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Renormalization invariants of the neutrino mass matrix

Sanghyeon Chang* and T. K. Kuo†
Physics Department, Purdue University, West Lafayette, Indiana 47907

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The renormalization evolution of all parameters in the neutrino mass matrix depends only on one variable, the energy scale. This fact, coupled with rephasing considerations, leads to a set of renormalization invariants, correlating the evolution of physical parameters. We obtain these invariants explicitly and discuss their implications.

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Recent results from the atmospheric and solar experiments have shown strong evidence for neutrino oscillations [1,2]. These observations indicate that neutrinos are not massless, and that two of the neutrino mixing angles are large, or even maximal. This is in contrast with the quark mixing angles, which are all small. Even with our limited knowledge, it seems clear that the neutrino mass matrix, just like their quark counterpart, has a rich structure, and it is urgent to have an understanding of its salient features. To account for the minuscule neutrino masses, the seesaw model [3] makes use of a heavy scale for the right-handed neutrinos. Thus, any theoretical understanding of the observed neutrino parameters necessarily involves two vastly different energy scales. This means that renormalization effects must be taken into account in any theoretical model of the neutrino mass matrix.

The renormalization of the neutrino mass matrix has been extensively discussed in the literature [4–6]. Although different models, such as the standard model (SM) or minimal supersymmetric standard model (MSSM), give rise to numerically distinct results for individual parameters, as we have shown for the two-flavor problem [6], there are renormalization group equation (RGE) invariants, correlating the evolution of the physical parameters. These invariants are the consequences of the general structure of the RGE, and remain the same for a class of models, including SM and MSSM.

In general, RGE evolution implies that each physical parameter becomes a function of the energy scale. Thus, if there are n independent parameters in the mass matrix, we might expect to have $(n-1)$ RGE invariants. However, the mass matrices are also subject to arbitrary rephasing transformations. In addition, such phases are generated by the renormalization transformation. As a result, the physical RGE invariants must also be rephasing invariants. For the three-flavor mass matrix, it turns out that there are three (complex) RGE and rephasing invariants, among its eight physical parameters. These invariants will be detailed later in this paper. Just as for the two-flavor problem, these invariants are the same for a class of models. As in earlier studies on renormalization, only one-loop effects are considered in this work.

In the study of a neutrino mass matrix, the importance of renormalization considerations is enhanced since the mixing angle exhibits sensitive resonance behavior. It would be most interesting if the observed large neutrino mixing has its origin in renormalization [4–6]. In this scenario, the RGE invariants can be used directly to tell us about neutrino masses through their correlation with the mixing angles.

Before we embark on a detailed discussion of renormalization, it is useful to introduce a general parametrization of the mass matrix which will facilitate the analysis. In this paper, we will consider the (symmetric) neutrino mass matrix to originate from a dimension-5 term in the effective Lagrangian, $L=f\nu^T\nu\langle\phi\rangle\langle\phi\rangle=\nu^T M_\nu^0\nu$. Here, $\langle\phi\rangle$ is the vacuum expectation value of the Higgs scalar, f is the coupling constant, M_ν^0 is the neutrino mass matrix, and ν is the neutrino wave function in the flavor basis.

We can write, in general,

$$M_\nu^0=UM^{\mathrm{diag}}U^T, \tag{1}$$

$$M^{\mathrm{diag}}=\begin{pmatrix} e^{2\eta_1} & & \\ & e^{2\eta_2} & \\ & & e^{2\eta_3}\end{pmatrix}, \tag{2}$$

$$U=Pe^{-i\epsilon_7\lambda_7}e^{-i\epsilon_5\lambda_5}e^{-i\epsilon_3\lambda_3}e^{-i\epsilon_2\lambda_2}P', \tag{3}$$

$$P=\begin{pmatrix} e^{i\alpha_1} & & \\ & e^{i\alpha_2} & \\ & & e^{i\alpha_3}\end{pmatrix},\quad P'=\begin{pmatrix} e^{i\gamma_1} & & \\ & e^{i\gamma_2} & \\ & & e^{i\gamma_3}\end{pmatrix}. \tag{4}$$

