Conformally flat stationary axisymmetric spacetimes

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It is shown that within conformally flat stationary axisymmetric spacetimes, in addition to the locally static family, there exists a new class of metrics, which is always stationary and axisymmetric. All these spacetimes, the static and the stationary ones, are endowed with an arbitrary function depending on the two non-Killingian coordinates. The explicit form of this function can be determined once the coupled matter, i.e., the energy-momentum tensor, is given. The locally static branch allows for horizons, and hence for the existence of black hole solutions, while the intrinsic stationary axisymmetric family does not permit black holes. Since both classes of metrics allow for surfaces possessing an extrinsic curvature proportional to their intrinsic metrics, there is room for the possibility of constructing two-branes in these spacetimes.

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I. INTRODUCTION

One of the challenging problems in general relativity is the search for interior solutions of isolated rotating bodies and the corresponding exterior solutions (of the vacuum Einstein equations). Usually, the description of the rotating masses is done by means of a perfect fluid energy-momentum tensor in the framework of stationary axisymmetric spacetimes. The first success in this respect was achieved at the very beginning of the Einstein theory by Schwarzschild in the case of static spherically symmetric interior (perfect fluid) and exterior spacetimes. The algebraic classification of the curvature tensor by Petrov [1] provided a powerful tool in the study of exact solutions. It was established that the exterior Schwarzschild solution belongs to the Petrov type-D family, while the interior Schwarzschild solution occurs as the conformally flat Petrov type 0 [2,3]. Later, in 1963, Kerr [4] published his famous rotating black hole solution, which, as is well known, belongs to the Petrov type-D class. To have a complete description of the gravitational field of an isolated rotating body it is necessary to establish the metric for the interior and the exterior solutions, taking care to satisfy the matching conditions on the boundary and the energy conditions as well. Thus, various researchers started to search for interior solutions to the Kerr solution, among others [5–7]. It is worth pointing out that the problem of which interior solution corresponds to (matches) to the Kerr solution still remains open. Attempts in this direction were accomplished mostly within Petrov type-D stationary axisymmetric spacetimes, in part because of the Collinson theorem [9] (see [8]).

In 1976, Collinson formulated the following theorem: Every conformally flat stationary axisymmetric spacetime is necessarily static. If the source of such spacetime is a perfect fluid, then the spacetime metric can be reduced to the usual Schwarzschild interior metric. The main goal of the present work is to establish the existence of a new class of intrinsically stationary axisymmetric spacetimes, which, at the same time, are conformally flat. The method followed by Collinson in the demonstration of his proposition, although correct, is incomplete: to prove his assertion, Collinson started with the equations arising from the vanishing of the conformal Weyl tensor, but he committed an unfortunate mistake at the stage of establishing his Eq. (2.12), losing the chance to consider the existing (as we establish in this work) stationary axisymmetric branch of metrics between the two possible ones. Hence, the Collinson theorem fails to be true in its first statement. This fact deserves a full and clear demonstration in its own right. Notice that to establish the conformally flat character of a metric, one does not need to satisfy the Einstein equations; the first question refers to the geometrical characterization of a given metric in the spirit of the Petrov classification, while the satisfaction of the Einstein equations deals with the allowable gravitational content. These aspects concern two different problems, which one can handle separately. The existence of a new branch of conformally flat stationary axisymmetric metrics suggests continuing the search for solutions of this class for relevant sources.

In what follows the complex coefficients associated with the conformal Weyl tensor for the general stationary axisymmetric metric admitting two-spaces orthogonal to the Killing vectors are given. Next, in Sec. II the general integrals for conformally flat spaces are presented, and we explicitly point out the flaw in Collinson's proof. Section III deals with conformally flat locally static spaces. As a by-product, we demonstrate here that one can isolate spacetimes of the form $\mathbb{R}\times$ Bañados-Teitelboim-Zanelli (BTZ) [10,12]; the results of [10] suggest the possibility of constructing four dimensional black holes bound to two-branes. In Sec. IV the general expression for conformally flat stationary axisymmetric metrics is given. Section V is devoted to the presentation of the algebraic classification of the energy-momentum tensor corresponding to the derived metric. Finally, some concluding remarks are stated.

The starting point in our study is the general stationary axisymmetric line element

$$ds^2=e^{-2Q(z,\bar{z})}dz\,d\bar{z}+\frac{e^{-2G(z,\bar{z})}}{a+b}[a(z,\bar{z})d\sigma+d\tau]\times[b(z,\bar{z})d\sigma-d\tau], \tag{1}$$

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where ∂_σ and ∂_τ are Killing vectors, such that one is spacelike and the second is timelike. Complex conjugation is denoted by an overbar. Notice that our function $a(z)$ differs from that of Collinson, $a_c(z)\rightarrow -a_{our}(z)$.

