can be represented by 3-vectors such as $\boldsymbol{a}=(a_u,a_d,a_s)$. Thus, matrix elements of the 1-body operator for charge radii generalize to

$$\sum_i^{N_c} a_i = \sum_\alpha N_\alpha a_\alpha, \tag{3.5}$$

while those of the 2-body operator may be expanded as

$$\left\langle \sum_{i\neq j} b_i c_j \ \ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \right\rangle = 4\sum_{\alpha,\beta} b_\alpha c_\beta \langle\alpha\beta\rangle^{(0)} - 3\sum_\alpha N_\alpha b_\alpha c_\alpha, \tag{3.6}$$

and those of the 3-body operator become

$$\left\langle \sum_{i\neq j\neq k} d_i e_j f_k \ \ \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j \right\rangle = +4\sum_{\alpha,\beta} d_\alpha e_\beta \ \langle\alpha\beta\rangle^{(0)} \sum_\gamma N_\gamma f_\gamma - 4\sum_{\alpha,\beta} d_\alpha e_\beta (f_\alpha + f_\beta)\langle\alpha\beta\rangle^{(0)} - 3\sum_\alpha N_\alpha d_\alpha e_\alpha \sum_\beta N_\beta f_\beta + 6\sum_\alpha N_\alpha d_\alpha e_\alpha f_\alpha, \tag{3.7}$$

where N_α is the number of quarks of type α (values given in Table I), which satisfies the constraints $N_u+N_d+N_s=N_c$ and $N_u-N_d=2I_3$.

TABLE I. Values of $N_{u,d,s}$ [whence $\langle \boldsymbol{S}_\alpha^2 \rangle = \langle \alpha\alpha \rangle^{(0)} = (N_\alpha/2)(N_\alpha/2+1)$] and matrix elements of the rank-0 tensors $\langle \alpha\beta \rangle^{(0)}$ with $\alpha \neq \beta$. Since spin is unchanged by these operators, the matrix elements vanish for all off-diagonal transitions except $\Sigma^0\Lambda$; in that case, the only nonvanishing entries are $\langle us \rangle^{(0)} = -\langle ds \rangle^{(0)} = -\frac{1}{8}\sqrt{(N_c-1)(N_c+3)}$.

State	N_u	N_d	N_s	$\langle ud \rangle^{(0)}$	$\langle us \rangle^{(0)}$	$\langle ds \rangle^{(0)}$
Δ^{++}	$\frac{1}{2}(N_c+3)$	$\frac{1}{2}(N_c-3)$	0	$-\frac{1}{16}(N_c-3)(N_c+7)$	0	0
Δ^{+}	$\frac{1}{2}(N_c+1)$	$\frac{1}{2}(N_c-1)$	0	$-\frac{1}{16}(N_c^2+4N_c-29)$	0	0
Δ^{0}	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c+1)$	0	$-\frac{1}{16}(N_c^2+4N_c-29)$	0	0
Δ^{-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c+3)$	0	$-\frac{1}{16}(N_c-3)(N_c+7)$	0	0
Σ^{*+}	$\frac{1}{2}(N_c+1)$	$\frac{1}{2}(N_c-3)$	1	$-\frac{1}{16}(N_c-3)(N_c+5)$	$+\frac{1}{16}(N_c+5)$	$-\frac{1}{16}(N_c-3)$
Σ^{*0}	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c-1)$	1	$-\frac{1}{16}(N_c^2+2N_c-19)$	$+\frac{1}{4}$	$+\frac{1}{4}$
Σ^{*-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c+1)$	1	$-\frac{1}{16}(N_c-3)(N_c+5)$	$-\frac{1}{16}(N_c-3)$	$+\frac{1}{16}(N_c+5)$
Ξ^{*0}	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c-3)$	2	$-\frac{1}{16}(N_c-3)(N_c+3)$	$+\frac{1}{12}(N_c+3)$	$-\frac{1}{12}(N_c-3)$
Ξ^{*-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c-1)$	2	$-\frac{1}{16}(N_c-3)(N_c+3)$	$-\frac{1}{12}(N_c-3)$	$+\frac{1}{12}(N_c+3)$
Ω^{-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c-3)$	3	$-\frac{1}{16}(N_c-3)(N_c+1)$	0	0
p	$\frac{1}{2}(N_c+1)$	$\frac{1}{2}(N_c-1)$	0	$-\frac{1}{16}(N_c-1)(N_c+5)$	0	0
n	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c+1)$	0	$-\frac{1}{16}(N_c-1)(N_c+5)$	0	0
Σ^{+}	$\frac{1}{2}(N_c+1)$	$\frac{1}{2}(N_c-3)$	1	$-\frac{1}{16}(N_c-3)(N_c+5)$	$-\frac{1}{8}(N_c+5)$	$+\frac{1}{8}(N_c-3)$
Σ^{0}	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c-1)$	1	$-\frac{1}{16}(N_c^2+2N_c-19)$	$-\frac{1}{2}$	$-\frac{1}{2}$
Λ	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c-1)$	1	$-\frac{1}{16}(N_c-1)(N_c+3)$	0	0
Σ^{-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c+1)$	1	$-\frac{1}{16}(N_c-3)(N_c+5)$	$+\frac{1}{8}(N_c-3)$	$-\frac{1}{8}(N_c+5)$
Ξ^{0}	$\frac{1}{2}(N_c-1)$	$\frac{1}{2}(N_c-3)$	2	$-\frac{1}{16}(N_c-3)(N_c+3)$	$-\frac{1}{6}(N_c+3)$	$+\frac{1}{6}(N_c-3)$
Ξ^{-}	$\frac{1}{2}(N_c-3)$	$\frac{1}{2}(N_c-1)$	2	$-\frac{1}{16}(N_c-3)(N_c+3)$	$+\frac{1}{6}(N_c-3)$	$-\frac{1}{6}(N_c+3)$

Matrix elements of the 2-and 3-body operators for the quadrupole moments expand as

$$\left\langle \sum_{i\neq j} b_i' c_j' (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \right\rangle = 4\sum_{\alpha,\beta} b_\alpha' \, c_\beta' \, \langle \alpha\beta \rangle^{(2)}, \tag{3.8}$$

and

$$\left\langle \sum_{i\neq j\neq k} d_i' e_j' f_k' (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \right\rangle = +4\sum_{\beta,\gamma} d_\beta' e_\gamma' \langle \beta\gamma \rangle^{(2)} \sum_\alpha N_\alpha f_\alpha' - 4\sum_{\alpha,\beta} d_\alpha' e_\beta' (f_\alpha' + f_\beta') \langle \alpha\beta \rangle^{(2)}. \tag{3.9}$$

The two examples of SU(3) symmetry breaking in the quadrupole operators considered in the $N_c=3$ work of Ref. [5] correspond to setting $b_\alpha' = Q_\alpha (m/m_\alpha)^n$, $c_\alpha' = e_\alpha' = m/m_\alpha$, $d_\alpha' = (m/m_\alpha)^n$, and $f_\alpha' = Q_\alpha$, where $m_u = m_d \equiv m$ and $m/m_s \equiv r$ represent SU(3) symmetry breaking arising from different quark masses. The quadratic (cubic) case in [5] is $n=1$ (2).

C. Reduction of the basis

Even with this decomposition, it is not necessary to compute the entire set of 12 matrix elements separately: They are related by a number of simple constraints. First, note that Eq. (3.1) is sensitive to the exchange of u and d quarks only through the second CG coefficient, and the factor obtained through this exchange is just $(-1)^{S_u+S_d-I}$. Of course, the values of S_α, which simply count one-half the number of quarks of flavor α in these baryons, remain unchanged from initial to final state. The same is true for $I_3 = S_u - S_d$, but the total isospin may change to a value I'. One thus finds for an operator $\mathcal{O}$ that

$$\langle I' I_3 | \mathcal{O}(u \leftrightarrow d) | I I_3 \rangle = (-1)^{I'-I} \langle I' - I_3 | \mathcal{O} | I - I_3 \rangle. \tag{3.10}$$

There are almost enough constraints to allow extraction of all the matrix elements $\langle \alpha\beta \rangle^{(0)}$ in terms of simple combinations of eigenvalues of the compatible operators of the system. To be precise, these are $\boldsymbol{J}^2 \to J(J+1)$, J_z, $\boldsymbol{S}_\alpha^2 \to (N_\alpha/2)(N_\alpha/2+1)$ for $\alpha = u,d,s$, and $\boldsymbol{I}^2 = \boldsymbol{S}_{ud}^2 = I(I+1)$. Note that the eigenvalue $I_3 = S_u - S_d$ is not independent. Clearly, $\langle \alpha\alpha \rangle^{(0)} = (N_\alpha/2)(N_\alpha/2+1)$. The other two constraints are

$$J(J+1) = \langle \boldsymbol{J}^2 \rangle = \langle (\boldsymbol{S}_u + \boldsymbol{S}_d + \boldsymbol{S}_s)^2 \rangle = \sum_{\alpha,\beta} \langle \alpha\beta \rangle^{(0)},$$

$$I(I+1) = \langle \boldsymbol{S}_{ud}^2 \rangle = \langle (\boldsymbol{S}_u + \boldsymbol{S}_d)^2 \rangle = \langle uu \rangle^{(0)} + 2\langle ud \rangle^{(0)} + \langle dd \rangle^{(0)}. \tag{3.11}$$

From these constraints it is clear that only one of the two matrix elements $\langle us \rangle^{(0)}$ or $\langle ds \rangle^{(0)}$ need be computed directly from Eq. (3.1), while the other can be obtained either from the inversion rule Eq. (3.10) or from the constraints Eqs. (3.11).

