Landau-Khalatnikov-Fradkin transformations and the fermion propagator in quantum electrodynamics

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We study the gauge covariance of the massive fermion propagator in three as well as four-dimensional quantum electrodynamics (QED). Starting from its value at the lowest order in perturbation theory, we evaluate a nonperturbative expression for it by means of its Landau-Khalatnikov-Fradkin (LKF) transformation. We compare the perturbative expansion of our findings with the known one-loop results and observe perfect agreement up to a gauge parameter independent term, a difference permitted by the structure of the LKF transformations.

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I. INTRODUCTION

In gauge field theory, Green functions transform in a specific manner under a variation of the gauge. In quantum electrodynamics (QED) these transformations carry the name Landau-Khalatnikov-Fradkin (LKF) transformations, [1–3]. These were derived also by Johnson and Zumino through functional methods [4,5].[1] LKF transformations are nonperturbative in nature and hence have the potential of playing an important role in addressing the problems of gauge invariance which plague the strong coupling studies of Schwinger-Dyson equations (SDE). In general, the rules governing these transformations are far from simple. The fact that they are written in coordinate space adds to their complexity. As a result, these transformations have played a less significant and practical role in the study of SDE than desired.

A consequence of gauge covariance is Ward-Green-Takahashi identities (WGTI) [7–9], which are simpler to use and, therefore, have been extensively implemented in the SDE studies which are based either upon gauge technique, e.g., [10–17], or upon making an *ansatz* for the full fermion-boson vertex, e.g., [18–25]. WGTI follow from the Becchi-Rouet-Stora-Tyutin (BRST) symmetry. One can enlarge these transformations by transforming also the gauge parameter ξ [26,27] to arrive at modified Ward identities, known as Nielsen identities (NI). An advantage of the NI over the conventional Ward identities is that $\partial/\partial\xi$ becomes a part of the new relations involving Green functions. This fact was exploited in [28,29] to prove the gauge independence of some of the quantities related to two-point Green functions at the one-loop level and to all orders in perturbation theory, respectively. As it is a difficult task to establish the gauge independence of physical observables in the study of SDE, NI may play a significant role in addressing this issue in addition to Ward identies and LKF transformations. However, in this paper, we concentrate only on the LKF transformations.

The LKF transformation for the three-point vertex is complicated and hampers direct extraction of analytical restrictions on its structure. Burden and Roberts [30] carried out a numerical analysis to compare the self-consistency of various *ansatze* for the vertex [18,19,31] by means of its LKF transformation. In addition to these numerical constraints, indirect analytical insight can be obtained on the nonperturbative structure of the vertex by demanding correct gauge covariance properties of the fermion propagator. In the context of gauge technique, examples are [32–34]. Concerning the works based upon choosing a vertex *ansatze*, Refs. [19,24,25,35–37] employ this idea.[2] However, all the work in the later category has been carried out for massless three-dimensional QED (QED_3) and four-dimensional QED (QED_4). The masslessness of the fermions implies that the fermion propagator can be written only in terms of one function, the so-called wave function renormalization $F(p)$. In order to apply the LKF transform, one needs to know a Green function at least in one particular gauge. This is a formidable task. However, one can rely on approximations based on perturbation theory. It is customary to take $F(p)=1$ in the Landau gauge, an approximation justified by a one-loop calculation of the massless fermion propagator in arbitrary dimensions, see for example, [39]. The LKF transformation then implies a power law for $F(p)$ in QED_4 and a simple trigonometric function in QED_3. To improve upon these results, one can take two paths: (i) incorporate the information contained in higher orders of perturbation theory and (ii) study the massive theory. As pointed out in [36], in QED_4, the power law structure of the wave function renormalization remains intact by increasing order of approximation in perturbation theory although the exponent of course gets a contribution from next to leading logarithms and so on.[3] In [36], constraint was obtained on the 3-point vertex by considering a power law where the exponent of this power law was not restricted only to the one-loop fermion propaga-

[1] Fukuda, Kubo, and Yokoyama have looked at possible formalisms where the wave function renormalization constants can actually be made gauge invariant [6].

[2] A criticism of the vertex construction in [37] was raised in [38].

[3] For the two-loop calculation of the fermion propagator, see for example [40].

tor. In QED_3 the two-loop fermion propagator was evaluated in [38,41,42], where it was explicitly shown that the the approximation $F(p)=1$ is only valid up to one loop, thus violating the *transversality condition* advocated in [37]. The result found there was used used in [43] to find the improved LKF transform.

