Surviving the renormalon in the heavy quark potential

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We show that the Borel resummed perturbative static potential at $N_f=0$ converges well, and is in remarkable agreement with the quenched lattice calculation at distances $1/r \gtrsim 660$ MeV. This shows that Borel resummation is very good at handling the renormalon in the static potential (and in the pole mass), and allows one to use the pole mass in the perturbative calculation of heavy quark physics.

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I. INTRODUCTION

The asymptotic freedom of quantum chromodynamics (QCD) allows one to calculate short distance physics accurately using perturbation. Unexpectedly, however, the perturbative expansion of the static potential between a quark-antiquark pair does not show convergence even at very short distances (see Fig. 1). Moreover, no agreement is seen with the accurate lattice calculations of the static potential. This led to the suggestion of a nonperturbative linear potential at short distance [1], which, if proven true, would violate the expectation of operator product expansion (OPE) that the nonperturbative effect at short distance is at most a quadratic potential.

On the other hand, the bad convergence behavior of the perturbative expansion of the potential is well understood to be caused by the infrared (IR) renormalon which induces a constant nonperturbative effect proportional to Λ_{QCD} [2]. This prompted several approaches to the problem. One is based on the observation that the force between a pair of static quarks is free from the leading renormalon. The potential obtained by integrating the force calculated in perturbation indeed agrees quite well at short distance with the lattice potential [3], up to an r independent constant. Another approach is the renormalon subtracted (RS) scheme [4], in which one subtracts order by order the renormalon contribution from the perturbative potential. The potential calculated in this way also shows an improved convergence and agreement with the lattice potential. Another idea is to employ the cancellation of the renormalons in the static potential and the pole mass of the heavy quark [5,6]. By expanding the pole mass and the static potential of a color singlet quarkonium in the running coupling $\alpha_s(\mu)$ and a short distance mass $m(\mu)$ one can avoid the renormalon problem, and indeed such an expansion shows an improved convergence [7–9].

In this paper we show that a more direct approach to the problem is possible via the Borel resummation of the perturbative potential. Since one might believe that the presence of an IR renormalon makes Borel resummation impossible, we state in advance that it is perfectly possible in this case. An IR renormalon in Borel resummation merely demands a corresponding nonperturbative effect, and since in this case it is a constant, the r dependence of the potential can be re-

summed with no difficulty. Moreover, this renormalon caused nonperturbative effect could be computed in the framework introduced in [10], where the nonperturbative effect is determined based on its conjectured analyticity in the complex coupling plane. An obvious advantage of the direct resummation is that the normalization of the potential can be fixed. In the approaches based on the renormalon cancellation/absence the potential can be fixed only up to an r independent constant.

As we shall see the Borel resummed potential at short distance converges quickly, and agrees remarkably well with the lattice calculation, in fact better than any other approach introduced so far. The implication of this is significant. In the perturbative calculation of a heavy quark system one does not have to give up the pole mass in favor of a short distance mass to avoid the renormalon problem, and still can have a tight control on the perturbative expansion.

Throughout the paper, unless stated otherwise, we consider pure QCD with no active quark flavors ($N_f=0$), and the perturbative expansions considered are assumed to be in the modified minimal subtraction ($\overline{\text{MS}}$) scheme. As for the renormalon, we restrict our attention to the leading infrared renormalon that is closest to the origin in the Borel plane.

II. BILOCAL EXPANSION OF THE BOREL TRANSFORM

In general, the perturbative expansion in weak coupling constant is an asymptotic expansion. When the large order behavior of the expansion is sign alternating like in ϕ^4
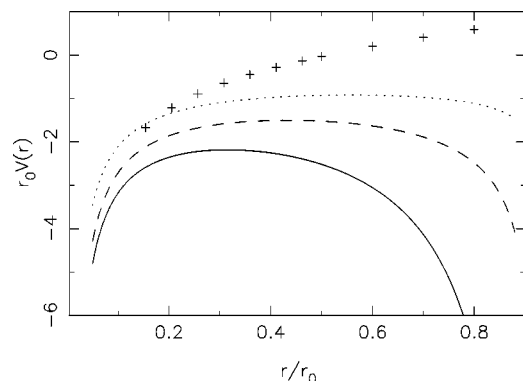

FIG. 1. The static potential at leading order (dotted), next-leading order (dashed), and next-next-leading order (solid). The data points denote lattice potential.

