Space of signed points and the self-dual model

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We study a generalization of the group of loops that is based on sets of signed points, instead of paths or loops. This geometrical setting incorporates the kinematical constraints of the sigma model, inasmuch as the group of loops does with Bianchi identities of Yang-Mills theories. We employ an Abelian version of this construction to quantize the self-dual model, which allows us to relate this theory with that of a massless scalar field obeying nontrivial boundary conditions.

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I. INTRODUCTION

It is well known that there is no natural way of defining non-Abelian theories of p forms, for $p>1$. This is closely related to the lack of the notion of order for p surfaces ($p>1$). Therefore, in the non-Abelian case only two geometric representations can be considered: the well known path-space representation [1,2] and a 0-surface or point-representation. In this paper, we shall study this problem. Despite the fact that these ideas are motivated by considerations about non-Abelian theories, we find it convenient to present, as an example of their application, the "signed-points" representation of the self-dual model (SDM) [3], since it appears that certain properties of this Abelian model are conveniently displayed in this geometrical framework. More precisely, we find that the SDM can be seen as the theory of a massless scalar field that obeys anyonic boundary conditions. This agrees with an earlier result about the Maxwell-Chern-Simons theory (MCST) [4] (which is dual to the SDM [5]), that was obtained by working in a path representation [6]. The latter representation has also been recently used with the SDM in order to study the geometrical content of the duality symmetry between this model and the Maxwell-Chern-Simons one [7].

The construction that we present is related with a recent proposal that introduces the concept of "point holonomies," which generalizes the "path holonomy" (or Wilson loop) of gauge theories in order to treat quantum Higgs fields on the same footing as gauge fields [14].

In the next section we present general ideas about the signed-point space. In the last one we discuss their application to the SDM, and present a brief discussion about the relationship between the "signed-point" representation and the "point-holonomy" concept introduced in Ref. [14].

II. THE SPACE OF SIGNED POINTS

Consider the space of ordered lists of points in R^n (the extension to general manifolds is immediate). We shall declare that there are two kinds of points, that we arbitrarily take as positive (or "points") and negative ("antipoints"). A typical element X of this space can be represented by

$$X=(x_1^{(s_1)},x_2^{(s_2)},x_3^{(s_3)},\ldots,x_r^{(s_r)}), \tag{1}$$

where the "sign" $s_a=\pm$ of each point x_a has been explicitly written. The number of points r is arbitrary. We define the composition XY of two lists as

$$XY=(x_1^{(s_1)},x_2^{(s_2)},\ldots,x_r^{(s_r)},y_1^{(t_1)},y_2^{(t_2)},\ldots,y_u^{(t_u)}). \tag{2}$$

The space of lists, with the composition defined above, can be endowed with a group structure as follows. First, we demand that pairs "point-antipoint" be annihilated if they meet at the same place and consecutively in a list. For instance, $(x_1^{(+)},x_2^{(+)},x_2^{(-)},x_3^{(+)})$ will be taken as $(x_1^{(+)},x_3^{(+)})$. Once all the consecutive and equally located pairs in X have been annihilated, we are left with a "reduced list" (RL) of "signed points" $R(X)$. The product of two RL's is then defined as the RL associated to their composition

$$R(X_1)\diamondsuit R(X_2)\equiv R(R(X_1)R(X_2)). \tag{3}$$

It can be seen that that the space of RL's forms a group under the multiplication defined by Eq. (3). The identity element results to be the empty list. The inverse element $R^{-1}(X)$ is the reduction of the list built by inverting the order and changing the signs of the points that appear in X:

$$R^{-1}(X)=R(x_r^{(-s_r)},x_{r-1}^{(-s_{r-1})},\ldots,x_1^{(-s_1)}). \tag{4}$$

Next, we are going to consider functionals $\Psi[R(X)]$ that depend on RL's. To simplify the notation, we shall label $R(X)$ simply as X, and the group product $\diamondsuit$ as the composition. This should not lead to confusion, since from now on we shall restrict ourselves to deal with RL's. We define an operator $a(Y)$ that appends a RL Y to the left of the argument X of the RL-dependent functional $\Psi(X)$

$$a(Y)\Psi(X)\equiv\Psi(Y^{-1}X). \tag{5}$$

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$$a(X_1)a(X_2)\Psi(X)=\Psi(X_2^{-1}X_1^{-1}X)$$
$$=a(X_1X_2)\Psi(X); \qquad (6)$$

