Quasinormal modes of near extremal black branes

Andrei O. Starinets*
Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195
(Received 8 September 2002; published 27 December 2002)

We find quasinormal modes of near extremal black branes by solving a singular boundary value problem for the Heun equation. The corresponding eigenvalues determine the dispersion law for the collective excitations in the dual strongly coupled $\mathcal{N}=4$ supersymmetric Yang-Mills theory at finite temperature.

DOI: 10.1103/PhysRevD.66.124013 PACS number(s): 04.70.Bw, 11.10.Wx, 11.27.+d

I. INTRODUCTION

The gauge-theory–gravity correspondence [1,2,3] provides useful insights into the properties of a strongly coupled supersymmetric Yang-Mills (SYM) theory at nonzero temperature. A well-known example is the finite-temperature $\mathcal{N}=4$ SU(N) SYM theory in 4D which in the large N, large 't Hooft coupling limit is dual to the gravitational background of N parallel near extremal three-branes, with temperature related to the parameter of nonextremality. For this theory, a number of quantities such as the free energy [4,5], the Wilson loop [6,7], the shear viscosity [8,9], the R-charge diffusion constant [9], and the Chern-Simons diffusion rate [10] have been computed using the methods of gauge-theory–gravity duality. Since supersymmetry is broken at finite temperature and thus no nonrenormalization theorem is expected to hold, quantities computed in the strong coupling regime using a gravity dual differ from their analogues obtained at weak coupling via perturbation theory. The results of a gravity calculation are then regarded as a prediction for the SYM theory, assuming that the AdS conformal field theory (CFT) correspondence is valid at finite temperature.[1]

Dynamical properties of a thermal gauge theory are encoded in its Green's functions. In the context of AdS/CFT, Minkowski space Green's functions can be computed from gravity using the recipe given in [10]. Unfortunately, for a nonextremal background, only approximate expressions for the correlators are usually obtained. For example, the retarded propagator of the gauge invariant local operator F^2 (dual to the dilaton) defined by

$$G^R(\omega,\mathbf{k}) = -i\int dt\, d^3x\, e^{-i\omega t+i\mathbf{kx}}\theta(t)\langle[F^2(x),F^2(0)]\rangle \tag{1.1}$$

can be explicitly computed only at zero or very high (with respect to the absolute value of momentum) temperature. At zero temperature, the retarded propagator (1.1) has a branch cut singularity for $|\omega|>|\mathbf{k}|$,

$$G^R(\omega,\mathbf{k}) = \frac{N^2(-\omega^2+\mathbf{k}^2)^2}{64\pi^2}(\ln|-\omega^2+\mathbf{k}^2| - i\pi\theta(\omega^2-\mathbf{k}^2)\operatorname{sgn}\omega). \tag{1.2}$$

In the high temperature limit $\omega/T \ll 1$, $|\mathbf{k}|/T \ll 1$, the propagator is analytic in the complex ω plane,

$$G^R(\omega,\mathbf{k}) = -\frac{N^2T^2}{16}(i2\pi T\omega+\mathbf{k}^2). \tag{1.3}$$

However, for generic values of ω and $\mathbf{k}$, we expect $G^R(\omega,\mathbf{k})$ to have poles corresponding to the spectrum of collective excitations of the SYM plasma.

One can compare the situation to the simpler case of the 2D CFT dual to the Bañados-Teitelboim-Zanelli (BTZ) black hole background. There, the retarded Green's functions can be computed exactly. For illustration, consider the case of the conformal dimension $\Delta=2$. Then

$$G^R_{2\mathrm{D}}(\omega,k) = \frac{\omega^2-k^2}{4\pi^2}\left[\psi\left(1-i\frac{\omega-k}{4\pi T}\right)+\psi\left(1-i\frac{\omega+k}{4\pi T}\right)\right], \tag{1.4}$$

where we have put $T_L=T_R$ and ignored the constant prefactor for simplicity. The high temperature limit of Eq. (1.4) is an analytic function of ω. In general, however, $G^R_{2\mathrm{D}}(\omega,k)$ has infinitely many poles located at