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theory it may be Borel resummed. However, when the expansion is of same sign at large orders Borel resummation demands a more careful treatment [11]. In the case of the latter, one can first do Borel resummation at an unphysical negative coupling, at which the series is sign alternating, and then do analytic continuation in the complex coupling plane to the physical positive coupling. The Borel resummed amplitude obtained in such a way, however, turns out to have a cut along the positive real axis in the coupling plane, and consequently has an ambiguous imaginary part at a physical coupling. In Borel integration this imaginary part arises precisely from the IR renormalon singularity of the Borel transform on the integration contour. This unphysical, ambiguous imaginary part then must be canceled by the nonperturbative effect corresponding to the renormalon. For further details we refer to [10].

Thus the static interquark potential $V(r)$,[1] which has an IR renormalon, can be written as the sum of the Borel integration with a contour on the upper (or lower) half plane and the nonperturbative effect [10],

$$V[r,\alpha_s(1/r)\pm i\epsilon] = \frac{1}{r\beta_0}\int_{0\pm i\epsilon}^{\infty\pm i\epsilon} e^{-b/\beta_0\alpha_s(1/r)}\tilde{V}(b)db$$
$$+ V_{\mathrm{NP}}[r,\alpha_s(1/r)\pm i\epsilon], \qquad (1)$$

where β_0 is the one loop coefficient of the QCD β function,

$$\beta(\alpha_s) = \mu^2\frac{d\alpha_s}{d\mu^2}$$
$$= -\alpha_s^2(\beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + \cdots), \qquad (2)$$

and $\tilde{V}(b)$ is the Borel transform that is given by

$$\tilde{V}(b) = \sum_{n=0}^{\infty}\frac{V_n}{n!}\left(\frac{b}{\beta_0}\right)^n, \qquad (3)$$

with V_n defined in the perturbative expansion of the potential,

$$V(r,\alpha_s) = \frac{1}{r}\sum_{n}^{\infty} V_n\alpha_s^{n+1}. \qquad (4)$$

V_{NP} denotes the renormalon caused nonperturbative effect. Since the imaginary parts in the first term in Eq. (1) and in V_{NP}, respectively, cancel, the potential can be written as

$$V[r,\alpha_s(1/r)] = \frac{1}{r\beta_0}\mathrm{Re}\left[\int_{0\pm i\epsilon}^{\infty\pm i\epsilon} e^{-b/\beta_0\alpha_s(1/r)}\tilde{V}(b)db\right]$$
$$+ \mathrm{Re}\{V_{\mathrm{NP}}[r,\alpha_s(1/r)\pm i\epsilon]\}. \qquad (5)$$

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[1]Because of its infrared sensitivity the static potential is dependent on the ultrasoft factorization scale beginning at next-next-next-leading order (NNNLO) [12,13]; however, to the order we are concerned (NNLO), this can be ignored.

Since V_{NP} is an r independent constant proportional to Λ_{QCD} we can ignore it as far as the r dependence of the potential is concerned.

The cancellation of the imaginary parts in the integral term and V_{NP} in Eq. (1) determines the renormalon singularity in the Borel transform $\tilde{V}(b)$. By comparing the functional form of

$$V_{\mathrm{NP}} \propto \Lambda_{\overline{\mathrm{MS}}}$$

$$\propto \frac{1}{r}\alpha_s(1/r)^{-\nu}e^{-1/2\beta_0\alpha_s(1/r)}\left[1 - \frac{1}{2}(\beta_2\beta_0\right.$$
$$\left. - \beta_1^2)/\beta_0^3\alpha_s(1/r) + \cdots\right] \qquad (6)$$

with the imaginary part of the Borel integration term in Eq. (1), one can see that $\tilde{V}(b)$ must have the singularity

$$\tilde{V}(b) = \frac{c_V}{(1-2b)^{1+\nu}}[1 + c_1(1-2b) + c_2(1-2b)^2 + \cdots]$$
$$+ (\text{analytic part}), \qquad (7)$$

where the "analytic part" denotes terms analytic around $b = 1/2$. The constants ν and c_i, which depend only on the coefficients of the β function, were first determined in [14], and can be computed up to c_2 from the known four loop β function [15]:

$$\nu = \frac{\beta_1}{2\beta_0^2}, \qquad c_1 = \frac{\beta_1^2 - \beta_0\beta_2}{4\nu\beta_0^4},$$

$$c_2 = \frac{\beta_1^4 + 4\beta_0^3\beta_1\beta_2 - 2\beta_0\beta_1^2\beta_2 + \beta_0^2(\beta_2^2 - 2\beta_1^3) - 2\beta_3\beta_0^4}{32\nu(\nu-1)\beta_0^8}. \qquad (8)$$