Here, the mass eigenvalues are given by $\exp(2\eta_i)$, α_i are the unphysical phases from the neutrino wave functions, $(\epsilon_2,\epsilon_5,\epsilon_7)$ are the physical neutrino mixing angles, ϵ_3 is a CP violating phase, the γ's are the intrinsic CP phases of the mass eigenvalues, and the λ's are the Gell-Mann matrices. To preserve the symmetry of the flavors, we will not use the diagonal λ's in M^{diag}. Note also that, since neutrino oscillations are governed by the effective Hamiltonian, $H=(M_\nu M_\nu^\dagger)/2E$, they are independent of the phases γ_i.

It is convenient to factor out the determinant (the overall scale) of M_ν^0 and define

$$M_\nu=(\det M_\nu^0)^{-1/3}M_\nu^0,\quad \det M_\nu=1. \tag{5}$$

The condition $\det M_\nu=1$ is obtained by imposing the follow-

*Email address: schang@physics.purdue.edu
†Email address: tkkuo@physics.purdue.edu

ing relations on M_ν^0: $\Sigma\eta_i=\Sigma\alpha_i=\Sigma\gamma_i=0$. This corresponds to the fact that, with $\det M_\nu=1$, M_ν only depends on the mass ratios $(\Delta\eta)$ and the relative phases ($\Delta\alpha$ and $\Delta\gamma$). Note that, with the parametrization in Eq. (2), M_ν can be analytically continued into an $SU(3)$ matrix ($\eta_j\to i\eta_j$), a fact which will be used later. Also, rephasing of the neutrino wave functions changes only α_j, while leaving all other parameters invariant.

It is also useful to write down the symmetric matrix M_ν explicitly:

$$M_\nu=\begin{pmatrix} c_{(5)}^2\overline{\chi_1}+s_{(5)}^2e^{2\overline{\eta_3}} & \begin{matrix} c_{(7)}c_{(5)}s_{2(2)}\Delta_{12}\\ -s_{(7)}s_{2(5)}\Delta_{13}\end{matrix} & \begin{matrix} s_{(7)}c_{(5)}s_{2(2)}\Delta_{12}\\ +c_{(7)}s_{2(5)}\Delta_{13}\end{matrix}\\ \begin{matrix} c_{(7)}c_{(5)}s_{2(2)}\Delta_{12}\\ -s_{(7)}s_{2(5)}\Delta_{13}\end{matrix} & \begin{matrix} c_{(7)}^2\overline{\chi_2}-s_{2(7)}s_{(5)}s_{2(2)}\Delta_{12}\\ +s_{(7)}^2(s_{(5)}^2\overline{\chi_1}+c_{(5)}^2e^{2\overline{\eta_3}})\end{matrix} & \begin{matrix} s_{2(7)}(\overline{\chi_2}-s_{(5)}^2\overline{\chi_1}+c_{(5)}^2e^{2\overline{\eta_3}})/2\\ +c_{2(7)}s_{(5)}s_{2(2)}\Delta_{12}\end{matrix}\\ \begin{matrix} s_{(7)}c_{(5)}s_{2(2)}\Delta_{12}\\ +c_{(7)}s_{2(5)}\Delta_{13}\end{matrix} & \begin{matrix} s_{2(7)}(\overline{\chi_2}-s_{(5)}^2\overline{\chi_1}+c_{(5)}^2e^{2\overline{\eta_3}})/2\\ +c_{2(7)}s_{(5)}s_{2(2)}\Delta_{12}\end{matrix} & \begin{matrix} s_{(7)}^2\overline{\chi_2}+s_{2(7)}s_{(5)}s_{2(2)}\Delta_{12}\\ +c_{(7)}^2(s_{(5)}^2\overline{\chi_1}+c_{(5)}^2e^{2\overline{\eta_3}})\end{matrix}\end{pmatrix}, \tag{6}$$

