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Perturbative calculation of $O(a)$ improvement coefficients

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We compute several coefficients needed for $O(a)$ improvement of currents in perturbation theory, using the Brodsky-Lepage-Mackenzie prescription for choosing an optimal scale q^*. We then compare the results to non-perturbative calculations. Normalization factors of the vector and axial vector currents show good agreement, especially when allowing for small two-loop effects. On the other hand, there are large discrepancies in the coefficients of $O(a)$ improvement terms. We suspect that they arise primarily from power corrections inherent in the nonperturbative methods.

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I. INTRODUCTION

During the past few years the Symanzik effective field theory has been an important focus of research in lattice gauge theory. Symanzik's idea is to describe cutoff effects in lattice field theory by a continuum effective field theory [1]. One writes [1,2]

$$\mathcal{L}_{\rm lat} \doteq \mathcal{L}_{\rm Sym}, \tag{1}$$

where the symbol $\doteq$ means that the lattice and Symanzik field theories have the same on-shell matrix elements. For lattice QCD with Wilson fermions [3] the Symanzik local effective Lagrangian (LE$\mathcal{L}$) is given by [4,5]

$$\begin{aligned}\mathcal{L}_{\rm Sym} = {} & \frac{1}{2g^2}\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] - \bar{q}(\not{D}+m)q \\ & + aK_{\sigma\cdot F}\bar{q}i\sigma_{\mu\nu}F^{\mu\nu}q + O(a^2),\end{aligned} \tag{2}$$

where g^2 is a renormalized coupling, m is a renormalized quark mass, and $aK_{\sigma\cdot F}$ is a short-distance coefficient. The effective field theory is useful when the scale of QCD in lattice units is small, $\Lambda a \ll 1$, and, as used in this paper, when $ma \ll 1$ also. With the description in hand, the lattice field theory can be adjusted so that it approaches its continuum limit more quickly. The effective theory shows that if $K_{\sigma\cdot F}$ is reduced for any given on-shell matrix element, then the $O(a)$ term in Eq. (2) makes commensurately smaller contributions to all other on-shell matrix elements. This application of the Symanzik effective field theory is called the Symanzik improvement program [2].

A similar correspondence is set up for the vector and axial vector currents (see below), introducing further short-distance coefficients. In the last several years methods have been devised to study all of them nonperturbatively [6–10]. The $O(a)$ discretization effects violate chiral symmetry, so the key idea is to ensure that violations of chiral symmetry are at least $O(a^2)$. On the other hand, because of asymptotic freedom and the success of perturbative QCD, even at GeV energies [11], one expects perturbation theory to yield accurate estimates of the short-distance coefficients. In this paper, we compare a perturbative calculation of the currents' short-distant coefficients to the nonperturbative results.

There are two issues that should be kept in mind when making such a comparison. First, the nonperturbative technique suffers from power corrections. Asymptotically, as $\Lambda a \to 0$ these are formally smaller than any error made from truncating the perturbation series. In practice, however, these effects can be significant.

Second, no two-loop results are available for the improvement coefficients considered here. Tests of perturbation theory are, therefore, not unambiguous, because different choices for the expansion parameter g^2 yield quantitatively different results. The bare coupling g_0^2 (for the Wilson gauge action) is an especially bad expansion parameter [12]. The obvious remedy is to rearrange the perturbative series, eliminating g_0^2 in favor of a renormalized (running) coupling, evaluated at a scale characteristic of the problem at hand. One is then faced, however, with many choices of renormalization scheme, and the question of how to determine the "characteristic scale." In this paper we choose the Brodsky-Lepage-Mackenzie (BLM) prescription [13,14]. Once this choice is made, little subjectivity remains, so one can ask quantitatively whether one-loop BLM perturbation theory agrees with the nonperturbative method.

In the BLM method, the characteristic scale is computed from Feynman diagrams. The new information presented in this paper consists of the calculations needed to determine the BLM scales of the normalization and improvement coefficients of the vector and axial vector currents for Wilson fermions with Sheikholeslami-Wohlert action. These calculations are a by-product of our recent work on the normalization and improvement of lattice currents with heavy quarks [15]. Details of the calculational method may be found there.

This paper is organized as follows. In Sec. II we define the lattice currents and review their description in the Symanzik effective field theory. Section III recalls the BLM prescription, focusing on points that are sometimes overlooked. Our new results for the BLM scales are given in Sec. IV. This paves the way for a systematic comparison with nonperturbative calculations of the same quantities in Sec. V. Section VI contains a few concluding remarks.

II. LATTICE CURRENTS

In this section we review the description of lattice currents with the Symanzik effective field theory. For quarks we take the Sheikholeslami-Wohlert Lagrangian [4], which has an improvement coupling c_{SW}. At the tree level $K_{\sigma\cdot F}^{[0]}=\frac{1}{4}(1-c_{\mathrm{SW}})$, so the improvement condition $K_{\sigma\cdot F}=0$ requires $c_{\mathrm{SW}}=1+O(g^2)$. For one-loop calculations, it is sufficient to specify c_{SW} at the tree level. For the nonperturbative calculations cited below, $c_{\mathrm{SW}}-1$ is determined nonperturbatively by the methods of Ref. [7].