$$\omega_n = \pm k - i4\pi T(n+1), \quad n=0,1,\ldots. \tag{1.5}$$

When $T\to 0$, the poles merge, forming branch cuts. One can use the asymptotic expansion $\psi(z)\sim \log z - 1/2z + \cdots$ (valid for $|\arg z|<\pi/2$) to find the zero-temperature limit of Eq. (1.4) (ignoring the contact terms):

$$G^R_{2\mathrm{D}} \sim \frac{\omega^2-k^2}{4\pi^2}\log|\omega^2-k^2| - i\frac{\omega^2-k^2}{4\pi}\theta(\omega^2-k^2)\operatorname{sgn}\omega. \tag{1.6}$$

For spacelike momenta $|\omega|<k$, the imaginary part of $G^R_{2\mathrm{D}}$ is exponentially suppressed.

Even though the retarded Green's function in 4D cannot be found explicitly, the location of its singularities and thus the "dispersion law" of thermal excitations at strong coupling can be determined precisely. As shown in [10], this amounts to finding the quasinormal frequencies of the dilaton's fluctuation in the dual near extremal black brane back-

*Email address: starina@phys.washington.edu

[1]A nontrivial check of the validity of the correspondence at finite temperature was made recently in [9].

ground as functions of the spatial momentum.[2] Recently, there has been considerable interest (inspired by the AdS/CFT duality conjecture) in studying quasinormal modes of Schwarzschild-AdS black holes[3] [12,13,14,15,16,17,18]. Here, we consider the noncompact case directly. This corresponds to finding quasinormal frequencies for an infinitely large AdS black hole.

Computing the scalar quasinormal frequencies in the black brane background is equivalent to solving the two-parameter singular spectral problem for an equation with four regular singularities (Heun equation). In this paper, we solve this problem by using an elegant method based on Pincherle's theorem. The eigenvalue equation is written in terms of continued fractions and solved numerically. A similar approach was used by Jaffé [19] in 1933 to find the spectrum of the hydrogen molecular ion. Later, it was applied to the problem of quasinormal modes in asymptotically flat spacetimes by Leaver [20].

The paper is organized as follows. In the next section, we formulate the problem of finding the quasinormal modes of a near extremal black three-brane in terms of the singular boundary value problem for the Heun equation. In Sec. III this boundary value problem is solved by analyzing the associated linear difference equation and applying Pincherle's theorem. The results [quasinormal frequencies ω_n and the "dispersion law" $\omega(\mathbf{k})$] are presented in Sec. IV. The discussion follows in Sec. V.

II. QUASINORMAL MODES AND THE BOUNDARY VALUE PROBLEM FOR THE HEUN EQUATION

The metric corresponding to the collection of N parallel nonextremal three-branes in the near horizon limit is given by

$$ds^2=\frac{r^2}{R^2}(-f\,dt^2+d\mathbf{x}^2)+\frac{R^2}{r^2}(f^{-1}dr^2+r^2d\Omega_5^2), \tag{2.1}$$

where $f(r)=1-r_0^4/r^4$. According to the gauge-theory–gravity correspondence, this background with the parameter of nonextremality r_0 is dual to the $\mathcal{N}=4$ SU(N) SYM at finite temperature $T=r_0/\pi R^2$ in the limit $N\to\infty$, $g_{\rm YM}^2N\to\infty$.

Using the new coordinate $z=1-r_0^2/r^2$ and the Fourier decomposition

$$\phi(z,t,\mathbf{x})=\int\frac{d^4k}{(2\pi)^4}e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}\phi_k(z), \tag{2.2}$$

the equation for the minimally coupled massless scalar in the background (2.1) reads

$$\phi_k''+\frac{1+(1-z)^2}{z(1-z)(2-z)}\phi_k'+\frac{\lambda^2}{4z^2(1-z)(2-z)^2}\phi_k-\frac{q^2}{4z(1-z)(2-z)}\phi_k=0, \tag{2.3}$$

where $\lambda=\omega/\pi T$, $q=|\vec{k}|/\pi T$. Equation (2.3) has four regular singularities at $z=0,1,2,\infty$, the corresponding pairs of characteristic exponents being, respectively, $\{-i\lambda/4,i\lambda/4\}$; $\{0,2\}$; $\{-\lambda/4,\lambda/4\}$; $\{0,0\}$.

