

Light-front field theories at finite temperature

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We study the question of generalizing light-front field theories to finite temperature. We show that the naive generalization has serious problems and we identify the source of the difficulty. We provide a proper generalization of these theories to finite temperature based on a relativistic description of thermal field theories, both in the real and the imaginary time formalisms. Various issues associated with scalar and fermion theories, such as nonanalyticity of self-energy, tensor decomposition are discussed in detail.

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I. INTRODUCTION

Quantum field theories are conventionally quantized on a spacelike surface. However, quantization on a lightlike surface provides an interesting alternative which dates back to the works of Dirac [1]. Light-front field theories [2,3], namely theories quantized on a light front, have found applications in various branches of physics such as QCD, string theories and membrane theories, among others [4]. In the case of QCD, for example, they provide a method for studying nonperturbative phenomena systematically, which is different from the usual lattice studies.

In fact, there is a lot of activity in the study of light-front field theories within the context of QCD. Many of the features of light-front theories are quite distinct from the conventional equal-time field theories—one of the most significant being that light-front theories are first order (as opposed to conventional theories which are second order) in time derivatives and, correspondingly, describe different degrees of freedom. While the behavior of conventional field theories at finite temperature are quite well understood by now [5–7], a systematic study of the thermal properties of light-front field theories is lacking so far. It is the purpose of this paper to work out the essential properties of scalar and fermion light-front field theories at finite temperature. We will defer the study of light-front gauge theories to a future publication.

It would seem that the generalization of the thermal field concepts to light-front theories should be straightforward. In fact, in n dimensions, if we define the light-front time variable as

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^{n-1}) \quad (1)$$

then the Hamiltonian of the system can be identified with

$$H = P^- = \frac{1}{\sqrt{2}}(P^0 - P^{n-1}). \quad (2)$$

This would suggest that the ensemble average of an arbitrary operator, \mathcal{O} , in the light-front field theory, should be defined as

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$$\langle \mathcal{O} \rangle_\beta = \text{Tr} e^{-\beta H} \mathcal{O} = \text{Tr} e^{-\beta P^-} \mathcal{O} \quad (3)$$

where β represents the inverse temperature in units where the Boltzmann constant is unity. This has been, in fact, the general thinking [8]. However, as we will show, this straightforward generalization is incorrect and leads to various problems. We will discuss the proper description of thermal field theories (for scalars and fermions), on the light front, both in the imaginary as well as the real time formalisms. In Sec. II, we will describe the naive generalization of the concepts of thermal field theories to light-front scalar field theories, both in the imaginary and the real time formalisms, and show that this leads to various problems. The source of the difficulty is identified in Sec. III, where we give the proper description of light-front scalar theories at finite temperature. We calculate various quantities of interest such as the thermal mass correction as well as the nonanalyticity in the self-energy. We show that there are more possible limits that can arise in these theories, in contrast to conventional thermal field theories. A short description of the tensor decomposition, which has a richer structure in such theories, is also given. In Sec. IV, we discuss briefly the generalization of light-front fermion theories to finite temperature. Here a new feature arises since these theories have only half the number of independent degrees of freedom [3]. The generalization of light-front gauge theories to finite temperature as well as various other applications are under study and will be described in a future publication.

II. NAIVE GENERALIZATION TO FINITE TEMPERATURE

In this section, we will discuss the naive generalization of the techniques of thermal field theory to the light-front scalar field theories. Let us briefly establish the notation. In n dimensions, we define

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{n-1}) \quad (4)$$

where x^+ is identified with the light-front time coordinate. Denoting the coordinate vector as $x^\mu = (x^+, x^-, \vec{x})$, where \vec{x} represent the $(n-2)$ transverse coordinates, it is easy to see that the nontrivial components of the metric in this basis have the form

$$\eta^{+-} = \eta^{-+} = 1, \quad \eta^{ij} = -\delta^{ij} \quad (5)$$

so that the scalar product of two vectors can be written as

$$A \cdot B = A^+ B^- + A^- B^+ - \vec{A} \cdot \vec{B}. \quad (6)$$

The momentum vector can also be written as $p^\mu = (p^+, p^-, \vec{p})$, where p^- can be identified with the energy variable. In the light-front variables, the Einstein relation takes a linear form

$$p^- = \frac{\vec{p}^2 + m^2}{2p^+}. \quad (7)$$

This is a major difference from the conventional quantization on an equal-time surface.

