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New approaches in $\mathcal{W}$ gravities

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We devote effort to studying some nonlinear actions, characteristic of $\mathcal{W}$ theories, in the framework of the soldering formalism. We disclose interesting new results concerning the embedding of the original chiral $\mathcal{W}$ particles in different metrical spaces in the final soldered action; i.e., the metric is modified by the soldering interference process. The results are presented in a weak field approximation for the $\mathcal{W}_N$ case when $N \geqslant 3$ and also in an exact way for $\mathcal{W}_2$. We promote a generalization of the interference phenomenon to $\mathcal{W}_N$ theories of different chiralities and show that the geometrical features introduced can yield a new understanding of the interference formalism in quantum field theories.

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I. INTRODUCTION

This paper is devoted to studying the effects of interference between the chiral modes carrying a representation of conformal spins of order higher than two in the context of the soldering formalism. These modes are described by chiral $\mathcal{W}_N$ gravities for $N \geqslant 2$. This study is a natural and deeper extension of [1] where the soldering of two chiral $\mathcal{W}_2$ particles (analogous to the Siegel particles [2]) was shown to produce the action for a nonchiral 2D scalar field coupled to a gravitational background. It is therefore natural to consider the possibility of soldering chiral modes carrying representations of higher conformal spins.

There are two main reasons for studying extended symmetries in conformal field theory [3]. Certain applications of conformal field theory in either string theory or statistical mechanics require some additional symmetry in addition to conformal invariance. Moreover, extended symmetries can help in the analysis of a large class of conformal field theories (called rational conformal field theories) and in classifying certain types of conformal field theories.

Conformal invariance in two dimensions is a powerful symmetry that allows certain two-dimensional quantum field theories to be solved exactly. Conformal field theories have found remarkable applications in string theory (see [4]). This study together with the investigation of critical phenomena in statistical mechanics (see [5] for selected reprints) has produced a large-scale study of conformal field theories in recent years. For example, the study of so-called perturbed conformal field theories has given rise to surprising new results for certain massive integrable quantum field theories [6].

Additional motivation for a detailed study of the infinite-dimensional algebraic structure of conformal field theories comes from the study of two-dimensional gravity which relies heavily on conformal field theory techniques and from two-dimensional topological quantum field theories. Infinite-dimensional symmetry algebras are known to play a central role in 2D physics. There is an intrinsic connection of these algebras with two-dimensional gauge theories or string theories, the most important example being the symmetry algebra of two-dimensional conformal field theories, i.e., the Virasoro algebra.

The Virasoro algebra admits higher spin extensions, known as $\mathcal{W}_N$ algebras, containing generators with conformal spins $2,3,4,\ldots,N$ [7,8]. A $\mathcal{W}$ algebra is an extended conformal algebra that satisfies the Jacobi identities and contains the Virasoro algebra as a subalgebra [3,9]. $\mathcal{W}$ algebras are infinite-dimensional symmetry algebras with the restriction that at least one of the generating currents has spin $s \geqslant 2$. This algebra is generated by a set of chiral currents which are our main interest in the present study.

In addition to the above motivation for the present study in extended conformal symmetry we would like to mention that in perturbed conformal field theories the presence of $\mathcal{W}$ symmetries in the original conformal field theory may lead to

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additional integrals of motion in the perturbed theory [10], and for the relation with topological field theories $N=2$ extended superconformal symmetry is essential [11–13]. Extended symmetries also appear to be particularly important for the coupling of conformal field theory "matter systems" to two-dimensional gravity [14]. Classical and quantum $\mathcal{W}$ gravity, in particular $\mathcal{W}_3$ gravity, have recently been studied by various groups [3].

In order to apply the soldering formalism to the opposite chiral $\mathcal{W}$ aspects we need an explicit field theoretical realization of these algebras. Realizations of the chiral $\mathcal{W}_N$ algebras have been constructed, for example in terms of $(N-1)$ free bosons via the Miura transformations [8]. Other realizations, using a generalizing Sugawara construction, have also been given [15]. This seems important since tentative extensions of string theory based on extra bosonic symmetry ($\mathcal{W}$ symmetry) on the worldsheet have been proposed and are called $\mathcal{W}$ strings [16–18]; these are higher spin generalizations of ordinary string theories, such that the string coordinates are coupled not only to the worldsheet metric but also to a set of higher spin worldsheet gauge fields (for a review see [19]). Since ordinary string theory can be considered as a gauge theory based on the Virasoro algebra, one can analogously define a $\mathcal{W}$ string theory as a gauge theory based on a $\mathcal{W}$ algebra [7,8,20] (or any other higher spin conformally extended algebra [19,21,22]). The bosonic representations of the chiral algebras will be the starting point for our application of the soldering formalism.

Recently, there has been great improvement in soldering together distinct manifestations of chiral and duality symmetries [23–32]. The procedure leads to new physical results that include the idea of the interference effect. The soldering formalism was introduced in [31,32] to solder together two chiral scalars by introducing a nondynamical gauge field to remove the degree of freedom that obstructs the vector gauge invariance. This is connected via chiral bosonization to the necessity that one has to have more than the direct sum of two fermion representations of the Kac-Moody algebra to describe a Dirac fermion. In other words, we can say that the equality of the weights in the two representations is physically connected with the necessity to abandon one of the two separate chiral symmetries and accept that only vector gauge symmetry should be maintained. In addition, being just an auxiliary field, it may be eliminated (integrated) *a posteriori* in favor of the physically relevant quantities. This restriction will force the two independent chiral representations to belong to the same multiplet, effectively soldering them together. This is the main motivation for the introduction of the soldering field which permeates to the case of higher conformal spin currents.

In Sec. II we give a review of $\mathcal{W}$ theories. The soldering formalism is briefly depicted in Sec. III. The fusion of chiral $\mathcal{W}_N$ particles is accomplished in Sec. IV. Conclusions and final remarks are given in Sec. V.

II. $\mathcal{W}$ GRAVITIES

In order to make this work self-contained, in this section we will make a brief review of the $\mathcal{W}$ realizations and gauging, following closely Refs. [9,33–35].

A. $\mathcal{W}$ algebra

Let us consider Lie algebras under Poisson brackets, with the generators t_a labeled by an index a (which may be of infinite range),

$$\{t_a,t_b\}=f^c_{ab}t_c+c_{ab}, \tag{1}$$

where f^c_{ab} are the structure constants and c_{ab} are constants defining the central extension of the algebra. However, for many $\mathcal{W}$ algebras, the Poisson brackets structures give a result nonlinear in the generators,

$$\{t_a,t_b\}=f^c_{ab}t_c+c_{ab}+g^{cd}_{ab}t_ct_d+\cdots=F_{ab}(t_c), \tag{2}$$

and the algebra is said to close in the sense that the right-hand side is a function of the generators. Most of the $\mathcal{W}$ algebras that are generated by a finite number of currents are nonlinear algebras of this type. At first sight, it appears that there might be a problem in trying to realize a nonlinear algebra in a field theory, as symmetry algebras are usually Lie algebras. However, as will be seen, a nonlinear algebra can be realized as a symmetry algebra for which the structure constants are replaced by field-dependent quantities.