We denote the lattice fermion field with ψ. The lattice vector and axial vector currents take the form

$$V_{\mathrm{lat}}^{\mu}=\bar{\psi}i\gamma^{\mu}\psi-ac_V\partial_{\nu\mathrm{lat}}\bar{\psi}\sigma^{\mu\nu}\psi, \tag{3}$$

$$A_{\mathrm{lat}}^{\mu}=\bar{\psi}i\gamma^{\mu}\gamma_5\psi+ac_A\partial_{\mathrm{lat}}^{\mu}\bar{\psi}i\gamma_5\psi. \tag{4}$$

The improvement couplings c_V and c_A should be chosen to reduce lattice artifacts, as discussed below.[1] In Symanzik's theory of cutoff effects, the lattice currents are described by operators in a continuum effective field theory [1,2,5,7]

$$V_{\mathrm{lat}}^{\mu}\doteq\bar{Z}_V^{-1}\bar{q}i\gamma^{\mu}q-aK_V\partial_{\nu}\bar{q}\sigma^{\mu\nu}q+\cdots, \tag{5}$$

$$A_{\mathrm{lat}}^{\mu}\doteq\bar{Z}_A^{-1}\bar{q}i\gamma^{\mu}\gamma_5q+aK_A\partial^{\mu}\bar{q}i\gamma_5q+\cdots, \tag{6}$$

where, as in Eq. (2), q is a continuum fermion field whose dynamics is defined by $\mathcal{L}_{\mathrm{QCD}}$. The ellipsis indicates operators of dimension 5 and higher. Further dimension-4 operators are omitted from Eqs. (5) and (6), because they are linear combinations of those listed and others that vanish by the equations of motion. The short-distance coefficients in the effective Lagrangian—$\bar{Z}_J$ and K_J ($J=V,A$)—are functions of g^2 and ma, and the improvement couplings c_{SW} and c_J.

Symanzik improvement is achieved by adjusting c_J so that $K_J=0$. Then $\bar{Z}_V V_{\mathrm{lat}}^{\mu}$ and $\bar{Z}_A A_{\mathrm{lat}}^{\mu}$ have the same matrix elements as $\bar{q}i\gamma^{\mu}q$ and $\bar{q}i\gamma^{\mu}\gamma_5q$, apart from lattice artifacts of order a^2. For light quarks one may expand $\bar{Z}_J$ in ma,

$$\bar{Z}_J=Z_J(1+mab_J), \tag{7}$$

and identify K_J with the zeroth order of a small ma expansion. At the tree level the coefficients of the normalization factor are $Z_J^{[0]}=1$, $b_J^{[0]}=1$. In addition, the coefficient of the lattice artifact is

$$K_J^{[0]}=c_J^{[0]}. \tag{8}$$

The improvement condition $K_J=0$ says that one should set $c_J^{[0]}=0$. Consequently, one-loop calculations are based solely on the first terms in Eqs. (3) and (4).

[1]The lattice currents in Eqs. (3) and (4) are useful for light quarks. For heavy quarks the "small" improvement terms become large, introducing unnecessary violations of heavy-quark symmetry. Better currents for heavy quarks are given in Refs. [15,16].

III. BRODSKY-LEPAGE-MACKENZIE PRESCRIPTION

In this section we review the BLM prescription, following the argumentation from Ref. [14]. This material should be familiar, but some of the literature on nonperturbative improvement blurs the difference between BLM perturbation theory and other topics, such as "tadpole improvement" and mean-field estimates of the renormalized coupling, which are also discussed in Ref. [14].

The problem is to find a reasonably accurate one-loop estimate of a quantity ζ, here $\bar{Z}_J$ or K_J. In these cases, one gluon with momentum k and propagator $D(k)$ appears. The contribution from the Feynman diagrams can be written

$$g_R^2\zeta^{[1]}(p)=g_0^2\int\frac{d^4k}{(2\pi)^4}D(k)f(k,p)+\cdots, \tag{9}$$

where p denotes k-independent parameters, such as external momenta. The ellipsis indicates higher-order terms that we would like to absorb into the renormalized coupling g_R^2. An important class of higher-order terms consists of the renormalization parts that dress the exchanged gluon. In the Fourier transform of the heavy-quark potential, for example, they turn $g_0^2D(k)$ into $g_V^2(k)D(k)$, where the potential $V(q)=-C_Fg_V^2(q)/q^2$. Thus,

$$g_R^2\zeta^{[1]}(p)=\int\frac{d^4k}{(2\pi)^4}g_V^2(k)D(k)f(k,p)+\cdots \tag{10}$$

sums the renormalization parts. Other ways of dressing the gluon would lead to other physical running couplings, but they all are the same at order β_0g^4 [13], where $\beta_0=11-\frac{2}{3}n_f$ is the one-loop coefficient of the β function for n_f light quarks.