In the present paper we calculate the LKF transformed fermion propagator in massive QED_3 and QED_4.[4] We start with the simplest input which corresponds to the lowest order of perturbation theory, i.e., $S(p)=1/i\not{p}-m$ in the Landau gauge. On LKF transforming, we find the fermion propagator in an arbitrary covariant gauge. In the case of QED_3, we obtain the result in terms of basic functions of momenta. In QED_4 the final expression is in the form of hypergeometric functions. Coupling α enters as a parameter of this transcendental function. A comparison with perturbation theory needs the expansion of the hypergeometric function in terms of its parameters. We use the technique developed by Moch *et al.* [44] for the said expansion. We compare our results with the one-loop expansion of the fermion propagator in QED_4 and QED_3 [45,46], and find perfect agreement up to terms independent of the gauge parameter at one loop, a difference permitted by the structure of the LKF transformations. We believe that the incorporation of LKF transformations, along with WGT identities, in the SDE can play a key role in addressing the problems of gauge invariance. For example, in the study of the SDE of the fermion propagator, only those assumptions should be permissible which keep intact the correct behavior of the Green functions under the LKF transformations, in addition to ensuring that the WGTI is satisfied. It makes it vital to explore how two- and three-point Green functions transform in a gauge covariant fashion. In this paper we consider only a two-point function, namely, the fermion propagator.

II. FERMION PROPAGATOR AND THE LKF TRANSFORMATION

We start by expanding out the fermion propagator, in momentum and coordinate spaces, respectively, in its most general form as follows:

$$S_F(p;\xi)=A(p;\xi)+i\,\frac{B(p;\xi)}{\not{p}}\equiv\frac{F(p;\xi)}{i\not{p}-\mathcal{M}(p;\xi)},\qquad(1)$$

$$S_F(x;\xi)=\not{x}X(x;\xi)+Y(x;\xi),\qquad(2)$$

where $F(p;\xi)$ is generally referred to as the wave function renormalization and $\mathcal{M}(p;\xi)$ as the mass function. ξ is the usual covariant gauge parameter. Motivated from the lowest order perturbation theory, we take

$$F(p;0)=1\quad\text{and}\quad\mathcal{M}(p;0)=m.\qquad(3)$$

[4]In the context of gauge technique, gauge covariance of the spectral functions in QED was studied in [32–34].

Perturbation theory also reveals that this result continues to hold true to one-loop order for the wave function renormalization. Equations (1), (2) are related to each other through the following Fourier transforms:

$$S_F(p;\xi)=\int d^dx e^{ip\cdot x}S_F(x;\xi),\qquad(4)$$

$$S_F(x;\xi)=\int\frac{d^dp}{(2\pi)^d}e^{-ip\cdot x}S_F(p;\xi),\qquad(5)$$

where d is the dimension of space-time. The LKF transformation relating the coordinate space fermion propagator in Landau gauge to the one in an arbitrary covariant gauge reads

$$S_F(x;\xi)=S_F(x;0)\mathrm{e}^{-i[\Delta_d(0)-\Delta_d(x)]},\qquad(6)$$

where

$$\Delta_d(x)=-i\xi e^2\mu^{4-d}\int_0^\infty\frac{d^dp}{(2\pi)^d}\frac{e^{-ip\cdot x}}{p^4}.\qquad(7)$$

e^2 is the dimensionless electromagnetic coupling. Taking ψ to be the angle between x and p, we can write $d^dp=dpp^{d-1}\sin^{d-2}\psi d\psi\Omega_{d-2}$, where $\Omega_{d-2}=2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$. Hence

$$\Delta_d(x)=-i\xi e^2\mu^{4-d}f(d)\int_0^\infty dpp^{d-5}\times\int_0^\pi d\psi\sin^{d-2}\psi e^{-ipx\cos\psi},\qquad(8)$$

where $f(d)=\Omega_{d-2}/(2\pi)^d$. Performing angular and radial integrations, we arrive at the following equation:

$$\Delta_d(x)=-\frac{i\xi e^2}{16(\pi)^{d/2}}(\mu x)^{4-d}\Gamma\left(\frac{d}{2}-2\right).\qquad(9)$$

With these tools at hand, the procedure now is as follows: Start with the lowest order fermion propagator and Fourier transform it to coordinate space, apply the LKF transformation law. Fourier transform the result back to momentum space.

III. THREE-DIMENSIONAL CASE

Employing Eqs. (1), (2), (3), (5), the lowest order three-dimensional fermion propagator in Landau gauge in the position space is given by

$$X(x;0)=-\frac{e^{-mx}(1+mx)}{4\pi x^3},$$

$$Y(x;0)=-\frac{me^{-mx}}{4\pi x}.\qquad(10)$$

Once in the coordinate space, we can apply the LKF transformation law using expression (9) explicitly in three dimensions:

$$\Delta_3(x) = -\frac{i\alpha\xi x}{2}, \tag{11}$$

where $\alpha = e^2/4\pi$. The fermion propagator in an arbitrary gauge is then

$$S_F(x;\xi) = S_F(x;0)\, e^{-(\alpha\xi/2)x}. \tag{12}$$

For Fourier transforming back to momentum space, we use

$$A(p;\xi) = -\frac{F(p;\xi)\mathcal{M}(p;\xi)}{p^2+\mathcal{M}^2(p;\xi)} = \int d^3x e^{ip\cdot x} Y(x;\xi),$$

$$iB(p;\xi) = -\frac{ip^2 F(p;\xi)}{p^2+\mathcal{M}^2(p;\xi)} = \int d^3x p\cdot x e^{ip\cdot x} X(x;\xi). \tag{13}$$