The easiest way known to derive the Einstein equations and the expressions for the curvature is based on the application of the Newman-Penrose formalism. In this approach the spacetime metric is given by

$$g=2e^1\otimes e^2-2e^3\otimes e^4, \tag{2}$$

where the null tetrad basis for the studied spacetime is

$$e^1=\frac{1}{\sqrt{2}}e^{-Q}dz,\quad e^3=\frac{1}{\sqrt{2}}\frac{e^{-G}}{\sqrt{a+b}}(d\tau-b\,d\sigma),$$
$$e^2=\frac{1}{\sqrt{2}}e^{-Q}d\bar{z},\quad e^4=\frac{1}{\sqrt{2}}\frac{e^{-G}}{\sqrt{a+b}}(d\tau+a\,d\sigma). \tag{3}$$

The evaluation of the Newman-Penrose curvature coefficients-Weyl complex components-yields the following nonvanishing quantities:

$$\Psi_0=\frac{e^{2Q}}{a+b}\left[2\frac{\partial a}{\partial z}\frac{\partial P}{\partial z}+\frac{\partial^2 a}{\partial z^2}-\frac{2}{a+b}\left(\frac{\partial a}{\partial z}\right)^2\right],$$
$$\bar{\Psi}_4=\frac{e^{2Q}}{a+b}\left[2\frac{\partial b}{\partial z}\frac{\partial P}{\partial z}+\frac{\partial^2 b}{\partial z^2}-\frac{2}{a+b}\left(\frac{\partial b}{\partial z}\right)^2\right],$$
$$6\Psi_2=\frac{e^{2Q}}{(a+b)^2}\left[2(a+b)^2\frac{\partial^2 P}{\partial z\partial\bar{z}}+5\frac{\partial a}{\partial z}\frac{\partial b}{\partial\bar{z}}-\frac{\partial a}{\partial\bar{z}}\frac{\partial b}{\partial z}\right], \tag{4}$$

where $P=P(z,\bar{z}):=Q-G$.

Incidentally, the above metric structure describes Petrov type-I gravitational fields for general Ψ's, Petrov type-D fields when the condition $\Psi_0\Psi_4=9\Psi_2^2$ is satisfied, and conformally flat spaces for vanishing Ψ's.

II. STATIONARY AXISYMMETRIC METRICS: GENERAL CASE $\partial a/\partial z\neq\partial b/\partial z\neq 0$

As we stated in the previous section, the demonstration of incorrectness of the Collinson theorem has to be done in a very clear and complete manner to avoid further misunderstandings, although this proof is time consuming and will involve lengthy mathematics. To start with, we shall find the general integrals of the equations arising from conformally flat conditions, i.e., we shall demand the vanishing of the Weyl tensor components. In the middle of the method, we shall explicitly notice where Collinson failed in his argument of assigning only one definite sign to an integration constant. According to our results, this constant can assume any real value. This gives room for the existence of stationary spacetimes in contrast to Collinson's conclusion about the static property of the considered spaces. To establish whether the derived metric is diagonalizable or not, we accomplish in Sec. II B linear transformations of the Killingian coordinates, and arrive at the conclusion that there exists a class of nondiagonal metrics, namely, the metric for conformally flat stationary axisymmetric spaces. Moreover, at the end of this section we derive the static metric.

Requiring the spacetime to be conformally flat, the Weyl tensor has to vanish, which is equivalent to satisfying the conditions $\Psi_0=\Psi_4=\Psi_2=0$. Accordingly, one has

$$\Psi_0=0\Rightarrow 2\frac{\partial a}{\partial z}\frac{\partial P}{\partial z}+\frac{\partial^2 a}{\partial z^2}-\frac{2}{a+b}\left(\frac{\partial a}{\partial z}\right)^2=0 \tag{5}$$

and

$$\Psi_4=0\Rightarrow 2\frac{\partial b}{\partial z}\frac{\partial P}{\partial z}+\frac{\partial^2 b}{\partial z^2}-\frac{2}{a+b}\left(\frac{\partial b}{\partial z}\right)^2=0. \tag{6}$$

Subtracting Eqs. (5) and (6), and dividing by $\partial a/\partial z-\partial b/\partial z\neq 0$, one obtains

$$\frac{\partial}{\partial z}\ln\left[\frac{e^{2P}}{(a+b)^2}\left(\frac{\partial a}{\partial z}-\frac{\partial b}{\partial z}\right)\right]=0; \tag{7}$$

its integration yields

$$\frac{\partial a}{\partial z}-\frac{\partial b}{\partial z}=\bar{g}(\bar{z})(a+b)^2e^{-2P}. \tag{8}$$

Next, dividing Eq. (5) by $\partial a/\partial z$ and Eq. (6) by $\partial b/\partial z$, and subsequently adding the resulting equations, one gets

$$\frac{\partial}{\partial z}\ln\left[\frac{e^{4P}}{(a+b)^2}\frac{\partial a}{\partial z}\frac{\partial b}{\partial z}\right]=0; \tag{9}$$

thus

$$\frac{\partial a}{\partial z}\frac{\partial b}{\partial z}=\bar{h}(\bar{z})(a+b)^2e^{-4P}. \tag{10}$$

Since a, b, and P are real functions, then from Eq. (8) one has

$$g(z)\frac{\partial}{\partial z}(a-b)=\bar{g}(\bar{z})\frac{\partial}{\partial\bar{z}}(a-b). \tag{11}$$

Using the freedom in the choice of the variable z, by introducing a new z, such that $g(z)\partial/\partial z\rightarrow\partial/\partial z$, one can set $g(z)=1$. Consequently Eq. (7) becomes

$$\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial\bar{z}}\right)(a-b)=0. \tag{12}$$

Hence $a-b=F(z+\bar{z})$.