Quasinormal modes are defined as solutions of Eq. (2.3) obeying the "incoming wave" boundary condition at the horizon $z=0$ and the vanishing Dirichlet boundary condition at spatial infinity $z=1$. The first condition singles out the exponent $\nu_0^{(1)}=-i\lambda/4$ at $z=0$.

The most straightforward way to find quasinormal modes would be to construct a local series solution $\varphi_{\rm loc}(z,\lambda)$ to Eq. (2.3) with the exponent $\nu_0^{(1)}$ near the origin, prove its convergence at $z=1$, and determine the eigenfrequencies $\lambda_n(q)$ by solving the equation $\varphi_{\rm loc}(1,\lambda)=0$ numerically. This approach works quite well for the low-level eigenfrequencies, and in fact it has been successfully used in a number of publications on quasinormal modes in asymptotically AdS space. Here we would like to solve the above eigenvalue problem in a somewhat different way which, in our opinion, is more appealing both analytically and numerically.

By making a transformation of the dependent variable

$$\phi(z)=z^{-i\lambda/4}(z-2)^{-\lambda/4}y(z), \tag{2.4}$$

Eq. (2.3) can be reduced to the standard form of the Heun equation [21,22],

$$y''+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-2}\right]y'+\frac{\alpha\beta z-Q}{z(z-1)(z-2)}y=0, \tag{2.5}$$

where $\alpha,\beta,\gamma,\epsilon$ depend on λ,

$$\alpha=\beta=-\frac{\lambda(1+i)}{4},\quad \gamma=1-\frac{i\lambda}{2},\quad \delta=-1,\quad \epsilon=1-\lambda/2, \tag{2.6}$$

and Q is the so-called "accessory parameter" given in our case by

$$Q=\frac{q^2}{4}-\frac{\lambda(1-i)}{4}-\frac{\lambda^2(2-i)}{8}. \tag{2.7}$$

We would like to determine values of λ and q for which Eq. (2.5) on the interval [0, 1] has solutions obeying the boundary conditions $y(0)=1$, $y(1)=0$.

Before turning to the solution, let us remark that Eq. (2.3) with $\lambda=0$ has received some attention previously in connection with the so called "glueball mass spectrum" in QCD_3 [23,24,25,26]. Approximate analytic expressions for the "glueball" masses squared $M_n^2=-q_n^2$ were found either via a "brute force" WKB calculation [26] or by appealing to the unpublished results of the RIMS group [23] (both ap-

---

[2] Note that the poles (1.5) of $G_{\rm 2D}^R(\omega,k)$ coincide with the quasinormal frequencies of a BTZ black hole [11].

[3] References to the early works on quasinormal modes in asymptotically AdS spacetime as well as to works considering bulk dimension other than five can be found in [10,12].

proaches give satisfactory agreement with the results obtained by numerical integration). Introducing a nonzero λ and using the "incoming wave" boundary condition makes the problem considerably more complicated, as is evident from Eqs. (2.5) and (2.6).

III. SOLVING THE BOUNDARY VALUE PROBLEM

A. Local solutions

The Frobenius set of local solutions near each of the singularities can be easily constructed.

At $z=0$, the local series solution corresponding to the index $\nu=0$ and normalized to 1 is given by

$$y_0(z)=\sum_{n=0}^{\infty} a_n(\lambda,k)z^n, \tag{3.1}$$

where $a_0=1$, $a_1=Q/2\gamma$, and the coefficients a_n with $n\geqslant 2$ obey the three-term recursion relation

$$a_{n+2}+A_n(\lambda)a_{n+1}+B_n(\lambda)a_n=0, \tag{3.2}$$

where

$$A_n(\lambda)=-\frac{(n+1)[2\delta+\epsilon+3(n+\gamma)]+Q}{2(n+2)(n+1+\gamma)}, \tag{3.3}$$

$$B_n(\lambda)=\frac{(n+\alpha)(n+\beta)}{2(n+2)(n+1+\gamma)}. \tag{3.4}$$

The series (3.1) is absolutely convergent for $|z|<1$ and, in general, is divergent for $|z|>1$. The condition for convergence at $|z|=1$ involves parameters of the equation and will be investigated below.