If there is a characteristic scale q^*, one can approximate

$$g_V^2(k)=\frac{g_V^2(q^*)}{1+(\beta_0/16\pi^2)g_V^2(q^*)\ln(k/q^*)^2} \tag{11}$$

$$=g_V^2(q^*)+\frac{\beta_0}{16\pi^2}g_V^4(q^*)\ln(q^*/k)^2+\cdots. \tag{12}$$

The aim is to choose q^* so that higher-order terms are small, particularly those of order $\beta_0g_V^4$, which could be enhanced by a foolish choice of q^*. Inserting Eq. (12) into Eq. (10) and setting the coefficient of $\beta_0g_V^4$ to zero yields

$$\ln q^*a=\frac{{}^*\zeta^{[1]}}{2\zeta^{[1]}}, \tag{13}$$

where a is a reference short-distance scale (namely, the lattice spacing), and

$${}^*\zeta^{[1]}(p)=\int\frac{d^4k}{(2\pi)^4}\ln(ka)^2D(k)f(k,p). \tag{14}$$

Thus, the BLM prescription is to set $g_R^2 = g_V^2(q^*)$ in the one-loop approximation.

If one prefers a different renormalized coupling, one must change the scale in the appropriate way. The coupling in scheme "S" is related to the V scheme by

$$\frac{1}{g_S^2(q)} = \frac{1}{g_V^2(q)} + \frac{\beta_0 b_S^{(1)} + b_S^{(0)}}{16\pi^2} + O(g^2), \tag{15}$$

where $b_S^{(0)}$ and $b_S^{(1)}$ are constants independent of n_f. The BLM scale q_S^* for this scheme is given by

$$\ln q_S^* = \ln q^* - \frac{1}{2} b_S^{(1)}. \tag{16}$$

For example, for the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme, $b_{\overline{\mathrm{MS}}}^{(0)} = -8$ and $b_{\overline{\mathrm{MS}}}^{(1)} = 5/3$, $q_{\overline{\mathrm{MS}}}^* = e^{-5/6} q^* = 0.435 q^*$. With Eq. (16) one recovers the summary statement of Ref. [13], namely to absorb into q_S^* the n_f dependence of the two-loop term, which enters through β_0.

The BLM prescription has several features that make it a natural choice in matching calculations, such as those considered in this paper. The effective field theory framework suggests using a renormalized coupling, in particular one that has a (quasi-)physical definition in both the underlying theory (here lattice gauge theory) and in the effective theory (here the Symanzik effective field theory). For quantitative purposes it is more interesting to note that

$$\frac{1}{g_S^2(q_S^*)} = \frac{1}{g_V^2(q^*)} + \frac{b_S^{(0)}}{16\pi^2} + O(g^2), \tag{17}$$

so the numerical difference in the BLM expansion parameters is small, as long as $g^2 b_S^{(0)}/16\pi^2$ is small.

IV. PERTURBATIVE RESULTS

In Ref. [15] we found for gauge group SU(3) and $c_{\mathrm{SW}} = 1$

$$Z_V^{[1]} = -0.129423(6), \tag{18}$$

$$Z_A^{[1]} = -0.116450(5), \tag{19}$$

in excellent agreement with previous work [17,18]. (Reference [18] gives precise results as a polynomial in c_{SW}.) We also found (with $c_J^{[0]} = 0$)

$$b_V^{[1]} = 0.153239(14), \tag{20}$$

$$b_A^{[1]} = 0.152189(14), \tag{21}$$

$$K_V^{[1]} = c_V^{[1]} + 0.016332(7), \tag{22}$$

$$K_A^{[1]} = c_A^{[1]} + 0.0075741(15), \tag{23}$$

which agree perfectly with Ref. [19]. Solving the improvement condition $K_J = 0$ at this order gives

$$c_V^{[1]} = -0.016332(7), \tag{24}$$

$$c_A^{[1]} = -0.0075741(15). \tag{25}$$

We also directly obtained

$$b_V^{[1]} - b_A^{[1]} = 0.0010444(16), \tag{26}$$

which is more accurate than the difference of the two numbers quoted above. In taking the difference, large contributions from the self-energy cancel, but, even so, the near equality of $b_V^{[1]}$ and $b_A^{[1]}$ is a bit astonishing. The mass dependence of $\overline{Z_J}$ shows that $b_V^{[1]} - b_A^{[1]}$ is not so small for the Wilson action [15].

In our method for computing the improvement coefficients it is easy to weight the integrands with $\ln(ka)^2$ and, thus, obtain the BLM scales. We find

$$^*Z_V^{[1]} = -0.270691(19), \tag{27}$$

$$^*Z_A^{[1]} = -0.243086(09), \tag{28}$$

$$^*b_V^{[1]} = 0.321556(35), \tag{29}$$

$$^*b_A^{[1]} = 0.318108(21), \tag{30}$$

$$^*b_V^{[1]} - {}^*b_A^{[1]} = 0.0034247(51), \tag{31}$$

$$^*c_V^{[1]} = -0.0222383(15), \tag{32}$$

$$^*c_A^{[1]} = -0.0147825(62), \tag{33}$$

and hence

$$q_{Z_V}^* a = 2.846, \tag{34}$$

$$q_{Z_A}^* a = 2.840, \tag{35}$$

$$q_{Z_A/Z_V}^* a = 2.898, \tag{36}$$

$$q_{b_V}^* a = 2.855, \tag{37}$$

$$q_{b_A}^* a = 2.844, \tag{38}$$

$$q_{b_V - b_A}^* a = 5.153, \tag{39}$$

$$q_{c_V}^* a = 1.975, \tag{40}$$

$$q_{c_A}^* a = 2.653. \tag{41}$$

The scales are in the expected range. The higher scale for $b_V - b_A$ means simply that the difference between these renormalization constants arises from very short distances. These numerical results are new; they have been obtained from two independent computer programs. As a further check, we have reproduced the values of $q_{Z_V}^* a$ and $q_{Z_A}^* a$ for the Wilson action ($c_{\mathrm{SW}} = 0$), given in Ref. [20].