A field theory with action S_0 and conserved symmetric tensor currents $T_{\mu\nu},W^A_{\mu_1\mu_2\cdots\mu_{s_A}}$ (where $A=1,2,\ldots$ labels the currents, which have spin s_A) will be invariant under global symmetries with constant parameters $k^\mu,\lambda_A^{\mu_1\mu_2\cdots\mu_{s_A-1}}$ (translations and $\mathcal{W}$ translations) generated by the Noether charges $P_\mu,Q^A_{\mu_1\mu_2\cdots\mu_{s_A-1}}$ (momentum and $\mathcal{W}$ momentum) given by $P_\mu=\int t dx^0 T_{0\mu}$ and $Q^A_{\mu_1\mu_2\cdots\mu_{s_A-1}}=\int dx^0 W^A_{\mu_1\mu_2\cdots\mu_{s_A-1}0}$. If the currents are traceless, then the theory is invariant under an infinite-dimensional extended conformal symmetry. The parameters $\lambda_A^{\mu_1\mu_2\cdots\mu_{s_A-1}}$ are then traceless and the corresponding transformations will be symmetries if the parameters are not constant but satisfy the conditions that the trace-free parts of $\partial^{(\nu}k^{\mu)},\partial^{(\nu}\lambda_A^{\mu_1\mu_2\cdots\mu_{s_A-1})}$ are zero. This implies that $\partial_\mp k^\pm=0$ and $\partial_\mp\lambda_A^{\pm\pm\cdots\pm}=0$ so that the parameters are semilocal, $k^\pm=k^\pm(x^\pm)$ and $\lambda_A^{\pm\pm\cdots\pm}=\lambda_A^{\pm\pm\cdots\pm}(x^\pm)$, which are the parameters of conformal and $\mathcal{W}$ conformal transformations.

The soldering formalism, to be developed in the next section, promotes the lift of these global symmetries to their gauge invariant version. The global symmetries corresponding to the currents $T_{\mu\nu},W^A_{\mu_1\mu_2\cdots\mu_{s_A}}$ are promoted to local ones by coupling to the $\mathcal{W}$ gravity gauge fields $h^{\mu\nu},B_A^{\mu_1\mu_2\cdots\mu_{s_A}}$, which are symmetric tensors transforming, to lowest order in the gauge fields, as

$$\delta h^{\mu\nu}=\partial^{(\nu}k^{\mu)}+\cdots,$$

$$\delta B_A^{\mu_1\mu_2\cdots\mu_s}=\partial^{(\nu}\lambda_A^{\mu_1\mu_2\cdots\mu_{s-1})}+\cdots. \tag{3}$$

The action is given by the Noether coupling

$$S=S_0+\int d^2x(h^{\mu\nu}T_{\mu\nu}+B_A^{\mu_1\mu_2\cdots\mu_{s_A}}W^A_{\mu_1\mu_2\cdots\mu_{s_A}})+\cdots \tag{4}$$

plus terms nonlinear in the gauge fields. If the currents $T_{\mu\nu},W^A_{\mu_1\mu_2\cdots\mu_s}$ are traceless, in the sense that there is an extended conformal symmetry, then the traces of the gauge fields decouple and the theory is invariant under Weyl and $\mathcal{W}$ Weyl transformations given to lowest order in the gauge fields by

$$\delta h^{\mu\nu}=\Omega\,\eta^{\mu\nu}+\cdots,$$

$$\delta B_A^{\mu_1\mu_2\cdots\mu_s}=\Omega_A^{(\mu_1\mu_2\cdots\mu_{s-2}}\eta^{\mu_{s-1}\mu_s)}+\cdots, \tag{5}$$

where $\Omega(x^\nu)$, $\Omega_A^{\mu_1\mu_2\cdots\mu_{s-2}}(x^\nu)$ are the local parameters. This defines the linearized coupling to $\mathcal{W}$ gravity. The full nonlinear theory is in general nonpolynomial in the gauge fields of spins 2 and higher, which makes matter harder. The nonlinear theory can be constructed to any given order using the Noether method, but to obtain the full nonlinear structure requires a deeper understanding of the geometry underlying $\mathcal{W}$ gravity, which is beyond the scope of this study.

B. $\mathcal{W}$ field theory

Consider a field theory in flat Minkowski space with metric $\eta_{\mu\nu}$ and coordinates x^0,x^1. The stress-energy tensor is a symmetric tensor $T_{\mu\nu}$ which, for a translation invariant theory, satisfies the conservation law

$$\partial^\mu T_{\mu\nu}=0. \tag{6}$$

A spin-s current in flat two-dimensional space is a rank-s symmetric tensor $W_{\mu_1\mu_2\cdots\mu_s}$ and will be conserved if

$$\partial^{\mu_1}W_{\mu_1\mu_2\cdots\mu_s}=0. \tag{7}$$

Recall that, in two dimensions, a rank-2 tensor can be decomposed as, e.g., $V_{\mu\nu}=V_{(\mu\nu)}+V\epsilon_{\mu\nu}$ where $V=\frac{1}{2}\epsilon^{\mu\nu}V_{\mu\nu}$. Thus, without loss of generality, all the conserved currents of a given theory can be taken to be symmetric tensors. A rank-s symmetric tensor transforms as the spin-s representation of the two-dimensional Lorentz group.

A theory is conformally invariant if the stress tensor is traceless, $T^\mu_\mu=0$. Introducing null coordinates $x^\pm=1/\sqrt{2}(x^0\pm x^1)$, the tracelessness condition becomes $T_{+-}=0$ and Eq. (6) then implies that the remaining components $T_{\pm\pm}$ satisfy

$$\partial_+T_{--}=0,\quad \partial_-T_{++}=0. \tag{8}$$

If a spin-s current $W_{\mu_1\mu_2\cdots\mu_s}$ is traceless, it will have only two nonvanishing components $W_{++\cdots+}$ and $W_{--\cdots-}$. The conservation condition (7) then implies that

$$\partial_-W_{++\cdots+}=0,\quad \partial_+W_{--\cdots-}=0 \tag{9}$$

so that $W_{++\cdots+}=W_{++\cdots+}(x^+)$ and $W_{--\cdots-}=W_{--\cdots-}(x^-)$ are right- and left-moving chiral currents, respectively. For a given conformal field theory, the set of all right-moving chiral currents generates a closed current algebra, the right-moving chiral algebra, and similarly for left movers. The right and left chiral algebras are examples of $\mathcal{W}$ algebras that are the main components of our construction here.

Consider a set S of right-moving chiral currents $T(x^+)=T_{++}(x^+),W(x^+),\ldots$ of spins $2,s_W,\ldots$. The main example that will be of interest here is that in which the currents arise from some field theory and the brackets are Poisson brackets in a canonical formalism in which x^- is regarded as the time variable. The current T satisfies the conformal algebra if

$$\{T(x^+),T(y^+)\}=-\delta'(x^+-y^+)[T(x^+)+T(y^+)], \tag{10}$$

in which case its modes L_n generate the Virasoro algebra. A current W is said to be primary of spin s_W if

$$\{T(x^+),W(y^+)\}=-\delta'(x^+-y^+)[W(x^+)+(s_W-1)W(y^+)]. \tag{11}$$

The set $\mathcal{S}$ of currents will generate a $\mathcal{W}$ algebra if the brackets of any two currents give a function of currents in $\mathcal{S}$ and if the brackets satisfy the Jacobi identities.