In the light-front variables, the Lagrangian density for a ϕ^4 theory, for example, takes the form

$$\mathcal{L} = \partial_+ \phi \partial_- \phi - \frac{1}{2} (\vec{\nabla} \phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4. \quad (8)$$

It is clear that the Euler-Lagrange equations, following from this, are only first order in the x^+ derivative, which is a distinctive feature of light-front theories. The quantization of this theory has been discussed quite a lot in the literature and without going into details, we simply note here that the Feynman rules for this theory, at zero temperature, take the form

$$= \frac{i}{2p^+ p^- - \vec{p}^2 - m^2 + i\epsilon}$$

$$= -i\lambda \delta^n(p + q + r + s) \quad (9)$$

where all the momenta, at the vertex, are assumed to be incoming.

In going to finite temperature, as is well known, the interaction vertices of the theory are unaffected, but the propagators modify to reflect the periodicity (or antiperiodicity) of the field variables [5–7]. Let us generalize the theory in Eq. (8) to finite temperature, following the naive identification

$$\langle \mathcal{O} \rangle_\beta = \text{Tr} e^{-\beta H} \mathcal{O} \equiv \text{Tr} e^{-\beta P^-} \mathcal{O}. \quad (10)$$

We can describe the resulting theory either in the real time formalism or in the imaginary time formalism and we discuss the two cases separately.

A. Real time formalism

Let us describe the propagators of the theory in the closed time path formalism [7,9] for simplicity. A similar structure for the propagators results in thermofield dynamics [7,10], which we do not go into. It is well known that, in the real time formalism, the field degrees of freedom double and the propagators have a 2×2 matrix structure. In the case of the light-front scalar field theory with the conventional generalization in Eq. (10), the propagators for the doubled degrees of freedom have the forms

$$iG_{++}(p) = \frac{i}{2p^+ p^- - \omega_p^2 + i\epsilon} + 2\pi n_B(|p^-|) \delta(2p^+ p^- - \omega_p^2)$$

$$iG_{+-}(p) = 2\pi [\theta(-p^-) + n_B(|p^-|)] \delta(2p^+ p^- - \omega_p^2)$$

$$iG_{-+}(p) = 2\pi [\theta(p^-) + n_B(|p^-|)] \delta(2p^+ p^- - \omega_p^2)$$

$$iG_{--}(p) = -\frac{i}{2p^+ p^- - \omega_p^2 - i\epsilon} + 2\pi n_B(|p^-|) \delta(2p^+ p^- - \omega_p^2) \quad (11)$$

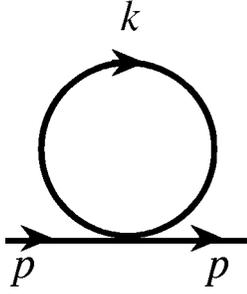
where we have introduced the bosonic distribution function

$$n_B(x) = \frac{1}{e^{\beta x} - 1}$$

and have defined

$$\omega_p^2 = \vec{p}^2 + m^2. \quad (12)$$

It is worth remembering that ω_p involves only $(n-2)$ transverse components of the momenta, as opposed to the conventional theories where it depends on all the $(n-1)$ spatial components of the momentum. We also note that the \pm subscripts refer to the “original” and the “doubled” degrees of freedom respectively. The propagators have the usual structure of a sum of the zero temperature part and the finite temperature part. However, the sign of trouble is already apparent in the form of the propagators in Eq. (11). The


 FIG. 1. One loop self-energy in ϕ^4 theory.

thermal distribution function does not seem to provide the necessary damping, as is usual in conventional thermal field theories. Namely, for $p^- = 0$ (which is allowed by the delta function constraint), the distribution function diverges. As we will see soon, more difficulties arise in actual calculations.

There are two kinds of vertices in the thermal field theory, “+” type and “-” type, with a relative sign difference between the two. However, at one loop, there is no mixing between the “original” and the “doubled” degrees of freedom. As a result, the one loop correction to the self-energy simply involves the tadpole graph (see Fig. 1), which can be readily evaluated.

$$i\Pi_{++}(p) = -\frac{i\lambda}{2} \int \frac{d^n k}{(2\pi)^n} iG_{++}(k). \quad (13)$$

Separating out the zero temperature part, the thermal correction to the self-energy can now be easily obtained. In fact, because of the delta function, the k^- integral can be trivially done leading to

$$\begin{aligned} i\Pi_{++}^{(\beta)}(p) &= -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^n k n_B(|k^-|) \delta(2k^+ k^- - \omega_k^2) \\ &= -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^{n-2} k \int_0^\infty \frac{dk^+}{k^+} n_B\left(\frac{\omega_k^2}{2k^+}\right). \end{aligned} \quad (14)$$