As for the matrix elements $\langle \alpha\beta \rangle^{(2)}$, the most obvious constraint reads

$$\langle 3J_z^2 - \boldsymbol{J}^2 \rangle = \langle 3(S_u^z + S_d^z + S_s^z)^2 - (\boldsymbol{S}_u + \boldsymbol{S}_d + \boldsymbol{S}_s)^2 \rangle = \sum_{\alpha,\beta} \langle \alpha\beta \rangle^{(2)}. \tag{3.12}$$

The left-hand side (l.h.s.) of this expression is of course simple to compute; in the stretched state ($J_z = J$) in which quadrupole moments are computed, it equals $J(2J-1)$ for diagonal matrix elements and vanishes for transitions.

Another constraint may be obtained by considering the combination

$$\begin{aligned}\langle 3(S_{ud}^z)^2 - \boldsymbol{S}_{ud}^2 \rangle &= \langle 3(J_z - S_s^z)^2 - \boldsymbol{I}^2 \rangle \\ &= 3J_z^2 - 6J_z \langle S_s^z \rangle + \langle ss \rangle^{(2)} \\ &\quad + \langle \boldsymbol{S}_s^2 \rangle - \langle \boldsymbol{I}^2 \rangle,\end{aligned} \tag{3.13}$$

whose l.h.s. is just

$$\langle uu \rangle^{(2)} + 2\langle ud \rangle^{(2)} + \langle dd \rangle^{(2)}. \tag{3.14}$$

Note that this constraint requires one to compute also the matrix element $\langle S_s^z \rangle$, but this calculation turns out in purely algebraic terms to be simpler than that of the rank-2 tensors. One still requires one more constraint to compute separate values for $\langle us \rangle^{(2)}$ and $\langle ds \rangle^{(2)}$; once one of these matrix elements is in hand, the other is obtained by using Eq. (3.10). The required constraint, again using $\langle \boldsymbol{S}_{ud}^2 \rangle = \langle \boldsymbol{I}^2 \rangle$, may be obtained from

$$\langle 3S_u^z J_z - \boldsymbol{S}_u \cdot \boldsymbol{J} \rangle = 3\langle S_u^z \rangle J_z - \langle us \rangle^{(0)} - \frac{1}{2}\langle \boldsymbol{I}^2 + \boldsymbol{S}_u^2 - \boldsymbol{S}_d^2 \rangle, \tag{3.15}$$

where the l.h.s. clearly equals

$$\langle uu \rangle^{(2)} + \langle ud \rangle^{(2)} + \langle us \rangle^{(2)}. \tag{3.16}$$

In summary, then, the charge radii matrix elements can be computed in complete generality using only eigenvalues and the calculation of $\langle us \rangle^{(0)}$, while the most general quadrupole moment matrix elements can be obtained from further explicitly calculating $\langle S_u^z \rangle$, $\langle S_s^z \rangle$, $\langle uu \rangle^{(2)}$, and $\langle ss \rangle^{(2)}$.

Another advantage of using the constraints (3.11)–(3.15) is that, by judicious choice of which matrix elements to compute explicitly, one may obtain results containing nothing more complicated than a $6j$ symbol. These appear due to the coupling of the quantum numbers of expansions for both the bra and ket [Eq. (3.1)] through the operator. For example, in the matrix elements of $\langle uu \rangle^{(2)}$ defined by Eq. (3.4), one uses the Wigner-Eckart theorem to obtain

$$\begin{aligned}&\langle S_u S_u^z | 3(S_u^z)^2 - \boldsymbol{S}_u^2 | S_u' S_u'^{\,z} \rangle \\ &= \delta_{S_u' S_u} \delta_{S_u'^{\,z} S_u^z} \sqrt{S_u(S_u+1)(2S_u-1)(2S_u+3)} \\ &\quad \times \left(\begin{matrix} 2 & S_u \\ 0 & S_u^z \end{matrix} \middle| \begin{matrix} S_u \\ S_u^z \end{matrix} \right),\end{aligned} \tag{3.17}$$

and the CG coefficient in this expression is linked to the those in the bra and ket of Eq. (3.1) by sums over S_u^z and $S_u'^{\,z}$. Four appropriately linked CG coefficients produce a $6j$ symbol, and six produce a $9j$ symbol. In our case, the matrix elements $\langle us \rangle^{(2)}$ and $\langle ds \rangle^{(2)}$ directly computed would produce $9j$ symbols. In fact, so do $\langle us \rangle^{(0)}$ and $\langle ds \rangle^{(0)}$, but they produce a $9j$ symbol with one zero argument, which can be written as the product of two $6j$ symbols, as seen below. Analytic forms for $6j$ symbols with one argument $\leqslant 2$ appear in Edmonds [19], which is precisely what is needed to compute the matrix elements of tensors up to rank 2.

The analytic forms of the matrix elements of interest are

$$\begin{aligned}\langle us \rangle^{(0)} = {}& \delta_{S'S} \delta_{S_z' S_z} \delta_{S_u' S_u} \delta_{S_d' S_d} \delta_{S_s' S_s} (-1)^{1+S+S_s-S_u-S_d} \sqrt{S_u(S_u+1)(2S_u+1)S_s(S_s+1)(2S_s+1)(2I'+1)(2I+1)} \\ & \times \begin{Bmatrix} S_d & S_u & I \\ 1 & I' & S_u \end{Bmatrix} \begin{Bmatrix} S & S_s & I \\ 1 & I' & S_s \end{Bmatrix},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\langle S_u^z \rangle = {}& \delta_{S_z' S_z} \delta_{S_u' S_u} \delta_{S_d' S_d} \delta_{S_s' S_s} (-1)^{S-S'+S_z+S_s+I'-I-S_u-S_d} \sqrt{S_u(S_u+1)(2S_u+1)(2I'+1)(2I+1)(2S'+1)(2S+1)} \\ & \times \begin{Bmatrix} S_d & S_u & I \\ 1 & I' & S_u \end{Bmatrix} \begin{Bmatrix} S_s & I & S \\ 1 & S' & I' \end{Bmatrix} \begin{pmatrix} 1 & S' & S \\ 0 & S_z & -S_z \end{pmatrix},\end{aligned} \tag{3.19}$$

$$\langle S_s^z \rangle = \delta_{S_z' S_z} \delta_{I' I} \delta_{S_u' S_u} \delta_{S_d' S_d} \delta_{S_s' S_s} (-1)^{1+S_z+S_s+I} \sqrt{S_s(S_s+1)(2S_s+1)(2S'+1)(2S+1)} \begin{Bmatrix} I & S_s & S \\ 1 & S' & S_s \end{Bmatrix} \begin{pmatrix} 1 & S' & S \\ 0 & S_z & -S_z \end{pmatrix}, \tag{3.20}$$

$$\begin{aligned}\langle uu \rangle^{(2)} = {}& \delta_{S_z' S_z} \delta_{I' I} \delta_{S_u' S_u} \delta_{S_d' S_d} \delta_{S_s' S_s} (-1)^{S-S'+S_z+S_s+I'-I-S_u-S_d} \\ & \times \sqrt{(2S_u-1)S_u(S_u+1)(2S_u+3)(2I'+1)(2I+1)(2S'+1)(2S+1)} \begin{Bmatrix} S_d & S_u & I \\ 2 & I' & S_u \end{Bmatrix} \begin{Bmatrix} S_s & I & S \\ 2 & S' & I' \end{Bmatrix} \begin{pmatrix} 2 & S' & S \\ 0 & S_z & -S_z \end{pmatrix},\end{aligned} \tag{3.21}$$

$$\langle ss\rangle^{(2)}=\delta_{S'_z S_z}\delta_{I'I}\delta_{S'_u S_u}\delta_{S'_d S_d}\delta_{S'_s S_s}(-1)^{S_z+S_s+I}\sqrt{(2S_s-1)S_s(S_s+1)(2S_s+1)(2S_s+3)(2S'+1)(2S+1)}\begin{Bmatrix} I & S_s & S \\ 2 & S' & S_s \end{Bmatrix} \times\begin{pmatrix} 2 & S' & S \\ 0 & S_z & -S_z \end{pmatrix}. \quad (3.22)$$

Note that, in the interest of exhibiting maximal symmetry, the remaining CG coefficients have been written as $3j$ symbols [19].

