Nicolai mapping versus exact chiral symmetry on the lattice

Yoshio Kikukawa* and Yoichi Nakayama[†]

Department of Physics, Nagoya University, Nagoya 464-8602, Japan
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The two-dimensional $N=2$ Wess-Zumino model is constructed on the lattice through Nicolai mapping with a Ginsparg-Wilson fermion. The Nicolai mapping requires a certain would-be surface term in the bosonic action which ensures the vacuum energy cancellation even on the lattice, but inevitably breaks chiral symmetry. With the Ginsparg-Wilson fermion, the holomorphic structure of the would-be surface term is maintained, leaving a discrete subgroup of the exact chiral symmetry intact for a monomial scalar potential. Through this feature both the boson and fermion can be kept massless on the lattice without any fine-tuning.

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I. INTRODUCTION

The recent rediscovery of the Ginsparg-Wilson relation [1–3] and the realization of exact chiral symmetry on the lattice [4] are interesting developments from the point of view of the constructive approach to quantum field theory.[1] It is a challenge to extend this idea to other aspects of quantum field theory. The construction of supersymmetric theories is one possibility in this direction, although it has been known to be difficult because of the lack of infinitesimal translation invariance on the lattice and the breakdown of the Leibniz rule [6–11]. Based on the domain wall fermion [12,13], the overlap formalism [14], and the Ginsparg-Wilson relation, there have been several attempts [15–22] so far.

Despite the difficulties, the two-dimensional $N=2$ Wess-Zumino model has been constructed successfully based on Nicolai mapping [23] in the Hamiltonian formalism by Cecotti and Girardello [24] and on a Euclidean lattice by Sakai and Sakamoto [25], respectively. The Nicolai mapping is the transformation of the bosonic field variables to the Gaussian stochastic variables whose Jacobian just reproduces the functional determinant of the fermions in the model. The Euclidean lattice version of the Nicolai mapping[2] produces a certain would-be surface term in the bosonic action and ensures the vacuum energy cancellation even on the lattice. Moreover, one special combination out of four supersymmetries of the $N=2$ model is manifest in the lattice action.[3],[4]

In this construction, however, the remaining three supersymmetries cannot be maintained. As clarified by Catterall and Karamov [28], the four different supersymmetries in the

original model can be associated with the four different methods to construct the Nicolai mapping. The resulted four different would-be surface terms reduce to surface terms in the continuum limit through the Leibniz rule, and then the four supersymmetries are realized at the same time. But at finite lattice spacing they define four different lattice models and in each model only one supersymmetry is realized.

Another unsatisfactory feature of the above construction is that chiral symmetry of the original model is not maintained and a fine-tuning is required to keep the degenerate boson and fermion light or massless. This is partly because the fermion theory obtained through the lattice Nicolai mapping turns out to be the Wilson-Dirac fermion. More seriously, the would-be surface term required in the bosonic action breaks chiral symmetry explicitly.

The purpose of this paper is to construct two-dimensional $N=2$ Wess-Zumino model with the Ginsparg-Wilson fermion and examine the above problems. We construct the lattice Nicolai mapping so that its Jacobian reproduces the functional determinant of the Ginsparg-Wilson fermion possessing Yukawa coupling with the exact chiral symmetry. We will see that the use of the Ginsparg-Wilson fermion improves the holomorphic structure of the would-be surface term. Although it still breaks chiral symmetry explicitly in general, but for monomial scalar potentials

$$W[\phi] = \lambda \phi^n, \quad n = 3,4,5,\ldots, \tag{1}$$

it leaves a discrete subgroup of exact chiral symmetry intact and both boson and fermion can be kept massless on the lattice without any fine-tuning.

We will also discuss how the asymmetric treatment between the field and antifield of the Ginsparg-Wilson fermion affects the structure of the Nicolai mapping. Actually, because of the asymmetric treatment, the Cauchy-Riemann condition can be satisfied for only two cases out of four possible Nicolai mappings discussed by Catterall and Karamov [28].