$$\overline{\eta_i}=\eta_i+i\gamma_i, \tag{7}$$

$$\chi_1=c_{(2)}^2e^{2\overline{\eta_1}}+s_{(2)}^2e^{2\overline{\eta_2}},\quad \overline{\chi_1}=e^{-2i\epsilon_3}\chi_1, \tag{8}$$

$$\chi_2=s_{(2)}^2e^{2\overline{\eta_1}}+c_{(2)}^2e^{2\overline{\eta_2}},\quad \overline{\chi_2}=e^{2i\epsilon_3}\chi_2, \tag{9}$$

$$\Delta_{12}=\tfrac{1}{2}(e^{2\overline{\eta_1}}-e^{2\overline{\eta_2}}), \tag{10}$$

$$\Delta_{13}=\tfrac{1}{2}(\overline{\chi_1}-e^{2\overline{\eta_3}}). \tag{11}$$

Here, we use the notation $s_{(2)}=\sin\epsilon_2, s_{2(2)}=\sin 2\epsilon_2$, etc. We have also set $\alpha_i=0$, without loss of generality.

The RGE for the effective neutrino mass matrix, M_ν, has been studied extensively. In the SM and MSSM, the RGE were obtained explicitly and can be written in the form [4–6]: $dM_\nu^0/dt=\kappa M_\nu^0+\{Q,M_\nu^0\}$, where κ is a constant, Q is a diagonal and traceless matrix $\mathrm{tr}\,Q=0$, t is the scale variable $t=(1/16\pi^2)\ln(\mu/\mu_0)$, with $(\mu,\mu_0)=$energy scale. The solution is given by

$$\det M_\nu^0(t)=e^{3\kappa t}\det M_\nu^0(0), \tag{12}$$

$$M_\nu(t)=e^{Qt}M_\nu(0)e^{Qt}. \tag{13}$$

The quantities κ and Q were given explicitly in terms of the leptonic Yukawa constants.

The effect of the operator, e^{Qt}, amounts to a change of relative scale (rescaling) between the different flavors. The close relation between rescaling (renormalization) and rephasing is revealed by considering pure imaginary values for Q, which turns Eq. (13) into a rephasing transformation. If, in addition, we consider $\eta_j\to i\eta_j$ in M_ν, then the equation becomes a rephasing transformation in $SU(3)$.

Equation (13) is a formal solution of RGE, since it only gives the t-dependence of the matrix elements of M_ν. One would really like to know the t-dependence of the physical parameters. To this end, we must reexpress M_ν in Eq. (13) in the form of Eq. (1),

$$M_\nu(t)=U(t)M^{\mathrm{diag}}(t)U^T(t), \tag{14}$$

and one needs to relate the physical parameters at scale t to those at $t=0$.

Mathematically, the RGE solution corresponds to the relation connecting the parameters, known as the Baker-Campbell-Hausdorff (BCH) formula, between different rearrangements [e.g. Eqs. (13) and (14)] of the noncommuting factors in an element of $SL(3,C)$. Since we are dealing with only exponential functions which are free of singularities, these relations should remain valid under analytic continuations. In particular, the BCH formulas for $SL(3,C)$ are the analytic continuation of those for $SU(3)$. These ideas can be implemented explicitly for the case of two flavors, which we will study first before we take on the full analysis of the three-flavor problem.