The vanishing of Ψ_2, namely, of its imaginary part, yields

$$\frac{\partial a}{\partial z}\frac{\partial b}{\partial \bar{z}}=\frac{\partial a}{\partial \bar{z}}\frac{\partial b}{\partial z}. \tag{13}$$

Introducing the real coordinates x and y through $z=x+iy$, denoting with overdots the derivatives with respect to x, and substituting $a=b+F(x)$ into Eq. (13), one obtains

$$\dot{F}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)b=0. \tag{14}$$

Therefore, two cases emerge from this condition: the general case with $\dot{F}\neq 0 \rightarrow b=b(z+\bar{z})$, $a=a(z+\bar{z})$ and the branch $\dot{F}=0$ which is static.

A. Metric for $\dot{F}\neq 0$, $b=b(z+\bar{z})$, and $a=a(z+\bar{z})$

To our mind this is the main core of our work. We shall carry out the integration of the Ψ equations in the most general case $b=b(x)\neq a=a(x)$ with all generality. Some particular branches arising from the metric obtained will be treated in detail.

Without loss of generality, Eq. (8) rewrites

$$\dot{a}-\dot{b}=(a+b)^2 e^{-2P}, \tag{15}$$

while Eq. (10), because of the real character of its left hand side, implies $h(z)=\overline{h(\bar{z})}$; hence $h(z)=$const. Therefore, in general one arrives at $h(z)=\epsilon k^2$, where $\epsilon=\pm 1$, and $k=$const. There are neither mathematical reasons nor physical ones to restrict oneself to only one of the signs of ϵ as was done by Collinson, who chose $\epsilon=1$ in our terminology. Recall that the function a of the Collinson work differs from our function a. It is just at this point where our general demonstration procedure and the proof by Collinson (restricted to only one possibility) take different courses of action.

Continuing with the general integration process, Eq. (10) amounts to

$$\dot{a}\dot{b}=\epsilon k^2(a+b)^2 e^{-4P}, \tag{16}$$

where the parameter $\epsilon=\pm 1$. Introducing new dependent functions $X=X(x)$ and $Y=Y(x)$ on the variable x according to

$$a+b=2kY, \quad a=k(Y+X),$$
$$a-b=2kX, \quad b=k(Y-X), \tag{17}$$

Eq. (15) becomes

$$\dot{X}=\frac{dX}{dx}=2kY^2 e^{-2P}\rightarrow dx=\frac{dX}{2kY^2}e^{2P}, \tag{18}$$

while Eq. (16) amounts to

$$\dot{Y}^2-\dot{X}^2=4\epsilon k^2 Y^2 e^{-4P}. \tag{19}$$

Substituting e^{-2P} from Eq. (18) into Eq. (19) one obtains

$$\dot{Y}^2-\dot{X}^2=\epsilon\frac{\dot{X}^2}{Y^2} \tag{20}$$

or, equivalently, dividing by $\dot{X}^2\neq 0$, one has

$$\left(\frac{dY}{dX}\right)^2=1+\frac{\epsilon}{Y^2}\rightarrow\frac{dY}{dX}=\nu\frac{\sqrt{Y^2+\epsilon}}{Y}, \quad \nu=\pm 1, \tag{21}$$

with the general integral

$$\nu(X-X_0)=\sqrt{Y^2+\epsilon},\rightarrow Y^2=(X-X_0)^2-\epsilon, \tag{22}$$

where X_0 is an integration constant.

Up to this stage, we have integrated the equations arising from $\Psi_0=0$, $\Psi_4=0$, and one of the conditions, namely, Eq. (13), arising from the vanishing of Ψ_2. Thus, it remains to integrate the equation

$$6\Psi_2=\frac{e^{2Q}}{Y^2}\left[2Y^2\ddot{P}+\epsilon\frac{\dot{X}^2}{Y^2}\right]=0. \tag{23}$$

Since the integral of Y is given through the variable X by Eq. (22), it is more convenient to proceed further with this new variable X. In relations of the form $d/dx=\dot{X}d/dX$, $\dot{P}=\dot{X}P_{,X}=:\dot{X}P'$, and $\ddot{P}=\dot{X}^2P''+\ddot{X}P'$, the derivative $\dot{X}$ has to be taken from Eq. (18); primes stand for derivatives with respect to X. Moreover, the second derivative $\ddot{X}(x)$ with respect to x yields $\ddot{X}=2\dot{X}^2(Y'/Y-P')$. Replacing all pertinent quantities in Eq. (23) one arrives at

$$P''-2P'^2+2\frac{Y'}{Y}P'+\frac{\epsilon}{2Y^4}=0. \tag{24}$$

Introducing the function $K=\exp(-2P)$, using Y' from Eq. (21) and Y from Eq. (22), Eq. (24) becomes

$$[(X-X_0)^2-\epsilon]^2K''+2[(X-X_0)^2-\epsilon](X-X_0)K'-\epsilon K=0. \tag{25}$$