It is clear that the k^+ integral is divergent at the ultraviolet limit, $k^+ \rightarrow \infty$, and needs to be regularized. Regularizing the power of k^+ in the denominator yields

$$\begin{aligned} i\Pi_{++}^{(\beta)}(p) &= \lim_{\epsilon \rightarrow 0} -\frac{i\lambda}{2(2\pi)^{n-1}} \\ &\quad \times \int d^{n-2} k \int_0^\infty \frac{dk^+}{(k^+)^{1+\epsilon}} n_B\left(\frac{\omega_k^2}{2k^+}\right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{i\lambda}{4(2\pi)^{n-1}} \int d^{n-2} k \\ &\quad \times \left(\frac{1}{\epsilon} - \mathbf{C} + \ln 2\pi + \ln \frac{2}{\beta \omega_k^2} + O(\epsilon) \right). \end{aligned} \quad (15)$$

The form of the thermal correction in Eq. (15) is exact and is quite interesting. It shows that even though this represents the thermal correction, it has a divergent part that is independent of temperature. Thus, it would seem that in a thermal background, the theory would require additional temperature independent counterterms beyond the ones needed for the regularization of the zero temperature theory. This is quite distinct from the behavior of the conventional thermal field theories and, if true, would cause enormous problems with the renormalizability properties of the light-front theories at finite temperature. Furthermore, since the integration over the $(n-2)$ transverse directions is yet to be performed, we see that the temperature dependent part of the amplitude is finite only in 1+1 dimensions. In any other dimension, the temperature dependent part diverges as well, requiring temperature dependent counterterms. We would like to emphasize here that, even though Eq. (15) represents the temperature dependent part of the one loop self-energy, it does not vanish when temperature vanishes, namely as $\beta \rightarrow \infty$. This is connected with the problem alluded to earlier, namely, since k^- can take a vanishing value, the limit $\beta \rightarrow \infty$ of the distribution function is ambiguous (it is nonanalytic at that point). Further problems arise when one studies the self-energy for the ϕ^3 theory on the light front, but we will not go into the details of these here. These are serious problems which suggest that the naive generalization of ideas from conventional thermal field theories may not be appropriate in the case of light-front field theories. In the next section, we will analyze the source of these problems and propose the appropriate generalization for such theories.

B. Imaginary time formalism

In the imaginary time formalism, we rotate the theory to Euclidean space (imaginary time) and assume that the energy variable takes discrete values. Consequently, the energy integrals are replaced by a sum over the discrete Matsubara frequencies. In the light-front theories, this translates to replacing, in the scalar propagator,

$$p^- \rightarrow 2i\pi n T \quad (16)$$

where $T = 1/\beta$ denotes temperature. As a result, the scalar propagator, in the imaginary time formalism, becomes

$$G(p) = \frac{1}{4i\pi n T p^+ - \omega_p^2}. \quad (17)$$

The tadpole diagram, Fig. 1, is now straightforward to evaluate in the imaginary time formalism,

$$\begin{aligned} -\Pi(p) &= -\frac{\lambda}{2} \int \frac{d^{n-1} k}{(2\pi)^{n-1}} T \sum_{n=-\infty}^{\infty} G(k) \\ &= -\frac{\lambda}{2(2\pi)^{n-1}} \int d^{n-1} k T \sum_{n=-\infty}^{\infty} \frac{1}{4i\pi n T k^+ - \omega_k^2} \\ &= \frac{\lambda}{4(2\pi)^{n-1}} \int d^{n-2} k \int_0^\infty \frac{dk^+}{k^+} \coth\left(\frac{\omega_k^2}{4T k^+}\right). \end{aligned} \quad (18)$$

Separating out the zero temperature contribution, we obtain

$$\begin{aligned}
-\Pi^{(\beta)}(p) &= \frac{\lambda}{4(2\pi)^{n-1}} \int d^{n-2}k \\
&\times \int_0^\infty \frac{dk^+}{k^+} \left[\coth\left(\frac{\omega_k^2}{4Tk^+}\right) - 1 \right] \\
&= \frac{\lambda}{2(2\pi)^{n-1}} \int d^{n-2}k \int_0^\infty \frac{dk^+}{k^+} n_B\left(\frac{\omega_k^2}{2k^+}\right).
\end{aligned} \tag{19}$$

This is exactly the same expression as in Eq. (14) and, therefore, all the subsequent analysis of the earlier subsection follows. The real time and the imaginary time formalisms give the same result which is, however, plagued by problems. We will discuss next the source of the problem.