Consider a general symmetric $SU(2)$ matrix (with two real parameters), which can be written as

$$\tilde{N}_1=e^{-i\beta\sigma_2}e^{2i\tilde{\gamma}\sigma_3}e^{i\beta\sigma_2}, \tag{15}$$

or

$$\tilde{N}_2=e^{i\tilde{\omega}\sigma_3}e^{2i\tilde{\tau}\sigma_1}e^{i\tilde{\omega}\sigma_3}. \tag{16}$$

The relations (BCH formula) between the two parametrizations $\tilde{N}_1$ and $\tilde{N}_2$ can be read off from the matrix elements as given in Eqs. (15),(16),

$$\cos 2\tilde{\gamma}=\cos 2\tilde{\tau}\cos 2\tilde{\omega}, \tag{17}$$

$$\sin 2\tilde{\gamma}\cos 2\beta=\cos 2\tilde{\tau}\sin 2\tilde{\omega}, \tag{18}$$

$$\sin 2\tilde{\gamma}\sin 2\beta=\sin 2\tilde{\tau}. \tag{19}$$

When $\tilde{N}$ is subject to a rephasing transformation,

$$\tilde{N}\to e^{-i\tilde{\alpha}\sigma_3}\tilde{N}e^{-i\tilde{\alpha}\sigma_3}, \tag{20}$$

it is obvious that parametrization Eq. (16) makes it trivial, resulting in $\tilde{\omega}\rightarrow\tilde{\omega}-\tilde{\alpha}$. But for $\tilde{N}_1$ in Eq. (15), it induces the change $\beta\rightarrow\beta', \tilde{\gamma}\rightarrow\tilde{\gamma}'$, satisfying

$$\cos 2\tilde{\gamma}' = \cos 2\tilde{\tau}\cos 2(\tilde{\omega}-\tilde{\alpha}), \tag{21}$$

$$\sin 2\tilde{\gamma}' \cos 2\beta' = \cos 2\tilde{\tau}\sin 2(\tilde{\omega}-\tilde{\alpha}), \tag{22}$$

$$\sin 2\tilde{\gamma}' \sin 2\beta' = \sin 2\tilde{\tau}. \tag{23}$$

It follows that

$$\tan 2\beta' = \frac{\sin 2\beta/\cos 2\tilde{\alpha}}{\cos 2\beta - \tan 2\tilde{\alpha}/\tan 2\tilde{\gamma}}. \tag{24}$$

In addition, $\tilde{\tau}$ is invariant under rephasing. Thus, we have the rephasing invariant, in terms of the parametrization in Eq. (15),

$$\sin 2\tilde{\gamma}\sin 2\beta = \sin 2\tilde{\gamma}' \sin 2\beta'. \tag{25}$$

Equations (24) and (25) are the solutions to the $SU(2)$ rephasing transformation, Eq. (20).

The same results can be taken over for symmetric mass matrices, where all variables $(\beta,\tilde{\gamma},\tilde{\omega},\tilde{\tau})$ are complex, with four real parameters. Let us first consider the case of real mass matrices, corresponding to pure imaginary $(\tilde{\gamma},\tilde{\omega},\tilde{\tau})$:

$$(\tilde{\gamma},\tilde{\omega},\tilde{\tau})\rightarrow(i\gamma,i\omega,i\tau). \tag{26}$$

The resulting mass matrix can be written in two alternative forms, $N_1 = e^{-i\beta\sigma_2}e^{-2\gamma\sigma_3}e^{i\beta\sigma_2}$, $N_2 = e^{-\omega\sigma_3}e^{-2\tau\sigma_1}e^{-\omega\sigma_3}$. With $\tilde{\alpha}\rightarrow i\alpha$, the rephasing transformation on $\tilde{N}$ becomes a renormalization (rescaling) transformation on N:

$$N\rightarrow e^{\alpha\sigma_3}Ne^{\alpha\sigma_3}. \tag{27}$$

The solution to the RGE is obtained directly from the BCH formula of $SU(2)$, Eq. (24), by analytic continuation:

$$\tan 2\beta' = \frac{\sin 2\beta/\cosh 2\alpha}{\cos 2\beta - \tanh 2\alpha/\tanh 2\gamma}. \tag{28}$$

This is the same relation obtained earlier for RGE evolution. At the same time, we have an RGE invariant [6]:

$$\sin 2\beta \sinh 2\gamma = \sin 2\beta' \sinh 2\gamma'. \tag{29}$$

Note that this is also automatically invariant under (relative) rephasing on the mass matrix N. In addition, while the value α in Eq. (27) is model dependent, Eq. (29) is not.