II. TWO-DIMENSIONAL $N=2$ WESS-ZUMINO MODEL— NICOLAI MAPPING AND SUPERSYMMETRY

The action of the two-dimensional $N=2$ Wess-Zumino model in the continuum limit is give by

$$S = S_B + S_F, \tag{2}$$

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*E-mail address: kikukawa@eken.phys.nagoya-u.ac.jp

[†]E-mail address: yoichi@eken.phys.nagoya-u.ac.jp

[1]Fujikawa has proposed a new class of Dirac operators by the algebraic extension of the Ginsparg-Wilson relation [5].

[2]The Nicolai mapping on the spacial lattice in the Hamiltonian formalism was first constructed by Cecotti and Girardello in Ref. [24]. The Nicolai mapping on the two-dimensional Euclidean lattice was obtained by Sakai and Sakamoto in Ref. [25].

[3]The lattice model with certain fermionic symmetry has recently been proposed by Itoh, Kato, Sawanaka, So, and Ukita [26].

[4]In the same spirit, but in a quite new approach, the construction of super-Yang-Mills theory on the spacial lattice has recently been proposed by Kaplan, Katz, and Unsal [27].

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$$S_B = \int d^2x \, \mathcal{L}_B(x) = \int d^2x \{ \partial_\mu \phi^* \partial_\mu \phi + W^{*\prime} W' \}, \quad (3)$$

$$S_F = \int d^2x \, \mathcal{L}_F(x) = \int d^2x \left\{ \bar{\psi} \gamma_\mu \partial_\mu \psi + \bar{\psi} W'' \frac{1+\gamma_3}{2} \psi \right.$$

$$\left. + \bar{\psi} W^{*\prime\prime} \frac{1-\gamma_3}{2} \psi \right\}. \quad (4)$$

This action is invariant under four independent supersymmetry transformations associated with four independent real Grassmann parameters. The Lagrangian is invariant up to terms which can be rewritten into a total divergence through the Leibniz rule. This property of the supersymmetry transformations immediately causes a trouble on the lattice, because the Leibniz rule does not hold for the field products of more than quadratic orders.

This model, however, possesses the so-called Nicolai mapping

$$M(x) = -\partial_1 A(x) - \partial_2 B(x) + U(x),$$

$$N(x) = -\partial_2 A(x) + \partial_1 B(x) + V(x), \quad (5)$$

where A, B and U, V are real and imaginary parts of ϕ and W', respectively,

$$\phi = \sqrt{\frac{1}{2}}(A + iB), \quad W' = \sqrt{\frac{1}{2}}(U + iV). \quad (6)$$

The Jacobian of this transformation of the bosonic field variables just coincides with the functional determinant of the fermion

$$\det \begin{pmatrix} \dfrac{\partial M}{\partial A} & \dfrac{\partial N}{\partial A} \\ \dfrac{\partial M}{\partial B} & \dfrac{\partial N}{\partial B} \end{pmatrix} = \det \left\{ \gamma_\mu \partial_\mu + W'' \frac{1+\gamma_3}{2} + W^{*\prime\prime} \frac{1-\gamma_3}{2} \right\}, \quad (7)$$

while the Gaussian weight for $M(x)$ and $N(x)$ reproduces the bosonic part of the Lagrangian $\mathcal{L}_B(x)$

$$\frac{1}{2} \{ M(x)^2 + N(x)^2 \} = \partial_\mu \phi^* \partial_\mu \phi + W^{*\prime} W' + W' \partial_{\bar{z}} \phi$$

$$+ W^{*\prime} \partial_z \phi^* \equiv \mathcal{L}_B'(x) \quad (8)$$

up to the surface terms $W' \partial_{\bar{z}} \phi + W^{*\prime} \partial_z \phi^* = \partial_{\bar{z}} W + \partial_z W^*$. The Gaussian path-integral of $M(x)$ and $N(x)$ can reproduce the partition function of the original model.