Performing the angular integration, we get

$$A(p;\xi) = -\frac{m}{p}\int_0^\infty dx \sin px\, \mathrm{e}^{-(m+\alpha\xi/2)x}, \tag{14}$$

$$B(p;\xi) = \frac{1}{p}\int_0^\infty \frac{dx}{x^2}(1+mx)[px\cos px - \sin px] \times \mathrm{e}^{-(m+\alpha\xi/2)x}, \tag{15}$$

and the radial integration then yields

$$A(p;\xi) = -\frac{4m}{4p^2+(2m+\alpha\xi)^2}, \tag{16}$$

$$B(p;\xi) = -\frac{4p^2+\alpha\xi(2m+\alpha\xi)}{4p^2+(2m+\alpha\xi)^2} + \frac{\alpha\xi}{2p}\arctan[2p/(2m+\alpha\xi)]. \tag{17}$$

One can now arrive at the following expressions for the wave function renormalization and the mass function, respectively:

$$F(p;\xi) = -\frac{\alpha\xi}{2p}\arctan[2p/(2m+\alpha\xi)] + \frac{2p(4p^2+\alpha^2\xi^2) - \alpha\xi[4p^2+\alpha\xi(2m+\alpha\xi)]\arctan[2p/(2m+\alpha\xi)]}{2p[4p^2+\alpha\xi(2m+\alpha\xi)] - \alpha\xi[4p^2+(2m+\alpha\xi)^2]\arctan[2p/(2m+\alpha\xi)]}, \tag{18}$$

$$\mathcal{M}(p;\xi) = \frac{8p^3 m}{2p[4p^2+\alpha\xi(2m+\alpha\xi)] - \alpha\xi[4p^2+(2m+\alpha\xi)^2]\arctan[2p/(2m+\alpha\xi)]}. \tag{19}$$

In the massless limit, one immediately recuperates the well-known results

$$F_{\text{massless}}(p;\xi) = 1 - \frac{\alpha\xi}{2p}\arctan\frac{2p}{\alpha\xi},$$

$$\mathcal{M}_{\text{massless}}(p;\xi) = 0. \tag{20}$$

In the weak coupling, we can expand out Eqs. (18), (19) in powers of α. To $\mathcal{O}(\alpha)$, we find

$$F(p;\xi) = 1 + \frac{\alpha\xi}{2p^3}\{(m^2-p^2)\arctan[p/m] - mp\}, \tag{21}$$

$$\mathcal{M}(p;\xi) = m\left[1 + \frac{\alpha\xi}{2p^3}\{(m^2+p^2)\arctan[p/m] - mp\}\right]. \tag{22}$$

Let us compare these results with the ones obtained in [46]:

$$F_{\text{1-loop}}(p;\xi) = 1 + \frac{\alpha\xi}{2p^3}\{(m^2-p^2)\arctan[p/m] - mp\}, \tag{23}$$

$$\mathcal{M}_{\text{1-loop}}(p;\xi) = m\left[1 + \frac{\alpha}{2p^3}\{[\xi(m^2+p^2)+4p^2] \times \arctan[p/m] - mp\}\right]. \tag{24}$$

We of course only expect the results to be in agreement up to a term proportional to $\alpha\xi^0$, as allowed by the structure of the LKF transformations. There is no such term in Eq. (23). Therefore, the agreement is exact. Equations (22) and (24) become identical only after we subract out the nonvanishing term in the Landau gauge from Eq. (24) to write out the *subtracted* mass function at one loop as

$$\mathcal{M}^S_{\text{1-loop}}(p;\xi) = m\left[1 + \frac{\alpha}{2p^3}\{\xi(m^2+p^2)\arctan[p/m] - mp\}\right]. \tag{25}$$

One can numerically check that without the above-mentioned subtraction, Eqs. (22), (24) approach the same value only in the large momentum regime.