III. THE PROPER GENERALIZATION TO FINITE TEMPERATURE

To understand the source of the problems in the last section, let us recapitulate briefly what happens in a conventional theory. In a conventional thermal field theory, the thermal part of the propagator represents the interactions of the particle with the thermal distribution of real particles in the medium. This is suppressed at high energies. In contrast, the propagators in Eq. (11), as we have argued, do not provide the necessary damping. The form of the propagators is, of course, derived from the assumption that the ensemble average, in light-front theories, is given by Eq. (10). This is, in fact, where the problem lies. In a conventional theory, when one assumes that the ensemble average has the form

$$\langle \mathcal{O} \rangle_\beta = \text{Tr} e^{-\beta H} \mathcal{O}, \tag{20}$$

it is understood that we are in a Lorentz frame where the heat bath is at rest. In fact, this is not a manifestly Lorentz covariant description. One can give a manifestly covariant description of thermal field theories [11,12] at the expense of introducing a velocity for the heat bath, u^μ , normalized to unity, namely

$$u \cdot u = u^\mu u_\mu = 1 \tag{21}$$

and generalizing the ensemble average to

$$\langle \mathcal{O} \rangle_\beta = \text{Tr} e^{-\beta u \cdot P} \mathcal{O} = \text{Tr} e^{-\beta u^\mu P_\mu} \mathcal{O}. \tag{22}$$

In a conventional thermal field theory, where the metric is diagonal and is of the form $(+, -, -, \dots, -)$, one can choose a rest frame of the heat bath corresponding to $u^\mu = (1, 0, 0, \dots, 0)$ consistent with Eq. (21) and, in this case, Eq. (22) would reduce to the conventional definition of ensemble average in Eq. (20).

In contrast, a light-front description of a theory is manifestly relativistic. Intuitively, it is clear that it is not possible to have a heat bath at rest on the light front. Equation (10), on the other hand, is a generalization of the rest frame en-

semble average (20) to light-front theories and, consequently, there are bound to be problems. Note that, for Eq. (22) to reduce to Eq. (10), we must have $u^\mu = (1, 0, 0, \dots, 0)$, which, with the light-front metric, gives

$$u^\mu u_\mu = 2u^+ u^- - \vec{u} \cdot \vec{u} = 0.$$

This is inconsistent with Eq. (21) and this is another way of saying that we cannot have a heat bath at rest on the light front.

It is clear, therefore, that in dealing with light-front field theories, we must use a manifestly covariant description of the thermal field theories. With this, let us now discuss the proper generalization of real time and imaginary time formalisms for light-front field theories separately. In this section, we will restrict ourselves to scalar field theories and describe fermion theories in the next section.

A. Real time formalism

Let us assume that the heat bath is moving with a velocity u^μ subject to Eq. (21). In that case, in the closed time path formalism, the propagators for the scalar field theory take the forms (for the doubled degrees of freedom)

$$\begin{aligned}
iG_{++}(p) &= \frac{i}{2p^+ p^- - \omega_p^2 + i\epsilon} \\
&\quad + 2\pi n_B(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2) \\
iG_{+-}(p) &= 2\pi [\theta(-u \cdot p) + n_B(|u \cdot p|)] \delta(2p^+ p^- - \omega_p^2) \\
iG_{-+}(p) &= 2\pi [\theta(u \cdot p) + n_B(|u \cdot p|)] \delta(2p^+ p^- - \omega_p^2) \\
iG_{--}(p) &= -\frac{i}{2p^+ p^- - \omega_p^2 - i\epsilon} \\
&\quad + 2\pi n_B(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2).
\end{aligned} \tag{23}$$

A simple choice for the velocity of the heat bath satisfying Eq. (21), for example, is (in the light-front basis)

$$u^\mu = \frac{1}{\sqrt{2}}(1, 1, 0, 0, \dots, 0) \tag{24}$$

in which case, we have

$$u \cdot p = \frac{1}{\sqrt{2}}(p^+ + p^-)$$

and the bosonic distribution function takes the form

$$n_B(|u \cdot p|) = n_B\left(\frac{1}{\sqrt{2}}|p^+ + p^-|\right).$$

It is easy to see that this distribution function provides the necessary damping, on-shell, both at $p^+ = 0$ and $p^+ \rightarrow \infty$.

With this modification of the propagators, we can now reevaluate the tadpole diagram (see Fig. 1). With Eq. (24), the temperature dependent part of the tadpole graph has the form

$$\begin{aligned}
 i\Pi_{++}^{(\beta)}(p) &= -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^n k n_B \\
 &\quad \times \left(\frac{1}{\sqrt{2}} |k^+ + k^-| \right) \delta(2k^+ k^- - \omega_k^2) \\
 &= -\frac{i\lambda}{2(2\pi)^{n-1}} \int d^{n-2} k \int_0^\infty \frac{dk^+}{k^+} n_B \\
 &\quad \times \left(\frac{\omega_k^2 + 2(k^+)^2}{2\sqrt{2}k^+} \right). \quad (25)
 \end{aligned}$$