At $z=1$, the difference of the exponents has an integer value, and we expect the local solution there to contain logarithms. Indeed, the set of local solutions is given by

$$y_1(z)=(1-z)^2[1+b_1^{(1)}(1-z)+b_2^{(1)}(1-z)^2+\cdots], \tag{3.5}$$

$$y_2(z)=1+b_1^{(2)}(1-z)+hy_1(z)\log(1-z)+b_2^{(2)}(1-z)^2+\cdots, \tag{3.6}$$

where $b_1^{(1)}=[q^2-\lambda^2+3\lambda(1-i)]/12$, $b_1^{(2)}=[-q^2+\lambda^2+\lambda(1-i)]/4$, $h=-(q^2-\lambda^2)^2/32$, and the coefficients $b_n^{(1,2)}$, $n\geqslant 2$, can be found recursively from the relation similar to the one in Eq. (3.2). Solving the differential equation (2.5) essentially means finding the connection between the sets of local solutions.

B. The connection problem

From the general theory of linear differential equations, it follows that three solutions $y_0(z)$, $y_1(z)$, $y_2(z)$ are connected on [0, 1] by the linear relation

$$y_0(z)=\mathcal{A}(\lambda,q)y_1(z)+\mathcal{B}(\lambda,q)y_2(z), \tag{3.7}$$

where $\mathcal{A},\mathcal{B}$ are independent of z. The vanishing Dirichlet boundary condition for quasinormal modes at $z=1$ implies $\mathcal{B}=0$, which is the equation for eigenfrequencies λ_n. If $\lambda=\lambda_n$, $y_0(z)$ is proportional to $y_1(z)$, and thus it is simultaneously a Frobenius solution about $z=0$ (with exponent 0) and a Frobenius solution about $z=1$ (with exponent 2). Such a solution is called a Heun function [27]. We learn that quasinormal modes of black branes are Heun functions. Unfortunately, the connection problem for the Heun equation remains unsolved, and explicit expressions for the coefficients $\mathcal{A},\mathcal{B}$ are unavailable, with the exception of some special cases.[4] There is, however, an indirect way of determining for which values of λ and q the connection coefficient $\mathcal{B}$ vanishes. This is achieved through an analysis of convergence.

C. The analysis of convergence

The convergence of the series (3.1) can be analyzed by studying the large n asymptotic behavior of the linear difference equation (3.2). One finds[5] that Eq. (3.2) possesses two linearly independent asymptotic solutions of the form

$$a_n^{(1)}\sim 2^{-n}n^{-1-\lambda/2}\sum_{s=0}^{\infty}\frac{c_s^{(1)}}{n^s}, \tag{3.8}$$

$$a_n^{(2)}\sim n^{-3}\sum_{s=0}^{\infty}\frac{c_s^{(2)}}{n^s}, \tag{3.9}$$

where the coefficients $c_s^{(1,2)}$ can be found recursively using the asymptotic expansion of $A(n),B(n)$. In particular, we have $c_1^{(1)}=[q^2-\lambda^2+\lambda(1+i)+2\lambda^2(1+i)]/4$, $c_1^{(2)}=[12-q^2+\lambda^2-3(1+i)\lambda]/4$. Equation (3.8) is called a *minimal* solution to Eq. (3.2), and Eq. (3.9) represents a *dominant* one. This distinction reflects the property

$$\lim_{n\to\infty}\frac{a_n^{(1)}}{a_n^{(2)}}=0, \tag{3.10}$$

and will be useful later on. Using Eqs. (3.8), (3.9), we obtain

$$\left|\frac{a_{n+1}^{(1)}}{a_n^{(1)}}\right|=\frac{1}{2}\left[1-\frac{2+\mathrm{Re}\,\lambda}{2n}+O\left(\frac{1}{n^2}\right)\right], \tag{3.11}$$

$$\left|\frac{a_{n+1}^{(2)}}{a_n^{(2)}}\right|=1-\frac{3}{n}+O\left(\frac{1}{n^2}\right). \tag{3.12}$$