To obtain the general solution of the above equation, one accomplishes the change $K(X)=M(X)[(X-X_0)^2-\epsilon]^{-1/2}$, which yields $M''=0\rightarrow M(X)=C_0+C_1X$. Therefore

$$e^{-2P}=(C_0+C_1X)/\sqrt{(X-X_0)^2-\epsilon}. \tag{26}$$

In terms of the new coordinates X, after trivial scaling and coordinate translations: $X-X_0\rightarrow X$, $\sqrt{2}(\tau+kX_0\sigma)\rightarrow\tau$, $\sqrt{2}k\sigma\rightarrow\sigma$, together with a redefinition of the function G, $G\rightarrow G-\frac{1}{4}\ln(X^2-\epsilon)-\ln|2k|$, the studied metric amounts to

$$ds^2=e^{-2G(X,y)}\left[\frac{dX^2}{(C_0+C_1X)(X^2-\epsilon)}+(C_0+C_1X)dy^2 + k(-\epsilon d\sigma^2-2Xd\sigma d\tau-d\tau^2)\right]. \tag{27}$$

From this expression one sees that only the sign of k might have some relevance; by additional scaling transformations of the Killingian coordinates τ and σ one can achieve that $k=\pm 1$. The choice of the sign of k determines the (timelike or spacelike) character of the Killing vectors.

B. Are these metrics diagonalizable?

To establish whether this metric can be diagonalized or not, one accomplishes in the Killingian metric sector

$$d\Sigma^2 := k(-\epsilon d\sigma^2 - 2X d\sigma d\tau - d\tau^2) \tag{28}$$

linear transformations of the Killingian variables τ and σ of the form

$$d\tau = \alpha d\tau' + \beta d\sigma',$$
$$d\sigma = \gamma d\tau' + \delta d\sigma', \alpha\delta - \beta\gamma \neq 0, \tag{29}$$

for real constants α, β, γ, and δ. The $g_{\tau'\sigma'}$ component of the metric sector $d\Sigma^2$ amounts to

$$g_{\tau'\sigma'} = -k[\epsilon\gamma\delta + X(\beta\gamma + \alpha\delta) + \alpha\beta]; \tag{30}$$

thus, $g_{\tau'\sigma'}$ may vanish if there exists a real solution of the equations

$$\beta\gamma + \alpha\delta = 0,$$
$$\alpha\beta + \epsilon\gamma\delta = 0. \tag{31}$$

The general solution of this system is given by

$$\alpha = \pm\gamma\sqrt{\epsilon}, \quad \gamma = \gamma,$$
$$\beta = \mp\delta\sqrt{\epsilon}, \quad \delta = \delta; \tag{32}$$

therefore these constants are real parameters only in the case $\epsilon=1$. Accordingly, the metric-sector components $g_{\tau'\tau'}$ and $g_{\sigma'\sigma'}$ acquire the form

$$g_{\tau'\tau'} = -2k\gamma^2(\epsilon \pm \sqrt{\epsilon}X),$$
$$g_{\sigma'\sigma'} = -2k\delta^2(\epsilon \mp \sqrt{\epsilon}X), \tag{33}$$

where the choice of the upper (lower) sign in $g_{\tau'\tau'}$ has to be accompanied by the choice of the upper (lower) sign in $g_{\sigma'\sigma'}$.

Only in the branch of metrics with $\epsilon=1$ can one carry out real linear transformations of the Killingian coordinates such that the metric sector $d\Sigma^2$ becomes diagonal, which in its turn implies the diagonal character of the whole metric ds^2.

The case $\epsilon=-1$ deserves special attention; the transformations (29) are purely imaginary ones, and the corresponding metric tensor components become complex, a fact which is forbidden in real Einstein relativity. This case corresponds to a completely new branch of metrics.

C. Conformally flat static metric

For completeness, we derive in this subsection the metric for conformally flat static spacetimes. Since $\dot{F}=0 \rightarrow F=F_0 =\text{const}$, then $a(z)=b(z)+F_0$. By linear transformations of the (τ,σ) variables, one achieves that $a(z)=b(z)$; therefore the resulting metric is static. Since this is the case, one has to return to the Ψ equations, and to substitute there $a(z) =b(z)$. From $\Psi_0=\Psi_4=0$, assuming $\partial a/\partial z\neq 0$, and using freedom in the choice of the z coordinate, without loss of generality, one concludes that $a=a(z+\bar{z})=a(x)$, and $P =P(x)$ which are constrained to satisfy

$$e^{-2P} = \frac{\dot{a}}{a}, \tag{34}$$

where the overdot denotes the derivative with respect to x. Replacing $\dot{a}/a$ into the expression for Ψ_2, one obtains $6\Psi_2=\exp(2Q)[2\ddot{P}+\exp(-4P)]=0$, which, multiplied by $\dot{P} \equiv dP/dx$, integrates as $(dP/dx)^2-\frac{1}{4}\exp(-4P)=K=\text{const}$. This equation gives rise to a relation between the x variable and an auxiliary variable $X:=e^{-2P}/2$, namely,

$$dx = \pm\frac{1}{2X}\frac{dX}{\sqrt{K+X^2}}. \tag{35}$$

In terms of this variable X Eq. (34) becomes

$$\frac{da}{a} = \pm\frac{dX}{\sqrt{K+X^2}}, \tag{36}$$

with general integrals

$$\beta a(X) = [X \pm \sqrt{K+X^2}],$$
$$2X(a) = \frac{1}{a}(K/\beta - \beta a^2). \tag{37}$$