The residue c_V becomes the normalization constant of the large order behavior of the expansion (4), and its exact value is not known, but it can be determined perturbatively using the method developed in [16,17]. Once c_V is known, we can combine the two expansions of the Borel transform, Eqs. (3) and (7), at $b=0$ and at $b=1/2$, respectively, to obtain an improved description of the Borel transform in the region between the origin and the renormalon location at $b=1/2$. There are, in principle, an infinite number of ways to interpolate the two expansions, but here we shall take a simple one which turns out to suffice our purpose very well. We write the Borel transform as a two point expansion, which we call a *bilocal expansion*,[2]

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[2]This was first introduced in [18] in a slightly different context.

$$\tilde{V}(b) = \lim_{N,M \to \infty} \tilde{V}_{N,M}(b)$$

$$= \lim_{N,M \to \infty} \left\{ \sum_{n=0}^{N} \frac{h_n}{n!} \left(\frac{b}{\beta_0} \right)^n + \frac{c_V}{(1-2b)^{1+\nu}} \left[1 + \sum_{i=1}^{M} c_i (1 \right. \right.$$

$$\left. \left. -2b)^i \right] \right\}. \tag{9}$$

By demanding that this bilocal expansion reproduce the expansion (3) around the origin the coefficients h_n can be determined in terms of V_n and c_i. This gives, for example, the first three coefficients as

$$h_0 = V_0 - c_V(1 + c_1 + c_2),$$

$$h_1 = V_1 - 2c_V\beta_0[1 - c_2 + \nu(1 + c_1 + c_2)],$$

$$h_2 = V_2 - 4c_V\beta_0^2[2 + \nu(3 + c_1 - c_2)$$

$$+ \nu^2(1 + c_1 + c_2)]. \tag{10}$$

For the bilocal expansion to work it is essential to have the residue c_V calculated to a good accuracy, which is the subject of the next section.

III. RENORMALON RESIDUE

The residue can be determined in perturbation using the method developed in [16,17]. It was shown in [4,19,20] that the residue in the case of the static potential can be calculated quite accurately. For completeness, we repeat the calculation here, and in the meantime obtain an improved estimate.

To compute c_V we first consider the function

$$R(b) \equiv (1 - 2b)^{1+\nu} \tilde{V}(b). \tag{11}$$

Then,

$$c_V = R\left(\frac{1}{2}\right). \tag{12}$$

$R(b)$ has a cut, but is bounded, at $b = 1/2$, and thus we can write c_V as a convergent series,

$$c_V = \sum_{n=0}^{\infty} r_n \left(\frac{1}{2} \right)^n, \tag{13}$$

where r_n are the coefficients of the power expansion of $R(b)$ at the origin. The first three r_n can be calculated from the known V_n up to next-next-leading order (NNLO) [21–23], and this gives

$$c_V \approx -1.33333 + 0.49943 - 0.33844 = -1.17234. \tag{14}$$

The convergence is not that rapid but the series is oscillating. An important observation made in [19] is that the reliability of this estimate can be checked by the mutual cancellation of the renormalons in the static potential and the pole mass.

In perturbation theory the heavy quark pole mass m_{pole} can be expanded as

$$m_{\text{pole}}[\alpha_s(m_{\overline{\text{MS}}})] = m_{\overline{\text{MS}}} \left[1 + \sum_{n=0}^{\infty} p_n \alpha_s(m_{\overline{\text{MS}}})^{n+1} \right], \tag{15}$$

where $m_{\overline{\text{MS}}}[\equiv m_{\overline{\text{MS}}}(m_{\overline{\text{MS}}})]$ denotes the $\overline{\text{MS}}$ mass. As in the case of the static potential the Borel resummed pole mass can be written as