From the structure of the above Nicolai mapping, it follows that the action is invariant under the following fermionic transformation [29,30]:

$$\delta A = \bar{\psi}_1 \xi, \quad \delta B = -i \bar{\psi}_2 \xi, \quad (9)$$

$$\delta \psi_1 = -\xi M, \quad \delta \psi_2 = i \xi N, \quad (10)$$

$$\delta \bar{\psi}_1 = 0, \quad \delta \bar{\psi}_2 = 0, \quad (11)$$

where ξ is a one-component Grassmann parameter and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2). \quad (12)$$

This transformation is a certain combination of the supersymmetry transformation of the $N=2$ model, which has a special feature: the total Lagrangian which includes the extra surface terms required by the Nicolai mapping $\mathcal{L}'(x) = \mathcal{L}_B'(x) + \mathcal{L}_F(x)$ is exactly invariant without leaving any surface term. See the Appendix for details. Therefore, this special supersymmetry has a fair chance to be realized on the lattice.

In fact, as shown by Sakai and Sakamoto [25], the Nicolai mapping can be constructed successfully on the two-dimensional Euclidean lattice. Their result reads

$$M(x) = (-\nabla_1^S - \nabla_1^A - \nabla_2^A) A(x) - \nabla_2^S B(x) + U(x),$$

$$N(x) = -\nabla_2^S A(x) + (\nabla_1^S - \nabla_1^A - \nabla_2^A) B(x) + V(x), \quad (13)$$

where $\nabla^{A,S}$ are defined by forward and backward differentials as

$$\nabla_j^S = \frac{1}{2}(\nabla_j^+ + \nabla_j^-), \quad \nabla_j^A = \frac{1}{2}(\nabla_j^+ - \nabla_j^-). \quad (14)$$

The Jacobian of this lattice Nicolai mapping reproduces the functional determinant of the Wilson-Dirac fermion with the Yukawa coupling

$$\det \begin{pmatrix} \dfrac{\partial M}{\partial A} & \dfrac{\partial N}{\partial A} \\ \dfrac{\partial M}{\partial B} & \dfrac{\partial N}{\partial B} \end{pmatrix}$$

$$= \det \left\{ \sum_\mu (\gamma_\mu \nabla_\mu^S - \nabla_\mu^A) + W'' \frac{1+\gamma_3}{2} + W^{*\prime\prime} \frac{1-\gamma_3}{2} \right\}, \quad (15)$$

while the bosonic action determined by the lattice Nicolai mapping contains the following "would-be surface terms:"

$$\phi(\nabla_1^S - i\nabla_2^S) W' + \phi^*(\nabla_1^S + i\nabla_2^S) W^{*\prime} - \phi(\nabla_1^A + \nabla_2^A) W^{*\prime}$$

$$- \phi^*(\nabla_1^A + \nabla_2^A) W'. \quad (16)$$

By virtue of these terms, the vacuum energy cancellation holds on the lattice. Moreover, the total action possesses a supersymmetry under the same transformation as Eqs. (9), (10), and (11).

III. NICOLAI MAPPING WITH GINSPARG-WILSON FERMION

Now we construct the two-dimensional $N=2$ Wess-Zumino model with the Ginsparg-Wilson fermion, relying on the existence of the Nicolai mapping as the guiding principle

to maintain supersymmetry as in Ref. [25]. Our strategy is as follows. First we fix the fermionic part of the action so that the Yukawa coupling possesses the exact chiral symmetry based on the Ginsparg-Wilson relation. Then we construct the Nicolai mapping so that its Jacobian reproduces the functional determinant of the Ginsparg-Wilson fermion with the Yukawa coupling. Finally, the bosonic part of the action is determined so that it coincides with the Gaussian weight for the Nicolai-mapped bosonic variables.

We take the following fermionic action:

$$S_F = \sum_x \bar{\psi}(D+F)\psi$$

$$= \sum_{x,y} \bar{\psi}(x)\left(D + \frac{1+\gamma_3}{2} W'' \frac{1+\hat{\gamma}_3}{2} \right.$$

$$\left. + \frac{1-\gamma_3}{2} W^{*\prime\prime} \frac{1-\hat{\gamma}_3}{2} \right)_{x,y} \psi(y), \quad (17)$$

where D is a lattice Dirac operator which satisfies the Ginsparg-Wilson relation

$$D\hat{\gamma}_3 + \gamma_3 D = 0, \quad \hat{\gamma}_3 = \gamma_3(1-aD). \quad (18)$$