Using a instead of x as a new coordinate, $dx =1/(2X)d\ln a$, and changing K and β correspondingly by new constants according to $K/\beta\rightarrow\alpha$ and $\beta\rightarrow-\beta$, the conformally flat static metric amounts to

$$ds^2 = e^{-2G(a,y)}\left[\frac{da^2}{a(\alpha+\beta a^2)} + \frac{\alpha+\beta a^2}{a}dy^2 + a\,d\sigma^2 - \frac{d\tau^2}{a}\right]. \tag{38}$$

In the derivation of this metric it has been assumed that $\partial a/\partial z\neq 0$. In the case $\partial a/\partial z=0$, the function $P(z,\bar{z}) =f(z)+\overline{f(z)}$ and the corresponding metric can be given directly as a product of a conformal factor function with the Minkowski metric.

III. METRIC FOR CONFORMALLY FLAT LOCALLY STATIC SPACETIMES, $\epsilon=1$

In the previous section, it was established that conformally flat stationary axisymmetric spacetimes for $\epsilon=1$ can be brought into the diagonal form. We shall give here a canonical representation corresponding to the $\epsilon=1$ static branch of the metric (27), and relate it to the metric (38). Moreover, we devote a subsection to an interesting local representation of the metric (27) with $\epsilon=1$, namely, the IR $\times$BTZ representation, which can be achieved when a negative cosmological constant is present. With respect to this form of the metric, an analysis about the existence of black holes is provided. These remarks on the possible existence of horizons are extended to the general metric (27). The existence of spacetimes locally equivalent to static spacetimes is noteworthy to point out, but they are such that the coordinate transformations that would relate them are not globally well defined; in terms of symmetries, one may characterize this feature by saying that under transformations and identifications of coordinates from all local symmetries only the Killing vectors corresponding to stationarity and axial symmetry remain, in the present context, as such in the whole space.

For $\epsilon=1$—the only case in which conformally flat stationary axisymmetric spacetimes can be locally diagonalized—with an additional scaling transformation of the form $\sqrt{2}\gamma\tau'\rightarrow\tau$ and $\sqrt{2}\delta\sigma'\rightarrow\sigma$, $(X\rightarrow x)$, the corresponding metric becomes

$$ds^2=e^{-2G(x,y)}\left[\frac{dx^2}{(C_0+C_1x)(x^2-1)}+(C_0+C_1x)dy^2-(1\mp x)d\sigma^2-(1\pm x)d\tau^2\right]. \tag{39}$$

This branch corresponds to the choice $k=1$; the case $k=-1$ yields similar final results. Considering the Killingian metric sector

$$d\Sigma^2:=-(1+x)d\sigma^2-(1-x)d\tau^2, \tag{40}$$

one is faced with two possibilities: (A) $x<-1$, ∂_τ timelike Killing vector, and ∂_σ spacelike Killing vector; (B) $x>1$, ∂_τ spacelike Killing vector, and ∂_σ timelike Killing vector.

Case A. Introducing a new coordinate A,

$$A^2=-\frac{1+x}{1-x},\quad 1+x<0,$$

$$x=-\frac{A^2+1}{1-A^2},\quad 1-A^2>0, \tag{41}$$

identifying $\sigma\rightarrow\phi$, $\tau\rightarrow t$, $C_0-C_1\rightarrow2\alpha$, $C_0+C_1\rightarrow-2\beta$, and $G(x,y)-\frac{1}{2}\ln(C_0+C_1x)\rightarrow G(A,y)$, one arrives at the expression

$$ds^2=e^{-2G(A,y)}\left[\frac{dA^2}{(\alpha+\beta A^2)^2}+dy^2+\frac{A^2}{\alpha+\beta A^2}d\phi^2-\frac{dt^2}{\alpha+\beta A^2}\right]. \tag{42}$$

Case B. Similarly to the treatment of the previous case, one introduces a new variable A, namely,

$$A^2=\frac{x-1}{x+1},\quad x-1>0,$$

$$x=\frac{1+A^2}{1-A^2},\quad 1-A^2>0, \tag{43}$$

identifying $\sigma\rightarrow t$, $\tau\rightarrow\phi$, $C_0+C_1\rightarrow2\alpha$, $C_1-C_0\rightarrow2\beta$, and $G(x,y)-\frac{1}{2}\ln(C_0+C_1x)\rightarrow G(A,y)$, one arrives at the expression (42).

This metric (42) reduces, by replacing $G\rightarrow G+\frac{1}{2}\ln[A/(\alpha+\beta A^2)]$ and $A\rightarrow a$, to the conformally static metric (38).