It is worth emphasizing that, unlike the corresponding expression with the naive generalization in Eq. (14), this integrand is well behaved in both the limits $k^+ = 0$ and $k^+ \rightarrow \infty$ as is expected of a thermal amplitude. As a result, it does not need any regularization. Furthermore, it vanishes at zero temperature, $\beta \rightarrow \infty$, as we would expect since it represents the thermal correction to the self-energy. However, in general, it cannot be evaluated in a closed form. In the high temperature limit, $\beta m \ll 1$, and in four space-time dimensions, the integral has the value,

$$i\Pi_{++}^{(\beta)}(p) \approx -\frac{i\lambda}{24\beta^2} + O(\beta m). \quad (26)$$

There are several things to note from this result. First of all, this yields a thermal mass correction which, in the high temperature limit, has the form

$$\Delta m_T^2 = \frac{\lambda}{24\beta^2} = \frac{\lambda T^2}{24} > 0. \quad (27)$$

Namely, the thermal mass correction is positive as is the case in conventional theories. This is, in fact, crucial for restoration of symmetry at finite temperature. More interestingly, we note that the thermal mass correction coincides exactly with that obtained in a conventional thermal (scalar) field theory in four dimensions.

Before closing this subsection on the real time formalism, let us note that the propagators (23) satisfy the usual relations

$$\begin{aligned}
 iG_{++}(p) + iG_{--}(p) &= iG_{+-}(p) + iG_{-+}(p) \\
 iG_{++}(p) - iG_{+-}(p) &= iG_{-+}(p) - iG_{--}(p) = iG_R(p) \\
 &= \frac{i}{(u \cdot p)^2 - (\bar{u} \cdot p)^2 - \omega_p^2 + i\epsilon \operatorname{sgn}(u \cdot p)} \\
 iG_{++}(p) - iG_{-+}(p) &= iG_{+-}(p) - iG_{--}(p) = iG_A(p) \\
 &= \frac{i}{(u \cdot p)^2 - (\bar{u} \cdot p)^2 - \omega_p^2 - i\epsilon \operatorname{sgn}(u \cdot p)} \quad (28)
 \end{aligned}$$

where G_R, G_A denote the retarded and the advanced propagators respectively and we have defined a vector \bar{u}^μ which is orthogonal to u^μ [as well as to any vector in the transverse $(n-2)$ dimensional space] and has a spacelike normalization, namely,

$$u \cdot \bar{u} = 0, \quad \bar{u} \cdot \bar{u} = -1. \quad (29)$$

For the choice in Eq. (24), $\bar{u}^\mu = 1/\sqrt{2}(1, -1, 0, \dots, 0)$.

For completeness, let us also note here the forms of the propagator in the formalism of thermofield dynamics in light-front scalar field theories, with proper generalization,

$$\begin{aligned}
 iG_{11}(p) &= \frac{i}{2p^+ p^- - \omega_p^2 + i\epsilon} + 2\pi n_B(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2) \\
 iG_{12}(p) &= 2\pi n_B(|u \cdot p|) e^{\beta|u \cdot p|/2} \delta(2p^+ p^- - \omega_p^2) \\
 iG_{21}(p) &= 2\pi n_B(|u \cdot p|) e^{\beta|u \cdot p|/2} \delta(2p^+ p^- - \omega_p^2) \\
 iG_{22}(p) &= -\frac{i}{2p^+ p^- - \omega_p^2 - i\epsilon} \\
 &\quad + 2\pi n_B(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2). \quad (30)
 \end{aligned}$$

Finally, let us note that since, in this case, we have two preferred vectors available, namely, u^μ and \bar{u}^μ , any given vector can be uniquely decomposed as

$$A^\mu = (A \cdot u)u^\mu - (A \cdot \bar{u})\bar{u}^\mu + A_T^\mu \quad (31)$$

where A_T^μ is transverse to both u, \bar{u} . Similarly, any higher order tensor structure can also be decomposed with respect to these two vectors. In this way, a richer tensor structure arises in light-front theories at finite temperature than in conventional thermal field theories.

B. Imaginary time formalism

In the imaginary time formalism, in the covariant description, it is the variable $(u \cdot p)$ that is rotated to Euclidean space and takes discrete values. Thus,

$$u \cdot p \rightarrow 2i\pi nT \quad (32)$$

which, with the choice of Eq. (24), leads to

$$p^- \rightarrow 2\sqrt{2}i\pi nT - p^+ = \bar{p}^- - p^+ \quad (33)$$

where we have identified

$$\bar{p}^- = 2\sqrt{2}i\pi nT. \quad (34)$$

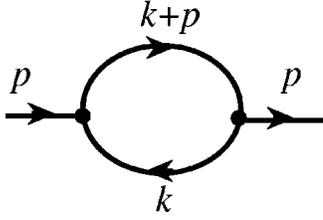


FIG. 2. One loop self-energy in ϕ^3 theory.