In general, all parameters $(\beta,\tilde{\gamma},\tilde{\omega},\tilde{\tau})$ are complex, so that $\tilde{N}_1$ and $\tilde{N}_2$ become complex mass matrices. As was shown in Ref. [7], we can demand that β and β' be real by adding rephasing factors to $\tilde{N}_1$. The resulting generalization of Eq. (24) coincides with Eq. (15) of Ref. [6]. In this connection, we emphasize that the renormalization transformation, Eq. (27), generates rephasing factors when N is complex, as was shown explicitly in Ref. [6].

Schematically, solutions to the rescaling transformations on mass matrices are the BCH formulas in $SL(2,C)$, which can be obtained from the rephasing transformations in $SU(2)$, by analytic continuation. We can represent this in a commutative diagram:

$$\begin{tikzcd} \tilde{N}_1 \arrow[r, leftrightarrow, "\text{BCH } SU(2)"] \arrow[d, leftrightarrow, "\text{Analytic continuation}"'] & \tilde{N}_2 \arrow[d, leftrightarrow, "\text{Analytic continuation}"] \\ N_1 \arrow[r, leftrightarrow, "\text{BCH } SL(2,C)"] & N_2 \end{tikzcd} \tag{30}$$

We now turn to the case of three flavors. A general symmetric $SU(3)$ matrix (with five parameters) can be parametrized in either of two ways

$$W_1 = VW^{\text{diag}}\ V^T, \tag{31}$$

$$W^{\text{diag}} = \begin{pmatrix} e^{i2\eta_1} & & \\ & e^{i2\eta_2} & \\ & & e^{i2\eta_3} \end{pmatrix}, \quad \sum \eta_i = 0 \tag{32}$$

$$V = e^{-i\epsilon_7\lambda_7}e^{-i\epsilon_5\lambda_5}e^{-i\epsilon_2\lambda_2}, \tag{33}$$

or

$$W_2 = Pe^{i(\xi_1\lambda_1+\xi_4\lambda_4+\xi_6\lambda_6)}P, \tag{34}$$

$$P = \begin{pmatrix} e^{i\delta_1} & & \\ & e^{i\delta_2} & \\ & & e^{i\delta_3} \end{pmatrix}, \quad \sum \delta_i = 0. \tag{35}$$

The order of the noncommuting matrix products is chosen so that W_1 corresponds to the usual mass matrix parametrization, while W_2 is most convenient for rephasing considerations.

The BCH formulas yield analytic relations between the two sets of parameters (η_i,ϵ_j) and (ξ_i,δ_j). A rephasing transformation would change $\delta_j\rightarrow\delta_j'$, and the corresponding transformation on (η_i,ϵ_j) can be calculated as in Eqs. (21)–(24). Also, the functions ξ_i in terms of (η_i,ϵ_j) are rephasing invariants. The functions for (η_i,ϵ_j) obtained would have provided explicit solutions to the RGE, as in Eq. (28). Unfortunately, owing to the complexity of the $SU(3)$ algebra, so far we are unable to solve for these functions explicitly.

When we let all parameters assume complex values, the matrix W turns into a symmetric mass matrix. A rescaling (renormalization) transformation corresponds to $\delta_j\rightarrow\delta_j + i\Delta_j$. Thus, the rescaling (RGE) invariants which are also rephasing invariants are precisely (ξ_1,ξ_4,ξ_6), as complex functions of (η_i,ϵ_j). Although we cannot obtain these functions explicitly, we can obtain the RGE invariants using the matrix elements as the variables. As before, we can arrive at the results by first studying the rephasing invariants in $SU(3)$.