IV. FOUR-DIMENSIONAL CASE

Employing Eqs. (1), (2), (3), (5), the lowest order four-dimensional fermion propagator in position space is given by

$$X(x;0) = -\frac{m^2}{4\pi^2 x^2} K_2(mx), \tag{26}$$

$$Y(x;0) = -\frac{m^2}{4\pi^2 x} K_1(mx), \tag{27}$$

where K_1 and K_2 are Bessel functions of the second kind. In order to apply the LKF transformation in four dimensions, we expand Eq. (9) around $d=4-\epsilon$ and use the following identities:

$$\Gamma\left(-\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon),$$

$$x^{\epsilon} = 1 + \epsilon \ln x + \mathcal{O}(\epsilon^2),$$

to obtain

$$\Delta_4(x) = i\frac{\xi e^2}{16\pi^{2-\epsilon/2}}\left[\frac{2}{\epsilon} + \gamma + 2\ln \mu x + \mathcal{O}(\epsilon)\right]. \tag{28}$$

Note that we cannot write a similar expression for $\Delta_4(0)$ because of the presence of the term proportional to $\ln x$. Therefore, we introduce a cut-off scale x_{min}. Now

$$\Delta_4(x_{min}) - \Delta_4(x) = -i\ln\left(\frac{x^2}{x_{min}^2}\right)^{\nu}, \tag{29}$$

where $\nu = \alpha\xi/4\pi$. Hence

$$S_F(x;\xi) = S_F(x;0)\left(\frac{x^2}{x_{min}^2}\right)^{-\nu}. \tag{30}$$

For Fourier transforming back to momentum space we use the following expressions:

$$A(p;\xi) = -\frac{F(p;\xi)\mathcal{M}(p;\xi)}{p^2+\mathcal{M}^2(p;\xi)} = \int d^4x e^{ip\cdot x} Y(x;\xi),$$

$$iB(p;\xi) = -\frac{ip^2 F(p;\xi)}{p^2+\mathcal{M}^2(p;\xi)} = \int d^4x p\cdot x e^{ip\cdot x} X(x;\xi). \tag{31}$$

On carrying out angular integration, we obtain

$$A(p;\xi) = -\frac{m^2}{p} x_{min}^{2\nu}\int_0^{\infty} dx x^{-2\nu+1} K_1(mx) J_1(px), \tag{32}$$

$$B(p;\xi) = -m^2\int_0^{\infty} dx x^{-2\nu+1} K_2(mx) J_2(px). \tag{33}$$

The radial integration then yields

$$A(p;\xi) = -\frac{1}{m}\left(\frac{m^2}{\Lambda^2}\right)^{\nu}\Gamma(1-\nu)\Gamma(2-\nu) \times {}_2F_1\left(1-\nu, 2-\nu; 2; -\frac{p^2}{m^2}\right), \tag{34}$$

$$B(p;\xi) = -\frac{p^2}{2m^2}\left(\frac{m^2}{\Lambda^2}\right)^{\nu}\Gamma(1-\nu)\Gamma(3-\nu) \times {}_2F_1\left(1-\nu, 3-\nu; 3; -\frac{p^2}{m^2}\right), \tag{35}$$

where we have identified $2/x_{min} \to \Lambda$. The above equations imply

$$F(p;\xi) = \frac{\Gamma(1-\nu)}{2m^2\Gamma(3-\nu)\,{}_2F_1(1-\nu,3-\nu;3;-p^2/m^2)} \times\left(\frac{m^2}{\Lambda^2}\right)^{\nu}\left[4m^2\Gamma^2(2-\nu)\,{}_2F_1^2\left(1-\nu,2-\nu;2; -\frac{p^2}{m^2}\right) + p^2\Gamma^2(3-\nu)\,{}_2F_1^2\left(1-\nu,3-\nu;3;-\frac{p^2}{m^2}\right)\right], \tag{36}$$

$$\mathcal{M}(p;\xi) = \frac{2m\,{}_2F_1(1-\nu,2-\nu;2;-p^2/m^2)}{(2-\nu)\,{}_2F_1(1-\nu,3-\nu;3;-p^2/m^2)}. \tag{37}$$

Equations (36), (37) constitute the LKF transformation of Eqs. (3). We shall now see that although Eqs. (3) correspond to the lowest order propagator, their LKF transformation, Eqs. (36), (37), is nonperturbative in nature and contains information of higher orders.

A. Case $\alpha=0$

Let us switch off the coupling and put $\alpha=0$ which implies $\nu=0$. Now using the identity

$${}_2F_1(1,2;2;-p^2/m^2) = {}_2F_1(1,3;3;-p^2/m^2) = (1+p^2/m^2)^{-1}, \tag{38}$$

it is easy to see that

$$F(p;\xi) = 1 \quad \text{and} \quad \mathcal{M}(p;\xi) = m, \tag{39}$$

which coincides with the lowest order perturbative result as expected.