$$m_{\text{pole}}[\alpha_s(m_{\overline{\text{MS}}}) \pm i\epsilon] = m_{\overline{\text{MS}}} \left[1 \right.$$

$$\left. + \frac{1}{\beta_0} \int_{0 \pm i\epsilon}^{\infty \pm i\epsilon} e^{-b/\beta_0 \alpha_s(m_{\overline{\text{MS}}})} \tilde{m}_{\text{pole}}(b) db \right]$$

$$+ m_{\text{NP}}[\alpha_s(m_{\overline{\text{MS}}}) \pm i\epsilon], \tag{16}$$

where the Borel transform $\tilde{m}_{\text{pole}}(b)$ has the perturbative expansion

$$\tilde{m}_{\text{pole}}(b) = \sum_{n=0}^{\infty} \frac{p_n}{n!} \left(\frac{b}{\beta_0} \right)^n, \tag{17}$$

and m_{NP} denotes the renormalon induced nonperturbative effect. The renormalon ambiguity in the pole mass proportional to $\Lambda_{\overline{\text{MS}}}$ gives rise to a renormalon singularity that has exactly the same form as Eq. (7) of the static potential,

$$\tilde{m}_{\text{pole}}(b) = \frac{c_m}{(1-2b)^{1+\nu}} [1 + c_1(1-2b) + c_2(1-2b)^2 + \cdots]$$

$$+ (\text{analytic part}). \tag{18}$$

Now the cancellation of the renormalons in $2m_{\text{pole}}$ and $V(r)$ [5,6] leads to

$$c_V + 2c_m = 0. \tag{19}$$

We shall now compute the residue c_m following the computation of c_V. Using the known coefficients up to NNLO [24–26] of expansion (15) we have

$$c_m \approx 0.42441 + 0.17473 + 0.02289 = 0.62203. \tag{20}$$

This time the convergence is quite good. With the two computed values we now have

$$\frac{c_V + 2c_m}{c_V - 2c_m} = 0.02968, \tag{21}$$

which shows a remarkable cancellation of the two residues. This gives an assurance on the accuracy of the calculated residues.

We shall now compute c_m in a slightly different way. As has been shown in solvable models [10], the knowledge on the renormalon locations in the Borel plane can be used in improving the residue calculation. Since we are interested in the power expansion of $R(b)$ around the origin, we can obtain, in principle, a better convergence by expanding it in a

new complex plane in which it is smoother around the origin [18]. This can be done by pushing the renormalon singularities save the first one away from the origin with a conformal mapping. Let us consider the mapping [18,27]

$$w = \frac{\sqrt{1+b} - \sqrt{1-2b/3}}{\sqrt{1+b} + \sqrt{1-2b/3}}, \quad (22)$$

which maps the first renormalon at $b = 1/2$ to $w = w_0$, where

$$w_0 = \frac{1}{5}, \quad (23)$$

and all other renormalons (at $b = -n$ and $b = 1/2 + n$ where $n = 1, 2, 3, \ldots$) onto the unit circle.

Expanding $R[b(w)]$ at the origin to $O(w^2)$ and evaluating it at $w = w_0$ we have a new estimate of c_m,

$$c_m \approx 0.42441 + 0.16774 + 0.03451 = 0.62667, \quad (24)$$

which is quite close to the previous one (20). This stability is reassuring that our computation is accurate.

Now we shall quantitatively estimate the error in the computed residue (24). We do this by computing c_m using an estimated NNNLO coefficient of the expansion (15). We first estimate the unknown NNNLO coefficient p_3 following the method developed in [27]. First, expand $R[b(w)]$ to $O(w^3)$ with p_3 included. This gives

$$R[b(w)] = 0.42441 + 0.83872w + 0.86284w^2 + (-129.2687$$
$$+ 3.43505 p_3) w^3. \quad (25)$$

Note that the p_3-independent constant term in the coefficient of w^3 is much larger than the coefficients of the lower orders. It turns out this is a generic feature of an asymptotic expansion with rapidly growing coefficients, and it can be used in estimating higher-order unknown coefficients. From the pattern of the known lower-order terms it appears quite reasonable to assume that the fourth coefficient is bounded by

$$|129.2687 - 3.43505 p_3| < 2. \quad (26)$$

This gives an estimate on p_3:

$$p_3 = 37.6322 \pm 0.58223. \quad (27)$$

With this result we can repeat the computation of c_m in w plane, now at NNNLO, to obtain

$$c_m = 0.62667 \pm 0.02553. \quad (28)$$

We thus conclude the error in the computed residue (24) is about 4%.