Charge radius and quadrupole transitions are diagonal in both charge and strangeness, and thus connect only states of the same values of I_3 and N_s, but do not necessarily conserve isospin. Furthermore, charge radius operators are scalars and thus connect only states of the same total spin J, but quadrupole operators are rank 2 and therefore can connect spin 3/2 to 3/2 or 1/2, but not 1/2 to 1/2. These selection rules are reflected by the transition matrix elements represented in the tables. Values at arbitrary N_c for the matrix elements $\langle\alpha\beta\rangle^{(0)}$ $(\alpha\neq\beta)$ for all relevant states are collected in Table I, for S^z_α in Table II, for $\langle\alpha\alpha\rangle^{(2)}$ in Table III, and $\langle\alpha\beta\rangle^{(2)}$ $(\alpha\neq\beta)$ in Table IV. One may obtain results for charge radii and quadrupole moments, including arbitrary SU(3) symmetry breaking, by combining the results of these tables using the expressions (3.5)–(3.9) derived in the previous subsection.

IV. RESULTS

We exhibit in the Appendixes expressions for the baryon charge radii (A) and quadrupole moments (B) under three sets of assumptions familiar to researchers in the quark model. The first is simply to assume no SU(3) flavor symmetry breaking except for that inherent from the quark charges. The second and third, which we call ‘‘quadratic’’ $(n=2)$ and ‘‘cubic’’ $(n=3)$ SU(3) symmetry breaking, respectively, correspond to modifying the spin-spin terms in the following way:

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j\rightarrow\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j m^n/m_i^{n-1}m_j\,, \quad (4.1)$$

where m_i denotes the constituent mass of quark i, and $m=m_u=m_d$ is the light quark constituent mass, thus fixing all parameters a-f and b'-f' defined in Eqs. (3.2) and (3.3). Let us henceforth abbreviate with $m/m_s\equiv r$ the degree of SU(3) flavor symmetry breaking. The quadratic mass dependence arises in a constituent quark model with the dominant interaction mechanism being one-gluon exchange, while the extra mass factor for cubic mass dependence arises from a quark propagator between photon absorption and gluon emission.

One can also mix these pictures so that, for example, the operator labeled by $B(B')$ uses cubic SU(3) symmetry breaking, while that labeled by $C(C')$ uses quadratic breaking. Note that no SU(3) symmetry breaking has been introduced in the one-body (A) operator.

We hasten to add that it is not necessary to adopt *any* model dependence beyond the single-photon exchange ansatz, since we have computed and tabulated all the relevant primitive matrix elements. However, the SU(3)-symmetric, quadratic, and cubic models provide a useful picture in which to investigate the consequences of the arbitrary N_c calculation, without adding any new parameters. Another interesting scheme for SU(3) symmetry breaking, but not investigated here, is that provided by chiral perturbation theory. In this case, loop graphs that include the octet of light mesons as Goldstone bosons produce terms with different r dependences, e.g., logarithmic.

We have also used expressions [20] for the quark charges that simultaneously guarantee chiral anomaly cancellation of the N_c-extended standard model and fix the total charges of the N_c-quark baryons to equal their $N_c=3$ values. These are

$$Q_{u,c,t}=(N_c+1)/2N_c\,,\quad Q_{d,s,b}=(-N_c+1)/2N_c\,. \quad (4.2)$$

Given the primitive matrix elements of Tables I–IV, one may alternately compute expressions using the strict $N_c=3$ values for the quark charges, with the caveats that the anomaly cancellation conditions are no longer satisfied and the baryon charges become N_c dependent.

One may choose any of a number of schemes for obtaining interesting predictions from our results. Rather than selecting just one and producing exhaustive results, we discuss several possibilities and exhibit illustrative examples.

First, one must differentiate predictions that hold in the physical case $N_c=3$ from those that hold in the $1/N_c\rightarrow 0$ limit. The former have the advantage of not depending on the quality of the $1/N_c$ expansion at $N_c=3$, but may depend upon delicate cancellations between terms at different powers in $1/N_c$. Thus, such predictions may hold well only for $N_c=3$ but not for 5,7, ... Conversely, the latter have the advantage of holding to a desired level of accuracy for small values of $1/N_c$, but these corrections may turn out to be numerically significant for $N_c=3$. As we exhibit below, a number of relations are found to hold independent of N_c and thus satisfy both criteria. Moreover, there are even relations that hold both for $N_c=3$ and $1/N_c\rightarrow 0$, but differ in between; we discuss one such example with the quadrupole moments below.

A. Charge radii

Baryon mean-square charge radii, denoted here by r_B^2, are defined by the l.h.s. of Eq. (3.2), divided by the total charge if it is nonzero. In all suggested substitutions for the coefficients $a_\alpha,\ldots,f_\alpha$, isospin violation is introduced via the single-photon exchange ansatz through a single power of the quark charge operator Q_α, which transforms as a combina-

TABLE II. Matrix elements of the operators S_u^z, S_d^z, and S_s^z in the state of maximal S_z.

State	$\langle S_u^z \rangle$	$\langle S_d^z \rangle$	$\langle S_s^z \rangle$
Δ^{++}	$+\frac{3}{20}(N_c+7)$	$-\frac{3}{20}(N_c-3)$	0
Δ^{+}	$+\frac{1}{20}(N_c+17)$	$-\frac{1}{20}(N_c-13)$	0
Δ^{0}	$-\frac{1}{20}(N_c-13)$	$+\frac{1}{20}(N_c+17)$	0
Δ^{-}	$-\frac{3}{20}(N_c-3)$	$+\frac{3}{20}(N_c+7)$	0
Σ^{*+}	$+\frac{1}{8}(N_c+5)$	$-\frac{1}{8}(N_c-3)$	$+\frac{1}{2}$
Σ^{*0}	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
Σ^{*-}	$-\frac{1}{8}(N_c-3)$	$+\frac{1}{8}(N_c+5)$	$+\frac{1}{2}$
Ξ^{*0}	$+\frac{1}{12}(N_c+3)$	$-\frac{1}{12}(N_c-3)$	$+1$
Ξ^{*-}	$-\frac{1}{12}(N_c-3)$	$+\frac{1}{12}(N_c+3)$	$+1$
Ω^{-}	0	0	$+\frac{3}{2}$
p	$+\frac{1}{12}(N_c+5)$	$-\frac{1}{12}(N_c-1)$	0
n	$-\frac{1}{12}(N_c-1)$	$+\frac{1}{12}(N_c+5)$	0
Σ^{+}	$+\frac{1}{12}(N_c+5)$	$-\frac{1}{12}(N_c-3)$	$-\frac{1}{6}$
Σ^{0}	$+\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{6}$
Λ	0	0	$+\frac{1}{2}$
Σ^{-}	$-\frac{1}{12}(N_c-3)$	$+\frac{1}{12}(N_c+5)$	$-\frac{1}{6}$
Ξ^{0}	$-\frac{1}{36}(N_c+3)$	$+\frac{1}{36}(N_c-3)$	$+\frac{2}{3}$
Ξ^{-}	$+\frac{1}{36}(N_c-3)$	$-\frac{1}{36}(N_c+3)$	$+\frac{2}{3}$
$\Delta^{+}p$	$+\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+5)}$	$-\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+5)}$	0
$\Delta^{0}n$	$+\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+5)}$	$-\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+5)}$	0
$\Sigma^{*+}\Sigma^{+}$	$+\frac{1}{12\sqrt{2}}(N_c+5)$	$-\frac{1}{12\sqrt{2}}(N_c-3)$	$-\frac{\sqrt{2}}{3}$
$\Sigma^{*0}\Sigma^{0}$	$+\frac{1}{3\sqrt{2}}$	$+\frac{1}{3\sqrt{2}}$	$-\frac{\sqrt{2}}{3}$
$\Sigma^{*0}\Lambda$	$+\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+3)}$	$-\frac{1}{6\sqrt{2}}\sqrt{(N_c-1)(N_c+3)}$	0
$\Sigma^{*-}\Sigma^{-}$	$-\frac{1}{12\sqrt{2}}(N_c-3)$	$+\frac{1}{12\sqrt{2}}(N_c+5)$	$-\frac{\sqrt{2}}{3}$
$\Xi^{*0}\Xi^{0}$	$+\frac{1}{9\sqrt{2}}(N_c+3)$	$-\frac{1}{9\sqrt{2}}(N_c-3)$	$-\frac{\sqrt{2}}{3}$
$\Xi^{*-}\Xi^{-}$	$-\frac{1}{9\sqrt{2}}(N_c-3)$	$+\frac{1}{9\sqrt{2}}(N_c+3)$	$-\frac{\sqrt{2}}{3}$
$\Sigma^{0}\Lambda$	$-\frac{1}{4\sqrt{6}}\sqrt{(N_c-1)(N_c+3)}$	$+\frac{1}{4\sqrt{6}}\sqrt{(N_c-1)(N_c+3)}$	0

tion of $I=0$ and $I=1$. Consequently, any combination of charge radii only sensitive to $I=2$ or $I=3$ operators must vanish. These are

$\underline{I=2}$:

$$0=2r^2_{\Delta^{++}}-3r^2_{\Delta^{+}}+3r^2_{\Delta^{0}}+r^2_{\Delta^{-}}, \tag{4.3}$$

$$0=r^2_{\Sigma^{+}}-2r^2_{\Sigma^{0}}-r^2_{\Sigma^{-}}, \tag{4.4}$$

$$0=r^2_{\Sigma^{*+}}-2r^2_{\Sigma^{*0}}-r^2_{\Sigma^{*-}}, \tag{4.5}$$

$\underline{I=3}$:

$$0=2r^2_{\Delta^{++}}-r^2_{\Delta^{+}}-r^2_{\Delta^{0}}-r^2_{\Delta^{-}}. \tag{4.6}$$

Equations (4.3) and (4.6) were first derived in Ref. [3], while Eqs. (4.4) and (4.5) are Σ equal-spacing rules, adjusted for the negative charge of $\Sigma^{(*)-}$. In addition, using Appendix A

TABLE III. The matrix elements $\langle\alpha\alpha\rangle^{(2)}$ for all relevant states.