B. Case $m \gg p$

In the limit $m \gg p$, the hypergeometric functions in Eqs. (36), (37) can be easily expanded in powers of p^2/m^2, using the identity

$$ {}_2F_1\left(\alpha,\beta;\gamma;-\frac{p^2}{m^2}\right)=1-\frac{\alpha\beta}{\gamma}\frac{p^2}{m^2}+\mathcal{O}\left(\frac{p^2}{m^2}\right)^2. \quad (40) $$

Retaining only $\mathcal{O}(p^2/m^2)$ terms, we arrive at

$$ F(p;\xi)=\frac{\Gamma(1-\nu)\Gamma(2-\nu)}{(1-\nu/2)}\left[1+\frac{2\nu}{3}\left(1-\frac{5\nu}{8}\right)\frac{p^2}{m^2}+\mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right]\left(\frac{m^2}{\Lambda^2}\right)^{\nu}, \quad (41) $$

$$ \mathcal{M}(p;\xi)=\frac{m}{(1-\nu/2)}\left[1+\frac{\nu}{6}(1-\nu)\frac{p^2}{m^2}+\mathcal{O}\left(\frac{p^2}{m^2}\right)^2\right]. \quad (42) $$

Now carrying out an expansion in α and substituting $\nu=\alpha\xi/4\pi$, we get the following $\mathcal{O}(\alpha)$ expressions:

$$ F(p;\xi)=1+\frac{\alpha\xi}{4\pi}\left[2\gamma-\frac{1}{2}+\frac{2p^2}{3m^2}+\ln\frac{m^2}{\Lambda^2}\right], \quad (43) $$

$$ \mathcal{M}(p;\xi)=m\left\{1+\frac{\alpha\xi}{8\pi}\left[1+\frac{p^2}{3m^2}\right]\right\}. \quad (44) $$

Let us now compare these expressions against the one-loop perturbative evaluation of the massive fermion propagator, see, e.g., [45]:

$$ F_{\text{1-loop}}(p;\xi)=1-\frac{\alpha\xi}{4\pi}\left[C\mu^{\epsilon}+\left(1-\frac{m^2}{p^2}\right)(1-L)\right], \quad (45) $$

$$ \mathcal{M}_{\text{1-loop}}(p;\xi)=m+\frac{\alpha m}{\pi}\left[\left(1+\frac{\xi}{4}\right)+\frac{3}{4}(C\mu^{\epsilon}-L)+\frac{\xi}{4}\frac{m^2}{p^2}(1-L)\right], \quad (46) $$

where

$$ L=\left(1+\frac{m^2}{p^2}\right)\ln\left(1+\frac{p^2}{m^2}\right), $$

$$ C=-\frac{2}{\epsilon}-\gamma-\ln\pi-\ln\left(\frac{m^2}{\mu^2}\right). $$

Knowing the fermion propagator even in one particular gauge is a prohibitively difficult task. Therefore, Eqs. (3) have to be viewed only as an approximation. For the wave function renormalization $F(p)$, this approximation is valid up to one-loop order, whereas for the mass function, it is true only to the lowest order. Therefore we cannot expect the LKF transform of Eqs. (3) to yield correctly each term in the perturbative expansion of the fermion propagator. However, it should correctly reproduce all those terms at every order of expansion which vanish in the Landau gauge at $\mathcal{O}(\alpha)$ and beyond. Therefore, we expect Eq. (45) to be exactly reproduced and Eq. (46) to be reproduced up to the terms which vanish in the Landau gauge at $\mathcal{O}(\alpha)$. After subtracting these terms, the resulting *subtracted* mass function is

$$ \mathcal{M}^{S}_{\text{1-loop}}(p;\xi)=m+\frac{\alpha\xi m}{4\pi}\left[1+\frac{m^2}{p^2}(1-L)\right]. \quad (47) $$

In the limit $m\to\infty$, the wave function renormalization acquires the form

$$ F(p;\xi)_{\text{1-loop}}=1+\frac{\alpha\xi}{4\pi}\left[-C\mu^{\epsilon}-\frac{1}{2}+\frac{2p^2}{3m^2}\right], \quad (48) $$

while the *subtracted* mass function is

$$ \mathcal{M}^{S}_{\text{1-loop}}(p;\xi)=m\left\{1+\frac{\alpha\xi}{8\pi}\left[1+\frac{p^2}{3m^2}\right]\right\}. \quad (49) $$

The last two expressions are in perfect agreement with Eqs. (43), (44) after we make the identification:

$$ -C\mu^{\epsilon}\to 2\gamma+\ln\frac{m^2}{\Lambda^2}. \quad (50) $$

C. Case of weak coupling

The case $m\gg p$ is relatively easier to handle as we merely have to expand ${}_2F_1(\beta,\gamma;\delta;x)$ in powers of x and retain only the leading terms. If we want to obtain a series in powers of the coupling alone, we need the expansion of the hypergeometric functions in terms of its parameters β and γ. We follow the technique developed in [44]. One of the mathematical objects we shall use for such an expansion are the Z sums, defined as

$$ Z(n;m_1,\ldots,m_k;x_1,\ldots,x_k)=\sum_{n\geqslant i_1>i_2>\ldots>i_k>0}\frac{x_1^{i_1}}{i_1^{m_1}}\cdots\frac{x_k^{i_k}}{i_k^{m_k}}. \quad (51) $$