The dominant contributor to the "large" one-loop normalization constants, Eqs. (18)–(21), is the tadpole diagram (in Feynman gauge) of the self-energy. One might expect perturbation theory to work better for quantities in which the effects of tadpole diagrams largely cancel (albeit in a gauge-invariant way). For example, Z_A/Z_V and b_V-b_A are tadpole-free and have smaller one-loop coefficients.

Another way to remove the tadpoles is to write

$$Z_J=u_0\tilde{Z}_J, \tag{42}$$

$$b_J=\tilde{b}_J/u_0, \tag{43}$$

where u_0 is any convenient tadpole-dominated quantity. Then one can take u_0 from a nonperturbative Monte Carlo calculation and use perturbation theory for $\tilde{Z}_J$ and $\tilde{b}_J$. The corresponding one-loop coefficients are

$$\tilde{Z}_J^{[1]}=Z_J^{[1]}-u_0^{[1]}, \tag{44}$$

$$\tilde{b}_J^{[1]}=b_J^{[1]}+u_0^{[1]}. \tag{45}$$

Similarly, to get the BLM scale

$$*\tilde{Z}_J^{[1]}=*Z_J^{[1]}-*u_0^{[1]}, \tag{46}$$

$$*\tilde{b}_J^{[1]}=*b_J^{[1]}+*u_0^{[1]}, \tag{47}$$

where $*u_0^{[1]}$ is the BLM numerator [cf. Eq. (13)] for u_0. Below we take u_0^4 to be the average value of the plaquette, with $u_0^{[1]}=-1/12=-0.08\overline{3}$ and $*u_0^{[1]}=-0.204049(1)$. A glance at Eqs. (44)–(47) shows immediately that tadpole improvement reduces the one-loop coefficients. With tadpole improvement the BLM scales become

$$q^*_{\tilde{Z}_V}a=2.061, \tag{48}$$

$$q^*_{\tilde{Z}_A}a=1.803, \tag{49}$$

$$q^*_{\tilde{b}_V}a=2.317, \tag{50}$$

$$q^*_{\tilde{b}_A}a=2.289. \tag{51}$$

The scales are lower than without tadpole improvement, but still ultraviolet.

It is perhaps worthwhile emphasizing the difference between tadpole improvement and the BLM prescription. The aim of tadpole improvement is to re-sum large contributions appearing at order g^2 and higher, replacing the sum with a nonperturbative estimate (u_0, for example). The aim of the BLM prescription is to re-sum potentially large renormalization parts into the renormalized coupling. Although the aims are similar, they are not identical. They are not mutually exclusive, and neither is a substitute for the other.

TABLE I. Comparison of perturbative and nonperturbative determinations of the improvement coefficients at $\beta=6.2$.

$\beta=6.2$	$\alpha_V(q^*)$	BLM	Refs. [21–23]	Ref. [24]
Z_V	0.1468	0.7612	0.7922(9)	0.7874(4)
Z_A	0.1469	0.7850	0.807(8)	0.818(5)
Z_A/Z_V	0.1461	1.0238	1.019(8)	1.039(5)
b_V	0.1467	1.2824	1.41(2)	1.42(1)
b_A	0.1468	1.2808		1.32(5)
b_V-b_A	0.1257	0.001649		0.11(5)
$-c_V$	0.1638	0.03361	0.21(7)	0.09(2)
$-c_A$	0.1498	0.01426	0.038(4)	0.032(7)
$u_0\tilde{Z}_V$	0.1616	0.8022	0.7922(9)	0.7874(4)
$u_0\tilde{Z}_A$	0.1686	0.8230	0.807(8)	0.818(5)
$\tilde{b}_V/u_0$	0.1559	1.2846	1.41(2)	1.42(1)
$\tilde{b}_A/u_0$	0.1565	1.2828		1.32(5)

V. COMPARISON TO NONPERTURBATIVE CALCULATIONS

With the BLM scales in hand we can compare the prediction of one-loop BLM-improved perturbation theory with nonperturbative determinations of the improvement coefficients. We shall make the comparison in two ways. First we compare the numerical values directly, at two values of the bare coupling. Here there are two methods in the literature, one based on finite-size techniques and the Schrödinger functional [21–23], and another based on large volumes with hadronic matrix elements [24]. The difference between these two illustrates how large power corrections to the improvement coefficients are. We also compare our results graphically, as a function of coupling, to Padé approximants given in Refs. [19,21,22]. These graphs are helpful for seeing whether discrepancies in the one-loop and nonperturbative estimates arise from two-loop or power corrections.