State	$\langle uu\rangle^{(2)}$	$\langle dd\rangle^{(2)}$	$\langle ss\rangle^{(2)}$
Δ^{++}	$+\frac{1}{40}(N_c+7)(N_c+9)$	$+\frac{1}{40}(N_c-5)(N_c-3)$	0
Δ^{+}	$-\frac{1}{40}(N_c-7)(N_c+7)$	$-\frac{1}{40}(N_c-3)(N_c+11)$	0
Δ^{0}	$-\frac{1}{40}(N_c-3)(N_c+11)$	$-\frac{1}{40}(N_c-7)(N_c+7)$	0
Δ^{-}	$+\frac{1}{40}(N_c-5)(N_c-3)$	$+\frac{1}{40}(N_c+7)(N_c+9)$	0
Σ^{*+}	$+\frac{1}{80}(N_c+5)(N_c+7)$	$+\frac{1}{80}(N_c-5)(N_c-3)$	0
Σ^{*0}	$-\frac{1}{40}(N_c-3)(N_c+5)$	$-\frac{1}{40}(N_c-3)(N_c+5)$	0
Σ^{*-}	$+\frac{1}{80}(N_c-5)(N_c-3)$	$+\frac{1}{80}(N_c+5)(N_c+7)$	0
Ξ^{*0}	0	0	$+1$
Ξ^{*-}	0	0	$+1$
Ω^{-}	0	0	$+3$
$\Delta^{+}p$	$+\frac{1}{20\sqrt{2}}(N_c+7)\sqrt{(N_c-1)(N_c+5)}$	$+\frac{1}{20\sqrt{2}}(N_c-3)\sqrt{(N_c-1)(N_c+5)}$	0
$\Delta^{0}n$	$-\frac{1}{20\sqrt{2}}(N_c-3)\sqrt{(N_c-1)(N_c+5)}$	$-\frac{1}{20\sqrt{2}}(N_c+7)\sqrt{(N_c-1)(N_c+5)}$	0
$\Sigma^{*+}\Sigma^{+}$	$+\frac{1}{40\sqrt{2}}(N_c+5)(N_c+7)$	$+\frac{1}{40\sqrt{2}}(N_c-5)(N_c-3)$	0
$\Sigma^{*0}\Sigma^{0}$	$-\frac{1}{20\sqrt{2}}(N_c-3)(N_c+5)$	$-\frac{1}{20\sqrt{2}}(N_c-3)(N_c+5)$	0
$\Sigma^{*0}\Lambda$	0	0	0
$\Sigma^{*-}\Sigma^{-}$	$+\frac{1}{40\sqrt{2}}(N_c-5)(N_c-3)$	$+\frac{1}{40\sqrt{2}}(N_c+5)(N_c+7)$	0
$\Xi^{*0}\Xi^{0}$	0	0	$-\sqrt{2}$
$\Xi^{*-}\Xi^{-}$	0	0	$-\sqrt{2}$

one finds 3 linear combinations of charge radii with N_c-independent coefficients that vanish for arbitrary values of N_c and r, either in the quadratic or cubic case of SU(3) symmetry breaking:

$$\begin{aligned} 0=&-20(r_p^2+r_n^2)+5(r_{\Sigma^+}^2-r_{\Sigma^0}^2+3r_{\Sigma^-}^2)\\ &+5r_\Lambda^2-4(4r_{\Delta^{++}}^2-r_{\Delta^+}^2-4r_{\Delta^0}^2+7r_{\Delta^-}^2)\\ &+10(r_{\Sigma^{*+}}^2-r_{\Sigma^{*0}}^2+3r_{\Sigma^{*-}}^2), \end{aligned} \tag{4.7}$$

$$\begin{aligned} 0=&4(r_p^2-5r_n^2)-(5r_{\Sigma^+}^2+3r_{\Sigma^0}^2-r_{\Sigma^-}^2)\\ &+35r_\Lambda^2+4(2r_{\Delta^{++}}^2+r_{\Delta^+}^2+r_{\Delta^0}^2-r_{\Delta^-}^2)\\ &-2(5r_{\Sigma^{*+}}^2+3r_{\Sigma^{*0}}^2-r_{\Sigma^{*-}}^2), \end{aligned} \tag{4.8}$$

$$\begin{aligned} 0=&-8(r_p^2+r_n^2)+(r_{\Sigma^+}^2-9r_{\Sigma^-}^2)+2r_\Lambda^2+16r_{\Xi^-}^2\\ &+2(r_{\Sigma^{*+}}^2-9r_{\Sigma^{*-}}^2)+32r_{\Xi^{*-}}^2-16r_{\Omega^-}^2. \end{aligned} \tag{4.9}$$

Since so many of these charge radii are currently unmeasured, these relations are presented merely for completeness.

A number of charge radii can be related if one permits N_c-dependent coefficients. A particularly pretty example is

$$r_\Lambda^2=\frac{3}{N_c+3}r_n^2. \tag{4.10}$$

The $N_c=3$ version of this relation, but applied to magnetic moments rather than charge radii, is known from the early days of SU(3) flavor (see, e.g., [21]). Using the measured value for the neutron charge radius (2.3) and $N_c=3$, one predicts $r_\Lambda^2=-0.057(3)$ fm^2. The expression (4.10) is especially interesting because it definitively predicts—independent of SU(3) symmetry breaking—that the sign of the charge radius of Λ is the same as that of the neutron, i.e., negative. This is not the case, for example, in the calculations of Ref. [22] (extrapolation from lattice results) or Ref. [23] (constituent quark model). For other comparisons, Refs. [23–25] also contain tabulated results of various authors' baryon charge radii calculations.

Another interesting result is that

$$r_{\Sigma^+}^2=r_p^2+O(1-r), \tag{4.11}$$

so that the two charge radii are equal in the SU(3) limit *for all values of* N_c, as is evident from Eqs. (A11) and (A13). One expects that this relationship holds to better accuracy

TABLE IV. The matrix elements $\langle\alpha\beta\rangle^{(2)}$, with $\alpha\neq\beta$, for all relevant states.

State	$\langle ud\rangle^{(2)}$	$\langle us\rangle^{(2)}$	$\langle ds\rangle^{(2)}$
Δ^{++}	$-\frac{1}{40}(N_c-3)(N_c+7)$	0	0
Δ^{+}	$+\frac{1}{40}(N_c^2+4N_c+19)$	0	0
Δ^{0}	$+\frac{1}{40}(N_c^2+4N_c+19)$	0	0
Δ^{-}	$-\frac{1}{40}(N_c-3)(N_c+7)$	0	0
Σ^{*+}	$-\frac{1}{80}(N_c-3)(N_c+5)$	$+\frac{1}{8}(N_c+5)$	$-\frac{1}{8}(N_c-3)$
Σ^{*0}	$+\frac{1}{40}(N_c^2+2N_c+5)$	$+\frac{1}{2}$	$+\frac{1}{2}$
Σ^{*-}	$-\frac{1}{80}(N_c-3)(N_c+5)$	$-\frac{1}{8}(N_c-3)$	$+\frac{1}{8}(N_c+5)$
Ξ^{*0}	0	$+\frac{1}{6}(N_c+3)$	$-\frac{1}{6}(N_c-3)$
Ξ^{*-}	0	$-\frac{1}{6}(N_c-3)$	$+\frac{1}{6}(N_c+3)$
Ω^{-}	0	0	0
$\Delta^{+}p$	$-\frac{1}{20\sqrt{2}}(N_c+2)\sqrt{(N_c-1)(N_c+5)}$	0	0
$\Delta^{0}n$	$+\frac{1}{20\sqrt{2}}(N_c+2)\sqrt{(N_c-1)(N_c+5)}$	0	0
$\Sigma^{*+}\Sigma^{+}$	$-\frac{1}{40\sqrt{2}}(N_c-3)(N_c+5)$	$-\frac{1}{8\sqrt{2}}(N_c+5)$	$+\frac{1}{8\sqrt{2}}(N_c-3)$
$\Sigma^{*0}\Sigma^{0}$	$+\frac{1}{20\sqrt{2}}(N_c^2+2N_c+5)$	$-\frac{1}{2\sqrt{2}}$	$-\frac{1}{2\sqrt{2}}$
$\Sigma^{*0}\Lambda$	0	$+\frac{1}{4\sqrt{2}}\sqrt{(N_c-1)(N_c+3)}$	$-\frac{1}{4\sqrt{2}}\sqrt{(N_c-1)(N_c+3)}$
$\Sigma^{*-}\Sigma^{-}$	$-\frac{1}{40\sqrt{2}}(N_c-3)(N_c+5)$	$+\frac{1}{8\sqrt{2}}(N_c-3)$	$-\frac{1}{8\sqrt{2}}(N_c+5)$
$\Xi^{*0}\Xi^{0}$	0	$+\frac{1}{6\sqrt{2}}(N_c+3)$	$-\frac{1}{6\sqrt{2}}(N_c-3)$
$\Xi^{*-}\Xi^{-}$	0	$-\frac{1}{6\sqrt{2}}(N_c-3)$	$+\frac{1}{6\sqrt{2}}(N_c+3)$

than $r^2_{\Xi^0}=r^2_n$, which requires *both* the SU(3) limit and $N_c=3$ to hold, so that both corrections of $O(1-r)$ and $O(1-3/N_c)$ occur.