Equations (3.11), (3.12) imply[6] that generically the series (3.1) converges absolutely for $|z|\leqslant 1$ for *any* value of λ. This proves that the equation

---

[4]Schäfke and Schmidt [28] give the connection coefficients in terms of the $n\to\infty$ limit of a_n obeying Eq. (3.2). Although the asymptotic behavior of a_n near $n=\infty$ can be studied by means of local analysis, it only determines a_n up to a λ-dependent coefficient which is essentially the very quantity we are looking for.

[5]Necessary information about second order linear difference equations can be found, for example, in the excellent review paper by Wong and Li [29].

[6]See [30], Sec. 2.37.

TABLE I. The lowest quasinormal frequencies λ_n for $q=0$.

n	Re λ_n	Im λ_n
1	±3.119452	−2.746676
2	±5.169521	−4.763570
3	±7.187931	−6.769565
4	±9.197199	−8.772481
5	±11.202676	−10.774162
6	±13.206247	−12.775239
7	±15.208736	−14.775979
8	±17.210558	−16.776515
9	±19.211943	−18.776919
10	±21.213025	−20.777232
11	±23.213896	−22.777489
12	±25.213896	−24.777489
13	±27.213896	−26.777489
14	±29.213896	−28.777489
15	±31.213896	−30.777489

$$\sum_{n=0}^{\infty} a_n(\lambda,k)=0 \tag{3.13}$$

can indeed be used to compute the quasinormal frequencies numerically. What is more interesting, however, is that in some cases the radius of convergence increases. It happens when the minimal solution (3.8) exists. In that case the series (3.1) converges absolutely for $|z|<2$ and thus represents an analytic function (the Heun function) in that region.

Thus, even though the connection coefficient $\mathcal{B}(\lambda,q)$ in Eq. (3.7) remains unknown, finding zeros of $\mathcal{B}(\lambda,q)$ is equivalent to finding a condition under which the minimal solution to Eq. (3.2) exists. Such a condition is conveniently supplied by Pincherle's theorem [31] which states that the minimal solution exists if and only if the continued fraction

$$-\frac{B_0(\lambda)}{A_0(\lambda)-}\,\frac{B_1(\lambda)}{A_1(\lambda)-}\,\frac{B_2(\lambda)}{A_2(\lambda)-}\cdots \tag{3.14}$$

converges. Moreover, in case of convergence one has

$$\frac{a_{n+1}}{a_n}=-\frac{B_n(\lambda)}{A_n(\lambda)-}\,\frac{B_{n+1}(\lambda)}{A_{n+1}(\lambda)-}\,\frac{B_{n+2}(\lambda)}{A_{n+2}(\lambda)-}\cdots. \tag{3.15}$$

The right hand side of Eq. (3.15) is generated by a simple algorithm. Define $r_n=a_{n+1}/a_n$. Then Eq. (3.2) can be written as

$$r_n=-\frac{B_n(\lambda)}{A_n(\lambda)+r_{n+1}}. \tag{3.16}$$

Applying Eq. (3.16) repeatedly, one gets Eq. (3.15). Now, setting $n=0$ in Eq. (3.15) we have

$$\frac{Q}{2-i\lambda}=-\frac{B_0(\lambda)}{A_0(\lambda)-}\,\frac{B_1(\lambda)}{A_1(\lambda)-}\,\frac{B_2(\lambda)}{A_2(\lambda)-}\cdots. \tag{3.17}$$

This is the transcendental eigenvalue equation which determines the quasinormal frequencies. Equation (3.17) can be solved numerically with great efficiency using a variety of methods (see [32], which also includes a discussion of the error analysis). Here we use a nonlinear backward recursion which amounts to breaking the continued fraction (3.16) by setting r_n to zero[7] for some large n_* and computing backward to get r_1. Stability can be checked by choosing a larger value of n_* and repeating the calculation.