For $x_1=\cdots=x_k=1$ the definition reduces to the Euler-Zagier sums [47,48]:

$$ Z(n;m_1,\ldots,m_k;1,\ldots,1)=Z_{m_1,\ldots,m_k}(n). \quad (52) $$

Euler-Zagier sums can be used in the expansion of gamma functions. For positive integers n we have [44]

$$ \Gamma(n+\epsilon)=\Gamma(1+\epsilon)\Gamma(n)[1+\epsilon Z_1(n-1)+\cdots+\epsilon^{n-1}Z_{11\ldots1}(n-1)]. \quad (53) $$

The first sum $Z_1(n-1)$, e.g., is just the $(n-1)$-th harmonic number, H_{n-1}, of order 1:

$$ Z_1(n-1)=\sum_{i=1}^{n-1}\frac{1}{i}\equiv H_{n-1}. \quad (54) $$

With these definitions in hand, we proceed to expand a hypergeometric function, ${}_2F_1(1+\varepsilon,2+\varepsilon;2;x)$, as an example, assuming $|x|<1$:

$$
\begin{aligned}
{}_2F_1&(1+\varepsilon,2+\varepsilon;2;x)\\
&=1+\frac{\Gamma(2)}{\Gamma(1+\varepsilon)\Gamma(2+\varepsilon)}\\
&\quad\times\sum_{n=1}^{\infty}\frac{\Gamma(1+\varepsilon+n)\Gamma(2+\varepsilon+n)}{\Gamma(2+n)}\frac{x^n}{n!}\\
&=1+\frac{1}{(1+\varepsilon)\Gamma^2(1+\varepsilon)}\\
&\quad\times\sum_{n=1}^{\infty}\frac{(1+\varepsilon+n)(\varepsilon+n)^2\Gamma^2(\epsilon+n)}{\Gamma(2+n)}\frac{x^n}{n!}.
\end{aligned}
$$

Employing Eq. (53), we can expand the last expression in powers of ϵ to any desired order of approximation. We shall be interested only in terms up to $\mathcal{O}(\alpha)$,

$$
\begin{aligned}
{}_2F_1&(1+\varepsilon,2+\varepsilon;2;x)\\
&=1+\sum_{n=1}^{\infty}x^n-\varepsilon\sum_{n=1}^{\infty}x^n+\varepsilon\sum_{n=1}^{\infty}\frac{2+3n}{n(n+1)}x^n\\
&\quad+2\varepsilon\sum_{n=1}^{\infty}H_{n-1}x^n. \qquad (55)
\end{aligned}
$$

Performing the summations, we obtain

$$
{}_2F_1(1+\varepsilon,2+\varepsilon;2;x)=\frac{1}{1-x}\left[1-\varepsilon\left\{1+\frac{1+x}{x}\ln(1-x)\right\}\right]. \qquad (56)
$$

Similarly,

$$
\begin{aligned}
{}_2F_1&(1+\varepsilon,3+\varepsilon;3;x)\\
&=\frac{1}{1-x}-\varepsilon\left\{\frac{1}{x}+\frac{3}{2}\frac{1}{1-x}+\left(\frac{1+x}{x^2}+\frac{2}{1-x}\right)\right.\\
&\quad\left.\times\ln(1-x)\right\}. \qquad (57)
\end{aligned}
$$

Substituting back into Eqs. (36), (37) and identifying $\varepsilon=-\nu$, we obtain

$$
F(p;\xi)=1-\frac{\alpha\xi}{4\pi}\left[-2\gamma-\ln\frac{m^2}{\Lambda^2}+\left(1-\frac{m^2}{p^2}\right)(1-L)\right],
$$

$$
\mathcal{M}(p;\xi)=m+\frac{\alpha\xi m}{4\pi}\left[1+\frac{m^2}{p^2}(1-L)\right], \qquad (58)
$$

which matches exactly onto the one-loop result of Eqs. (45), (47) after the same identification as before, i.e., Eq. (50). Therefore, we have seen that the LKF transformation of the bare propagator contains important information of higher orders in perturbation theory.

V. CONCLUSIONS

We have studied the gauge covariance of the massive fermion propagator in three- as well as four-dimensional QED through its LKF transformation, starting from its lowest order approximation. Equations (18), (19), (36), (37) form the main result of this paper. In the three-dimensional case the LKF transformation consists of basic functions of the momentum variable, whereas in the four-dimensional case hypergeometric functions arise with electromagnetic coupling as parameter of these functions. Although our input is only the bare propagator, the corresponding LKF transformation, being nonperturbative in nature, contains useful information of higher orders in perturbation theory. For example, we have shown that a perturbative expansion of our results matches onto the known one-loop results up to gauge-independent terms at this order. This slight difference arises due to our approximated input and can be corrected systematically at the cost of increasing complexity of the integrals involved. We intend to carry out a similar exercise for the three-point fermion-boson vertex. LKF transformations of the propagator and the vertex impose useful constraints on the SDE and we believe that these transformations can be of immense help in addressing the problems of gauge invariance in the related studies.