There is a relation of the latter type between the 3 measured charge radii:

$$r_p^2-r_{\Sigma^-}^2+r_n^2=O(1-r)+O(1-3/N_c). \qquad (4.12)$$

Or, if one allows N_c-dependent coefficients,

$$\tfrac{1}{2}(N_c-1)(r_p^2-r_{\Sigma^-}^2)+r_n^2=O[(1-r)(1-3/N_c)]. \qquad (4.13)$$

Numerically, the l.h.s. nearly vanishes: One finds $-0.06(15)$ fm^2. However, the uncertainty is the figure of merit here, since it may be used to gauge the typical size of a first-order SU(3)-breaking effect.

Yet one more interesting result is

$$r^2_{\Xi^-}=\frac{1}{2}(r_p^2+r_{\Sigma^-}^2+r_n^2)+O[(1-r)^2]+O(1-3/N_c), \qquad (4.14)$$

in both the quadratic and cubic forms of SU(3) symmetry breaking. The second of the order terms implies that the result holds exactly only for $N_c=3$, but this can be pushed to higher order if one allows N_c-dependent coefficients on the l.h.s., just as in Eq. (4.13). The virtue of this expression as it stands is that it predicts $r^2_{\Xi^-}$ up to second order in SU(3) symmetry breaking entirely in terms of the known charge radii, with simple coefficients. One finds

$$r^2_{\Xi^-}=0.64(8)\ \ \text{fm}^2, \qquad (4.15)$$

where the uncertainty is dominated by that of the $r^2_{\Sigma^-}$ measurement, which is much larger than the piece of the uncertainty one would estimate from the second-order SU(3) symmetry breaking.

Lastly, our expressions for charge radii depend upon only four parameters, $A,B,C,$ and r [and a choice of scheme for SU(3) breaking]. Once one additional charge radius is measured, it will be possible to solve for all the parameters and predict all of the other charge radii. In this sense, $r^2_{\Xi^-}$ was chosen in the previous paragraph because it could be predicted in terms of just the measured charge radii, with little sensitivity to the SU(3)-breaking scheme or the parameter r. Conversely, if it becomes the fourth measured baryon charge radius, it will not effectively constrain the parameter r.

On the other hand, it turns out that the value of $r^2_{\Omega^-}$ (to give one example) cannot be predicted in terms of $r^2_{p,\,n,\,\Sigma^-}$, even in the SU(3) limit. Turning this unfortunate observation around means that a measurement of $r^2_{\Omega^-}$ would provide an extremely sensitive probe of SU(3) symmetry breaking, and permit high-quality predictions of all other charge radii.

Using the three measured charge radii and the assumption of either quadratic $(n=1)$ or cubic $(n=2)$ SU(3) symmetry breaking and an assumed value of r, one can obtain at least ranges for the values of the parameters A, B, and C. Since these parameters have an expansion in $1/N_c^m$ starting with $m=0$, the values obtained for $N_c=3$ hold for arbitrary N_c. One finds

$$(1-2r-2r^n)(r_p^2+r_n^2)+3r_{\Sigma^-}^2=+2(2-r-r^n)A,$$

$$r_p^2-r_{\Sigma^-}^2+\frac{1}{3}(5-r-r^n)r_n^2=-\frac{2}{3}(2-r-r^n)B,$$

$$r_p^2-r_{\Sigma^-}^2-\frac{1}{3}(1-2r-2r^n)r_n^2=-\frac{4}{9}(2-r-r^n)C. \tag{4.16}$$

Note that these all become Eq. (4.12) in the SU(3) limit. The uncertainties in these expressions are dominated by that of $r^2_{\Sigma^-}$, even more than by that of n. Including only the former and assuming the value $r=330$ MeV/540 MeV suggested by the constituent quark model, one computes

$$A=+0.56\pm0.29 \;\; \text{fm}^2,$$

$$B=-0.05\pm0.29 \;\; \text{fm}^2,$$

$$C=-0.33\pm0.44 \;\; \text{fm}^2. \tag{4.17}$$

With these central values one predicts $r^2_{\Delta^+}<r^2_p$ for $N_c=3$, contrary to the physical picture that the Δ is an excited state of the nucleon and hence is more extended in space. However, it is well within the error bars of $r^2_{\Sigma^-}$ for B to be substantial and for C to nearly vanish. For example, with $r^2_{\Sigma^-}=0.73$ fm^2 as suggested by the upper value of the range of statistical uncertainty in Eq. (2.4) and r as above one obtains $A=0.79$ fm^2, $B=0.18$ fm^2, and $C=0.02$ fm^2, a hierarchy of parameters leading to the reasonable conclusion $r^2_{\Delta^+}>r^2_p$.

B. Quadrupole moments

The operator (3.3) defining quadrupole moments shares with the charge radius operator (3.2) the feature in the single-photon exchange ansatz that isospin violation enters only through a single power of the quark charge operator Q_α. Thus, as before, combinations sensitive only to $I=2$ or $I=3$ operators must vanish. These are

$\underline{I=3}$:

$$0=Q_{\Delta^{++}}-3Q_{\Delta^+}+3Q_{\Delta^0}-Q_{\Delta^-}, \tag{4.18}$$

$\underline{I=2}$:

$$0=Q_{\Delta^{++}}-Q_{\Delta^+}-Q_{\Delta^0}+Q_{\Delta^-}, \tag{4.19}$$

$$0=Q_{\Delta^+p}-Q_{\Delta^0n}, \tag{4.20}$$

$$0=Q_{\Sigma^{*+}}-2Q_{\Sigma^{*0}}+Q_{\Sigma^{*-}}, \tag{4.21}$$

$$0=Q_{\Sigma^{*+}\Sigma^+}-2Q_{\Sigma^{*0}\Sigma^0}+Q_{\Sigma^{*-}\Sigma^-}. \tag{4.22}$$

The first three of these expressions were obtained in Ref. [2], while the last two are $\Sigma^{(*)}$ equal-spacing rules, obtained for $N_c=3$ in [26] and [5]. In addition, there is precisely one linear relation with N_c-independent coefficients that holds for all values of N_c in all cases of SU(3) symmetry breaking studied here:

$$0=Q_{\Xi^{*-}}-Q_{\Omega^-}-\sqrt{2}Q_{\Xi^{*-}\Xi^-}. \tag{4.23}$$

Unlike the charge radius case, only the $N\to\Delta$ quadrupole transition matrix element has been measured (via photoproduction experiments [27]), and even here the extraction of $Q_{N\to\Delta}$ is plagued by a large model dependence:

$$Q_{N\to\Delta}=-0.108\pm0.009 \;\; (\text{stat}+\text{syst}) \pm0.034 \;\; (\text{model}) \;\; \text{fm}^2. \tag{4.24}$$

Since there are 3 undetermined parameters $[B',C'$, and r, as well as a choice of SU(3)-breaking scheme], we do not attempt to predict any of the other quadrupole moments numerically. However, we can still make a number of interesting observations based on the structure of the expressions in Appendix B.

First note that each quadrupole moment expression is $O(N_c^0)$ or smaller in the $1/N_c$ expansion. Indeed, only one coefficient, B', contributes to this leading order [in the case of charge radii, both A (for charged baryons only) and B contribute at leading order]. A few moments' study will confirm that the diagonal quadrupole moments are given by the expression

$$Q(I_3,Y)=I_3[1-Y(4-Y)/15]B'+O(1/N_c). \tag{4.25}$$

Proportionality to I_3 also holds for the leading terms of the transition quadrupole moments when the initial and final baryon states have the same value of I ($\Sigma^*\Sigma$ and $\Xi^*\Xi$). This behavior, due to the dominance of the isovector portion of the quadrupole operator, is familiar from the Skyrme model and a variety of other model calculations [28,29]. Taken at face value, it predicts an appreciable quadrupole moment for Δ^0, as well as $Q_{\Omega^-}=0$. However, the subleading corrections in $1/N_c$, which interpolate between the extreme $N_c\to\infty$ and $N_c=3$ cases, soften this behavior. In particular, the diagonal quadrupole moments in the strict $N_c=3$ case with no SU(3) flavor symmetry breaking obey

$$Q=4q/3(B'+C'/3), \tag{4.26}$$

where q is the baryon charge.