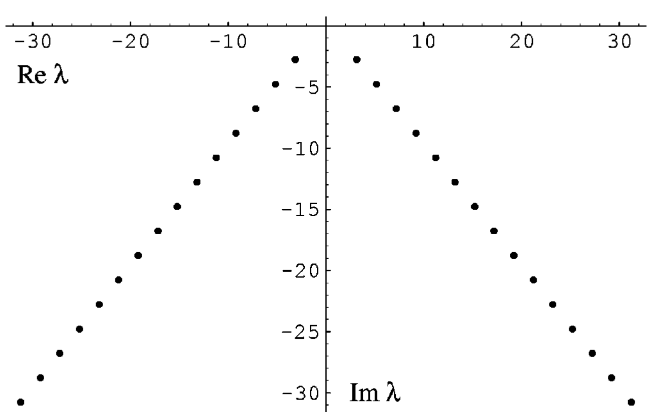

FIG. 1. The lowest 15 quasinormal frequencies in the complex λ plane for $q=0$.

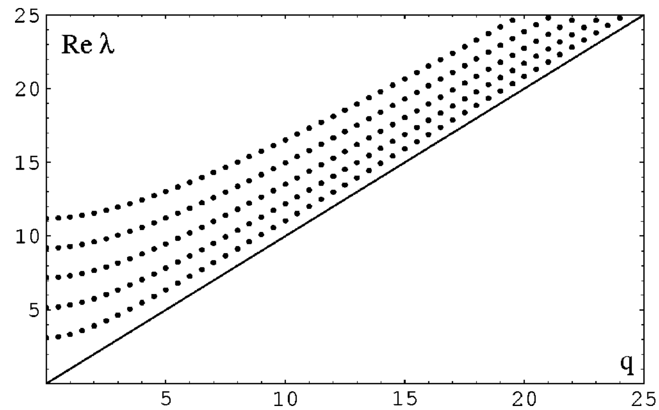

FIG. 2. Re λ vs q (with the interval $\Delta q=0.5$) for the lowest five quasinormal frequencies. The zeros approach the line Re $\lambda=q$ as $q\to\infty$.

IV. QUASINORMAL FREQUENCIES

For $q=0$, the lowest 15 quasinormal frequencies λ_n obtained by solving Eq. (3.17) numerically are listed in Table I and shown in Fig. 1. The results suggest that the number of frequencies is infinite, and that their large n asymptotic behavior is given by the simple formula

[7] To improve the convergence of continued fractions, one can instead set r_n to its asymptotic value at n_*, $r_n=1/2-(2+\lambda)/4n_*+\cdots$, as suggested by Nollert [33].

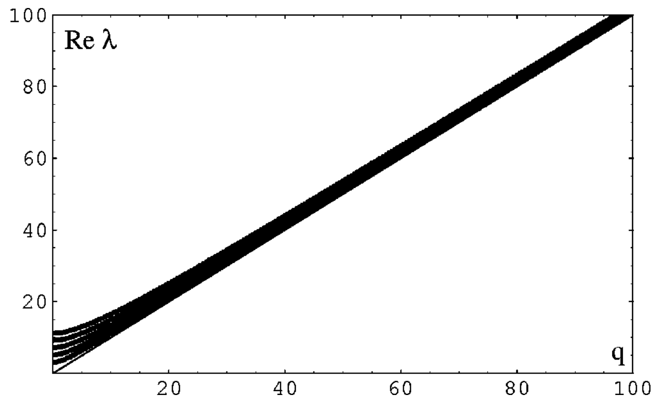

FIG. 3. Re λ vs q (with the interval $\Delta q=1$) for the lowest five quasinormal frequencies.