For the numerical analysis in Sec. V we use the exact relation (19) and the pole mass residue (24) to compute c_V. Since the convergence in the calculation of the pole mass residue is better than that of the potential, we would have a more accurate value this way. We thus have

$$c_V = -1.25334 \pm 0.05106. \quad (29)$$

IV. DETERMINATION OF THE NONPERTURBATIVE EFFECT

In this section we give an evaluation of the renormalon caused nonperturbative effect V_{NP} using the method developed in [10]. As mentioned in Sec. II, the role of V_{NP} in Borel resummation is to cancel the imaginary part arising from the renormalon singularity in the Borel integration of the static potential. This means that, in principle, the imaginary part of V_{NP} is calculable from perturbation theory. However, the real part, which is the physical quantity, cannot be directly calculated perturbatively without further input.

The method for computing the real part relies on the analyticity of V_{NP} in the complex α_s plane. As mentioned, the potential obtained by Borel resumming the asymptotic expansion has a cut along the positive real axis in the α_s plane, and from this cut the imaginary part of the perturbative term, the integral term in Eq. (1), arises. To cancel this imaginary part it is thus plausible to demand that $V_{NP}(r, \alpha_s)$ also have a cut only along the positive real axis in the coupling plane. This then relates the real part to the perturbatively calculable imaginary part (we refer the reader to [10] for details). For convenience, we shall call this method of determining the nonperturbative effect (along with the Borel integration of the perturbation series) "analytic Borel resummation" (ABR). Some nonperturbative effects in solvable models were shown to be calculable in ABR [10].

For ABR to work it is essential to have the functional form of the nonperturbative effect beforehand. In the case of the static potential it is provided by the renormalization group equation. Since V_{NP} in the $\overline{MS}$ scheme should be a constant proportional to $\Lambda_{\overline{MS}}$, where

$$\Lambda_{\overline{MS}} = \frac{1}{r} [\beta_0 \alpha_s(1/r)]^{-\nu} e^{-1/2\beta_0 \alpha_s(1/r)} \exp\left\{ -\frac{1}{2} \int_0^{\alpha_s(1/r)} \left[\frac{1}{\beta(x)} \right. \right.$$
$$\left. \left. + \frac{1}{\beta_0 x^2} - \frac{\beta_1}{\beta_0^2 x} \right] dx \right\}, \quad (30)$$

we can write, by demanding V_{NP} have a cut only along the positive real axis,

$$V_{NP}[r, \alpha_s(1/r)] = \frac{C}{r} [-\alpha_s(1/r)]^{-\nu} e^{-1/2\beta_0 \alpha_s(1/r)}$$
$$\times \exp\left\{ -\frac{1}{2} \int_0^{\alpha_s(1/r)} \left[\frac{1}{\beta(x)} + \frac{1}{\beta_0 \alpha_s^2} \right. \right.$$
$$\left. \left. - \frac{\beta_1}{\beta_0^2 x} \right] dx \right\}, \quad (31)$$

with C an undetermined real constant. Note that a cut can arise only from the prefactor in Eq. (30) with a noninteger ν. Now the cancellation of the imaginary part in $V_{NP}[r, \alpha_s(1/r) \pm i\epsilon]$ with the corresponding imaginary part in the Borel integration term in Eq. (1) fixes the constant C:

$$C = \frac{c_V \Gamma(-\nu)}{(2\beta_0)^{1+\nu}}. \qquad (32)$$

The real part of V_{NP} is then given by

$$\text{Re}[V_{\text{NP}}(\alpha_s \pm i\epsilon)] = \frac{c_V \Gamma(-\nu)}{2^{1+\nu}\beta_0} \cos(\nu\pi)\Lambda_{\overline{\text{MS}}}. \qquad (33)$$

With the calculated residue c_V in Eq. (29), we find at $N_f = 0$,

$$\text{Re}[V_{\text{NP}}(\alpha_s \pm i\epsilon)] = 0.477\Lambda_{\overline{\text{MS}}}. \qquad (34)$$

In the numerical analysis in the next section we will combine this result with the Borel integration of the perturbative expansions.