We obtain $\alpha_V(q^*)$ as follows. First we compute

$$\alpha_{1\times 1}=-\frac{3}{4\pi}\ln\langle\Box\rangle, \tag{52}$$

where $\langle\Box\rangle$ is the ensemble average of the plaquette. Then we follow Ref. [14] and take α_V to be

$$\alpha_V(3.402/a)\equiv\frac{2\alpha_{1\times 1}}{1+\sqrt{1-4.741\alpha_{1\times 1}}}, \tag{53}$$

which agrees with the standard definition of α_V with an accuracy of order α_s^3. The scale $3.402/a$ is the BLM scale for $\langle\Box\rangle$. We then run from $3.402/a$ to q^* with the two-loop evolution equation. Of course, once two-loop perturbation theory is available, one would have to extend the accuracy of Eq. (53) and of the evolution.

Table I gives results from our perturbative calculation with nonperturbative results from the Alpha Collaboration [21–23] and from Bhattacharya *et al.* [24], at $\beta=6.2$. Table II gives the same comparison at $\beta=6.0$. Above (below) the horizontal line, we have applied the BLM prescription with-

TABLE II. Comparison of perturbative and nonperturbative determinations of the improvement coefficients at $\beta=6.0$.

$\beta=6.0$	$\alpha_V(q^*)$	BLM	Refs. [21–23]	Ref. [24]
Z_V	0.1602	0.7394	0.7809(6)	0.770(1)
Z_A	0.1603	0.7654	0.791(9)	0.807(8)
Z_A/Z_V	0.1593	1.0260	1.012(9)	1.048(8)
b_V	0.1601	1.3082	1.54(2)	1.52(1)
b_A	0.1603	1.3065		1.28(5)
b_V-b_A	0.1352	0.001774		0.24(5)
$-c_V$	0.1808	0.03711	0.32(7)	0.107(17)
$-c_A$	0.1638	0.01559	0.083(5)	0.037(9)
$u_0\tilde{Z}_V$	0.1782	0.7872	0.7809(6)	0.770(1)
$u_0\tilde{Z}_A$	0.1868	0.8095	0.791(9)	0.807(8)
$\tilde{b}_V/u_0$	0.1712	1.3105	1.54(2)	1.52(1)
$\tilde{b}_A/u_0$	0.1719	1.3087		1.28(5)

out (with) tadpole improvement. The error bars on the entries from Refs. [21–24] are statistical, and compiled in Ref. [24].

For the normalization factors Z_V and Z_A, BLM perturbation theory and the nonperturbative methods agree well, within 3–4%. The difference between the two nonperturbative values of Z_V exceeds the reported errors, but is easily explained by power correction of order $(\Lambda a)^2$. For the tadpole-free ratio Z_A/Z_V and for the tadpole-improved quantities $u_0\tilde{Z}_J$, BLM perturbation theory lies very close to the nonperturbative range. These impressions are strengthened by Fig. 1, which shows Z_V and Z_A as functions of g_0^2. Circles show BLM perturbation theory, and the thin solid (dashed) lines indicate how two-loop contributions of $\pm\alpha_V^2$ ($\pm 2\alpha_V^2$) could modify the result. We show the result with and without tadpole improvement in Figs. 1(b,d) and 1(a,c), respectively. For the nonperturbative method, a heavy line shows the Padé approximants [22]