The interior Schwarzschild perfect fluid solution is described by the above metric with $\alpha=1$, $\beta=-1$, and the function $G(A,y)=F(A)+H(y)$, such that $F(A)=p/(1-A^2)$, and $H(y)=q\cos y$, where p and q are constants. One distinguishes two branches of solutions satisfying the energy conditions; for details see [3,9].

Metric for conformally flat locally static spacetimes, $\epsilon=1$, a $\mathbb{R}\times$BTZ representation

We would like to point out an alternative formulation of the above metric (27) when a negative cosmological constant is present, $\lambda\sim1/l^2$. One achieves an $\mathbb{R}\times$BTZ representation of the quoted metric by subjecting the Killingian coordinates to linear transformations accompanied with a transformation of the X coordinate.

For this purpose, we first replace in the metric (27) $X\rightarrow x$, $G(X,y)\rightarrow G(x,y)+\frac{1}{2}\ln(C_0+C_1x)$, $\epsilon=1$, and $k=1$. Secondly, in the resulting metric one accomplishes a SL(2,IR) transformation of the Killingian coordinates of the form

$$d\tau=\alpha dt+\beta d\phi,$$

$$d\sigma=\gamma dt+\delta d\phi,\quad \alpha\delta-\beta\gamma\neq0, \tag{44}$$

where

$$\alpha=-\frac{1}{\sqrt{2l}}(b_0r_++a_0r_-),\quad \beta=\frac{1}{\sqrt{2}}(a_0r_++b_0r_-),$$

$$\gamma=\frac{1}{\sqrt{2l}}(b_0r_+-a_0r_-),\quad \delta=\frac{1}{\sqrt{2}}(a_0r_+-b_0r_-),$$

$$r_\pm=\sqrt{\frac{l}{2}\sqrt{Ml\pm\sqrt{M^2l^2-J^2}}}. \tag{45}$$

Moreover, the constants C_0 and C_1 satisfy the relations

$$C_0+C_1=-2a_0^2,\quad C_0-C_1=-2b_0^2. \tag{46}$$

The parameters l, M, and J stand, respectively, for the inverse of the cosmological constant, the global mass, and the total angular momentum of the BTZ solution.

Finally, subjecting the x coordinate to the transformation

$$x=\frac{b_0^2(r^2-r_-^2)+a_0^2(r^2-r_+^2)}{b_0^2(r^2-r_-^2)-a_0^2(r^2-r_+^2)}, \tag{47}$$

one arrives at an $\mathbb{R}\times$BTZ representation of the studied metric, namely,

$$\begin{aligned} ds^2&=e^{-2G(r,y)}(dy^2+ds^2{}_{BTZ})\\ &=e^{-2G(r,y)}\Bigg[dy^2+\left(\frac{J^2}{4r^2}-M+\frac{r^2}{l^2}\right)^{-1}dr^2\\ &\quad+r^2\left(d\phi-\frac{J}{2r^2}dt\right)^2-\left(\frac{J^2}{4r^2}-M+\frac{r^2}{l^2}\right)dt^2\Bigg]. \end{aligned} \tag{48}$$

As was demonstrated in [10], the Weyl zero Plebański-Demiański (W0PD) metric [see Eq. (5.5) in [10]] with negative cosmological constant can be thought of as a rotating BTZ black string. One reproduces the W0PD metric of [10], if in our Eq. (48) the function G is replaced by $G=-\ln R$, and the coordinate y is transformed according to $y=1/\sqrt{\lambda}\arctan\sqrt{R^2/l_4^2-\lambda}$, where R has been introduced instead of the r coordinate used in [11] to avoid confusion, λ is related to our l through $\lambda=1/l^2$, and $l_4^2=-3/\Lambda_4$, where Λ_4 denotes the corresponding cosmological constant of $(3+1)$ gravity. The results of [10] suggest the possibility of constructing four dimensional black holes bound to two-branes.

Following [10] (see also [11]), a brane can be introduced in the spacetime if there exists a surface whose extrinsic curvature K_{ab} is proportional to the intrinsic metric g_{ab}, $K_{ab}\sim g_{ab}$, where the italic subscripts a, b run values denoting coordinates defined on the surface. Introducing the normal to the surface unit vector n^μ, the extrinsic curvature can be evaluated according to $K_{ab}=\nabla_a n_b$. For the spacetime (48), the slices at constant y with normal vector $n^\mu=-\delta^\mu_y\exp G$ satisfy

$$K_{ab}=-\frac{1}{2}e^G\frac{\partial g_{ab}}{\partial y}=e^G\frac{\partial G}{\partial y}g_{ab}; \tag{49}$$

therefore, one has the possibility of constructing solutions describing black holes on a two-brane. The comment on the construction of branes holds also for our general metric (27); for slices $y=$const with normal vector $n^\mu=-\delta^\mu_y/\sqrt{C_0+C_1X}\exp G$, the extrinsic curvature is proportional to the intrinsic metric of the surface, namely,

$$K_{ab}=\frac{e^G}{\sqrt{C_0+C_1X}}\frac{\partial G}{\partial y}g_{ab}; \tag{50}$$

hence, in principle, by using this metric one might construct two-branes.