As discussed in Sec. III, the charge radius and quadrupole operators are very similar in that both represent spin-dependent electromagnetic couplings to baryons. It thus should not be surprising that their coefficients are related in a typical model. In Ref. [2] we saw that the one-gluon exchange picture gives rise to the relation

$$Q_{\Delta^+ p}=\frac{1}{\sqrt{2}}r_n^2\frac{N_c}{N_c+3}\sqrt{\frac{N_c+5}{N_c-1}}, \tag{4.27}$$

for which the factor on the r.h.s. following r_n^2 equals unity both for $N_c=3$ and $N_c\to\infty$. In fact, this result can be obtained in a much more general setting. The key constraints needed to obtain Eq. (4.27) are $B'=-2B$, $C'=-2C$, and as argued in Ref. [18] for $N_c=3$, the same relation may be derived using not only one-gluon exchange, but one-pion exchange or scalar exchanges, or a mixture of these.

A particularly useful measurement for determining the coefficients in the quadrupole sector would be that of Q_{Ω^-}. As we see in Appendix B, the value of Q_{Ω^-} is very sensitive to the precise nature of SU(3) symmetry breaking, even more so than $r^2_{\Omega^-}$. We discussed in Ref. [2] ideas in the literature for the experimental determination of Q_{Ω^-}; while such experiments are challenging, they appear to be feasible. The value of Q_{Ω^-} would teach us much about the shape of baryon wave functions and the nature of SU(3) flavor symmetry.

V. CONCLUSIONS

We have presented techniques that permit the calculation of matrix elements of operators acting upon 3-flavor baryon states with arbitrary N_c. The approach requires only SU(2) Clebsch-Gordan coefficients and combinations of them in the form of $6j$ symbols. We tabulated the values of a set of primitive operators for all relevant states and demonstrated how they can be combined to give results for charge radii or quadrupole moments, using any chosen pattern of SU(3) (or isospin) flavor symmetry breaking. In particular, we presented in the Appendixes results using the single-photon exchange ansatz, augmented by either no SU(3) symmetry breaking, or one of two popular types of SU(3) symmetry breaking suggested by (but not limited to) the quark model with gluon exchange. We obtained a large number of interesting predictions and demonstrated how many others can be made either working in some particular model, or once a small number of additional baryon charge radii or quadrupole moments are experimentally measured.

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APPENDIX A: CHARGE RADIUS EXPRESSIONS

We use the subscripts 0, Q, and C to denote expressions with zero, quadratic, and cubic SU(3) flavor symmetry breaking via constituent quark masses [see Eq. (4.1)] in addition to that provided by the quark charge operator. In the SU(3) symmetry limit ($r=1$) the expressions with subscripts Q and C reduce to those with subscript 0:

$$r^2_{0,Q,C}(\Delta^{++})=A+B\ \frac{3(N_c^2-2N_c+5)}{4N_c^2}-C\ \frac{3(3N_c^2-12N_c+5)}{2N_c^3}, \tag{A1}$$

$$r^2_{0,Q,C}(\Delta^{+})=A+B\ \frac{N_c^2-4N_c+15}{2N_c^2}-C\ \frac{(N_c-1)(4N_c-15)}{N_c^3}, \tag{A2}$$

$$r^2_{0,Q,C}(\Delta^{0})=-\left(B-\frac{2C}{N_c}\right)\frac{(N_c-3)(N_c-5)}{2N_c}, \tag{A3}$$

$$r^2_{0,Q,C}(\Delta^{-})=A+B\ \frac{3(N_c^2-5)}{2N_c^2}-C\ \frac{3(2N_c^2-5N_c-5)}{N_c^3}, \tag{A4}$$

$$r^2_{0}(\Sigma^{*+})=A+B\ \frac{5N_c^2-17N_c+30}{4N_c^2}-C\ \frac{11N_c^2-47N_c+30}{2N_c^3},$$

$$r_Q^2(\Sigma^{*+}) = A + \frac{B}{4}\left[4 + r - \frac{14+3r}{N_c} + \frac{2(11+4r)}{N_c^2}\right] - \frac{C}{2N_c}\left[10 + r - \frac{36+11r}{N_c} + \frac{2(11+4r)}{N_c^2}\right],$$

$$r_C^2(\Sigma^{*+}) = A + \frac{B}{4}\left[4 + r - \frac{14-r+4r^2}{N_c} + \frac{2(11+2r+2r^2)}{N_c^2}\right] - \frac{C}{4N_c}\left[20 + r + r^2 - \frac{72+11r+11r^2}{N_c} + \frac{4(11+2r+2r^2)}{N_c^2}\right], \tag{A5}$$

$$r_0^2(\Sigma^{*0}) = -\left(B - \frac{2C}{N_c}\right)\frac{5(N_c-3)}{2N_c^2},$$

$$r_Q^2(\Sigma^{*0}) = -\left(B - \frac{2C}{N_c}\right)\frac{1}{2N_c}\left(3 + 2r - \frac{11+4r}{N_c}\right),$$

$$r_C^2(\Sigma^{*0}) = -\frac{B}{2N_c}\left(3 + 2r^2 - \frac{11+2r+2r^2}{N_c}\right) + \frac{C}{N_c^2}\left(3 + r + r^2 - \frac{11+2r+2r^2}{N_c}\right), \tag{A6}$$

$$r_0^2(\Sigma^{*-}) = A + B\,\frac{5N_c^2+3N_c-30}{4N_c^2} - C\,\frac{11N_c^2-27N_c-30}{2N_c^3},$$

$$r_Q^2(\Sigma^{*-}) = A + \frac{B}{4}\left[4 + r - \frac{2-5r}{N_c} - \frac{2(11+4r)}{N_c^2}\right] - \frac{C}{2N_c}\left[10 + r - \frac{3(8+r)}{N_c} - \frac{2(11+4r)}{N_c^2}\right],$$

$$r_C^2(\Sigma^{*-}) = A + \frac{B}{4}\left[4 + r - \frac{2-r-4r^2}{N_c} - \frac{2(11+2r+2r^2)}{N_c^2}\right] - \frac{C}{4N_c}\left[20 + r + r^2 - \frac{3(16+r+r^2)}{N_c} - \frac{4(11+2r+2r^2)}{N_c^2}\right], \tag{A7}$$

$$r_0^2(\Xi^{*0}) = +\left(B - \frac{2C}{N_c}\right)\frac{5(N_c-3)^2}{6N_c^2},$$

$$r_Q^2(\Xi^{*0}) = +\left(B - \frac{2C}{N_c}\right)\frac{1}{6}\left[3 + 2r - \frac{6(3+r+r^2)}{N_c} + \frac{3(9+4r+2r^2)}{N_c^2}\right],$$

$$r_C^2(\Xi^{*0}) = +\frac{B}{6}\left[3 + 2r - \frac{6(3+r^2+r^3)}{N_c} + \frac{3(9+2r+2r^2+2r^3)}{N_c^2}\right] + \frac{C}{3N_c}\left[3 + r + r^2 - \frac{3(6+r+r^2+2r^3)}{N_c} + \frac{3(9+2r+2r^2+2r^3)}{N_c^2}\right], \tag{A8}$$

$$r_0^2(\Xi^{*-}) = A + B\,\frac{5N_c^2+12N_c-45}{6N_c^2} - C\,\frac{14N_c^2-33N_c-45}{3N_c^3},$$

$$r_Q^2(\Xi^{*-}) = A + \frac{B}{6}\left[3 + 2r + \frac{6r(1+r)}{N_c} - \frac{3(9+4r+2r^2)}{N_c^2}\right] - \frac{C}{3N_c}\left[2(6+r) - \frac{3(9+2r)}{N_c} - \frac{3(9+4r+2r^2)}{N_c^2}\right],$$

$$r_C^2(\Xi^{*-}) = A + \frac{B}{6}\left[3 + 2r + \frac{6r^2(1+r)}{N_c} - \frac{3(9+2r+2r^2+2r^3)}{N_c^2}\right] - \frac{C}{3N_c}\left[12 + r + r^2 - \frac{3(9+r+r^2)}{N_c} - \frac{3(9+2r+2r^2+2r^3)}{N_c^2}\right], \tag{A9}$$

$$r_0^2(\Omega^-)=A+B\ \frac{3(3N_c-5)}{2N_c^2}-C\ \frac{3(N_c^2-2N_c-5)}{N_c^3},$$

$$r_Q^2(\Omega^-)=A+\frac{3B}{2N_c}\left(1+2r^2-\frac{3+2r^2}{N_c}\right)-\frac{3C}{N_c}\left(1-\frac{2}{N_c}-\frac{3+2r^2}{N_c^2}\right),$$