ing $\bar{u} \cdot p$ as an independent variable.] As a result, the scalar propagator, in the imaginary time formalism, takes the form

$$G(p) = \frac{1}{2(\bar{p}^- - p^+)p^+ - \omega_p^2} = \frac{1}{4\sqrt{2}i\pi n T p^+ - [\omega_p^2 + 2(p^+)^2]}. \quad (35)$$

[An alternate way of doing the rotation is to decompose the momentum vector as in Eq. (31) and rotate $u \cdot p$ while treat-

With this, let us calculate the tadpole diagram, Fig. 1, in the ϕ^4 theory,

$$\begin{aligned} -\Pi(p) &= -\frac{\lambda}{2(2\pi)^{n-1}} \int d^{n-1}k \sqrt{2}T \sum_{n=-\infty}^{\infty} G(k,n) = -\frac{\lambda}{2(2\pi)^{n-1}} \int d^{n-1}k \sqrt{2}T \sum_{n=-\infty}^{\infty} \frac{1}{4\sqrt{2}i\pi n T k^+ - [\omega_k^2 + 2(k^+)^2]} \\ &= \frac{\lambda}{4(2\pi)^{n-1}} \int d^{n-2}k \int_0^\infty \frac{dk^+}{k^+} \coth\left(\frac{\omega_k^2 + 2(k^+)^2}{4\sqrt{2}k^+ T}\right). \end{aligned} \quad (36)$$

Here, the factor of $\sqrt{2}$ arises from the Jacobian (because of the particular choice of the unit vector). Separating out the zero temperature part, then, leads to the thermal correction to the self-energy,

$$-\Pi^{(\beta)}(p) = \frac{\lambda}{2(2\pi)^{n-1}} \int d^{n-2}k \int_0^\infty \frac{dk^+}{k^+} n_B\left(\frac{[\omega_k^2 + 2(k^+)^2]}{2\sqrt{2}k^+}\right) \quad (37)$$

which coincides with the result calculated earlier in the real time formalism in Eq. (25) and all the subsequent analysis carries through.

As another example, let us calculate the one loop scalar self-energy in a massive ϕ^3 theory in 3+1 dimensions (see Fig. 2). In the imaginary time formalism, this has the form

$$\begin{aligned} -\Pi(p) &= \frac{g^2}{2(2\pi)^3} \int d^3k \sqrt{2}T \sum_m G(k,m) G(k+p,m) \\ &= \frac{g^2}{2(2\pi)^3} \int d^3k \sqrt{2}T \sum_m \frac{1}{2(2\sqrt{2}i\pi m T - k^+)k^+ - \omega_k^2} \frac{1}{2(2\sqrt{2}i\pi m T + \bar{p}^- - k^+ - p^+)(k^+ + p^+) - \omega_{k+p}^2}. \end{aligned} \quad (38)$$

The sum can be evaluated using standard formulas and leads to

$$\begin{aligned} -\Pi(p) &= -\frac{g^2}{16(2\pi)^3} \int \frac{d^3k}{k^+(k^+ + p^+)} \frac{1}{\frac{\omega_{k+p}^2 + 2(k^+ + p^+)^2}{2(k^+ + p^+)} - \frac{\omega_k^2 + 2(k^+)^2}{2k^+} - \bar{p}^-} \\ &\quad \times \left[\coth\frac{\omega_k^2 + 2(k^+)^2}{4\sqrt{2}k^+ T} - \coth\left(\frac{\omega_{k+p}^2 + 2(k^+ + p^+)^2}{4\sqrt{2}(k^+ + p^+)T} - \frac{\bar{p}^-}{2\sqrt{2}T}\right) \right]. \end{aligned} \quad (39)$$

If we use here the fact that $\bar{p}^- = 2\sqrt{2}i\pi n T$ as well as the periodicity of the hyperbolic function, the self-energy becomes (upon rotation to real time)

$$\begin{aligned}
 -\Pi(p) = & -\frac{g^2}{16(2\pi)^3} \int \frac{d^3k}{k^+(k^++p^+)} \frac{1}{\frac{\omega_{k+p}^2+2(k^++p^+)^2}{2(k^++p^+)} - \frac{\omega_k^2+2(k^+)^2}{2k^+} - (p^-+p^+)} \\
 & \times \left[\coth \frac{\omega_k^2+2(k^+)^2}{4\sqrt{2}k^+T} - \coth \left(\frac{\omega_{k+p}^2+2(k^++p^+)^2}{4\sqrt{2}(k^++p^+)T} \right) \right]. \quad (40)
 \end{aligned}$$