Consider rephasing transformations on a general $SU(3)$ matrix (with eight parameters), which can be written in the form $V = Pe^{-i\epsilon_7\lambda_7}e^{-i\epsilon_5\lambda_5}e^{-i\epsilon_3\lambda_3}e^{-i\epsilon_2\lambda_2}P'$, where P is given in Eq. (35) and P' is obtained from P by the substitution $\delta_j \to \delta_j'$. This is precisely the parametrization for the CKM matrix and $(\epsilon_2, \epsilon_5, \epsilon_3, \epsilon_7)$ are rephasing invariants. In terms of the matrix elements of V, V_{IJ}, rephasing transformation gives $V_{IJ} \to e^{i(\delta_I + \delta_J')} V_{IJ}$. Thus, the rephasing invariants are $|V_{IJ}|^2$. When we let the parameters be complex, V analytically continues into M, the mass matrix. The rephasing and rescaling invariants are now given by $M_{IJ}^{-1} M_{IJ}$.

Besides $|V_{IJ}|^2$, another familiar form of the rephasing invariant [8] is given by $\mathcal{J}_{IJKL} = V_{IJ} V_{KL} V_{IL}^* V_{KJ}^*$. When we impose the condition $\det V = 1$, only relative rephasings, but no overall phases, are admitted. The rephasing invariant takes a simpler form $I_s = e_{IJK} e_{I'J'K'} V_{II'} V_{JJ'} V_{KK'}$. There are six different ways to arrange the indices so that we may label I by s, which is an element of the permutation group S_3,

$$s = \begin{pmatrix} I & J & K \\ I' & J' & K' \end{pmatrix}, \quad \text{denoting } (I \to I', J \to J', K \to K').$$

Note that

$$\sum_s I_s = \det V = 1. \tag{36}$$

Also, I_s has a simple relation to the familiar rephasing invariant $\mathcal{J}_{IJKL}$. For a unitary V with $\det = 1$, its minors are just the complex conjugated elements. For instance, $V_{11} = V_{22}^* V_{33}^* - V_{23}^* V_{32}^*$ or $V_{11} V_{22} V_{33} = |V_{22}|^2 |V_{33}|^2 - V_{22} V_{33} V_{23}^* V_{32}^*$. When we let the indices take on different values, it is easy to show that all of the products I_s are similarly related to $\mathcal{J}_{IJKL}$, with $\operatorname{Im} I_S = -\operatorname{Im}(\mathcal{J}_{IJKL})$, independent of the indices.

The analytic continuation of the rephasing invariants I_s turns them into rescaling and rephasing invariants for the mass matrices:

$$\mathcal{J}_s = e_{IJK} e_{LMN} M_{IL} M_{JM} M_{KN}. \tag{37}$$

As we discussed in the previous section, the physical RGE invariants are the rephasing and rescaling invariants of the mass matrix. There are two equivalent forms of these invariants. (i) $M_{IJ}^{-1} M_{IJ}$; (ii) $e_{IJK} e_{LMN} M_{IL} M_{JM} M_{KN}$. When we convert these into physical variables, it turns out that the former is more convenient, which will be presented in the following.

Let us define $I_{IJ} = M_{IJ}^{-1} M_{IJ}$. These invariants are not independent, since $M_{IJ} = M_{JI}$, and

$$\sum_I I_{IJ} = \sum_J I_{IJ} = 1. \tag{38}$$

So there are altogether three independent (complex) invariants, which we can take to be

$$I_1 = I_{11} - 1, \quad I_2 = I_{12} - I_{13} \text{ and } I_3 = I_{23}. \tag{39}$$

Explicitly, we find

$$\begin{aligned} I_1 = M_{11} M_{11}^{-1} - 1 = {} & c_{(5)}^4 s_{2(2)}^2 \sinh^2(\overline{\eta}_1 - \overline{\eta}_2) \\ & + s_{2(5)}^2 c_{(2)}^2 \sinh^2(\overline{\eta}_1 - \overline{\overline{\eta}}_3) + s_{2(5)}^2 s_{(2)}^2 \sinh^2(\overline{\eta}_2 - \overline{\overline{\eta}}_3), \end{aligned}$$