$$Z_V=\frac{1-0.7663g_0^2+0.0488g_0^4}{1-0.6369g_0^2}, \tag{54}$$

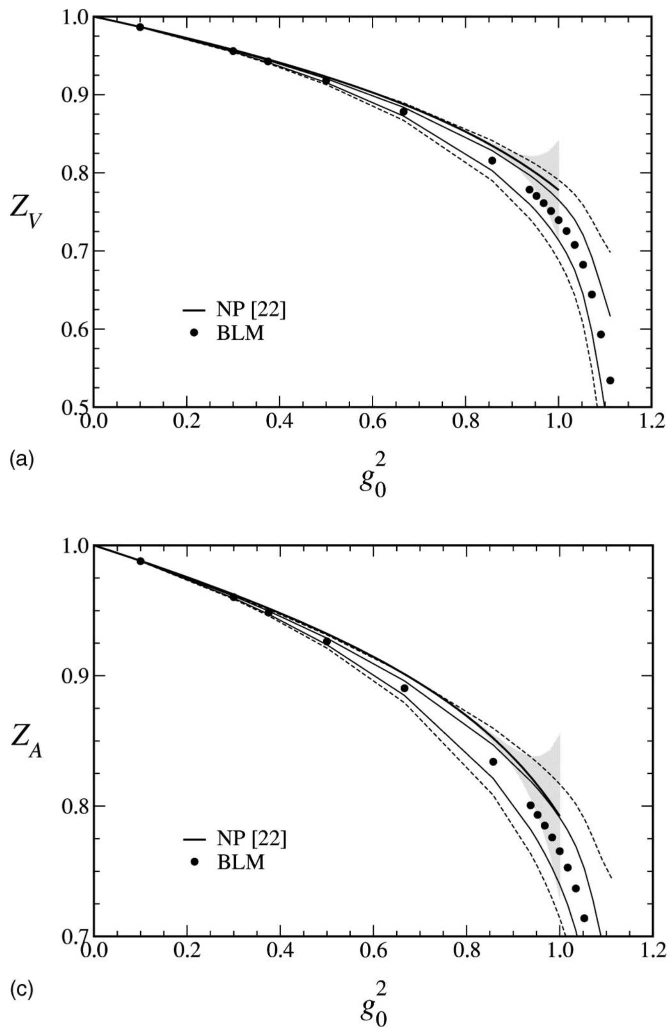

FIG. 1. Z_V and Z_A vs g_0^2. Heavy lines show the nonperturbative results, Eqs. (54) and (55), and shading possible corrections of order $\pm(\Lambda a)^2$. Circles show BLM perturbation theory, with thin and dashed lines to indicate a two-loop term $\pm\alpha_V^2$ or $\pm 2\alpha_V^2$. (a) and (c) No tadpole improvement, $Z_J^{\mathrm{BLM}}=1+g_V^2(q_{Z_J}^*)Z_J^{[1]}$; (b) and (d) with tadpole improvement, $Z_J^{\mathrm{BLM}}=u_0[1+g_V^2(q_{\tilde{Z}_J}^*)\tilde{Z}_J^{[1]}]$.

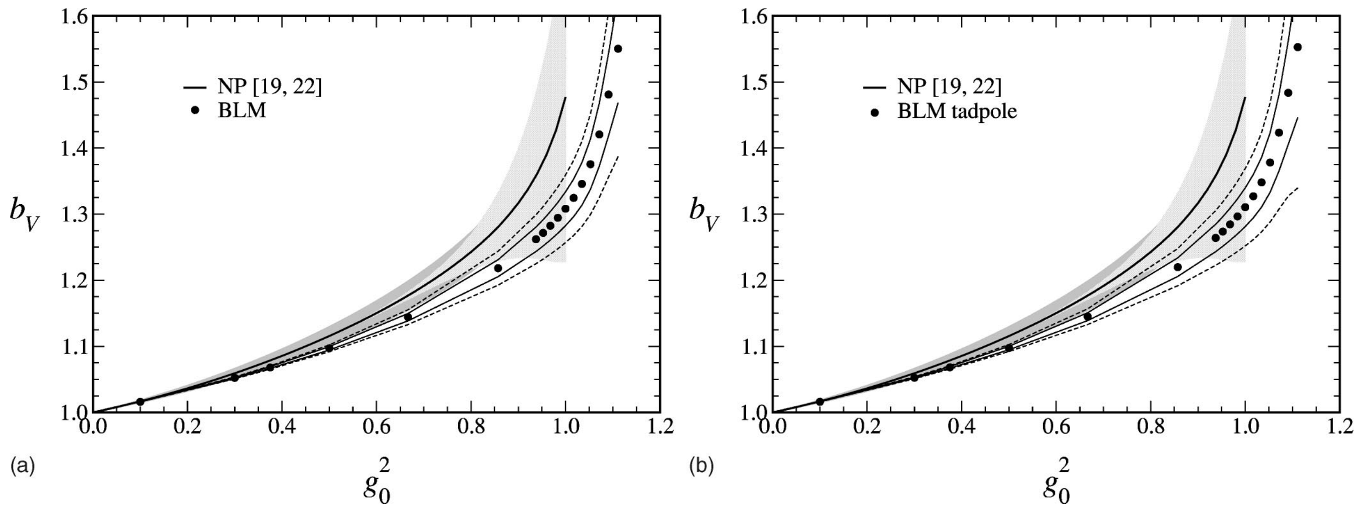

FIG. 2. b_V vs g_0^2. (a) No tadpole improvement, $b_V^{\rm BLM}=1+g_V^2(q^*_{b_V})b_V^{[1]}$; (b) with tadpole improvement, $b_V^{\rm BLM}=[1+g_V^2(q^*_{\tilde{b}_V})\tilde{b}_V^{[1]}]/u_0$. Light gray shading indicates power corrections to b_V of order $\pm\Lambda a$; darker gray shading power corrections to b_V-1 of order $\pm a/L$.

$$Z_A=\frac{1-0.8496g_0^2+0.0610g_0^4}{1-0.7332g_0^2}, \tag{55}$$

which deviate from the underlying calculations negligibly for $g_0^2\leqslant 1$. The shaded bands behind the Padé curves show a power correction of $\pm(\Lambda a)^2$, with $\Lambda\sim 500$ GeV. The finite-volume result also suffers from power corrections of order $(a/L)^2$. They are estimated to be small by comparing calculations on lattices with $a/L=1/8$ and 1/12 [22]. Also, they are parametrically smaller, because Ref. [22] holds $L\Lambda\sim 2$ for all g_0^2.