Consider first the case in which there are just two currents T and W, where W is primary of spin $s=s_W$, and where the $\{W,W\}$ brackets take the form

$$\{W(x^+),W(y^+)\}=-2\kappa\delta'(x^+-y^+)[\Lambda(x^+)+\Lambda(y^+)] \tag{12}$$

for some Λ, where κ is a constant. If the algebra is to close, the current Λ must be a function of the currents T,W and their derivatives. If $s=3$, then Λ is a spin-4 current and the Jacobi identities are satisfied if

$$\Lambda=TT. \tag{13}$$

The algebra then closes nonlinearly. Notice that in the limit $\kappa\to 0$ this contracts to a linear algebra. For $s>3$, the algebra will again close and satisfy the Jacobi identities if Λ depends on T,W but not on their derivatives. If s is even, the most general such Λ is of the form

$$\Lambda=\alpha T^{s-1}+\beta WT^{s/2-1} \tag{14}$$

for some constants α,β, while if s is odd, such a Λ must be of the form (14) with $\beta=0$. The algebra given by Eqs. (10), (11), (12), and (14) is the algebra $W_{s/s-2}$ of Ref. [35].

A large number of $\mathcal{W}$ algebras are now known. The $\mathcal{W}_N$ algebra [36] has currents of spins $2,3,\ldots,N$ (so that $\mathcal{W}_2$ is the Virasoro algebra), the $\mathcal{W}_\infty$ [37,38] algebra has currents of spins $2,3,\ldots,\infty$, while the $\mathcal{W}_{1+\infty}$ algebra [38] has currents of spins $1,2,3,\ldots,\infty$.

Consider a theory of D free scalar fields ϕ^i ($i=1,\ldots,D$) with action

$$S_0 = \int d^2x \partial_+ \phi^i \partial_- \phi^i \tag{15}$$

where the two-dimensional space-time has null coordinates $x^\mu = (x^+, x^-)$ which are related to the usual Cartesian coordinates by $x^\pm = 1/\sqrt{2}(x^0 \pm x^1)$. The stress-energy tensor

$$T_{++} = \frac{1}{2}\partial_+ \phi^i \partial_+ \phi^i \tag{16}$$

is conserved, $\partial_- T_{++} = 0$, and generates the Poisson bracket algebra (10) (in a canonical treatment regarding x^- as time [32]), which is the conformal algebra with vanishing central charge. For any rank-s constant symmetric tensor $d_{i_1 i_2 \cdots i_s}$ one can construct a current

$$W_{++\cdots+} = \frac{1}{s} d_{i_1 i_2 \cdots i_s} \partial_+ \phi^{i_1} \partial_+ \phi^{i_2} \cdots \partial_+ \phi^{i_s}, \tag{17}$$

which is conserved, $\partial_- W = 0$, and which is a spin-s classical primary field—its Poisson brackets with T are given by Eq. (11). The Poisson brackets of two W's are Eq. (12), where Λ is given by

$$\Lambda = \frac{1}{4\kappa} d^m_{i\cdots j} d_{k\cdots lm} \partial_+ \phi^i \cdots \partial_+ \phi^j \partial_+ \phi^k \cdots \partial_+ \phi^l \tag{18}$$

(the indices $i, j, \cdots$ are raised and lowered with the flat metric δ_{ij}).

Consider first the case $s = 3$. In general, closing the algebra generated by T, W will lead to an infinite sequence of currents $(T, W, \Lambda, \ldots)$. However, if $\Lambda = T^2$, for some constant κ, then the algebra closes nonlinearly on T and W, to give the so-called classical $\mathcal{W}_3$ algebra depicted above.

In [34], it was shown that for any number D of bosons, the necessary and sufficient condition for Eq. (13) to be satisfied and hence for the classical $\mathcal{W}_3$ algebra to be generated is that the "structure constants" d_{ijk} satisfy

$$d^m_{(ij} d_{k)lm} = \kappa \delta_{(ij} \delta_{k)l}. \tag{19}$$

This rather striking algebraic constraint has an interesting algebraic interpretation.[1] It implies that the d_{ijk} are the structure constants for a Jordan algebra (of degree 3) [40], which is a commutative algebra for which Eq. (19) plays the role of the Jacobi identities. Moreover, the set of all such algebras has been classified [41], allowing one to write down the general solution to Eq. (19) [40]. In particular, Eq. (19) has a solution for any number D of bosons. Examples of solutions to Eq. (19) are given for $D = 1$ by $d_{111} = \kappa$ and for $D = 8$ by taking d_{ijk} proportional to the d symbol for the group SU(3) [32]. For $D = 2$, the construction of [20] gives a solution of Eq. (19) in which the only nonvanishing components of d_{ijk} are given by $d_{112} = -\kappa$ and $d_{222} = \kappa$, together with those related to these by symmetry. The conserved currents T, W correspond to the invariance of the free action S_0 under the conformal symmetries

$$\delta\phi^i = k_- \partial_+ \phi^i + \lambda_{--} d^i_{jk} \partial_+ \phi^j \partial_+ \phi^k, \tag{20}$$

where the parameters satisfy

$$\partial_- k_- = 0, \quad \partial_- \lambda_{--} = 0. \tag{21}$$

Symmetries of this kind whose parameters are functions only of x^+ (or only of x^-) will be referred to here as semilocal. The symmetry algebra closes to give

$$[\delta_{k_1} + \delta_{\lambda_1}, \delta_{k_2} + \delta_{\lambda_2}] = \delta_{k_3} + \delta_{\lambda_3} \tag{22}$$

where

$$k_3 = [k_2 \partial_+ k_1 + 4\kappa(\lambda_2 \partial_+ \lambda_1) T_{++}] - (1 \leftrightarrow 2),$$
$$\lambda_3 = [2\lambda_2 \partial_+ k_1 + k_2 \partial_+ \lambda_1] - (1 \leftrightarrow 2). \tag{23}$$

In particular, the commutator of two λ transformations is a field-dependent k transformation, which is precisely the transformation generated by the spin-4 current $\Lambda = TT$. The gauge algebra structure "constants" are not constant but depend on the fields ϕ through the current T, reflecting the TT term in the current algebra.

To gauge these symmetries to a local $\mathcal{W}$ diffeomorphism, the spin-2 and spin-3 conformal Noether currents above are introduced in the model with the corresponding Lagrange multiplier fields h_{--} (the graviton) and B_{---} (the $\mathcal{W}$ graviton), leading to the Lagrangian of the right $\mathcal{W}_3$ model as

$$\mathcal{L} = \partial_+ \phi^i \partial_- \phi^i + \frac{h_{--}}{2} \partial_+ \phi^i \partial_+ \phi^i + \frac{B_{---}}{3} d_{ijk} \partial_+ \phi^i \partial_+ \phi^j \partial_+ \phi^k, \tag{24}$$

which is the result of the Noether couplings. Notice that the free action is already invariant under "right-moving" transformations.