It is now easy to take various limits of this expression. In fact, in the present case, we have more possibilities of taking limits than in a conventional thermal field theory [12,13]. For example, we note that if we set $p^+ = \vec{p} = 0$ and take the limit $p^- \rightarrow 0$, we obtain

$$-\Pi(p^+ = 0, p^- \rightarrow 0, \vec{p} = 0) = 0. \quad (41)$$

On the other hand, if we set $p^- = \vec{p} = 0$ and take the limit $p^+ \rightarrow 0$, we obtain

$$\begin{aligned}
 -\Pi(p^+ \rightarrow 0, p^- = 0, \vec{p} = 0) \\
 = & \frac{g^2}{256\sqrt{2}\pi^3 T} \int d^3k \left(\frac{1}{(k^+)^2} - \frac{2}{\omega_k^2} \right) \\
 & \times \operatorname{cosech}^2 \frac{\omega_k^2+2(k^+)^2}{4\sqrt{2}k^+T}. \quad (42)
 \end{aligned}$$

Finally, we can also set $p^- = p^+ = 0$ and take the limit $\vec{p} \rightarrow 0$. In this limit, we obtain

$$\begin{aligned}
 -\Pi(p^+ = 0, p^- = 0, \vec{p} \rightarrow 0) \\
 = & \frac{g^2}{256\sqrt{2}\pi^3 T} \int d^3k \frac{1}{(k^+)^2} \operatorname{cosech}^2 \frac{\omega_k^2+2(k^+)^2}{4\sqrt{2}k^+T}. \quad (43)
 \end{aligned}$$

This shows that the three different ways of approaching the origin in the energy-momentum space lead to quite different results. Thus, light-front theories have a richer structure than the conventional thermal field theories also in this sense. Note, however, that as $T \rightarrow 0$, all three limits lead to a vanishing result, as would be expected in a zero temperature theory. Furthermore, an interesting question arises as to whether the three limits would lead to newer definitions of masses in light-front theories (in a conventional thermal field theory, we have only the screening mass and the plasmon mass corresponding to the two possible limits that are allowed). Even the question of what would correspond to the screening and the plasmon masses in such theories remains an open question.

IV. FERMION THEORIES

The fermion theories on the light front are more tricky simply because the number of degrees of freedom decreases

in this case. Let us consider, for example, a free massive fermion theory on the light front described by the Lagrangian density,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi. \quad (44)$$

We can, of course, add interactions, but, as we know, interaction vertices are not modified at finite temperature. Therefore, it is sufficient to look at the free theory to determine the propagators at finite temperature.

Let us define the light-front gamma matrices

$$\gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^0 \pm \gamma^{n-1}). \quad (45)$$

These are, in fact, nilpotent matrices, namely,

$$(\gamma^\pm)^2 = 0. \quad (46)$$

Defining the projection operators,

$$P^{(\pm)} = \frac{1}{2} \gamma^\mp \gamma^\pm = \frac{1}{2} (1 \pm \alpha_{n-1}) \quad (47)$$

where $\vec{\alpha}$ represents the Dirac matrices, it is easy to check that

$$(P^{(\pm)})^2 = P^{(\pm)},$$

$$P^{(+)}P^{(-)} = 0 = P^{(-)}P^{(+)},$$

$$P^{(+)} + P^{(-)} = 1. \quad (48)$$

With these projection operators, let us define

$$\psi^{(\pm)} = P^{(\pm)}\psi. \quad (49)$$

Then, it follows from the properties of the gamma matrices that

$$\gamma^- \psi^{(+)} = 0 = \gamma^+ \psi^{(-)}. \quad (50)$$

This allows us to write the Lagrangian density in the light-front variables as

$$\begin{aligned} \mathcal{L} = & \sqrt{2} \left[\psi^{(+)\dagger} i \partial_+ \psi^{(+)} + \psi^{(-)\dagger} i \partial_- \psi^{(-)} \right. \\ & - \frac{1}{2} \psi^{(+)\dagger} \gamma^- (i \vec{\gamma} \cdot \vec{\nabla} + m) \psi^{(-)} \\ & \left. - \frac{1}{2} \psi^{(-)\dagger} \gamma^+ (i \vec{\gamma} \cdot \vec{\nabla} + m) \psi^{(+)} \right]. \end{aligned} \quad (51)$$

It is clear now that only the $\psi^{(+)}, \psi^{(+)\dagger}$ degrees of freedom are dynamical. The other degrees of freedom are related to these and can be eliminated.