Next let us turn to the $O(ma)$ corrections to the normalization factors, b_V and b_A. There is only one calculation of b_A [24], so let us concentrate first on b_V. The two nonperturbative results for b_V agree perfectly with each other (see the tables), but they deviate significantly from one-loop BLM perturbation theory. Some insight can be gleaned from Fig. 2, which shows b_V as a function of g_0^2. The nonperturbative method is represented with the Padé approximant [19]

$$b_V=\frac{1-0.7613g_0^2+0.0012g_0^4-0.1136g_0^6}{1-0.9145g_0^2}, \tag{56}$$

with light shading for a power correction $\pm\Lambda a$. In finite volume there is also a power correction to b_V of order a/L; by construction it applies to b_V-1 [22], but now L with a varies such that $a/L=1/8$ for all g_0^2. We model this effect as $(b_V-1)(1\pm\frac{1}{8})$, shown in the darker shading in Fig. 2. Judging from its size and shape, the deviation seen in Fig. 2 looks less like a two-loop effect than a combination of power corrections of order a/L and Λa. (Similar conclusions are reached in Ref. [24].) There is almost no difference whether one applies tadpole improvement to b_V or not, once the BLM prescription is applied. These two approximations truncate higher orders of the perturbation series differently, substantiating the idea that the discrepancy is a power correction.

The nonperturbative calculation of b_A agrees with one-loop BLM perturbation theory. Note, however, that Ref. [24] obtains b_V and b_V-b_A directly, and then $b_A=b_V-(b_V-b_A)$. The agreement between BLM perturbation theory and the nonperturbative results for b_V and b_V-b_A is not good, so the agreement for b_A may be an accident. Since the coefficient $b_V^{[1]}-b_A^{[1]}$ in Eq. (26) is remarkably small, the two-loop contribution could be as large as the one-loop term. Furthermore, inspection of Fig. 14 in Ref. [24] suggests that a fit to the three smallest masses would yield a smaller value of b_V-b_A. We consider the comparison of b_A and b_V-b_A to be unsettled pending a two-loop calculation and a more robust nonperturbative calculation.

In any case, the mild disagreement on b_V and b_V-b_A is not of much practical importance. For the sake of argument, suppose $ma<0.1$, which holds for the light quarks for which the currents were designed. Then power corrections in b_J, at fixed a, lead to an uncertainty in a decay constant or a form factor of only a few per cent. After a continuum limit extrapolation, these uncertainties will not be important.

Now let us turn to the coefficients c_J of the improvement terms in Eqs. (3) and (4). At the tabulated values of β, the nonperturbative and BLM calculations of c_A do not agree at all. At $\beta=6.0$ (Table II) the two nonperturbative calculations also do not agree with each other. Fig. 3(a) shows c_A as a function of g_0^2, using the Padé approximant [21]

$$c_A=-0.00756g_0^2\frac{1-0.748g_0^2}{1-0.977g_0^2} \tag{57}$$

to represent the nonperturbative calculations. The disagreement between BLM perturbation theory and Eq. (57) sets in for $g_0^2>0.9$. There are two reasons to suspect that the discrepancy stems from a power correction of order Λa to the results of Ref. [21]. First, Fig. 3(a) shows that it has the shape and size of such a power correction. Second, the extracted value of c_A depends on the lattice derivative used to

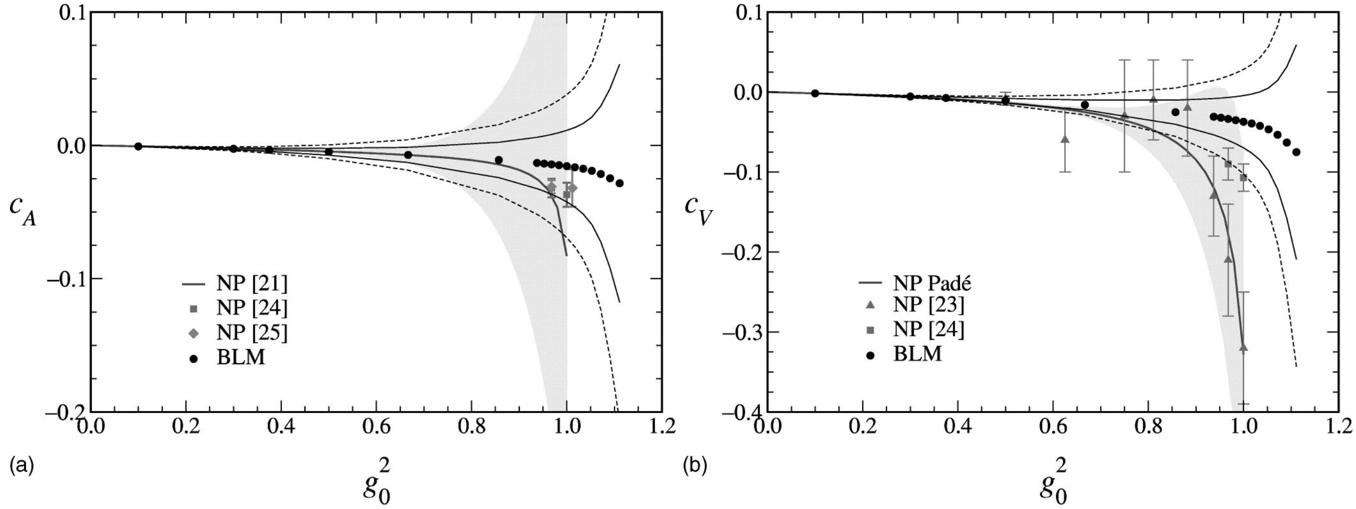

FIG. 3. c_A and c_V vs g_0^2. Shading shows power corrections of order $\pm\Lambda a$ to (a) Eq. (57), (b) Eq. (58). Points with error bars are from (a) Refs. [24,25], (b) Refs. [23,24].

define the current [25]. Note [24] that errors in c_A propagate to c_V, because in the Ward identities c_A is multiplied by large hadronic matrix elements such as $am_K^2/m_s \sim a\times 2.5$ GeV. This enhancement also explains why Eq. (57) leads to worse scaling in f_π [25]. Figure 3(a) also includes the nonperturbative results of Refs. [24,25]. The difference between those points and BLM perturbation theory could be a modest two-loop effect or a small power correction.