hence, $a(X)$ constitute a representation of the group of RL's, acting on RL-dependent functionals. Furthermore, since

$$a(X)=a(x_1^{(s_1)})a(x_2^{(s_2)})\cdots a(x_r^{(s_r)}), \qquad (7)$$

the a's depending on a single point generate the representation. Observe that

$$a(x^{(+)})=a^{-1}(x^{(-)}); \qquad (8)$$

thus, we can adopt the notation $a(x)=a(x^{(+)})$ [and $a^{-1}(x)=a(x^{(-)})$] without ambiguity. It is worth observing that besides being RL-dependent operators, the a's are ordinary functions of the variables x_a, $a=1,\ldots,r$, and can be, for instance, derived with respect to them.

As in the group of loops case [1], there exists a sort of infinitesimal generators that we define as follows. Take the RL δY, consisting on a pair point-antipoint, separated by an infinitesimal vector u

$$\delta Y=((x+u)^{(+)},x^{(-)}). \qquad (9)$$

In the limit $|u|\to 0$, this ''dipole list'' reduces to the identity. In this sense it is an infinitesimal element of the group of RL's. We define $\delta_\mu(x)$ as the operator that measures the response of $\Psi(X)$ when its argument X is slightly changed by appending the ''dipole'' δY at x

$$\Psi(\delta YX)-\Psi(X)\equiv u^\mu\delta_\mu(x)\Psi(X), \qquad (10)$$

up to first order in u. From Eqs. (5), (8), (9), (10), one has

$$\begin{aligned}[1+u^\mu\delta_\mu(x)]\Psi(X)&=a(x)a^{-1}(x+u)\Psi(X)\\ &=a(x)\left(a^{-1}(x)+u^\mu\frac{\partial}{\partial x^\mu}a^{-1}(x)\right)\Psi(X)\\ &=\left(1+u^\mu a(x)\frac{\partial}{\partial x^\mu}a^{-1}(x)\right)\Psi(X);\end{aligned} \qquad (11)$$

thus, we obtain the identity

$$\delta_\mu(x)\equiv a(x)\frac{\partial}{\partial x^\mu}a^{-1}(x), \qquad (12)$$

and we see that the ''dipole derivative'' $\delta_\mu(x)$ can be analyzed in terms of the elementary generators $a(x)$. From Eq. (12) it is immediate to obtain

$$\partial_\mu\delta_\nu(x)-\partial_\nu\delta_\mu(x)+[\delta_\mu(x),\delta_\nu(x)]=0, \qquad (13)$$

which is just the kinematical constraint obeyed by chiral fields. This should be compared with the Bianchi identity obeyed by the area derivative in the loop-space formulation of Gambini-Trías [8,9]. There is a geometric construction underlying identity (13) that deserves to be pointed out. Take an infinitesimal parallelogram of sides u,v, centered at x. At each vertex, put a pair point-antipoint. The resulting configuration is then equal to the empty list which in turn, is the identity of the group. On the other hand, the same configuration can also be reached by a successive pasting of ''dipoles,'' the first one with its point at, say, $x+u$, and its antipoint at x, the second one consisting of a point at $x+u+v$ and an antipoint at $x+u$, and so on. Since the two constructions correspond to the same RL, namely, the identity of the group, one has

$$[1+u^\mu\delta_\mu(x)][1+v^\mu\delta_\mu(x+u)][1-u^\mu\delta_\mu(x+u+v)]\times[1-v^\mu\delta_\mu(x+v)]=1, \qquad (14)$$

and it is a trivial matter to see that this is the same as Eq. (13), up to first order in the area of the parallelogram expanded by u and v.

Summarizing, we see that the RL's or ''signed-points'' space encodes, through its infinitesimal generators, the kinematical properties of chiral (or sigma model) fields. This results as a consequence of the group structure, and follows strictly from geometrical considerations.

This construction, as with the very definition of the space of RL's and its generators, is very close to the loop-space construction of Gambini and Trías [8,9], which is the basis for the present formulation.