As an explicit example, we adopt the overlap Dirac operator given by Neuberger [3]:

$$D = \begin{pmatrix} T+S_1 & iS_2 \\ -iS_2 & T-S_1 \end{pmatrix}, \quad (19)$$

where T, S_1, S_2 are defined as

$$T = \frac{1}{a}\left(1 - \frac{1}{\sqrt{X^\dagger X}} \right) - \frac{\nabla_1^A + \nabla_2^A}{\sqrt{X^\dagger X}} = {}^tT, \quad (20)$$

$$S_j = \frac{\nabla_j^S}{\sqrt{X^\dagger X}} = -{}^tS_j, \quad j=1,2 \quad (21)$$

$$X = 1 - aD_W. \quad (22)$$

In this notation, the Ginsparg-Wilson relation can be written as

$$a(T^2 - S_1^2 - S_2^2) = 2T. \quad (23)$$

By construction, the fermionic part of the action (17) is invariant under lattice chiral rotation [4]

$$\psi \to \exp(i\theta\hat{\gamma}_3)\psi, \quad \bar{\psi} \to \bar{\psi}\exp(i\theta\gamma_3),$$

$$W'' \to W''\exp(2i\theta), \quad W^{*\prime\prime} \to W^{*\prime\prime}\exp(-2i\theta). \quad (24)$$

By inserting the Dirac operator (19) into (17), we obtain

$$D+F = \begin{pmatrix} T+S_1 & iS_2 \\ -iS_2 & T-S_1 \end{pmatrix} + \begin{pmatrix} \frac{\partial U}{\partial A}\left(1-\frac{a}{2}(T+S_1)\right) - \frac{\partial V}{\partial A}\frac{a}{2}S_2 & i\left\{ \frac{\partial V}{\partial A}\left(1-\frac{a}{2}(T-S_1)\right) - \frac{\partial U}{\partial A}\frac{a}{2}S_2 \right\} \\ -i\left\{ \frac{\partial U}{\partial B}\left(1-\frac{a}{2}(T+S_1)\right) - \frac{\partial V}{\partial B}\frac{a}{2}S_2 \right\} & \frac{\partial V}{\partial B}\left(1-\frac{a}{2}(T-S_1)\right) - \frac{\partial U}{\partial B}\frac{a}{2}S_2 \end{pmatrix}, \quad (25)$$

where A,B,U,V are real and imaginary parts of ϕ, W'

$$\phi = \sqrt{\frac{1}{2}}(A+iB), \quad W' = \sqrt{\frac{1}{2}}(U+iV). \quad (26)$$

Then the Nicolai mapping should solve the differential equation

$$D+F = \begin{pmatrix} \frac{\partial M}{\partial A} & i\frac{\partial N}{\partial A} \\ -i\frac{\partial M}{\partial B} & \frac{\partial N}{\partial B} \end{pmatrix}. \quad (27)$$

We can find a solution to this equation as follows:

$$M = A(T+S_1) + BS_2 + U\left(1-\frac{a}{2}(T+S_1)\right) - V\frac{a}{2}S_2, \quad (28)$$

$$N = AS_2 + B(T-S_1) + V\left(1-\frac{a}{2}(T-S_1)\right) - U\frac{a}{2}S_2, \quad (29)$$

where M,N,A,B,U,V are functions of x and difference operators T,S_1,S_2 are multiplied from the right. As to other possible solutions, we will discuss later.

We now evaluate the bosonic part of the action implied by the above Nicolai mapping

$$S_B = \frac{1}{2} \sum_x \{M^2+N^2\}. \quad (30)$$

The Ginsparg-Wilson relation plays an important role through the calculation: as an illustrative example, we show $A \times U$ term and $B \times V$ term

$$A\left(S_1+T-\frac{a}{2}(T^2-S_1^2-S_2^2)\right)U$$

$$+B\left(-S_1+T-\frac{a}{2}(T^2-S_1^2-S_2^2)\right)V$$

$$=\phi^*\left(T-\frac{a}{2}(T^2-S_1^2-S_2^2)\right)W'$$

$$+\phi\left(T-\frac{a}{2}(T^2-S_1^2-S_2^2)\right)W^{*\prime}+\phi S_1 W'+\phi^* S_1 W^{*\prime}.$$

(31)

Here we note that the combination $T-a/2\,(T^2-S_1^2-S_2^2)$ is equal to zero by Eq. (23). We finally obtain the bosonic part of the action as

$$S_B=\sum_x\left\{\phi^*\Delta\phi+W^{*\prime}\left(1-\frac{a^2}{4}\Delta\right)W'+W'(-S_1+iS_2)\phi\right.$$

$$\left.+W^{*\prime}(-S_1-iS_2)\phi^*\right\},$$

(32)

where Δ is defined by $D^\dagger D=\Delta\cdot\mathbf{1}$ and $\Delta=(T^2-S_1^2-S_2^2)$ $=2T/a$.