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APPENDIX

Most of the integrals involved in this paper are listed below for a quick reference [49,50]:

$$
\int_0^{\pi}d\psi\sin^{d-2}\psi\cos\psi e^{-ipx\cos\psi}=-i\sqrt{\pi}\left(\frac{px}{2}\right)^{1-(d/2)}\Gamma\left(\frac{d-1}{2}\right)J_{d/2}(px), \qquad (A1)
$$

$$
\int_0^{\infty}x^{d/2-1}J_{d/2}(ax)=\frac{\Gamma(d/2)}{2^{1-d/2}a^{d/2}}. \qquad (A2)
$$

For the three-dimensional case, the needed integrals are

$$\int_0^\pi d\theta \sin\theta e^{-ipx\cos\theta} = \frac{2\sin px}{px}, \tag{A3}$$

$$\int_0^\pi d\theta \cos\theta \sin\theta e^{-ipx\cos\theta} = 2i\left[\frac{\cos px}{px} - \frac{\sin px}{(px)^2}\right], \tag{A4}$$

$$\int_0^\infty dp \frac{p^3}{(p^2+m^2)}\left[\frac{\cos px}{px} - \frac{\sin px}{(px)^2}\right] = -\frac{\pi}{2}\frac{(1+mx)}{x^2}e^{-mx}, \tag{A5}$$

$$\int_0^\infty dp \frac{p \sin px}{(p^2+m^2)} = \frac{\pi}{2}e^{-mx}, \tag{A6}$$

$$\frac{1}{p}\int_0^\infty \frac{dx}{x^2}e^{-ax}[px\cos px - \sin px] = -1 + \frac{a}{p}\arctan\frac{p}{a}, \tag{A7}$$

$$\frac{1}{p}\int_0^\infty \frac{dx}{x}e^{-ax}[px\cos px - \sin px] = \frac{a}{a^2+p^2} - \frac{1}{p}\arctan\frac{p}{a}, \tag{A8}$$

$$\int_0^\infty dx \sin px e^{-(m+\alpha\xi/2)x} = \frac{p}{(m+\alpha\xi/2)^2+p^2}. \tag{A9}$$

For the four-dimensional case, we used the following integrals in particular:

$$\int_0^\pi d\theta \sin^2\theta e^{-ipx\cos\theta} = \frac{\pi}{px}J_1(px), \tag{A10}$$

$$\int_0^\infty dp \frac{p^{\nu+1}J_\nu(px)}{(p^2+m^2)^{\mu+1}} = \frac{m^{\nu-\mu}x^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(mx), \tag{A11}$$

$$\int_0^\infty dx x^{-\lambda}K_\mu(ax)J_\nu(bx) = \frac{a^{\lambda-\nu-1}b^\nu}{2^{\lambda+1}\Gamma(1+\nu)}\Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right) \times {}_2F_1\left(\frac{\nu-\lambda+\mu+1}{2},\frac{\nu-\lambda-\mu+1}{2};\nu+1;-\frac{b^2}{a^2}\right). \tag{A12}$$

Some of the series used in our calculation are as follows:

$$\sum_{n=1}^\infty H_{n-1}x^n = -\frac{x\ln(1-x)}{1-x}, \tag{A13}$$

$$\sum_{n=1}^\infty \frac{n+1}{n(n+2)}x^n = -\frac{2+x}{4x} - \frac{(1+x^2)\ln(1-x)}{2x^2}, \tag{A14}$$

$$\sum_{n=1}^\infty \frac{1}{(n+1)(n+2)}x^n = \frac{2-x}{2x} + \frac{(1-x)\ln(1-x)}{x^2}. \tag{A15}$$