III. AN APPLICATION: SELF-DUAL MODEL AND THE ABELIAN GROUP OF SIGNED POINTS

In this section we present a simple application of the ideas discussed above. In a recent article that deals with the quantization of the Maxwell-Chern-Simons theory (MCST) in a geometric representation [3], it was mentioned that an appropriate geometrical setting that would serve to relate the topological interaction provided by the Chern-Simons term, with certain anyonic behavior of the wave functional of the theory, should be one of ''points and antipoints'' (in the sense discussed before), both for the (MCST) and its dual model (which is the SDM). This conclusion was reached after solving the ''Gauss constraint'' of the MCST in a path representation, and noticing that the feature of the paths that survives in the reduced phase space is precisely the distribution of their ending points. These boundary points acquire a long-range interaction due to the topological term. Now we address this point in some detail, providing an example of how the ideas of the preceding section could be useful in field theory. We shall restrict ourselves to the SDM (dual to the MCST), since it is in this model where the RL's representation can be implemented in a more natural and geometrically appealing form.

We start from the SD action in the Stueckelberg form

$$S=\int d^3x\left(\frac{k}{2}\varepsilon^{\alpha\beta\gamma}\partial_\alpha A_\beta A_\gamma+\frac{1}{2}(A_\alpha+\partial_\alpha\varphi)(A^\alpha+\partial^\alpha\varphi)\right), \tag{15}$$

which is invariant under the gauge transformations

$$A_\alpha\to A_\alpha+\partial_\alpha\Lambda, \tag{16}$$

$$\varphi\to\varphi-\Lambda. \tag{17}$$

The equations of motion that follow from varying S with respect to φ are nothing but consistence equations for the true equations of motion, that result when varying with respect to A_α. This reflects the unphysical character of the Stueckelberg field φ, which could be set equal to zero by a gauge choice, accordingly with Eq. (17). Instead, we are interested in ‘‘gauging away’’ the Chern-Simons field, in a sense that will be clear soon, and within the spirit of what is a common procedure in ordinary quantum mechanics of particles with Chern-Simons interactions [10].

The quantization in the Dirac manner produces the following results. The canonical commutators are

$$[\varphi(\vec{x}),\Pi(\vec{y})]=i\,\delta^2(\vec{x}-\vec{y}), \tag{18}$$

$$[A_i(\vec{x}),A_j(\vec{y})]=\frac{i}{k}\varepsilon_{ij}\delta^2(\vec{x}-\vec{y}), \tag{19}$$

and the Hamiltonian is

$$H=\int d^2x\frac{1}{2}[\Pi^2+(A_i+\partial_i\varphi)(A_i+\partial_i\varphi)]. \tag{20}$$

There is also a first class constraint

$$k\varepsilon^{ij}\partial_i A_j+\Pi=0, \tag{21}$$

that generates the time-independent gauge transformations on the canonical variables. At this point, it is worth comparing the SDM with the $(2+1)$-dimensional massless scalar field theory, whose action and Hamiltonian can be obtained by putting $A_\mu=0$ in Eqs. (15) and (20), respectively. Also, the canonical commutators of the scalar theory are just given by Eq. (18). Since in this case the gauge symmetry is absent, there are no constraints [it would be incorrect to set $A_i=0$ in Eq. (21) and to say that $\Pi=0$ is a constraint in this case]. We shall exploit these apparent simmilarities by working in a geometric representation based on the RL's space and employing old ideas borrowed from the loop representation formulation of gauge theories. Since the theories we are considering are both Abelian, we need to ‘‘abelianize’’ the group of RL's. To this end, we choose the following route. Given a RL [as in Eq. (1)], we define its ‘‘form factor’’

$$\rho(\vec{x},X)\equiv\sum_{a=1}^{r}s_a\delta^2(\vec{x}-\vec{x}_a), \tag{22}$$

which allows us to group the RL's accordingly with the following rule: two RL's are said to be equivalent if they share the same form factor. It can be easily checked that this indeed defines an equivalence relation. Moreover, since

$$\rho(\vec{x},XY)=\rho(\vec{x},X)+\rho(\vec{x},Y), \tag{23}$$

each equivalence class of RL's defines an element of an Abelian group. What we have done is to relax the condition that demanded points and antipoints to be consecutive (apart from being at the same place) in order to annihilate each other. In other words, with this further identification we are not concerned about the order of the points in the list. Within this geometric setting the quantum algebra of the massless scalar field theory can be realized as follows:

$$\exp[-i\varphi(\vec{x})]\Psi(X)=\exp[-i\varphi(\vec{x})]\Psi(\vec{x}_1^{(s_1)},\vec{x}_2^{(s_2)},\ldots,\vec{x}_r^{(s_r)})$$
$$=\Psi(\vec{x}^{(+)},\vec{x}_1^{(s_1)},\vec{x}_2^{(s_2)},\ldots,\vec{x}_r^{(s_r)}), \tag{24}$$