It is well known that this model is invariant under the transformations (20) and together with the symmetries

$$\delta h_{--} = \partial_- k_- + k_- \partial_+ h_{--} - h_{--} \partial_+ k_- + 2\kappa(\lambda_{--} \partial_+ B_{---} - B_{---} \partial_+ \lambda_{--}) T_{++},$$
$$\delta B_{---} = \partial_- \lambda_{--} + 2\lambda_{--} \partial_+ h_{--} - h_{--} \partial_+ \lambda_{--} - 2B_{---} \partial_+ k_- + k_- \partial_+ B_{---}, \tag{25}$$

extends the original theory to a $\mathcal{W}$ gravity, so that the original semilocal conformal symmetries are promoted to a local $\mathcal{W}$ diffeomorphism. In a similar way we may also gauge the left-handed $\mathcal{W}$ algebra generated by T_{--} and W_{---} with analogous definitions and results. The terms with H_{--} and k_- are analogous to the gauging of the right-handed Virasoro algebra. Hence we can see expressions similar to those for

[1]This identity has in fact occurred at least once before in the physics literature, in the study of five-dimensional supergravity theories [39].

two-dimensional gravity in the chiral gauge [42] or from Siegel's analysis for the chiral boson [2].

The situation is similar for $s>3$. The algebra will close, i.e., Eq. (14) will be satisfied, if the d tensor in Eq. (17) satisfies a quadratic constraint [35], and again this constraint has an algebraic interpretation [35]. The k and λ transformations become

$$\delta\phi^i = k\partial_+\phi^i + \lambda d^i_{i_1\cdots i_{s-1}}\partial_+\phi^{i_1}\cdots\partial_+\phi^{i_{s-1}}, \tag{26}$$

where the parameters satisfy $\partial_- k=0$, $\partial_-\lambda=0$. The symmetry algebra again has field-dependent structure "constants." More generally, any set of constant symmetric tensors $d^A_{ij\cdots k}$ labeled by some index A can be used to construct a set of conserved currents

$$W^A_{++\cdots+} = d^A_{ij\cdots k}\partial_+\phi^i\partial_+\phi^j\cdots\partial_+\phi^k \tag{27}$$

which are classical primary fields, i.e., their Poisson brackets with T are given by Eq. (11). The current algebra will close if the d^A tensors satisfy certain algebraic constraints and the Jacobi identities will automatically be satisfied as the algebra occurs as a symmetry algebra. In this way, a large class of classical $\mathcal{W}$ algebras can be constructed by seeking d^A tensors satisfying the appropriate constraints. D boson realizations of the $\mathcal{W}_N$ algebras were constructed in this way in Ref. [35], where it was shown that the $\mathcal{W}_N$ d tensor constraints had an interpretation in terms of Jordan algebras of degree N, and this again allowed the explicit construction of solutions to the d tensor constraints. These realizations of classical $c=0$ algebras can be generalized to ones with $c>0$ by introducing a background charge a_i, so that the stress tensor becomes $T=\partial_+\phi^i\partial_+\phi^i + a_i\partial^2_+\phi^i$ and adding appropriate higher derivative terms (i.e., ones involving $\partial^m_+\phi^i$ for $m>1$) to the other currents. The classical central charge becomes $c=a^2/12$, and, for the $N-1$ boson realization of $\mathcal{W}_N$, the structure of the higher derivative terms in the currents W_n can be derived using Miura transform methods [36,8].

Another important realization of classical $\mathcal{W}$ algebras is as the Casimir algebra of Wess-Zumino-Witten-Novikov (WZWN) models [33]. For the WZWN model corresponding to a group G, the Lie algebra valued currents $J_+=g^{-1}\partial_+ g$ generate a Kac-Moody algebra and are (classically) primary with respect to the Sugawara stress tensor $T=\frac{1}{2}\mathrm{tr}(J_+J_+)$. Similarly, the higher order Casimir operators allow a generalized Sugawara construction of higher spin currents $\mathrm{tr}(J^n_+)$. For example, for $G=SU(N)$ the set of currents $W_n=1/n\,\mathrm{tr}(J^n_+)$ for $n=2,3,\ldots,N$ generates a closed algebra which is a classical W_N algebra [33]; similar results hold for other groups.

Quantum mechanically, however, the Sugawara expressions for the currents need normal ordering and must be rescaled [15,43]. For example, in the case of SU(3), the quantum Casimir algebra leads to a closed $\mathcal{W}$ algebra (after a certain truncation) only in the case in which the Kac-Moody algebra is of level 1 [15].

$\mathcal{W}$ algebras also arise as symmetry algebras of many other field theories, including Toda theories [36], free-fermion theories [33], and nonlinear sigma models [34,44], giving corresponding realizations of $\mathcal{W}$ algebras.

III. THE INTERFERENCE OF CHIRAL $\mathcal{W}$ THEORIES

A. $\mathcal{W}_2$ gravity in the weak field approximation

Let us analyze the $\mathcal{W}_2$ model for right-handed chirality, which is obtained from Eq. (24) by making $d_{ijk}\to 0$, i.e.,

$$\mathcal{L}^0_+ = \partial_+\phi^i\partial_-\phi^i + \frac{h_{--}}{2}\partial_+\phi^i\partial_+\phi^i. \tag{28}$$

The soldering transformation to be gauged, as described in the last section, is

$$\phi^i \to \phi^i + \alpha^i, \tag{29}$$

where α^i is the semilocal gauge parameter.

The corresponding variation of the model under this transformation is

$$\delta\mathcal{L}^0_+ = J^+_i\partial_+\alpha^i, \tag{30}$$

where J^+_i is the left Noether current given by

$$J^+_i = 2\partial_-\phi^i + h_{--}\partial_+\phi^i. \tag{31}$$

Following the soldering algorithm and computing only the final steps, we have after two iterations that

$$\delta\mathcal{L}^2_+ = -2A^i_+\delta A^i_-, \tag{32}$$

where $A^i_\pm$ are the soldering fields.

For the left chirality we can write

$$\mathcal{L}^0_- = \partial_+\rho^i\partial_-\rho^i + \frac{h_{++}}{2}\partial_-\rho^i\partial_-\rho^i. \tag{33}$$

Analogously, the variation of the model under this is

$$\rho^i \to \rho^i + \alpha^i \tag{34}$$

and

$$\delta\mathcal{L}^0_- = J^-_i\partial_-\alpha^i, \tag{35}$$

where J^-_i is the right Noether current

$$J^-_i = 2\partial_+\rho^i + h_{++}\partial_-\rho^i. \tag{36}$$

Again, after two iterations we have that

$$\delta\mathcal{L}^2_- = -2A^i_-\delta A^i_+. \tag{37}$$

We can see easily that the final soldered action is

$$\mathcal{L}_{FINAL} = \mathcal{L}^2_+ + \mathcal{L}^2_- + 2A^i_-A^i_+, \tag{38}$$

which has the desired vectorial gauge invariance, i.e., $\delta\mathcal{L}=0$, as can be easily checked. Substituting all the $\mathcal{L}^{(N)}_\pm$ previously computed, we can write the final form of the action explicitly as

$$\mathcal{L}_{FINAL}=\mathcal{L}_{+}^{0}+\mathcal{L}_{-}^{0}-A_{+}^{i}J_{i}^{+}-A_{-}^{i}J_{i}^{-}+\frac{h_{--}}{2}(A_{+}^{i})^{2}+\frac{h_{++}}{2}(A_{-}^{i})^{2}+2A_{-}^{i}A_{+}^{i}. \quad (39)$$

Next, by solving the equations of motion for the soldering fields we have

$$A_{+}^{i}=\frac{1}{2}J_{i}^{-}-\frac{1}{2}h_{++}A_{-}^{i}, \quad (40)$$

$$A_{-}^{i}=\frac{1}{2}J_{i}^{+}-\frac{1}{2}h_{--}A_{+}^{i} \quad (41)$$

and these fields can be eliminated in favor of the other variables.