Thanks to the existence of the Nicolai mapping, (28) and (29), it is ensured that all the nice features of the construction by Sakai and Sakamoto [25] are maintained in our construction. The total action $S=S_B+S_F$ given by Eqs. (32) and (17), possesses a supersymmetry under the transformation

$$\delta A=\bar{\psi}_1\xi,\quad \delta B=-i\bar{\psi}_2\xi,$$

(33)

$$\delta\psi_1=-\xi M,\quad \delta\psi_2=i\xi N,$$

(34)

$$\delta\bar{\psi}_1=0,\quad \delta\bar{\psi}_2=0,$$

(35)

where ξ is a one-component Grassmann parameter and

$$\psi=\begin{pmatrix}\psi_1\\\psi_2\end{pmatrix},\quad \bar{\psi}=(\bar{\psi}_1,\bar{\psi}_2).$$

(36)

The vacuum energy cancellation also holds even at the finite lattice spacing. (One may verify through explicit calculations that the vacuum energy is canceled exactly in any orders of the lattice perturbation theory.)

IV. CHIRAL SYMMETRY IN THE SUPERSYMMETRIC ACTION

Now let us examine the chiral properties of the lattice action of the two-dimensional $N=2$ Wess-Zumino model obtained in the previous section. The fermionic part of the action (17) respects the exact chiral symmetry on the lattice by our construction. Then the question is the chiral properties of the bosonic part of the action (32).

First of all, the bosonic part of the action (32) should be compared with the counterpart in the construction by Sakai and Sakamoto (16) or Eq. (3.6) in Ref. [25]. An important

difference is in that the terms with the structures, $W'\times\phi^*$ and $W^{*\prime}\times\phi$, do not appear in Eq. (32), and this implies that the holomorphic structure of the would-be surface terms is maintained just as in the continuum theory. As we have seen explicitly in Eq. (31), these terms vanish identically by virtue of the Ginsparg-Wilson relation. Thus the use of the Ginsparg-Wilson fermion can improve the holomorphic structure of the would-be surface term.

The would-be surface terms in Eq. (32) still break the exact chiral symmetry on the lattice explicitly. They cannot be eliminated, because these terms are playing a crucial role in order to maintain the supersymmetry of the action. Therefore the breakdown of the exact chiral symmetry on the lattice seems inevitable.

Thanks to the improved holomorphic structure, however, if one assumes that the superpotential is a monomial

$$W(\phi)=\lambda\phi^n,\quad n=3,4,5,\ldots,$$

(37)

then the total action is invariant under the discrete chiral rotation with the angle $\theta=\pi k/n$ for arbitrary integer k. By this remaining discrete exact chiral symmetry, both boson and fermion can be kept massless on the lattice without any fine-tuning. We would have the same situation in the continuum theory if we keep the total divergence term implied by the Nicolai mapping in the action so that an exact supersymmetry is maintained at the Lagrangian level. So, we think, it is not quite a lattice artifact.

It is not difficult to prove in any order of the lattice perturbation expansion that the fermion mass term would not be produced in this lattice model with a monomial potential. The possible coupling terms appear in the following combinations:

$$\phi^{n-1}\phi^{*n-1},\ \phi^n,\ \phi^{*n},\ \bar{\psi}_L\phi^{n-2}\psi_R,\ \bar{\psi}_R\phi^{*n-2}\psi_L,\quad (38)$$