For c_V, the two nonperturbative results agree neither with each other, nor with BLM perturbation theory. The Alpha Collaboration has only a preliminary calculation [23]. We have taken the liberty of extracting results from Fig. 3 of Ref. [23] and fitting them to a Padé formula. The leading behavior is fixed to Eq. (24), and we obtain

$$c_V = -0.01633 g_0^2 \frac{1-0.257g_0^2}{1-0.963g_0^2}. \tag{58}$$

Figure 3(b) plots Eq. (58), the underlying points [23], the nonperturbative results from hadronic correlation functions [24], and BLM perturbation theory. As usual we show possible power corrections to Eq. (58) of order $\pm\Lambda a$, as well as the size of typical two-loop effects. At small g_0^2, there is good agreement with (BLM) perturbation theory, but once $g_0^2 > 0.9$, there is a sharp turnover. It is probably a power correction, possibly exacerbated by power corrections to c_A as modeled by Eq. (57). With hadronic correlation functions [24] the nonperturbative value of c_V is half or a third as large. It is not clear at present whether the discrepancy between Ref. [24] and BLM perturbation theory is a power correction to the former or a sizable two-loop correction to the latter.

We should also mention that BLM perturbation theory works better than several forms of mean-field perturbation theory (let alone bare perturbation theory). In Table III we list several choices for α_s:

$$\alpha_0 = g_0^2/4\pi, \tag{59}$$

$$\tilde{\alpha}_0 = \alpha_0/u_0, \tag{60}$$

as well as $\alpha_{1\times 1}$ [Eq. (52)] and $\alpha_{\overline{\rm MS}}(q^*_{\overline{\rm MS}})$ [Eq. (17)]. With only one-loop expansions available, the mean-field choices $\tilde{\alpha}_0$ and $\alpha_{1\times 1}$ give smaller corrections, and one-loop perturbation theory falls short even when power corrections are negligible. The consistency of BLM-V perturbation theory for Z_V, Z_A, and Z_A/Z_V indicates that the BLM prescription does indeed re-sum an important class of higher-order contributions. On the other hand, the coupling $\alpha_{\overline{\rm MS}}(q^*_{\overline{\rm MS}})$ seems, empirically, to work less well. In continuum perturbative QCD, it usually does not matter whether one adopts $\alpha_V(q^*_V)$, $\alpha_{\overline{\rm MS}}(q^*_{\overline{\rm MS}})$ or some other renormalized coupling (at the BLM scale), once two-loop effects are included. It would not be surprising for the same to hold for short-distance quantities in lattice gauge theory, such as improvement coefficients.

VI. CONCLUSIONS

In this paper we have compared nonperturbative calculations of several improvement coefficients to perturbation theory with the BLM prescription. Previously this could not be done, because the "BLM numerators" in Eqs. (27)–(33)

TABLE III. Expansion parameters for perturbation theory.

β	α_0	$\tilde{\alpha}_0$	$\alpha_{1\times 1}$	$\alpha_V(q^*_{Z_V})$	$\alpha_{\overline{\rm MS}}(q^*_{Z_V})$
6.0	0.0796	0.1340	0.1245	0.1602	0.1784
6.2	0.0770	0.1255	0.1166	0.1468	0.1619
6.4	0.0746	0.1183	0.1101	0.1362	0.1491
7.0	0.0682	0.1016	0.0951	0.1134	0.1222
9.0	0.0531	0.0702	0.0667	0.0748	0.0786

were not available. We find that, for the scale-independent quantities considered here, the integration of the $\log k^2$-weighted integrals is numerically straightforward.

BLM perturbation theory for the current normalization factors Z_J agrees very well with nonperturbative calculations of the same quantities. Here the leading power correction is only of order $(\Lambda a)^2$, and the small deviations can probably be removed with a two-loop calculation. Note that generalizations of the BLM method for higher-order perturbation theory have been considered in continuum perturbative QCD [26] and in lattice gauge theory [27].

For the improvement coefficients b_J and c_J, the leading power corrections are of order Λa (and in the Schrödinger functional also of order $a/L = 1/8$), while some of the one-loop coefficients are small. It is consequently difficult to diagnose the discrepancies. By noting the size and dependence on g_0^2 of the differences, we concur with the authors of Refs. [24,25], namely, that power corrections contaminate the nonperturbative results. In particular, it seems unlikely that higher orders in perturbative series could explain all discrepancies between one-loop BLM perturbation theory and the results from Refs. [21–23].

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