Susbstituting the $A_{\pm}^{i}$ defined in Eq. (41) into Eq. (40) and solving the system iteratively we obtain

$$A_{+}^{i}=\frac{1}{2}J^{-}-\frac{1}{4}h_{++}J^{+}+h^{2}A_{+}^{i}=A_{+}^{i(0)}+h^{2}A_{+}^{i}, \quad (42)$$

where $h^{2}=\frac{1}{4}h_{++}h_{--}$. Now substituting the second equation in the first and so on we have

$$A_{+}^{i}=A_{+}^{i(0)}+h^{2}[A_{+}^{i(0)}+A_{+}^{i}] \; \vdots \; =f_{\infty}A^{i(0)} \quad (43)$$

where

$$f_{\infty}=1+h^{2}+h^{4}+h^{6}+\cdots=\frac{1}{1-h^{2}}. \quad (44)$$

Using the same procedure for A_{-}^{i} and using Eqs. (31) and (36),

$$A_{\pm}^{i}=\frac{1}{1-h^{2}}\left[\frac{1}{2}J_{\pm}^{i}-\frac{1}{4}h_{\pm\pm}J_{\mp}^{i}\right]. \quad (45)$$

Hence, bringing these results back into Eq. (39) we have finally that

$$\mathcal{L}=\frac{1+h^{2}}{1-h^{2}}\partial_{+}\Phi^{i}\partial_{-}\Phi^{i}+\frac{h_{--}}{2(1-h^{2})}\partial_{+}\Phi^{i}\partial_{+}\Phi^{i}+\frac{h_{++}}{2(1-h^{2})}\partial_{-}\Phi^{i}\partial_{-}\Phi^{i}, \quad (46)$$

where $\Phi^{i}=\phi^{i}-\rho^{i}$. In other words

$$\mathcal{L}=\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_{\mu}\Phi^{i}\partial_{\nu}\Phi^{i} \quad (47)$$

where

$$\sqrt{-g}g^{\mu\nu}=\frac{1}{1-h^{2}}\begin{pmatrix} h_{--} & 1+h^{2} \\ 1+h^{2} & h_{++}\end{pmatrix}, \quad (48)$$

and the metric has been modified by a constructive interference phenomenon.

To promote a profound investigation into this constructive interference, let us consider a perturbative solution for this problem. To this end let us write Eqs. (40) and (41) as

$$A_{+}^{i}=\frac{1}{2}J_{+}^{i}-\frac{1}{4}h_{++}J_{-}^{i}+h^{2}A_{+}^{i},$$

$$A_{-}^{i}=\frac{1}{2}J_{-}^{i}-\frac{1}{4}h_{--}J_{+}^{i}-h^{2}A_{-}^{i}, \quad (49)$$

and consider the weak field approximation (WFA) where terms of $O(h^{2})\rightarrow 0$. Notice that with this procedure Eq. (49) gives the same result as Eq. (45). To simplify the notation we introduce the vector in the internal space as $\boldsymbol{\phi}=\phi^{i}\hat{e}_{i}$, $\mathbf{A}_{\pm}=A_{\pm}^{i}\hat{e}_{i}$, etc., where $\hat{e}_{i}\hat{e}_{j}=\delta_{ij}$ is an orthogonal basis. Expanding these equations in powers of h^{2} we have, in the zeroth order approximation,

$$\mathbf{A}_{+}^{(0)}=\partial_{+}\rho+\frac{1}{2}h_{++}(\partial_{-}\rho-\partial_{-}\phi),$$

$$\mathbf{A}_{-}^{(0)}=\partial_{+}\phi-\frac{1}{2}h_{--}(\partial_{+}\rho-\partial_{+}\phi). \quad (50)$$

The Lagrangian (39) and the Noether currents are, respectively,

$$\mathcal{L}=\mathcal{L}_{+}^{0}+\mathcal{L}_{-}^{0}-\mathbf{A}_{+}\mathbf{J}_{-}-\mathbf{A}_{-}\mathbf{J}_{+}+\frac{h_{--}}{2}(\mathbf{A}_{+})^{2}+\frac{h_{++}}{2}(\mathbf{A}_{-})^{2}+2\mathbf{A}_{-}\mathbf{A}_{+} \quad (51)$$

and

$$J_{i}^{+}=2\partial_{-}\phi^{i}+h_{--}\partial_{+}\phi^{i}, \quad (52)$$

$$J_{i}^{-}=2\partial_{+}\rho^{i}+h_{++}\partial_{-}\rho^{i}. \quad (53)$$

Substituting all the values in Eq. (51) we have that

$$\mathcal{L}_{WFA}=\partial_{+}\mathbf{\Phi}\partial_{-}\mathbf{\Phi}+\frac{h_{--}}{2}\partial_{+}\mathbf{\Phi}\partial_{+}\mathbf{\Phi}+\frac{h_{++}}{2}\partial_{-}\mathbf{\Phi}\partial_{-}\mathbf{\Phi}, \quad (54)$$

where, as usual, $\mathbf{\Phi}=\Phi^{i}\hat{e}^{i}=(\phi^{i}-\rho^{i})\hat{e}^{i}$. Therefore,

$$\mathcal{L}_{WFA}=\frac{1}{2}\sqrt{-g_{(0)}}g_{(0)}^{\mu\nu}\partial_{\mu}\mathbf{\Phi}\partial_{\nu}\mathbf{\Phi}, \quad (55)$$

which is the result obtained from Eq. (47) when $\sqrt{-g}g^{\mu\nu} \xrightarrow{h^2\to 0} \sqrt{-g_{(0)}}g^{\mu\nu}_{(0)}$. Considering the inclusion of higher orders of the h^2 term in W_2 ($d_{ijk}\to 0$) in

$$A^i_\pm = \frac{1}{1-h^2}\left[\frac{1}{2}J^\mp_i - \frac{1}{4}h_{\pm\pm}J^\pm_i\right], \tag{56}$$

we obtain from Eqs. (56) and (51)

$$\mathcal{L}_{h^2} = \frac{1+h^2}{1-h^2}\partial_+\Phi^i\partial_-\Phi^i + \frac{h_{--}}{2(1-h^2)}\partial_+\Phi^i\partial_+\Phi^i + \frac{h_{++}}{2(1-h^2)}\partial_-\Phi^i\partial_-\Phi^i. \tag{57}$$

We can see clearly that Eq. (54) is the zeroth order approximation of the action (46) with $h^2\to 0$: i.e.,

$$\mathcal{L}_{WFA} = \mathcal{L}_{h^2}(h^2\to 0). \tag{58}$$

Hence we can assume that the perturbative procedure in the soldering fields has disclosed an interesting behavior. The zeroth order approximation written in Eq. (50) showed that in Eq. (54) the interference between the two distinct chiralities in $\mathcal{W}_2$ is, disregarding a cross term, the simple sum of both actions. However, taking into consideration the h^2 terms we see that such a behavior is not true any longer. As we have stressed, Eq. (57) is not a trivial result: both chiral particles are now parts of the same multiplet and we have a modification of the metric through a constructive interference. We can see clearly that Eq. (54) is just a $h^2\to 0$ approximation of the exact action written in Eq. (57) or compactly written in Eq. (46). Next we will prove in a precise way that this behavior can be seen in all spin-s $\mathcal{W}$ theories.