where we omit derivatives and proportional factors. In perturbation expansion, we should consider all possible diagrams produced by the product of those couplings. The mass term must have the external legs $\bar{\psi}_L$-ψ_R (or $\bar{\psi}_R$-ψ_L), while the $n-2$ legs of scalar field coming from the combination $(\bar{\psi}_L\phi^{n-2}\psi_R)^{(l+1)}(\bar{\psi}_R\phi^{*n-2}\psi_L)^l\ (l=0,1,2,\ldots)$ cannot be closed by $-n$ legs coming from ϕ^{*n} or by any other product of the interaction terms.[5] Therefore we can conclude that the fermion mass term would not be generated in our model. Then the supersymmetry implies that the boson would not acquire mass, neither. Here we should emphasize that the same result cannot be obtained in the case of the Wilson fermion, because there are no mechanisms to suppress non-holomorphic scalar self-interaction.

V. SOLUBILITY OF NICOLAI MAPPINGS

Two-dimensional $N=2$ Wess-Zumino model is invariant under four supersymmetry transformations which can be related to four types of the Nicolai mappings as clarified in Ref. [28]. In the case with Wilson-Dirac fermions, we can actually obtain all the four mappings.

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[5] $-jn=n-2$ cannot be satisfied by any integer j for $n=3,4,5,\ldots$.

In the case with Ginsparg-Wilson fermions, however, the situation differs due to the asymmetric choice of chiral projectors (17). The four differential equations corresponding to the four Nicolai mappings are given by

$$
D+F=\begin{pmatrix} \dfrac{\partial M}{\partial A} & i\dfrac{\partial N}{\partial A} \\[2ex] -i\dfrac{\partial M}{\partial B} & \dfrac{\partial N}{\partial B} \end{pmatrix}, \tag{39}
$$

$$
D+F=\begin{pmatrix} \dfrac{\partial M}{\partial B} & -i\dfrac{\partial N}{\partial B} \\[2ex] i\dfrac{\partial M}{\partial A} & \dfrac{\partial N}{\partial A} \end{pmatrix}, \tag{40}
$$

$$
D+F=\begin{pmatrix} \dfrac{\partial M}{\partial A} & -i\dfrac{\partial M}{\partial B} \\[2ex] i\dfrac{\partial N}{\partial A} & \dfrac{\partial N}{\partial B} \end{pmatrix}, \tag{41}
$$

$$
D+F=\begin{pmatrix} \dfrac{\partial M}{\partial B} & i\dfrac{\partial M}{\partial A} \\[2ex] -i\dfrac{\partial N}{\partial B} & \dfrac{\partial N}{\partial A} \end{pmatrix}. \tag{42}
$$

The solution of the first one (39) is the solution given in Sec. III. The solution of the second one (40) is obtained in the similar manner using $\partial U/\partial A = \partial V/\partial B$ and $\partial U/\partial B = -\partial V/\partial A$.[6] However, the rest two cases cannot be solved. The Cauchy-Riemann condition, which is the necessary condition for the solubility, does not hold for the latter two cases. For example, the Cauchy-Riemann condition for the third one (41) is evaluated as

$$
\frac{\partial}{\partial B}(D+F)_{11} - i\frac{\partial}{\partial A}(D+F)_{12}
$$
$$
= -a\left\{ \frac{\partial^2 U}{\partial A\,\partial B}S_1 + \frac{\partial^2 V}{\partial A\,\partial B}S_2 \right\} \neq 0. \tag{43}
$$

This violation of the Cauchy-Riemann condition is the consequence of the asymmetric choice of the chiral projectors. Therefore the Nicolai mappings related to the other two supersymmetries have no solutions.

If we perform singular change of the field variables as

$$
\psi' = \left(1 - \frac{a}{2}D\right)\psi, \quad \bar\psi' = \bar\psi\left(1 - \frac{a}{2}D\right)^{-1}, \tag{44}
$$

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[6]The bosonic action given by the solution of Eq. (40) has the form (32) with the sign of S_j reversed.

then we can solve the differential equations which correspond to Eqs. (41) and (42), while the Cauchy-Riemann conditions for the equations which correspond to Eqs. (39) and (40) break down.