V. COMPARISON WITH LATTICE CALCULATION

The static potential in lattice calculation is extracted from the Wilson line of a static quark-antiquark pair, computed in Monte Carlo simulation. The recent calculations [28–31] employing large lattices up to 64^4 achieved a remarkable accuracy, and can probe a short distance where perturbative QCD should be applicable. It is thus an ideal place where perturbative QCD can be compared with lattice calculations.

As we mentioned in the Introduction, the truncated power series of the perturbative expansion fails even at a very short distance. We shall now see that this problem can be cured by Borel resummation.

The numerical integration of the Borel integral in Eq. (5) can be done easily in w plane defined by the mapping (22). Using the Cauchy's theorem, the integration contour, for example, on the upper half plane in w plane can be deformed to a ray off the origin to the unit circle in the first quadrant. This trick allows us to avoid the renormalon singularity on the integration contour, and makes the computation easy. For details we refer the reader to [18].

For comparison with lattice calculation we take the accurate data of the recent computation employing large lattices [28]. All the dimensional quantities are in units of the Sommer scale $r_0(\approx 0.5 \text{ fm})$ [32], where r_0 in terms of $\Lambda_{\overline{\text{MS}}}$ (≈ 238 MeV) is determined in lattice computation [33] to be

$$r_0 \Lambda_{\overline{\text{MS}}} = 0.602(48). \qquad (35)$$

On the side of the perturbative potential, the Borel integration in Eq. (5) was done using the Borel transform $\tilde{V}_{0,2}, \tilde{V}_{1,2}$, and $\tilde{V}_{2,2}$ in the bilocal expansion (9). The coupling constant $\alpha_s(1/r)$ was computed by numerically solving Eq. (30) employing the four loop β function [15]. Because of the divergent quark self-energy the lattice potential is determined only up to an r independent constant, so we subtracted such a constant from the lattice data so that the lattice potential and the NNLO perturbative potential agree exactly at $r/r_0 = 0.30798$.

The result is in Fig. 2. Notice the rapid convergence of the resummed potential at distances $r \lesssim 0.6r_0$ [$\approx (660 \text{ MeV})^{-1}$], and the excellent agreement of the
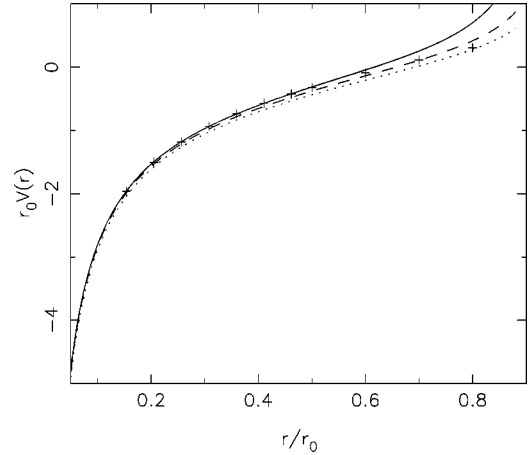

FIG. 2. The lattice potential vs the Borel resummed potential using $\tilde{V}_{0,2}$ (dotted), $\tilde{V}_{1,2}$ (dashed), and $\tilde{V}_{2,2}$ (solid).

NNLO potential with the lattice data. The potential at leading order already fits the lattice values quite well. It is remarkable that perturbative QCD is applicable at distances as large as $r = (660 \text{ MeV})^{-1}$.

VI. DISCUSSION AND SUMMARY

The first thing we can learn from our result is that in the static potential the leading renormalon is overwhelmingly dominant at short distances and there cannot be any significant nonperturbative effect other than that caused by the renormalon. As already observed in [3,4], large linear potentials at short distances like those proposed in [1,34,35] are excluded.