$$\begin{aligned} I_2 = M_{12}^{-1} M_{12} - M_{13}^{-1} M_{13} = {} & s_{2(2)} s_{2(5)} c_{(5)} s_{2(7)} (\Delta_{13} \Delta_{12}^{-} \\ & + \Delta_{13}^{-} \Delta_{12}) + c_{2(7)} (s_{2(2)}^2 c_{(5)}^2 \Delta_{12} \Delta_{12}^{-} - s_{2(5)}^2 \Delta_{13} \Delta_{13}^{-}), \end{aligned} \tag{40}$$

$$\begin{aligned} I_3 = M_{23} M_{23}^{-1} = {} & \Big[\frac{s_{2(7)}}{2} (\overline{\chi}_2 - s_{(5)}^2 \overline{\chi}_1 - c_{(5)}^2 e^{2\overline{\eta}_3}) \\ & + s_{2(2)} s_{(5)} c_{2(7)} \Delta_{12} \Big] \Big[\frac{s_{2(7)}}{2} (\overline{\chi}_2^{-} - s_{(5)}^2 \overline{\chi}_1^{-} - c_{(5)}^2 e^{-2\overline{\eta}_3}) \\ & + s_{2(2)} s_{(5)} c_{2(7)} \Delta_{12}^{-} \Big]. \end{aligned}$$

Here we have used the notations in Eqs. (6)–(11). In addition, $\overline{\overline{\eta}}_3 = e^{2i\epsilon_3} \overline{\eta}_3$, $\Delta_{12}^{-} = \frac{1}{2}(e^{-2\overline{\eta}_1} - e^{-2\overline{\eta}_2})$, $\Delta_{13}^{-} = \frac{1}{2}(\overline{\chi}_1^{-} - \Delta_{13}^{-} = \frac{1}{2} 12 \ (\overline{\chi}_1^{-} - \Delta_{13}^{-} = \frac{1}{2} 12 \ (\overline{\chi}_1^{-} - \Delta_{13}^{-} = \frac{1}{2} 12 \ (\overline{\chi}_1^{-} - \Delta_{13}^{-} = \frac{1}{2} 12 \ (\overline{\chi}_1^{-} - e^{-2\eta_3})$,

$$\overline{\chi}_1^{-} = e^{2i\epsilon_3} (c_{(2)}^2 e^{-2\overline{\eta}_1} + s_{(2)}^2 e^{-2\overline{\eta}_2}), \tag{41}$$

$$\overline{\chi}_2^{-} = e^{-2i\epsilon_3} (s_{(2)}^2 e^{-2\overline{\eta}_1} + c_{(2)}^2 e^{-2\overline{\eta}_2}). \tag{42}$$

These invariants show that the physical parameters are intricately correlated during the RGE evolution. In general, we cannot single out one or two variables which evolve independently of the others. However, in limited regions when certain conditions are satisfied, we do get simplified relations between a subset of the parameters. We will highlight some of these relations in the following.

(i) Real mass matrix. In this case, all physical phases vanish, $\epsilon_3 = \gamma_i = 0$. The RGE invariants $I_{1,2,3}$ are all real so that renormalization does not generate any physical phases, as expected from Eq. (13).

(ii) Two-flavor solutions. The three-flavor problem reduces to that of two flavor under certain conditions. This happens when two of the three mixing angles vanish. Thus, if $s_{(5)} = s_{(7)} = 0$, we find that $I_1 = I_2 = s_{2(2)}^2 \sinh^2(\overline{\eta}_1 - \overline{\eta}_2), I_3 = 0$. Note that the condition $s_{(5)} = s_{(7)} = 0$ is RGE stable. Similarly, if $s_{(2)} = s_{(7)} = 0$, or if $s_{(2)} = s_{(5)} = 0$, the result is genuine two-flavor solutions.