VI. SUMMARY

We have constructed two-dimensional $N=2$ Wess-Zumino model on the lattice which possesses both the supersymmetry based on the Nicolai mapping and the exact chiral symmetry based on the Ginsparg-Wilson relation. The Nicolai mapping ensures that the vacuum energy cancellation holds and boson and fermion are degenerate. The use of the Ginsparg-Wilson fermion maintains the holomorphic structure of the would-be surface term, leaving a discrete subgroup of the exact chiral symmetry intact for a monomial scalar potential. Thus both boson and fermion can be kept massless on the lattice without any fine-tuning.

APPENDIX: NICOLAI MAPPING AND SUPERSYMMETRY

In this appendix we examine the properties of the supersymmetry which follows from the Nicolai mapping in the continuum theory. Supercharges in two-dimensional $N=2$ theory are written as

$$
Q_+ = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\bar\theta_-} - \theta_-\frac{\partial}{\partial z} \right), \quad \bar Q_+ = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta_-} - \bar\theta_-\frac{\partial}{\partial z} \right), \tag{A1}
$$

$$
Q_- = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta_+} - \bar\theta_+\frac{\partial}{\partial\bar z} \right), \quad \bar Q_- = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\bar\theta_+} - \theta_+\frac{\partial}{\partial\bar z} \right). \tag{A2}
$$

These Q's satisfy following SUSY algebra:

$$
\{Q_+,\bar Q_+\} = -\frac{\partial}{\partial z}, \quad \{Q_-,\bar Q_-\} = -\frac{\partial}{\partial\bar z}. \tag{A3}
$$

We can define the chiral superfield in such theory as

$$
\bar D_\pm \Phi = 0, \tag{A4}
$$

where

$$
\bar D_+ = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta_-} + \bar\theta_-\frac{\partial}{\partial z} \right), \quad \bar D_- = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\bar\theta_+} + \theta_+\frac{\partial}{\partial\bar z} \right). \tag{A5}
$$

The form of the chiral superfield is

$$\Phi = \phi(z + \bar{\theta}_-\theta_-, \bar{z} - \bar{\theta}_+\theta_+) + \sqrt{2}\,\bar{\theta}_-\psi_+(z + \bar{\theta}_-\theta_-, \bar{z}$$
$$- \bar{\theta}_+\theta_+) + \sqrt{2}\,\theta_+\bar{\psi}_-(z + \bar{\theta}_-\theta_-, \bar{z} - \bar{\theta}_+\theta_+)$$
$$+ 2\theta_+\bar{\theta}_- D(z + \bar{\theta}_-\theta_-, \bar{z} - \bar{\theta}_+\theta_+), \tag{A6}$$

where $\psi_\pm$ and $\bar{\psi}_\pm$ are chiral components of the Dirac fermion

$$\psi_\pm = \frac{1 \pm \gamma_3}{2}\psi, \qquad \bar{\psi}_\mp = \bar{\psi}\frac{1 \pm \gamma_3}{2}. \tag{A7}$$

On the other hand, the antichiral superfield is written as

$$\bar{\Phi} = \phi^*(z - \bar{\theta}_-\theta_-, \bar{z} + \bar{\theta}_+\theta_+) - \sqrt{2}\,\theta_-\bar{\psi}_+(z - \bar{\theta}_-\theta_-, \bar{z}$$
$$+ \bar{\theta}_+\theta_+) - \sqrt{2}\,\bar{\theta}_+\psi_-(z - \bar{\theta}_-\theta_-, \bar{z} + \bar{\theta}_+\theta_+)$$
$$- 2\bar{\theta}_+\theta_- D^*(z - \bar{\theta}_-\theta_-, \bar{z} + \bar{\theta}_+\theta_+). \tag{A8}$$

By calculating $\epsilon Q_+ \Phi$, $\epsilon Q_+ \bar{\Phi}$, we introduce supersymmetry transformation as

$$\delta_1 \phi = \epsilon \psi_+, \quad \delta_1 \phi^* = 0,$$
$$\delta_1 \psi_+ = 0, \qquad \delta_1 \psi_- = 0,$$
$$\delta_1 \bar{\psi}_- = \epsilon D, \qquad \delta_1 \bar{\psi}_+ = -\epsilon \partial_z \phi^*, \tag{A9}$$
$$\delta_1 D = 0, \qquad \delta_1 D^* = \epsilon \partial_z \psi_-,$$