$$\Pi(\vec{x}\Psi(X)=\rho(\vec{x},X)\Psi(X), \tag{25}$$

as can be verified. Equation (24) is equivalent to set

$$\partial_i\varphi(\vec{x})\to i\,\delta_i(\vec{x}). \tag{26}$$

From Eqs. (24), (25) we see that in the space of functionals that depend on Abelian RL's, the operator $\exp[\mp i\varphi(\vec{x})]$ appends a ‘‘positive’’ (‘‘negative’’) point to the list X, while $\Pi(\vec{x})$ displays the form factor of X. It should be observed that it is the derivative of the field operator (and not the field itself) which enters in the expressions for the observables of the theory. This is reminiscent of the invariance of the theory under the shift $\varphi\to\varphi+\text{const}$. This derivative is realized as i times the ‘‘dipole’’ derivative discussed before, accordingly with Eq. (26). In terms of X-dependent functionals, the Schrödinger equation of the massles scalar theory becomes

$$i\frac{\partial}{\partial t}\Psi(X,t)=\int d^2x\frac{1}{2}[\rho^2(\vec{x},X)-\delta_i(\vec{x})\delta_i(\vec{x})]\Psi(X,t). \tag{27}$$

The Hamiltonian comprises a ‘‘dipole’’ Laplacian, together with a potential term ρ^2 which should be regularized, since it is essentially the square of Dirac's delta functions.

At this point, we turn back to the SDM, and try to realize its quantum algebra in the RL's representation. First of all, notice that we dispense with realizing gauge-dependent operators. Hence, we focus on the algebra of the basic gauge-invariant ones

$$[\Pi(\vec{x}),(A_i+\partial_i\varphi)(\vec{y})]=-i\frac{\partial}{\partial y^i}\delta^2(\vec{x}-\vec{y}), \tag{28}$$

$$[(A_i+\partial_i\varphi)(\vec{x}),(A_j+\partial_j\varphi)(\vec{y})]=\frac{i}{k}\varepsilon_{ij}\delta^2(\vec{x}-\vec{y}). \tag{29}$$

It can be seen that the prescriptions

$$\Pi(\vec{x})\to\rho(\vec{x},X), \tag{30}$$

$$(A_i+\partial_i\varphi)(\vec{x})\rightarrow iD_i(\vec{x}) \equiv i\,\delta_i(\vec{x})+\frac{1}{2\pi k}\sum_a s_a\varepsilon_{ij}\frac{(x-x_a)^j}{|\vec{x}-\vec{x}_a|^2}, \quad (31)$$

verify Eqs. (28),(29) when acting on (Abelian) RL-dependent wave functionals $\Psi(X)$. It should be noticed that the second terms on the right-hand side of Eq. (31) is a genuine RL-dependent quantity. This is mandatory in order to have a consistent realization of the quantum algebra. On the other hand, it must be said that this term already appears in earlier discussions about anyons in ordinary quantum mechanics [10].

Using the Abelian version of Eq. (13)

$$\varepsilon^{ij}\partial_i\delta_j(\vec{x})=0, \quad (32)$$

it can be shown that the gauge constraint (21) is automatically satisfied. It could be interesting to compare this feature with what occurs in other gauge theories. In the loop-space formulation of Maxwell theory [11], it is found that the very introduction of loops (i.e., closed Faradays lines) suffices to solve the Gauss constraint. The introduction of point sources, demands that there must be open Faradays lines, starting or ending at points where charges (that must be quantized) are placed [12]. A similar result holds when the Proca-Stueckelberg model is quantized in an appropriate geometric space [13]. In the MCSM (that is dual to the SDM that we are considering), however, it was found that the quantization in path space does not lead to convert the gauge constraint in an identity [6]. Nevertheless, after solving this constraint in path space it was seen that the property of the paths that really matters is the winding number of the open curves around their boundaries. Then, performing a certain unitary transformation, it was obtained that this dependence can be rewritten as a functional dependence in the boundaries of the paths, together with the inclusion in the Hamiltonian of a term describing a long-range interaction between these boundaries. This is precisely what we have obtained in the present approach, using a different way. In fact, substituting Eqs. (30) and (31) into the Hamiltonian (20), we can write the Schrödinger equation of the SDM, in the RL's representation as