B. Weak field approach to the soldering of chiral $\mathcal{W}_3$

Let us next analyze the $\mathcal{W}_3$ model for the right-handed chirality, Eq. (24),

$$\mathcal{L}^0_+ = \partial_+\phi^i\partial_-\phi^i + \frac{h_{--}}{2}\partial_+\phi^i\partial_+\phi^i + \frac{B_{---}}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k; \tag{59}$$

this is the action for lowest nonminimal coupling [45] with k, the expansion parameter for the Noether method, equal to -1.

The gauge transformation is

$$\phi^i \to \phi^i + \alpha^i, \tag{60}$$

leading to a gauge variation of the model as

$$\delta\mathcal{L}^0_+ = J^+_i\partial_+\alpha^i, \tag{61}$$

where J^+_i is the left Noether current

$$J^+_i = 2\partial_-\phi^i + h_{--}\partial_+\phi^i + B_{---}d_{ijk}\partial_+\phi^j\partial_+\phi^k. \tag{62}$$

Following again the soldering algorithm, we have after two iterations that

$$\delta\mathcal{L}^2_+ = -2A^i_+\delta A^i_-, \tag{63}$$

where $A^i_\pm$ are the soldering fields. For the left chirality we can write

$$\mathcal{L}^0_- = \partial_+\rho^i\partial_-\rho^i + \frac{h_{++}}{2}\partial_-\rho^i\partial_-\rho^i + \frac{B_{+++}}{3}d_{ijk}\partial_-\rho^i\partial_-\rho^j\partial_-\rho^k. \tag{64}$$

The gauge variation of the model is

$$\delta\mathcal{L}^0_- = J^-_i\partial_-\alpha^i, \tag{65}$$

where J^-_i is the right Noether current

$$J^-_i = 2\partial_+\rho^i + h_{++}\partial_-\rho^i + B_{+++}d_{ijk}\partial_-\phi^j\partial_-\phi^k. \tag{66}$$

Again, after two iterations we have that

$$\delta\mathcal{L}^2_- = -2A^i_-\delta A^i_+, \tag{67}$$

and the final soldered action is

$$\mathcal{L}_{FINAL} = \mathcal{L}^2_+ + \mathcal{L}^2_- + 2A^i_-A^i_+ \tag{68}$$

which has a vectorial gauge invariance, i.e., $\delta\mathcal{L}=0$.

Substituting all the $\mathcal{L}^{(N)}_\pm$ we can write the final form of the action explicitly as

$$\mathcal{L}_{FINAL} = \mathcal{L}^0_+ + \mathcal{L}^0_- - A^i_+J^+_i - A^i_-J^-_i + \frac{h_{--}}{2}(A^i_+)^2 + \frac{h_{++}}{2}(A^i_-)^2 + B_{---}d_{ijk}\left[A^i_+A^j_+\partial_+\phi^k - \frac{1}{3}A^i_+A^j_+A^k_+\right] + B_{+++}d_{ijk}\left[A^i_-A^j_-\partial_-\rho^k - \frac{1}{3}A^i_-A^j_-A^k_-\right] + 2A^i_-A^i_+. \tag{69}$$

Solving the equations of motion for the soldering fields, these can be eliminated in favor of the other variables,

$$A^i_+ = \frac{1}{2}J^-_i - \frac{1}{2}h_{++}A^i_- + B_{+++}d_{ijk}A^j_-\left(\frac{A^k_-}{6} - \frac{\partial_+\rho^k}{2}\right), \tag{70}$$

$$A^i_- = \frac{1}{2} J_i^+ - \frac{1}{2} h_{--} A^i_+ + B_{---} d_{ijk} A^j_+ \left(\frac{A^k_+}{6} - \frac{\partial_+ \phi^k}{2} \right). \tag{71}$$

Substituting Eq. (71) in Eq. (70) and, to be concise, writing only the solution for the A^i_+, we have

$$\begin{aligned} A^i_+ = {} & \frac{1}{2} J_i^- - \frac{1}{4} h_{++} J_i^+ + \frac{1}{24} B_{+++} d_{ijk} J_j^+ J_k^+ - \frac{1}{4} B_{+++} d_{ijk} J_j^+ J_k^+ \partial_- \rho^k + \frac{1}{4} h_{++} h_{--} A^i_+ + \frac{1}{4} h_{++} B_{---} d_{ijk} A^j_+ \partial_+ \phi^k \\ & - \frac{1}{12} h_{--} B_{+++} d_{ijk} J_j^+ A^k_+ - \frac{1}{12} B^2 d_{ijk} d_{kmn} J_j^+ A^m_+ \partial_+ \phi^n + \frac{1}{4} h_{--} B_{+++} d_{ijk} A^j_+ \partial_- \rho^k + \frac{1}{4} B^2 d_{ijk} d_{jmn} A^m_+ \partial_+ \phi^n \partial_- \rho^k \\ & - \frac{1}{12} h_{++} B_{---} d_{ijk} A^j_+ A^k_+ + \frac{1}{36} d_{ijk} d_{kmn} J_j^+ A^m_+ A^n_+ + \frac{1}{24} h^2_{--} B_{+++} d_{ijk} A^j_+ A^k_+ + \frac{1}{12} h_{--} B^2 d_{ijk} d_{kmn} A^j_+ A^m_+ \partial_+ \phi^n \\ & + \frac{1}{24} B_{---} B^2 d_{ijk} d_{jmn} d_{kpq} A^m_+ A^p_+ \partial_+ \phi^n \partial_+ \phi^q - \frac{1}{12} B^2 d_{ijk} d_{jmn} A^m_+ A^n_+ \partial_- \rho^k - \frac{1}{36} h_{--} B^2 d_{ijk} d_{jmn} A^k_+ A^m_+ A^n_+ \\ & - \frac{1}{72} B_{---} B^2 d_{ijk} d_{jmn} d_{kpq} A^m_+ A^p_+ A^n_+ (\partial_+ \phi^q A^n_+ + \partial_+ \phi^n A^q_+) + \frac{1}{216} B_{---} B^2 d_{ijk} d_{jmn} d_{kpq} A^m_+ A^p_+ A^n_+ A^q_+ , \end{aligned} \tag{72}$$

where $B^2 = B_{+++} B_{---}$.