The fermion propagator, at zero temperature, has been derived long ago [3] and has the form

$$\begin{aligned} iS_F(x-y) &= \langle 0 | T^+ (\psi(x) \bar{\psi}(y)) | 0 \rangle \\ &= \sqrt{2} i \int \frac{d^n p}{(2\pi)^n} e^{-ip \cdot (x-y)} \left(\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} - \frac{\gamma^+}{2p^+} \right) \end{aligned} \quad (52)$$

where “ T^+ ” denotes ordering with respect to x^+ . Thus, we can identify

$$S_F(p) = \sqrt{2} \left(\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} - \frac{\gamma^+}{2p^+} \right) = \sqrt{2} \frac{\bar{p} + m}{p^2 - m^2 + i\epsilon} \quad (53)$$

where \bar{p}^μ denotes an on-shell momentum, namely,

$$\bar{p}^+ = p^+, \quad \bar{p}^- = \vec{p}^2 + m^2, \quad \bar{p}^- = \frac{\vec{p}^2 + m^2}{2p^+}. \quad (54)$$

The second form of the propagator makes it very clear that, when properly normalized, it can be thought of as a projection operator and, consequently, its inverse does not exist. This is, therefore, not a suitable structure to generalize to finite temperature. On the other hand, the singular structure of the zero temperature propagator simply reflects the fact that there are constraints in the theory, namely, that not all the degrees of freedom are dynamical. If we eliminate the nondynamical degrees of freedom, the entire theory can be recast in terms of $\psi^{(+)}, \psi^{(+)\dagger}$ variables. Therefore, the relevant propagator, from the point of view of the theory, is

$$\begin{aligned} \bar{S}_F(x-y) &= \langle 0 | T^+ [\psi^{(+)}(x) \psi^{(+)\dagger}(y)] | 0 \rangle \\ &= \langle 0 | P^{(+)} T^+ [\psi(x) \bar{\psi}(y)] \gamma^0 P^{(+)} | 0 \rangle \\ &= \int \frac{d^n p}{(2\pi)^n} e^{-ip \cdot (x-y)} \bar{S}_F(p). \end{aligned} \quad (55)$$

This can, in fact, be calculated from Eq. (53) in a simple manner and is determined to be

$$\bar{S}_F(p) = \frac{\sqrt{2} p^+}{p^2 - m^2 + i\epsilon} = \frac{\sqrt{2} p^+}{2p^+ p^- - \omega_p^2 + i\epsilon}. \quad (56)$$

This is well behaved with the two point function, in this projected space, corresponding to

$$\frac{p^2 - m^2}{\sqrt{2} p^+} = \frac{2p^+ p^- - \omega_p^2}{\sqrt{2} p^+}$$

and this propagator can be easily generalized to finite temperature. We note that this form of the propagator has also been calculated earlier directly from the field decomposition in [14].

The fermion propagators at finite temperature, in the real time formalism (closed time path), now take the forms

$$\begin{aligned} i\bar{S}_{++}(p) &= \sqrt{2} p^+ \left(\frac{i}{2p^+ p^- - \omega_p^2 + i\epsilon} \right. \\ &\quad \left. - 2\pi n_F(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2) \right) \\ i\bar{S}_{+-}(p) &= -2\sqrt{2} \pi p^+ [n_F(|u \cdot p|) - \theta(-u \cdot p)] \\ &\quad \times \delta(2p^+ p^- - \omega_p^2) \\ i\bar{S}_{-+}(p) &= -2\sqrt{2} \pi p^+ [n_F(|u \cdot p|) - \theta(u \cdot p)] \\ &\quad \times \delta(2p^+ p^- - \omega_p^2) \\ i\bar{S}_{--}(p) &= \sqrt{2} p^+ \left(-\frac{i}{2p^+ p^- - \omega_p^2 - i\epsilon} \right. \\ &\quad \left. - 2\pi n_F(|u \cdot p|) \delta(2p^+ p^- - \omega_p^2) \right) \end{aligned} \quad (57)$$

where n_F represents the fermion distribution function

$$n_F(x) = \frac{1}{e^{\beta x} + 1}.$$

The propagators in thermofield dynamics can similarly be obtained and we do not go into this here. In the imaginary time formalism, the fermion propagator, for the independent degrees of freedom, takes the form

$$\bar{S}(p) = \frac{\sqrt{2} p^+}{2\sqrt{2} i (2n+1) \pi p^+ T - [\omega_p^2 + 2(p^+)^2]}. \quad (58)$$

V. SUMMARY

In this paper, we have described how light-front field theories can be generalized to finite temperature. We have shown that the naive generalization leads to problems and the origin of the difficulty is identified. Since light-front field theories describe relativistic systems, a covariant description of thermal field theories becomes necessary for the proper formulation of thermal light-front theories. We discuss scalar and fermion light-front field theories at finite temperature in detail, including issues such as nonanalyticity of self-energy

and tensor decomposition. Several open questions are also discussed. Light-front gauge theories at finite temperature as well as further applications are presently under study and will be reported later.

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