$$i\frac{\partial}{\partial t}\Psi(X,t)=\int d^2x\frac{1}{2}[\rho^2(\vec{x},X)-D_i(\vec{x})D_i(\vec{x})]\Psi(X,t), \quad (33)$$

which differs from Eq. (27) in the appearance of the covariant derivative $D_i(\vec{x})$ that encodes the Chern-Simons interaction. As in the MCSM, one can perform the singular gauge transformation

$$\Psi(X)\rightarrow\overline{\Psi}(X)\equiv\exp[i\Lambda(X)]\Psi(X), \quad (34)$$

with

$$\Lambda(X)=\frac{1}{4\pi k}\int d^2x\int d^2y\,\rho(X,\vec{x})\,\theta(\vec{x}-\vec{y})\,\rho(X,\vec{y}) =\frac{1}{4\pi k}\sum_a\sum_{a'} s_a s_{a'}\,\theta(\vec{x}_a-\vec{x}_{a'}), \quad (35)$$

to convert the covariant derivative $D_i(\vec{x})$ into an ordinary "dipole derivative" $\delta_i(\vec{x})$, as it appears in the Schrödinger equation of the massless scalar field. In the last equation, $\theta(\vec{x})$ is the angle that $\vec{x}$ makes with the x axis. The price for this simplification is that the resulting wave functional $\overline{\Psi}(X)$ is multivalued, due precisely to this dependence in the angle. [In Eq. (35) "self-interaction" terms appear proportional to $\theta(\vec{0})$, that are not well defined. We shall ignore these regularization issues in this paper (for further details see Ref. [7]).] Thus, we see that the SDM can be seen as the theory of a massless scalar field obeying anyonic boundary conditions, as happens with the MCST [6].

It should be understood that despite the appearances, there is a fundamental difference between the two "gauges" that admits the SDM. In the usual one, the Stueckelberg fields φ are eliminated by means of a legitimate gauge transformation. Instead, in the second one, the vector field A_μ is eliminated, but by means of a *singular* gauge transformation. Nevertheless, there is nothing wrong with this last point of view, as long as we keep in mind that the wave functionals become multivalued. It should be stressed that this second approach, which is well known for the case of particles in Chern-Simons interactions [10], becomes quite natural in the SDM thanks to the introduction of the RL's formalism.

To conclude, let us make some remarks about the relation between the "signed-points" representation and the "point holonomies" of Ref. [14] that we briefly mentioned in the Introduction. It is well known that starting from the Schrödinger representation of gauge theories, one can construct the loop representation by making a "generalized Fourier transform" (the so-called loop transform [15,16]) on the connection-dependendent functional $\Psi(A)$ to obtain the loop-dependent one $\Psi(\gamma)$:

$$\Psi(\gamma)=\int DA\,W_A(\gamma)\Psi(A), \quad (36)$$

where $W_A(\gamma)$ is the Wilson loop or holonomy of the gauge theory and plays the role of the "plane wave" basis that relates the connection and loop representations. On the other hand, in the Schrödinger representation of the scalar field φ, the basic operators $\partial_i\varphi$ and Π act as

$$\partial_i\varphi_{\text{operator}}\rightarrow\partial_i\varphi, \quad (37)$$

$$\Pi\rightarrow -i\frac{\delta}{\delta\varphi}, \quad (38)$$

over the field-dependent functional $\Psi(\varphi)$. From these equations it can be seen that one can turn to the RL representation, characterized by the realization of the canonical algebra given by Eqs. (25) and (26), if we define the X-dependent wave functional $\Psi(X)$ through the transformation

$$\Psi(X)=\int D\varphi U_{\varphi}(X)\Psi(\varphi), \tag{39}$$

where

$$U_{\varphi}(X)\equiv\exp\left(-i\sum_{a=1}^{r} s_a\varphi(\vec{x}_a)\right) \tag{40}$$

is just the (Abelian) ''point holonomy'' of Ref. [14]. Thus we see that these point holonomies are the bridge between the Schrödinger representation and the RL representation introduced in this paper. Although we have restricted ourselves to discussing this relationship in the Abelian case, it seems that these considerations could also be extended to the non-Abelian one. It should be noticed that both the path and the loop holonomies are diffeomorphism invariant objects. This is a nice property in order to deal with quantization in general space-times [14].

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