Now we can solder the Lagrangian with $d_{ijk} \neq 0$ and take the terms of zero order ($h^2 \to 0$) of A^i_+ in Eq. (72), with the Noether currents

$$\begin{aligned} A^i_+ = {} & \partial_+ \rho^i + \frac{1}{2} h_{++} (\partial_- \rho^i - \partial_- \phi^i) \\ & + \frac{1}{2} B_{+++} d_{ijk} \partial_- \rho^k (\partial_- \rho^j - \partial_- \phi^j) \\ & + \frac{1}{6} B_{+++} d_{ijk} \partial_- \phi^j \partial_- \phi^k, \end{aligned} \tag{73}$$

$$\begin{aligned} A^i_- = {} & \partial_- \phi^i + \frac{1}{2} h_{--} (\partial_+ \rho^i - \partial_+ \phi^i) \\ & + \frac{1}{2} B_{---} d_{ijk} \partial_+ \phi^k (\partial_+ \phi^j - \partial_- \rho^j) \\ & + \frac{1}{6} B_{---} d_{ijk} \partial_+ \rho^j \partial_+ \rho^k. \end{aligned} \tag{74}$$

Substituting in Eq. (69) and computing the soldering fields through the equations of motion we have the soldered action for the WFA $\mathcal{W}_3$ model and, after an arduous algebra where we have considered the h^2 terms, with $\Phi^i = \phi^i - \rho^i$,

$$\begin{aligned} \mathcal{L}_{WFA} = {} & \partial_+ \Phi^i \partial_- \Phi^i + \frac{1}{2} h_{--} \partial_+ \Phi^i \partial_+ \Phi^i \\ & + \frac{1}{2} h_{++} \partial_- \Phi^i \partial_- \Phi^i + \frac{1}{3} B_{+++} d_{ijk} \partial_- \Phi^i \partial_- \Phi^j \partial_- \Phi^k \\ & + \frac{1}{3} B_{---} d_{ijk} \partial_+ \Phi^i \partial_+ \Phi^j \partial_+ \Phi^k \end{aligned} \tag{75}$$

$$\begin{aligned} \mathcal{L}_{h^2} = {} & \frac{1+h^2}{1-h^2} \partial_+ \Phi^i \partial_- \Phi^i + \frac{h_{--}}{2(1-h^2)} \partial_+ \Phi^i \partial_+ \Phi^i \\ & + \frac{h_{++}}{2(1-h^2)} \partial_- \Phi^i \partial_- \Phi^i \\ & + \frac{1}{3(1-h^2)} B_{+++} d_{ijk} \partial_- \Phi^i \partial_- \Phi^j \partial_- \Phi^k \\ & + \frac{1}{3(1-h^2)} B_{---} d_{ijk} \partial_+ \Phi^i \partial_+ \Phi^j \partial_+ \Phi^k, \end{aligned} \tag{76}$$

where we can see again that

$$\mathcal{L}_{WFA} = \mathcal{L}_{h^2}(h^2 \to 0), \tag{77}$$

demonstrating that what occurred to the $\mathcal{W}_2$ theory happened to $\mathcal{W}_3$, and again we have a constructive interference modifying the metric.

The first-order action (75) is similar to that found by Schoutens, Sevrin, and van Nieuwenhuizen (SSN) [45] for the spin-s theory (the SSN action), to describe a $\mathcal{W}$ string propagating on a flat background spacetime metric. The $\partial_\pm \Phi^i$ substitutes the so-called "nested covariant derivatives." In addition we have also obtained a reduction in the infinite nonlinearity. The soldered action couples both chiral scalar fields to a dynamical gauge field. This action is characteristic of an interference process which leads to the new and nontrivial result of the modification of the metric by constructive interference, and proves that the SSN action can be an approximation of a more general action. Next we will look at the spin- $s > 3$ generalization of the SSN-like action.

C. $\mathcal{W}_N$, $N\geqslant 4$

In this section we will write only the final results of the interference mechanism, i.e., for the $\mathcal{W}_4$ algebra, as we know,

$$\mathcal{L}_+^0=\partial_+\phi^i\partial_-\phi^i+\frac{h_{--}}{2}\partial_+\phi^i\partial_+\phi^i+\frac{B_{---}}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k+\frac{B_{----}}{4}d_{ijkl}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k\partial_+\phi^l, \tag{78}$$

$$\mathcal{L}_-^0=\partial_+\rho^i\partial_-\rho^i+\frac{h_{++}}{2}\partial_-\rho^i\partial_-\rho^i+\frac{B_{+++}}{3}d_{ijk}\partial_-\rho^i\partial_-\rho^j\partial_-\rho^k+\frac{B_{++++}}{4}d_{ijkl}\partial_-\rho^i\partial_-\rho^j\partial_-\rho^k\partial_-\rho^l, \tag{79}$$

where d_{ijkl} is another symmetric tensor [35].

After applying the whole interference method, the WFA and h^2-term actions are, respectively,

$$\mathcal{L}_{WFA}=\partial_+\Phi^i\partial_-\Phi^i+\frac{1}{2}h_{--}\partial_+\Phi^i\partial_+\Phi^i+\frac{1}{2}h_{++}\partial_-\Phi^i\partial_-\Phi^i++\frac{1}{3}B_{+++}d_{ijk}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k+\frac{1}{3}B_{---}d_{ijk}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k+\frac{1}{4}B_{++++}d_{ijkl}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k\partial_-\Phi^l+\frac{1}{4}B_{----}d_{ijkl}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k\partial_+\Phi^l, \tag{80}$$

$$\mathcal{L}_{h^2}=\frac{1+h^2}{1-h^2}\partial_+\Phi^i\partial_-\Phi^i+\frac{h_{--}}{2(1-h^2)}\partial_+\Phi^i\partial_+\Phi^i+\frac{h_{++}}{2(1-h^2)}\partial_-\Phi^i\partial_-\Phi^i+\frac{1}{3(1-h^2)}B_{+++}d_{ijk}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k+\frac{1}{3(1-h^2)}B_{---}d_{ijk}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k+\frac{B_{++++}}{4(1-h^2)}d_{ijkl}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k\partial_-\Phi^l+\frac{B_{----}}{4(1-h^2)}d_{ijkl}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k\partial_+\Phi^l. \tag{81}$$