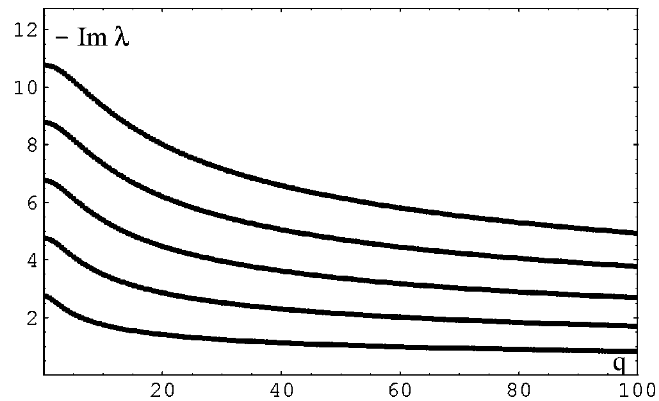

FIG. 4. $-\mathrm{Im}\,\lambda$ vs q (with the interval $\Delta q=1$) for the lowest five quasinormal frequencies.

$$\lambda_n^{\pm}=\lambda_0^{\pm}\pm 2n(1\mp i), \tag{4.1}$$

where $\lambda_0^{\pm}\approx\pm 1.2139-0.7775i$. In terms of the original variable ω, Eq. (4.1) is written as

$$\omega_n^{\pm}=\omega_0^{\pm}\pm 2\pi Tn(1\mp i), \tag{4.2}$$

where the coefficient in front of the parentheses is the bosonic Matsubara frequency and $\omega_0^{\pm}=\pi T\lambda_0^{\pm}$.

The lowest eigenfrequency in Table I can be compared with the result of Horowitz and Hubeny [12] for a large 5D Schwarzschild-AdS black hole. Normalizing the $r_+=100$ entry in Table I of [12] appropriately ($\lambda=\omega R^2/r_+ , R=1$), we have $\lambda^{(100)}\approx 3.119627-2.746655i$, which is fairly close to our result $\lambda^{(\infty)}\approx 3.119452-2.746667i$.

The dependence of the lowest five quasinormal frequencies on q for $q\in[0,100]$ is shown in Figs. 2–4. All five branches stay above the line $\mathrm{Re}\,\lambda=q(\mathrm{Re}\omega=|\mathbf{k}|)$, slowly approaching it in the low-temperature limit $q\to\infty$. The imaginary part of the branches tends to zero in the same limit. Figures 2–4 bear resemblance to the dispersion law of thermal excitations in a weakly coupled Yang-Mills plasma.[8]

All quasinormal frequencies have negative imaginary parts. This fact can be proven rigorously using an argument based on the "energy-type" integral [34] or by writing Eq. (2.3) in the Regge-Wheeler form and essentially repeating the proofs for AdS black holes given recently in [12,14]. It reflects the stability of the near extremal metric against a scalar perturbation. On the field theory side, the negative sign corresponds to the damping of the plasma excitations.

The computed frequencies appear to be symmetric with respect to the imaginary λ axis, i.e., the eigenvalue equation (3.17) seems to give complex-conjugate pairs of solutions in terms of the variable $i\lambda$. This symmetry does not seem to be explicit in Eq. (3.17).[9]

[8]Note, however, that in this paper we consider correlators of the gauge-invariant operators rather than the gluon propagator.

[9]The symmetry, however, is obvious in the original form of Eq. (2.3). I thank A. V. Shchepetilov for pointing this out to me.

V. DISCUSSION OF RESULTS

The results are compatible with expectations outlined in the Introduction. The retarded Green's function has infinitely many poles in the complex ω plane whose location depends on the spatial momentum. There is a "mass gap:" for small enough values of λ the propagator $G^R(\omega,\mathbf{k})$ is analytic in agreement with Eq. (1.3). In the low-temperature limit $\lambda\to\infty$, $q\to\infty$ singularities merge, forming branch cuts of the zero-temperature propagator[10] (1.2) as in the BTZ case.

The distribution of quasinormal frequencies in the complex ω plane for an asymptotically AdS background appears to follow a much simpler pattern than the one corresponding to the asymptotically flat case (compare Fig. 1 in [20]). In particular, Chandrasekhar's "algebraically special" solution is absent: there are no frequencies with $\mathrm{Re}\,\lambda=0$. Obviously, all statements about the behavior of higher-order modes are conjectural. It would be very desirable to confirm the asymptotic formula (4.1) analytically, possibly by using the complex WKB method. We remark, however, that the analogous problem remains unsolved even in the much studied case of a Schwarzschild black hole in flat space.