The rapid convergence of the perturbative potential in ABR allows one to use the pole mass in the perturbative calculation of heavy quarkonium physics. Because of the bad convergence of the truncated power series of the static potential, there was a limit in the precision achievable with perturbative QCD in quarkonium physics [36,37]. But, it was soon realized that the cancellation of renormalons in the pole mass and the static potential can be used to alleviate the problem [5,6]. Instead of using the pole mass directly, one can achieve an improved convergence by simultaneously expanding the pole mass and static potential in the heavy quark Hamiltonian in terms of the running coupling $\alpha_s(\mu)$ and a short distance mass such as the $\overline{\text{MS}}$ mass [8,9]. Although this approach *avoids* the renormalon problem, there could be large logs in the perturbative expansion which could, in principle, spoil the convergence. Since the expansion involves two far-separated scales, the heavy quark mass and $1/r$ ($\approx mv$, where v is the heavy quark velocity) large logs like $\ln(r\mu)$ and/or $\ln(m/\mu)$ could survive for any choice of μ, which in practice is typically taken as values in between the two scales. With our resummation of the static potential, the convergence problem at short distance is solved, so the pole mass need not be abandoned in favor of a short distance mass. Once the pole mass is extracted by comparing, say, a calculated quarkonium spectrum to an experimental value, the $\overline{\text{MS}}$ mass can be obtained from the pole mass by resumming the quark mass expansion (15) in ABR. Since the

renormalon in the pole mass is essentially same as that in the static potential, we can expect a rapid convergence of the Borel resummation of the mass expansion, and we have checked that this is indeed the case. As an example, for the bottom quark ($N_f=4$) with $\alpha_s(m_{\overline{\text{MS}}})=0.22$ the ''Borel resummed'' (BR) mass m_{BR}, which is defined as the real part of the integral term in Eq. (16), converges as

$$m_{\text{BR}}=m_{\overline{\text{MS}}}(1+0.15769+0.00409-0.00028). \quad (36)$$

Notice the rapid convergence. The renormalon caused nonperturbative effect m_{NP} in Eq. (16) can be determined in ABR, and its real part equals to $-\text{Re}[V_{\text{NP}}]/2$ that is given in Eq. (33). An obvious advantage of the direct resummation of the renormalons is the separation of scales. The perturbative expansions for the pole mass and the static potential are resummed at their optimal scales $\mu=m_{\overline{\text{MS}}}$ and $\mu=1/r$, respectively, and there is no mixing of these scales as in the above implementation of renormalon cancellation using a short distance mass. The absence of large logs and the excellent convergence of the resummed mass and potential are expected to provide a new level of precision calculation for heavy quarkonium.

It is worthwhile to mention that the nonperturbative effects V_{NP} and m_{NP} may actually decouple completely from the quarkonium system. The renormalon cancellation between the pole mass and the static potential means that the ambiguous imaginary parts in these quantities cancel without the introduction of the nonperturbative effects. This implies that the nonperturbative effects are actually spurious, appearing only at an intermediate step in Borel resummation, and physical observables are completely independent of them. Specifically, we may write the Hamiltonian of a heavy quarkonium system as

$$H=2m_{\text{pole}}+\frac{\vec{p}^2}{m_{\text{pole}}}+V[r,\alpha_s(1/r)]. \quad (37)$$

Putting

$$m_{\text{pole}}=m_{\text{BR}}[m_{\overline{\text{MS}}},\alpha_s(m_{\overline{\text{MS}}})]+\text{Re}[m_{\text{NP}}],$$

$$V[r,\alpha_s(1/r)]=V_{\text{BR}}[r,\alpha_s(1/r)]+\text{Re}[V_{\text{NP}}], \quad (38)$$

where the BR potential V_{BR} denotes the real part of the integral term in Eq. (1), and using the cancellation of $2\text{Re}[m_{\text{NP}}]$ with $\text{Re}[V_{\text{NP}}]$ in ABR,[3] we can write H in terms of the BR quantities only:

$$H=2m_{\text{BR}}+\frac{\vec{p}^2}{m_{\text{BR}}}+V_{\text{BR}}[r,\alpha_s(1/r)]+O(\vec{p}^2\text{Re}[m_{\text{NP}}]/m_{\text{BR}}^2). \quad (39)$$

The remaining dependence on the nonperturbative effect suppressed by an inverse power of the quark mass is expected to cancel when higher-order terms in quark mass expansion of the Hamiltonian are taken into account. This
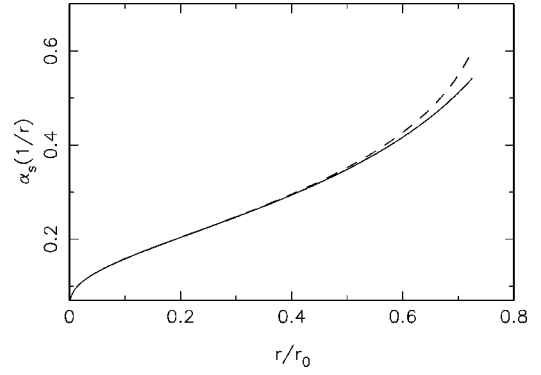
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[3]This cancellation is not automatic but a feature of ABR.