Finally, for a $\mathcal{W}$ gravity of spin s, for both chiralities, respectively,

$$\mathcal{L}_+^0=\partial_+\phi^i\partial_-\phi^i+\frac{h_{--}}{2}\partial_+\phi^i\partial_+\phi^i+\frac{B_{---}}{3}d_{ijk}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k+\frac{1}{4}B_{----}d_{ijkl}\partial_+\phi^i\partial_+\phi^j\partial_+\phi^k\partial_+\phi^l+\cdots+\frac{1}{s}B_{--\ldots-}d_{i_1i_2\cdots i_s}\partial_+\phi^{i_1}\partial_+\phi^{i_2}\cdots\partial_+\phi^{i_s}, \tag{82}$$

$$\mathcal{L}_-^0=\partial_+\rho^i\partial_-\rho^i+\frac{h_{++}}{2}\partial_-\rho^i\partial_-\rho^i+\frac{B_{+++}}{3}d_{ijk}\partial_-\rho^i\partial_-\rho^j\partial_-\rho^k+\frac{1}{4}B_{++++}d_{ijkl}\partial_-\rho^i\partial_-\rho^j\partial_-\rho^k\partial_-\rho^l+\cdots+\frac{1}{s}B_{++\ldots+}d_{i_1i_2\cdots i_s}\partial_-\phi^{i_1}\partial_-\phi^{i_2}\cdots\partial_-\phi^{i_s}, \tag{83}$$

and the final actions are

$$\mathcal{L}_{WFA}=\partial_+\Phi^i\partial_-\Phi^i+\frac{1}{2}h_{--}\partial_+\Phi^i\partial_+\Phi^i+\frac{1}{2}h_{++}\partial_-\Phi^i\partial_-\Phi^i+\frac{1}{3}B_{+++}d_{ijk}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k+\frac{1}{3}B_{---}d_{ijk}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k+\frac{1}{4}B_{++++}d_{ijkl}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k\partial_-\Phi^l+\frac{1}{4}B_{----}d_{ijkl}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k\partial_+\Phi^l+\cdots+\frac{B_{++\cdots+}}{s}d_{i_1i_2\cdots i_s}\partial_-\Phi^{i_1}\partial_-\Phi^{i_2}\cdots\partial_-\Phi^{i_s}+\frac{B_{--\cdots-}}{s}d_{i_1i_2\cdots i_s}\partial_+\Phi^{i_1}\partial_+\Phi^{i_2}\cdots\partial_+\Phi^{i_s}, \tag{84}$$

$$\mathcal{L}_{h^2}=\frac{1+h^2}{1-h^2}\partial_+\Phi^i\partial_-\Phi^i+\frac{h_{--}}{2(1-h^2)}\partial_+\Phi^i\partial_+\Phi^i+\frac{h_{++}}{2(1-h^2)}\partial_-\Phi^i\partial_-\Phi^i+\frac{1}{3(1-h^2)}B_{+++}d_{ijk}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k$$
$$+\frac{1}{3(1-h^2)}B_{---}d_{ijk}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k+\frac{1}{4(1-h^2)}B_{++++}d_{ijkl}\partial_-\Phi^i\partial_-\Phi^j\partial_-\Phi^k\partial_-\Phi^l$$
$$+\frac{1}{4(1-h^2)}B_{----}d_{ijkl}\partial_+\Phi^i\partial_+\Phi^j\partial_+\Phi^k\partial_+\Phi^l+\cdots+\frac{B_{++\ldots+}}{s(1-h^2)}d_{i_1i_2\cdots i_s}\partial_-\Phi^{i_1}\partial_-\Phi^{i_2}\cdots\partial_-\Phi^{i_s}$$
$$+\frac{B_{--\ldots-}}{s(1-h^2)}d_{i_1i_2\cdots i_s}\partial_+\Phi^{i_1}\partial_+\Phi^{i_2}\cdots\partial_+\Phi^{i_s}. \tag{85}$$

Now we have a first-order action for spin $s\geqslant 4$ theories. Hence, we can conjecture if the SSN action is an $h^2\to 0$ approximation of a more general action as can be seen by comparing Eqs. (84) and (85).

IV. FINAL REMARKS AND PERSPECTIVES

The quantization of such a system of matter coupled to gravity defines a string theory. This interesting behavior warrants a study of the fusion of W algebras coupled to gravity. We have obtained an action similar to that obtained by SSN for spin-3 gravities. The result showed us that the SSN action can be an approximation of a more general action where the metric is modified. We have demonstrated in a precise way that this behavior is confirmed in spin $s>3$ gravities.

As a final remark, in particular for further studies, we can analyze chiral W_3 gravity, where to cancel the anomaly we have to add finite local counterterms. Considering the nonchiral W_3 gravity, the relation between the dynamical (chiral) decomposition and the factorization that occurs in a closed W_3 string can be analyzed. There, the Hilbert space factorizes as usual into a tensor product of the Hilbert spaces of the left-moving states with those of the right-moving states. The Hilbert space H of the left movers is then the product of the single-boson Fock space F_ϕ with the Hilbert space of the effective conformal field theory $\widetilde{H}$, and $H=F_\phi\otimes\widetilde{H}$.

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APPENDIX: A REVIEW OF THE INTERFERENCE MECHANISM

The technique of soldering (interference) essentially comprises simultaneously lifting the gauging of global symmetry of two self-dual and anti-self-dual actions to their local version [31,29]. We remark that the direct sum of the classical actions depending on different fields would not give anything new. It is the soldering process that leads to a new and nontrivial result.

The basic idea of the soldering procedure is to raise a global Noether symmetry of the constituent actions into a local one, but for an effective composite system, consisting of the dual components and an interference term. This algorithm, consequently, defines the soldered action. Here we shall adopt an iterative Noether procedure to lift the global symmetries. Therefore, assume that the symmetries in question are being described by the local actions $S_\pm(\phi_\pm^\eta)$, invariant under the global multiparametric transformation

$$\delta\phi_\pm^\eta=\alpha^\eta. \tag{A1}$$

Here η represents the tensorial character of the basic fields in the dual actions $S_\pm$ and, for notational simplicity, will be dropped from now on. Now, under local transformations these actions will not remain invariant, and Noether counterterms become necessary to reestablish the invariance, along with appropriate compensatory soldering fields $B^{(N)}$,

$$S_\pm(\phi_\pm)^{(0)}\to S_\pm(\phi_\pm)^{(N)}$$
$$=S_\pm(\phi_\pm)^{(N-1)}-B^{(N)}J_\pm^{(N)}. \tag{A2}$$

Here $J_\pm^{(N)}$ are the Noether currents, and N is the iteration number. For the self- and anti-self-dual systems we must have in mind that this iterative gauging procedure is (intentionally) constructed not to produce invariant actions for any finite number of steps. However, if after N repetitions the noninvariant piece ends up being dependent only on the gauging parameters, but not on the original fields, there will exist the possibility of mutual cancellation, if both self- and anti-self-gauged systems are put together. Then, suppose that after N repetitions we arrive at the following simultaneous conditions:

$$\delta S_\pm(\phi_\pm)^{(N)}\neq 0,$$
$$\delta S_B(\phi_\pm)=0, \tag{A3}$$

with

$$S_B(\phi_\pm)=S_+^{(N)}(\phi_+)+S_-^{(N)}(\phi_-)+\text{contact terms}; \tag{A4}$$

then we can immediately identify the (soldering) interference term as

$$S_{int}=\text{contact terms}-\sum_N B^{(N)}J_\pm^{(N)}, \tag{A5}$$

where the contact terms are generally higher order functions of the soldering fields. Incidentally, these auxiliary fields $B^{(N)}$ may be eliminated, in some cases, from the resulting effective action in favor of the physically relevant degrees of freedom. It is important to notice that, after elimination of the soldering fields, the resulting effective action will not depend on either self- or anti-self-dual fields $\phi_\pm$ but only in some collective field, say Φ, defined in terms of the original ones in a (Noether) invariant way:

$$S_B(\phi_\pm)\rightarrow S_{eff}(\Phi). \tag{A6}$$

Once such an effective action has been established, the physical consequences of the soldering are readily obtained by simple inspection.

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