[1] L.D. Landau and I.M. Khalatnikov, Zh. Eksp. Teor. Fiz. **29**, 89 (1956).
[2] L.D. Landau and I.M. Khalatnikov, Sov. Phys. JETP **2**, 69 (1956).
[3] E.S. Fradkin, Sov. Phys. JETP **2**, 361 (1956).
[4] K. Johnson and B. Zumino, Phys. Rev. Lett. **3**, 351 (1959).
[5] B. Zumino, J. Math. Phys. **1**, 1 (1960).
[6] T. Fukuda, R. Kubo, and K. Yokoyama, Prog. Theor. Phys. **63**, 1384 (1980).
[7] J.C. Ward, Phys. Rev. **78**, 182 (1950).
[8] H.S. Green, Proc. Phys. Soc., London, Sect. A **66**, 873 (1953).
[9] Y. Takahashi, Nuovo Cimento **6**, 371 (1957).
[10] A. Salam, Phys. Rev. **130**, 1287 (1963).
[11] A. Salam and R. Delbourgo, Phys. Rev. **135**, 1398 (1964).
[12] J. Strathdee, Phys. Rev. **135**, 1428 (1964).
[13] R. Delbourgo and P. West, J. Phys. A **10**, 1049 (1977).
[14] R. Delbourgo and P. West, Phys. Lett. **72B**, 96 (1977).
[15] R. Delbourgo, Nuovo Cimento A **49**, 484 (1979).
[16] R. Delbourgo and B.W. Keck, J. Phys. G **6**, 275 (1980).
[17] R. Delbourgo, Aust. J. Phys. **52**, 681 (1999).
[18] J.S. Ball and T.-W. Chiu, Phys. Rev. D **22**, 2542 (1980).
[19] D.C. Curtis and M.R. Pennington, Phys. Rev. D **42**, 4165 (1990).
[20] D.C. Curtis and M.R. Pennington, Phys. Rev. D **44**, 536 (1991).
[21] D.C. Curtis and M.R. Pennington, Phys. Rev. D **48**, 4933 (1993).
[22] D. Atkinson, J.C.R. Bloch, V.P. Gusynin, M.R. Pennington, and M. Reenders, Phys. Lett. B **329**, 117 (1994).
[23] D. Atkinson, V.P. Gusynin, and P. Maris, Phys. Lett. B **303**, 157 (1993).
[24] A. Bashir and M.R. Pennington, Phys. Rev. D **50**, 7679 (1994).
[25] A. Bashir and M.R. Pennington, Phys. Rev. D **53**, 4694 (1996).
[26] N.K. Nielsen, Nucl. Phys. **B101**, 173 (1975).
[27] O. Piguet and K. Sibold, Nucl. Phys. **B253**, 517 (1985).
[28] J.C. Breckenridge, M.J. Lavelle, and T.G. Steele, Z. Phys. C **65**, 155 (1995).
[29] P. Gambino and P.A. Grassi, Phys. Rev. D **62**, 076002 (2000).
[30] C.J. Burden and C.D. Roberts, Phys. Rev. D **47**, 5581 (1993).
[31] B. Haeri, Phys. Rev. D **43**, 2701 (1991).
[32] R. Delbourgo and B.W. Keck, J. Phys. A **13**, 701 (1980).
[33] R. Delbourgo, B.W. Keck, and C.N. Parker, J. Phys. A **14**, 921 (1981).

[34] A.B. Waites and R. Delbourgo, Int. J. Mod. Phys. A **7**, 6857 (1992).
[35] Z. Dong, H.J. Munczek, and C.D. Roberts, Phys. Lett. B **333**, 536 (1994).
[36] A. Bashir, A. Kızılersü, and M.R. Pennington, Phys. Rev. D **57**, 1242 (1998).
[37] C.J. Burden and P.C. Tjiang, Phys. Rev. D **58**, 085019 (1998).
[38] A. Bashir, A. Kızılersü, and M.R. Pennington, Phys. Rev. D **62**, 085002 (2000).
[39] A.I. Davydychev, P. Osland, and L. Saks, Phys. Rev. D **63**, 014022 (2001).
[40] E.G. Floratos, D.A. Ross, and C.T. Sachrajda, Nucl. Phys. **B129**, 66 (1977).
[41] A. Bashir, A. Kızılersü, and M.R. Pennington, ''Analytic Form of the One Loop Vertex and the Two Loop Fermion Propagator in 3-Dimensional Massless QED'' Adelaide University preprint no. ADP-99-8/T353, Durham University preprint no. DTP-99/76, hep-ph/9907418.
[42] A. Bashir, ''Perturbation Theory Constraints on the 3-Point Vertex in massless QED_3'' *Proceedings of the Workshop on Light-cone QCD and Nonperturbative Hadron Physics* (World Scientific, Singapore, 2000), pp. 227–232.
[43] A. Bashir, Phys. Lett. B **491**, 280 (2000).
[44] S. Moch, P. Uwer, and S. Weinzierl, J. Math. Phys. **43**, 3363 (2002).
[45] A. Kızılersü, M. Reenders, and M.R. Pennington, Phys. Rev. D **52**, 1242 (1995).
[46] A. Bashir and A. Raya, Phys. Rev. D **64**, 105001 (2001).
[47] L. Euler, Novi Comm. Acad. Sci. Petropol **20**, 140 (1775).
[48] D. Zagier, *First European Congress of Mathematics* (Birkhauser, Boston, 1994), Vol. II, p. 497.
[49] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th ed. (Academic Press, Orlando, FL, 2000).
[50] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).