Spectra of excitations in more realistic theories can be similarly studied provided their gravitational duals are known explicitly. It would also be interesting to determine the poles of the current and energy-momentum tensor correlators at finite temperature, whose hydrodynamic limit has recently been computed in [9].

Finally, studying gravitational quasinormal modes may be important in investigating the stability of the nonextremal black brane background.

ACKNOWLEDGMENTS

It is a pleasure to thank P. K. Kovtun, D. T. Son, and L. G. Yaffe for discussions. I would also like to thank A. Nuñes-Nikitin and A. V. Shchepetilov for valuable comments on the manuscript. This work is supported in part by DOE Grant No. DOE-ER-41132.

[10]The zero-temperature limit (1.2) of $G^R(\omega,\mathbf{k})$ can be obtained from the Heun equation using the Langer-Olver asymptotic expansion (see [10]).

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)].
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] S. S. Gubser, I. R. Klebanov, and A. W. Peet, Phys. Rev. D **54**, 3915 (1996).
[5] S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, Nucl. Phys. **B534**, 202 (1998).
[6] S. J. Rey, S. Theisen, and J. T. Yee, Nucl. Phys. **B527**, 171 (1998).
[7] A. Brandhuber, N. Itzhaki, J. Sonnenschein, and S. Yankielowicz, Phys. Lett. B **434**, 36 (1998).
[8] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001).
[9] G. Policastro, D. T. Son, and A. O. Starinets, J. High Energy Phys. **09**, 043 (2002).
[10] D. T. Son and A. O. Starinets, J. High Energy Phys. **09**, 042 (2002).
[11] D. Birmingham, I. Sachs, and S. N. Solodukhin, Phys. Rev. Lett. **88**, 151301 (2002).
[12] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D **62**, 024027 (2000).
[13] B. Wang, C. Y. Lin, and E. Abdalla, Phys. Lett. B **481**, 79 (2000).
[14] V. Cardoso and J. P. Lemos, Phys. Rev. D **64**, 084017 (2001).
[15] W. T. Kim and J. J. Oh, Phys. Lett. B **514**, 155 (2001).
[16] V. Cardoso and J. P. Lemos, Class. Quantum Grav. **18**, 5257 (2001).
[17] I. G. Moss and J. P. Norman, Class. Quantum Grav. **19**, 2323 (2002).
[18] R. A. Konoplya, Phys. Rev. D **66**, 044009 (2002).
[19] G. Jaffé, Z. Phys. **87**, 535 (1933).
[20] E. W. Leaver, Proc. R. Soc. London **A402**, 285 (1985).
[21] K. Heun, Math. Ann. **33**, 61 (1989).
[22] *Heun's Differential Equation*, edited by A. Ronveaux and F. Arscott (Oxford University Press, Oxford, 1995).
[23] C. Csaki, H. Ooguri, Y. Oz, and J. Terning, J. High Energy Phys. **01**, 017 (1999).
[24] R. de Mello Koch, A. Jevicki, M. Mihailescu, and J. P. Nunes, Phys. Rev. D **58**, 105009 (1998).
[25] M. Zyskin, Phys. Lett. B **439**, 373 (1998).
[26] J. A. Minahan, J. High Energy Phys. **01**, 020 (1999).
[27] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill, New York, 1955), Vol. III.
[28] R. Schäfke and D. Schmidt, SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **11**, 848 (1980).
[29] R. Wong and H. Li, J. Comput. Appl. Math. **41**, 65 (1992).
[30] E. T. Whittaker and G. N. Watson, A *Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1927).
[31] S. Pincherle, Mat. Battaglini **32**, 209 (1894).
[32] *Theory and Application of Special Functions*, edited by R. Askey (Academic, New York, 1975).
[33] H.-P. Nollert, Phys. Rev. D **47**, 5253 (1993).
[34] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).