$$r_C^2(\Omega^-)=A+\frac{3B}{2N_c}\left(1+2r^3-\frac{3+2r^3}{N_c}\right)-\frac{3C}{N_c}\left(1-\frac{2}{N_c}-\frac{3+2r^3}{N_c^2}\right), \tag{A10}$$

$$r_{0,Q,C}^2(p)=A+B\ \frac{(N_c-3)(N_c-1)}{2N_c^2}-C\ \frac{(N_c-1)(4N_c-3)}{N_c^3}, \tag{A11}$$

$$r_{0,Q,C}^2(n)=-\left(B-\frac{2C}{N_c}\right)\frac{(N_c-1)(N_c+3)}{2N_c^2}, \tag{A12}$$

$$r_0^2(\Sigma^+)=A+B\ \frac{(N_c-3)(N_c-1)}{2N_c^2}-C\ \frac{(N_c-1)(4N_c-3)}{N_c^3},$$

$$r_Q^2(\Sigma^+)=A+\frac{B}{2}\left(2-r-\frac{7-3r}{N_c}+\frac{11-8r}{N_c^2}\right)-\frac{C}{N_c}\left(5-r-\frac{18-11r}{N_c}+\frac{11-8r}{N_c^2}\right),$$

$$r_C^2(\Sigma^+)=A+\frac{B}{2}\left(2-r-\frac{7+r-4r^2}{N_c}+\frac{11-4r-4r^2}{N_c^2}\right)-\frac{C}{2N_c}\left[10-r-r^2-\frac{36-11r-11r^2}{N_c}+\frac{2(11-4r-4r^2)}{N_c^2}\right], \tag{A13}$$

$$r_0^2(\Sigma^0)=+\left(B-\frac{2C}{N_c}\right)\frac{N_c+3}{2N_c^2},$$

$$r_Q^2(\Sigma^0)=-\left(B-\frac{2C}{N_c}\right)\frac{1}{2N_c}\left(3-4r-\frac{11-8r}{N_c}\right),$$

$$r_C^2(\Sigma^0)=-\frac{B}{2N_c}\left(3-4r^2-\frac{11-4r-4r^2}{N_c}\right)+\frac{C}{N_c^2}\left(3-2r-2r^2-\frac{11-4r-4r^2}{N_c}\right), \tag{A14}$$

$$r_{0,Q,C}^2(\Lambda)=-\left(B-\frac{2C}{N_c}\right)\frac{3(N_c-1)}{2N_c^2}, \tag{A15}$$

$$r_0^2(\Sigma^0\Lambda)=-\left(B-\frac{2C}{N_c}\right)\frac{\sqrt{(N_c-1)(N_c+3)}}{2N_c},$$

$$r_Q^2(\Sigma^0\Lambda)=-r\left(B-\frac{2C}{N_c}\right)\frac{\sqrt{(N_c-1)(N_c+3)}}{2N_c},$$

$$r_C^2(\Sigma^0\Lambda)=-r\left[B-\frac{C(1+r)}{N_c}\right]\frac{\sqrt{(N_c-1)(N_c+3)}}{2N_c}, \tag{A16}$$

$$r_0^2(\Sigma^-)=A+B\ \frac{N_c^2-6N_c-3}{2N_c^2}-C\ \frac{4N_c^2-9N_c-3}{N_c^3},$$

$$r_Q^2(\Sigma^-) = A + \frac{B}{2}\left(2 - r - \frac{1+5r}{N_c} - \frac{11-8r}{N_c^2}\right) - \frac{C}{N_c^2}\left[5 - r - \frac{3(4-r)}{N_c} - \frac{11-8r}{N_c^2}\right],$$

$$r_C^2(\Sigma^-) = A + \frac{B}{2}\left(2 - r - \frac{1+r+4r^2}{N_c} - \frac{11-4r-4r^2}{N_c^2}\right) - \frac{C}{2N_c^2}\left[10 - r - r^2 - \frac{3(8-r-r^2)}{N_c} - \frac{2(11-4r-4r^2)}{N_c^2}\right], \tag{A17}$$

$$r_0^2(\Xi^0) = -\left(B - \frac{2C}{N_c}\right)\frac{N_c^2+12N_c-9}{6N_c^2},$$

$$r_Q^2(\Xi^0) = +\left(B - \frac{2C}{N_c}\right)\frac{1}{6}\left[3 - 4r - \frac{6(3-2r+r^2)}{N_c} + \frac{3(9-8r+2r^2)}{N_c^2}\right],$$

$$r_C^2(\Xi^0) = +\frac{B}{6}\left[3 - 4r - \frac{6(1+r)(3-3r+r^2)}{N_c} + \frac{3(9-4r-4r^2+2r^3)}{N_c^2}\right] - \frac{C}{3N_c}\left[3 - 2r - 2r^2 - \frac{6(3-r-r^2+r^3)}{N_c} + \frac{3(9-4r-4r^2+2r^3)}{N_c^2}\right], \tag{A18}$$

$$r_0^2(\Xi^-) = A - B\ \frac{(N_c+3)^2}{6N_c^2} - C\ \frac{8N_c^2-15N_c-9}{3N_c^3},$$

$$r_Q^2(\Xi^-) = A + \frac{B}{6}\left[3 - 4r - \frac{6r(2-r)}{N_c} - \frac{3(9-8r+2r^2)}{N_c^2}\right] - \frac{C}{3N_c}\left[4(3-r) - \frac{3(9-4r)}{N_c} - \frac{3(9-8r+2r^2)}{N_c^2}\right],$$

$$r_C^2(\Xi^-) = A + \frac{B}{6}\left[3 - 4r - \frac{6r^2(2-r)}{N_c} - \frac{3(9-4r-4r^2+2r^3)}{N_c^2}\right] - \frac{C}{3N_c}\left[2(6-r-r^2) - \frac{3(9-2r-2r^2)}{N_c} - \frac{3(9-4r-4r^2+2r^3)}{N_c^2}\right]. \tag{A19}$$

APPENDIX B: QUADRUPOLE MOMENT EXPRESSIONS

We use the subscripts 0, Q, and C to denote expressions with zero, quadratic, and cubic SU(3) breaking via constituent quark masses [see Eq. (4.1)] in addition to that provided by the quark charge operator. In the SU(3) symmetry limit ($r=1$) the expressions with subscripts Q and C reduce to those with subscript 0:

$$Q_{0,Q,C}(\Delta^{++}) = +B'\frac{6(N_c^2+2N_c+5)}{5N_c^2} - C'\frac{12(N_c^2-8N_c+5)}{5N_c^3}, \tag{B1}$$

$$Q_{0,Q,C}(\Delta^{+}) = +B'\ \frac{2(N_c^2+2N_c+15)}{5N_c^2} - C'\frac{4(N_c^2-13N_c+15)}{5N_c^3}, \tag{B2}$$

$$Q_{0,Q,C}(\Delta^{0}) = -\left(B' - \frac{2C'}{N_c}\right)\frac{2(N_c-3)(N_c+5)}{5N_c^2}, \tag{B3}$$

$$Q_{0,Q,C}(\Delta^{-}) = -B'\ \frac{6(N_c^2+2N_c-5)}{5N_c^2} + C'\frac{12(N_c^2-3N_c-5)}{5N_c^3}, \tag{B4}$$

$$Q_0(\Sigma^{*+}) = +B' \frac{N_c^2 - N_c + 6}{N_c^2} - C' \frac{2(N_c-6)(N_c-1)}{N_c^3},$$

$$Q_Q(\Sigma^{*+}) = +\frac{B'}{2}\left[1 + r + \frac{1-3r}{N_c} + \frac{4(1+2r)}{N_c^2}\right] - \frac{C'}{N_c}\left[1 + r - \frac{3+11r}{N_c} + \frac{4(1+2r)}{N_c^2}\right],$$

$$Q_C(\Sigma^{*+}) = +\frac{B'}{2}\left[1 + r + \frac{1+r-4r^2}{N_c} + \frac{4(1+r+r^2)}{N_c^2}\right] - \frac{C'}{2N_c}\left[2 + r + r^2 - \frac{6+11r+11r^2}{N_c} + \frac{8(1+r+r^2)}{N_c^2}\right], \tag{B5}$$

$$Q_0(\Sigma^{*0}) = -\left(B' - \frac{2C'}{N_c}\right)\frac{2(N_c-3)}{N_c^2},$$

$$Q_Q(\Sigma^{*0}) = -\left(B' - \frac{2C'}{N_c}\right)\frac{2}{N_c}\left(r - \frac{1+2r}{N_c}\right),$$

$$Q_C(\Sigma^{*0}) = -\frac{2B'}{N_c}\left(r^2 - \frac{1+r+r^2}{N_c}\right) + \frac{2C'}{N_c^2}\left[r(1+r) - \frac{2(1+r+r^2)}{N_c}\right], \tag{B6}$$

$$Q_0(\Sigma^{*-}) = -B' \frac{N_c^2 + 3N_c - 6}{N_c^2} + C' \frac{2(N_c^2 - 3N_c - 6)}{N_c^3},$$