As is well known the BTZ solution has outer and inner event horizons respectively at $r=r_+$ and $r=r_-$. Substituting these values in Eq. (47), one obtains $x=1$ for the black hole horizon, while for the inner horizon one gets $x=-1$.

Therefore, one might ask oneself if the existence of horizons is a generic feature of our metric (27). Assuming in the metric (27) that the function $G(X,y)$ is smooth in the ranges of definition of the X and y variables, singularities could emerge in the metric component g^{XX}. In fact, let us consider the normal vector $\mathbf{n}$, $n_\mu=\delta^X_\mu$, to the hypersurface $X=$const, Its norm $n_\mu n^\mu=g^{XX}=\exp(2G)(C_0+C_1X)(X^2-\epsilon)$ could vanish at $X^2-\epsilon=0$ or for $C_0+C_1X=0$. Consequently, the metric becomes singular on these hypersurfaces. For $\epsilon=1$, one might have inner and outer horizons at $X=-1$ and $X=1$, respectively. The singularity at $X=-C_0/C_1$ can be thought of as a coordinate singularity at the origin of the x coordinate $x:=C_0+C_1X$. For $\epsilon=-1$ there are no horizons. Incidentally, one arrives at the above conclusions on the existence of horizons, if any, by establishing the existence of null hypersurfaces $S(x^\mu)=$const, satisfying the equation $g^{\mu\nu}(\partial_\mu S)(\partial_\nu S)=0$.

On the other hand, the Weyl zero Plebański-Carter [A] metric with negative cosmological constant [12] allows for a coordinate transformation to a locally anti–de Sitter representation. However, the Jacobian of this transformation blows up at the same set of points at which the Jacobian of the transformation from the BTZ solution to the locally $(2+1)$ anti–de Sitter space becomes singular, namely, along

$$r_\pm=\sqrt{\frac{l}{2}}\sqrt{Ml\pm\sqrt{M^2l^2-J^2}},$$

i.e., just at the horizons of the metrics. The same behavior under transformation is exhibited by the W0PD metric. Therefore, one can consider that at each surface $y=$const the Penrose diagram for BTZ space occurs in our metric (48).

IV. GENERAL METRIC FOR CONFORMALLY FLAT STATIONARY AXISYMMETRIC SPACETIMES

From the metric (27) when $\epsilon=-1$, one arrives at a new result: there is no way to carry out a diagonalization of the whole metric ds^2; it remains *locally stationary axisymmetric*. This conclusion contradicts the theorem by Collinson, which asserts that "every conformally flat stationary axisymmetric spacetime is necessarily static." Introducing $\{x,y,\phi,t\}$ typing in the metric (27) such that for positive k $\sqrt{k}\tau\to t$, $\sqrt{k}\sigma\to\phi$, while for k negative, $k=-\kappa$, $\sqrt{\kappa}\tau\to\phi$, $\sqrt{\kappa}\sigma\to -t$, one arrives at the canonical form of conformally flat stationary axisymmetric spacetimes:

$$\begin{aligned} ds^2=e^{-2G(x,y)}\Bigg[&\frac{dx^2}{(C_0+C_1x)(x^2+1)}\\ &+(C_0+C_1x)dy^2+(x^2+1)d\phi^2-(dt+x\,d\phi)^2\Bigg]. \end{aligned} \tag{51}$$

Since we started with a general form for the metric of stationary axisymmetric spacetimes, and arrived at the above expression in the case of conformal flatness through the determination of the general solution of the zero Weyl tensor equations, the above metric is the more general form for conformally flat stationary axisymmetric spacetimes. Of course, one can give other representations of this metric by using coordinate transformations of the variable x and y; for instance, representations in terms of trigonometric or hyperbolic functions. The explicit expression for the factor function $G(x,y)$ depends on the matter-field content, i.e., on the energy-momentum tensor for different kinds of fields in the Einstein equations; different energy tensors will give rise to different $G(x,y)$.

V. ALGEBRAIC STRUCTURE OF THE ENERGY-MOMENTUM TENSOR

In this section the algebraic structure of the energy-momentum tensor for our general metric is presented. The corresponding classification yields the best insight about possible sources which could be coupled to the studied metric (27). For definiteness we assume that $k=1$, and that the coordinate X has been replaced by x. For $k=-1$ the treatment will give similar results. We use the orthonormal tetrad formalism, in which the metric is given by $g=\omega^{1^2}+\omega^{2^2}+\omega^{3^2}-\omega^{4^2}$, where the orthonormal tetrad ω^a, $a=1,\ldots,4$, is given by

$$\omega^1=e^{-G(x,y)}\sqrt{C_0+C_1x}dy,$$

$$\omega^2=e^{-G(x,y)}\frac{dx}{\sqrt{C_0+C_1x}\sqrt{x^2-\epsilon}},$$

$$\omega^3=e^{-G(x,y)}\sqrt{x^2-\epsilon}d\sigma,$$

$$\omega^4=e^{-G(x,y)}(d\tau+xd\sigma). \tag{52}$$

The evaluation, with respect to the above ω tetrad, of the energy-momentum tensor T_{ab} by means of the Einstein equations, namely, $G_{ab}=8\pi T_{ab}$, yields the following energy matrix:

$$(T^a{}_b):=\begin{bmatrix} T_{11} & T_{12} & 0 & 0 \\ T_{12} & T_{22} & 0 & 0 \\ 0 & 0 & T_{33} & T_{34} \\ 0 & 0 & -T_{34} & -T_{44} \end{bmatrix}. \tag{53}$$