FIG. 3. The strong couplings obtained by employing the four loop β function (solid) and its [2/3] Padé approximant (dashed).

shows that the Hamiltonian in BR scheme is formally same as that in the on-shell scheme with the on-shell quantities m_{pole} and $V(r)$ replaced by the corresponding BR quantities. Thus, for physical observables the specific form of the nonperturbative effects is not necessary.

The perturbative potential and the lattice values in Fig. 2 begin to deviate at $r\approx0.6r_0$, which we regard as the failure of the perturbative potential at these distances. It is interesting to observe that this deviation occurs approximately at the same position where the four loop β function fails. The couplings $\alpha_s(1/r)$ obtained by running with the four loop β function and its [2/3] Padé approximant, which differs from the former only at orders higher than four loop, are plotted in Fig. 3. Notice that they begin to deviate approximately at the same distance where the perturbative potential begins to fail. At $r=0.6r_0$ [$\alpha_s(1/0.6r_0)=0.417$] the β function has the expansion

$$\beta=-0.152(1+0.308+0.143+0.097+\cdots), \quad (40)$$

which shows the convergence is quite slow at this distance. It seems the coupling grows too fast at these distances, since a more slowly growing coupling would fit the lattice data. This simultaneous deviation could be a coincidence, but a more plausible explanation would be that the failure of the β function at these distances results in an unreliable coupling, which then causes the deviation. The β function would not be all that fails the perturbative potential. Since there is a renormalon singularity at $b=3/2$, the bilocal expansion (9) at a finite order would certainly fail around $b\gtrsim3/2$. This does not cause any serious problem at small couplings, but as the coupling increases this becomes problematic because the integral term in Eq. (5) receives a sizable contribution from the region far from the origin. By varying the upper bound of the integration in Eq. (5) one can easily check that the resummed potential at $r\gtrsim0.6r_0$ is indeed sensitive on the Borel transform at $b\gtrsim3/2$. This argument suggests that the applicability of the Borel resummed perturbative potential could be extended to larger distances once we have a better control over the β function and the Borel transform at such distances.

Lastly, the convergence problem of the truncated power series in the perturbative potential is only one example, although a very conspicuous one, of the problem of the QCD

expansions in general, especially, at low energies of a few GeVs. The problem was not so visible in these expansions, since many were considered at a fixed scale, not like the perturbative potential considered here where a continuum of scale is involved. Conventionally, in the OPE approach, in these low-energy expansions the physical quantity is organized as the sum of a truncated power series and power corrections. Any difference between the truncated power series and the (unknown) true value is swept over to the power corrections. Clearly, this approach fails in the static potential because the potential of the OPE approach is just the truncated power series plus an *r* independent constant, which we know has a bad convergence and disagrees with the lattice calculation. As already discussed more extensively in the Gross–Llewellyn Smith sum rule [38], the solution to the problem is the Borel resummation that properly accounts for the renormalon. Without Borel resummation the bad convergence in the truncated power series results in wide fluctuations in the power corrections as the order of perturbation varies, which is observed in many cases. See [39,40] for some examples.

To summarize, we have shown that the Borel resummation with a proper account of the renormalon singularity in the Borel plane can resolve the convergence problem of the perturbative static potential and the pole mass, and the potential obtained in such a way is in excellent agreement with the lattice calculation. Consequently, any significant nonperturbative effect at short distance other than the renormalon effect is excluded, and the pole mass can be used in the perturbative calculation of heavy quarkonium physics. The advantages of the direct resummation of the renormalons include rapid convergence of the summations and absence of large logs, and these can open a new level of precision calculation for heavy quarkonium. We also calculated in the framework of ABR the renormalon caused nonperturbative effects in the static potential and the pole mass. The resummation method developed here may be applied to the computation of heavy quarkonium spectra in an approach similar to that employed in [41], where the perturbative potential at short distance is combined with the phenomenological potential at large distance. Also, it may be employed in the top threshold production.

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