$$Q_Q(\Sigma^{*-}) = -\frac{B'}{2}\left[1 + r + \frac{1+5r}{N_c} - \frac{4(1+2r)}{N_c^2}\right] + \frac{C'}{N_c}\left[1 + r - \frac{3(1+r)}{N_c} - \frac{4(1+2r)}{N_c^2}\right],$$

$$Q_C(\Sigma^{*-}) = -\frac{B'}{2}\left[1 + r + \frac{1+r+4r^2}{N_c} - \frac{4(1+r+r^2)}{N_c^2}\right] + \frac{C'}{2N_c}\left[2 + r + r^2 - \frac{3(2+r+r^2)}{N_c} - \frac{8(1+r+r^2)}{N_c^2}\right], \tag{B7}$$

$$Q_0(\Xi^{*0}) = +\left(B' - \frac{2C'}{N_c}\right)\frac{2(N_c-3)^2}{3N_c^2},$$

$$Q_Q(\Xi^{*0}) = +\left(B' - \frac{2C'}{N_c}\right)\frac{2r}{3}\left[1 - \frac{3(1+r)}{N_c} + \frac{3(2+r)}{N_c^2}\right],$$

$$Q_C(\Xi^{*0}) = +\frac{2r}{3}\left\{B'\left[1 - \frac{3r(1+r)}{N_c} + \frac{3(1+r+r^2)}{N_c^2}\right] - \frac{C'}{N_c}\left[1 + r - \frac{3(1+r+2r^2)}{N_c} + \frac{6(1+r+r^2)}{N_c^2}\right]\right\}, \tag{B8}$$

$$Q_0(\Xi^{*-}) = -B' \frac{2(N_c^2 + 6N_c - 9)}{3N_c^2} + C' \frac{4(N_c^2 - 3N_c - 9)}{3N_c^3},$$

$$Q_Q(\Xi^{*-}) = -\frac{2r}{3}\left\{B'\left[1 + \frac{3(1+r)}{N_c} - \frac{3(2+r)}{N_c^2}\right] - \frac{2C'}{N_c}\left[1 - \frac{3}{N_c} - \frac{3(r+2)}{N_c^2}\right]\right\},$$

$$Q_C(\Xi^{*-}) = -\frac{2r}{3}\left\{B'\left[1 + \frac{3r(1+r)}{N_c} - \frac{3(1+r+r^2)}{N_c^2}\right] - \frac{C'}{N_c}\left[(1+r) - \frac{3(1+r)}{N_c} - \frac{6(1+r+r^2)}{N_c^2}\right]\right\}, \tag{B9}$$

$$Q_0(\Omega^-) = -B' \frac{6(N_c-1)}{N_c^2} - \frac{12C'}{N_c^3},$$

$$Q_Q(\Omega^-) = -\frac{6r^2}{N_c}\left[B'\left(1-\frac{1}{N_c}\right)+\frac{2C'}{N_c^2}\right],$$

$$Q_C(\Omega^-) = -\frac{6r^3}{N_c}\left[B'\left(1-\frac{1}{N_c}\right)+\frac{2C'}{N_c^2}\right], \tag{B10}$$

$$Q_{0,Q,C}(\Delta^+ p) = +\left(B'-\frac{2C'}{N_c}\right)\sqrt{\frac{(N_c-1)(N_c+5)}{2N_c^2}}, \tag{B11}$$

$$Q_{0,Q,C}(\Delta^0 n) = +\left(B'-\frac{2C'}{N_c}\right)\sqrt{\frac{(N_c-1)(N_c+5)}{2N_c^2}}, \tag{B12}$$

$$Q_0(\Sigma^{*+}\Sigma^+) = +\left(B'-\frac{2C'}{N_c}\right)\frac{N_c+5}{2N_c\sqrt{2}},$$

$$Q_Q(\Sigma^{*+}\Sigma^+) = +\frac{B'}{2\sqrt{2}}\left[2-r+\frac{2+3r}{N_c}+\frac{8(1-r)}{N_c^2}\right]-\frac{C'}{N_c\sqrt{2}}\left[2-r-\frac{6-11r}{N_c}+\frac{8(1-r)}{N_c^2}\right],$$

$$Q_C(\Sigma^{*+}\Sigma^+) = +\frac{1}{2\sqrt{2}}\left\{B'\left[2-r+\frac{2-r+4r^2}{N_c}+\frac{4(1-r)(2+r)}{N_c^2}\right]\right.$$
$$\left.-\frac{C'}{N_c}\left[4-r-r^2-\frac{12-11r-11r^2}{N_c}+\frac{8(1-r)(2+r)}{N_c^2}\right]\right\}, \tag{B13}$$

$$Q_0(\Sigma^{*0}\Sigma^0) = +\left(B'-\frac{2C'}{N_c}\right)\frac{\sqrt{2}}{N_c},$$

$$Q_Q(\Sigma^{*0}\Sigma^0) = +\left(B'-\frac{2C'}{N_c}\right)\frac{\sqrt{2}}{N_c}\left[r+\frac{2(1-r)}{N_c}\right],$$

$$Q_C(\Sigma^{*0}\Sigma^0) = +\frac{\sqrt{2}}{N_c}\left\{B'\left[r^2+\frac{(1-r)(2+r)}{N_c}\right]-\frac{C'}{N_c}\left[r(1+r)+\frac{2(1-r)(2+r)}{N_c}\right]\right\}, \tag{B14}$$

$$Q_0(\Sigma^{*0}\Lambda) = +\left(B'-\frac{2C'}{N_c}\right)\sqrt{\frac{(N_c-1)(N_c+3)}{2N_c^2}},$$

$$Q_Q(\Sigma^{*0}\Lambda) = +r\left(B'-\frac{2C'}{N_c}\right)\sqrt{\frac{(N_c-1)(N_c+3)}{2N_c^2}},$$

$$Q_C(\Sigma^{*0}\Lambda) = +r\left[B'-\frac{C'(1+r)}{N_c}\right]\sqrt{\frac{(N_c-1)(N_c+3)}{2N_c^2}}, \tag{B15}$$

$$Q_0(\Sigma^{*-}\Sigma^-) = -\left(B'-\frac{2C'}{N_c}\right)\frac{N_c-3}{2N_c\sqrt{2}},$$

$$Q_Q(\Sigma^{*-}\Sigma^-) = -\frac{1}{2\sqrt{2}}\left\{B'\left[2-r+\frac{2-5r}{N_c}-\frac{8(1-r)}{N_c^2}\right]-\frac{2C'}{N_c}\left[2-r-\frac{3(2-r)}{N_c}-\frac{8(1-r)}{N_c^2}\right]\right\},$$

$$Q_C(\Sigma^{*-}\Sigma^-) = -\frac{1}{2\sqrt{2}}\left\{B'\left[2-r+\frac{2-r-4r^2}{N_c}-\frac{4(1-r)(2+r)}{N_c^2}\right]-\frac{C'}{N_c}\left[4-r-r^2-\frac{3(4-r-r^2)}{N_c}-\frac{8(2+r)(1-r)}{N_c^2}\right]\right\}, \tag{B16}$$

$$Q_0(\Xi^{*0}\Xi^0) = +\left(B'-\frac{2C'}{N_c}\right)\frac{\sqrt{2}(N_c+3)}{3N_c},$$

$$Q_Q(\Xi^{*0}\Xi^0) = +\left(B'-\frac{2C'}{N_c}\right)\frac{\sqrt{2}r}{3}\left[1-\frac{3(1-2r)}{N_c}+\frac{6(1-r)}{N_c^2}\right],$$

$$Q_C(\Xi^{*0}\Xi^0) = +\frac{\sqrt{2}r}{3}\left\{B'\left[1-\frac{3r(1-2r)}{N_c}+\frac{3(1-r)(1+2r)}{N_c^2}\right]-\frac{C'}{N_c}\left[1+r-\frac{3(1+r-4r^2)}{N_c}+\frac{6(1-r)(1+2r)}{N_c^2}\right]\right\}, \tag{B17}$$

$$Q_0(\Xi^{*-}\Xi^-) = -\left(B'-\frac{2C'}{N_c}\right)\frac{\sqrt{2}(N_c-3)}{3N_c},$$

$$Q_Q(\Xi^{*-}\Xi^-) = -\frac{\sqrt{2}r}{3}\left\{B'\left[1+\frac{3(1-2r)}{N_c}-\frac{6(1-r)}{N_c^2}\right]-\frac{2C'}{N_c}\left[1-\frac{3}{N_c}-\frac{6(1-r)}{N_c^2}\right]\right\},$$

$$Q_C(\Xi^{*-}\Xi^-) = -\frac{\sqrt{2}r}{3}\left\{B'\left[1+\frac{3r(1-2r)}{N_c}-\frac{3(1-r)(1+2r)}{N_c^2}\right]-\frac{C'}{N_c}\left[1+r-\frac{3(1+r)}{N_c}-\frac{6(1-r)(1+2r)}{N_c^2}\right]\right\}. \tag{B18}$$

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