The eigenvalues ν associated with this matrix, satisfying the secular equation, amount to

$$\nu_{1,2}=\frac{1}{2}(T_{11}+T_{22})\pm\frac{1}{2}\sqrt{(T_{11}-T_{22})^2+4T_{12}^2},$$

$$\nu_{3,4}=\frac{1}{2}(T_{33}-T_{44})\pm\frac{1}{2}\sqrt{(T_{33}+T_{44})^2-4T_{34}^2}. \tag{54}$$

We shall follow the Plebański notation [13], in which for real eigenvalues one denotes the eigenvectors through T, S, and N in correspondence with their timelike, spacelike, or null character, while for the single complex eigenvalue, the corresponding eigenvectors are denoted by Z and $\overline{Z}$. If multiplicity of eigenvalues occurs, a number corresponding to the multiplicity will appear in front of the eigenvector symbols.

For the studied metric, the energy-momentum tensor allows for the following algebraic general types:

$$A_1=[S_1-S_2-S_3-T],$$

$$A_2=[S_1-S_2-Z-\overline{Z}],$$

$$A_3=[S_1-S_2-2N]. \tag{55}$$

Energy-momentum tensors of type A_1 describe matter whose movement velocity is less than the velocity of light; in consequence they could be used to represent the usual matter. Energy-momentum tensors of type A_2 and its degeneration $[2S-Z-\overline{Z}]$ do not represent standard matter since they do not satisfy energy conditions. The class of tensors of type A_3 are associated with radiation processes.

The subtypes of A_1 are given by $[S_1-S_2-2T]$, $[2S_1-S_2-T]$, $[2S-2T]$, $[S-3T]$, $[3S-T]$, $[4T]$. Commonly one deals with fields described by tensors of the classes $[2S-2T]$, $[3S-T]$, and $[4T]$, which physically describe general electromagnetic fields, perfect fluids, and the cosmological constant term, respectively. Of course, within type-A_1 tensors, one might search for gravitational solutions coupled to different tensor combinations, for instance, charged perfect fluids in the presence of the cosmological constant, or charged $[S_1-S_2-S_3-T]$ anisotropic fluids.

It is clear then that our conformally flat stationary axisymmetric metric and its static subclass do not allow solutions for a strict vacuum (the irreducible decomposition of the Riemann tensor involves the Weyl tensor and the Ricci and scalar curvatures, since in this case they are all zero; then the space reduces to the flat Minkowski space). In the case of a pure cosmological constant term the solution corresponds to locally (anti–)de Sitter spaces. For perfect fluids with cosmological constant one obtains three classes of Schwarszchild-like solutions, from which, in the static spherical subclass, one recognizes the Schwarszchild interior solutions with lambda. A detailed analysis of solutions belonging to tensor type A_1 will be given elsewhere.

VI. CONCLUSIONS

In the light of the present results, we conclude that the Collinson theorem is wrong. In addition to the locally static class, there exists a branch of spacetimes which are conformally flat and, at the same time, are stationary and axisymmetric. The conformal factor function $G(x,y)$ of both classes of metrics is a function of the non-Killingian variables x and y; its explicit expression depends on the sources of the Einstein equations. The family of spacetimes with $\epsilon=-1$ does not allow for black hole solutions. On the contrary, the locally static branch permits the existence of black holes. Moreover, since for both classes of metrics one can determine a surface possessing an extrinsic curvature tensor proportional to its intrinsic metric, one comes to a conclusion about the possibility of constructing domain walls.

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[1] A.Z. Petrov, *Einstein Spaces* (Pergamon Press, Oxford, 1969).
[2] H. Takeno, S.R.R.I.T.P. Hiroshima University **5**, 32 (1966).
[3] M. Gurses and Y. Gursey, Nuovo Cimento Soc. Ital. Fis., B **25**, 786 (1975).
[4] R.P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
[5] H.D. Wahlquist, Phys. Rev. **172**, 1291 (1968).
[6] J. Wainwright, Gen. Relativ. Gravit. **8**, 797 (1977).
[7] J.M.M. Senovilla, Phys. Lett. A **123**, 211 (1987).
[8] J.M.M. Senovilla, *In Rotating Objects and Relativistic Physics*, Lecture Notes in Physics Vol. 73, edited by F.J. Chinea *et al.* (Springer-Verlag, Berlin, 1993).
[9] C.D. Collinson, Gen. Relativ. Gravit. **7**, 419 (1976).
[10] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 021 (2000).
[11] R. Emparan, G.T. Horowitz, and R.C. Myers, J. High Energy Phys. **01**, 007 (2000).
[12] A.A. García, in *Exact Solutions and Scalar Fields in Gravity: Recent Developments*, edited by A. Macias *et al.* (Kluwer Academic/Plenum Publishers, New York, 2001).
[13] J.F. Plebański, Acta Phys. Pol. **26**, 963 (1964).