(iii) In regions when one angle is small. It may happen that, in a certain range of t, one of the mixing angles can be small. In general, such conditions can only be fulfilled in a limited region. There will then be approximate invariant combinations from a reduced set of parameters. For instance, in a region where $s_{(2)} \to 0$, we find $(\overline{\overline{\eta}}_1 = \overline{\eta}_1 - i\epsilon_3, \overline{\overline{\eta}}_2 = \overline{\eta}_2 + i\epsilon_3)$

$$I_1 \to s_{2(5)}^2 \sinh^2(\overline{\eta}_1 - \overline{\eta}_3), \quad I_2 \to -c_{2(7)} s_{2(5)}^2 \sinh^2(\overline{\overline{\eta}}_1 - \overline{\eta}_3)$$

$$I_3 \to s_{2(7)}^2 [-s_{(5)}^2 \sinh^2(\overline{\overline{\eta}}_2 - \overline{\overline{\eta}}_1) - c_{(5)}^2 \sinh^2(\overline{\overline{\eta}}_2 - \overline{\eta}_3) + s_{(5)}^2 c_{(5)}^2 \sinh^2(\overline{\overline{\eta}}_1 - \overline{\eta}_3)]. \tag{43}$$

This means that, in the limit $s_{(2)} \to 0$, the $(1-3)$ sector behaves like a two-flavor problem. At the same time, there is a correlation between the mixing angle ϵ_7 and the *CP* phase, ϵ_3. However, if $\epsilon_3 = 0$, then the approximation $s_{(2)} \to 0$ is consistent only if $\epsilon_7 \to 0$. Similar conclusions can be reached for the case $s_{(5)} \to 0$ and $s_{(7)} \to 0$. In addition, the cases when two masses are nearly degenerate, or when one mass value dominates, can also be analyzed along these lines.

In this paper, we have studied the properties of the three-flavor neutrino mass matrix under RGE evolution. Unlike the two-flavor problem, where we obtained exact analytic solutions, the algebra of the 3×3 matrices is formidable, and we are only able to find three (complex) RGE invariants which correlate the evolutions of the many physical parameters.

The RGE evolution of a (symmetric) mass matrix with $\det M = 1$, for a class of theories (including the SM and MSSM), is given in the form $M \to e^Q M e^Q$, $Q=$ real diagonal and $\mathrm{tr}\, Q = 0$. That is, renormalization amounts to a relative rescaling between the different flavors. At the same time, M is subject to arbitrary rephasing transformations, which correspond to taking Q to be an arbitrary pure imaginary matrix. The combined rescaling and rephasing transformation is thus given by $M \to e^{\overline{Q}} M e^{\overline{Q}}$, $\overline{Q} =$ complex. Since $\overline{Q}$ contains only two (complex) variables, we expect RGE invariants formed from the many physical variables in M. By means of an analytic continuation, these considerations are the same as those used in obtaining rephasing invariants of the CKM matrix. We can thus write down three (complex) RGE and rephasing invariants explicitly. Our arguments also make it clear that these invariants are independent of the specific values of Q, and are the same for the SM and MSSM.

Since exact solutions for the three-flavor problem are not available, a number of approximate solutions have been considered in the literature [5]. The RGE invariants can be used to check the consistency of these approximations and to suggest viable new ones. The structure of these invariants also shows that, while the two-flavor approximation is natural in a number of situations, their validity can only be established for a limited range of t. For large t, when the parameters also vary considerably, the two-flavor approximation is viable only under very stringent conditions.

With minor changes, most of the arguments in this work can be adapted to the study of quark mass matrices. In fact [6], it was shown that the infrared fixed point for two-flavor RGE evolution corresponds to $\beta \to 0, m_2/m_1 \to \infty$. The approach to the fixed point, however, is governed by Eq. (36), giving $\beta\sqrt{m_2/m_1} \to \mathrm{const}$. This suggests that the quark mass matrices are the results of large RGE evolution, and that the well-known empirical relations between mixing angle and mass rations, $\theta_{ij} \sim \sqrt{m_i/m_j}$, may have a dynamical origin. We plan to apply our analysis to a detailed study of the quark sector in the future.

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