and from $\epsilon \bar{Q}_- \Phi$, $\epsilon \bar{Q}_- \bar{\Phi}$, introduce another one

$$\delta_2 \phi = 0, \qquad \delta_2 \phi^* = -\epsilon \psi_-,$$
$$\delta_2 \psi_+ = 0, \qquad \delta_2 \psi_- = 0,$$
$$\delta_2 \bar{\psi}_- = \epsilon \partial_{\bar{z}} \phi, \qquad \delta_2 \bar{\psi}_+ = -\epsilon D^*,$$
$$\delta_2 D = -\epsilon \partial_{\bar{z}} \psi_+, \quad \delta_2 D^* = 0. \tag{A10}$$

Now we take Lagrangian $\mathcal{L}$ as

$$[\bar{\Phi}\Phi]_{D\,\text{term}} + ([W(\Phi)]_{F\,\text{term}} + \text{H.c.})$$
$$\cong \bar{\psi}_+ \partial_{\bar{z}}\psi_+ + \bar{\psi}_- \partial_z\psi_- + \partial_z\phi^* \partial_{\bar{z}}\phi - D^*D + \bar{\psi}_- W''\psi_+$$
$$- W'D + \bar{\psi}_+ W^{*\prime\prime}\psi_- - W^{*\prime}D'$$
$$\equiv \mathcal{L}, \tag{A11}$$

where we have arranged total divergence terms appropriately. The variation of this Lagrangian under δ_1 gives total divergence term

$$-\epsilon(\psi_- \partial_z W^{*\prime} + W^{*\prime}\partial_z\psi_-). \tag{A12}$$

On the other hand, the variation of $W^{*\prime}\partial_z\phi^*$ under $-\delta_2$ gives

$$-\delta_2(W^{*\prime}\partial_z\phi^*) = -W^{*\prime}\partial_z(-\epsilon\psi_-) - (-\epsilon\psi_-)W^{*\prime\prime}\partial_z\phi^*$$
$$= \epsilon(\psi_- \partial_z W^{*\prime} + W^{*\prime}\partial_z\psi_-). \tag{A13}$$

So if we redefine Lagrangian including $W^{*\prime}\partial_z\phi^*$ and its complex conjugate

$$\tilde{\mathcal{L}} \equiv \mathcal{L} + W'\partial_{\bar{z}}\phi + W^{*\prime}\partial_z\phi^*, \tag{A14}$$

then we have the symmetry under $\delta_1 - \delta_2$ at the Lagrangian level.

Now let us see the relation between this symmetry and the Nicolai mapping. The Nicolai mapping in continuum is written as

$$M = \partial_1 A - \partial_2 B + U, \tag{A15}$$

$$N = -\partial_2 A - \partial_1 B + V \tag{A16}$$

so that fermionic action is given by

$$\bar{\psi}(D + F)\psi = (\bar{\psi}_1\, \bar{\psi}_2) \begin{pmatrix} \dfrac{\partial M}{\partial A} & -i\dfrac{\partial M}{\partial B} \\[2ex] i\dfrac{\partial N}{\partial A} & \dfrac{\partial N}{\partial B} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{A17}$$

where A and B are real and imaginary part (normalized by $1/\sqrt{2}$) of ϕ and U and V are those of W'. The total action implied by the Nicolai mapping is just equal to Eq. (A14). We can also see that the supersymmetry transformation implied by the Nicolai mapping

$$\delta A = \epsilon \psi_1, \qquad \delta B = -i\epsilon\psi_2,$$
$$\delta\psi_1 = 0, \qquad \delta\psi_2 = 0, \tag{A18}$$
$$\delta\bar{\psi}_1 = -\epsilon M, \qquad \delta\bar{\psi}_2 = i\epsilon N$$

is nothing but the transformation $\delta_1 - \delta_2$ described above. Actually, by inserting

$$\psi_+ = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad \psi_- = \frac{1}{\sqrt{2}}(\psi_1 - \psi_2), \tag{A19}$$

$$\bar{\psi}_+ = \frac{1}{\sqrt{2}}(\bar{\psi}_1 - \bar{\psi}_2), \quad \bar{\psi}_- = \frac{1}{\sqrt{2}}(\bar{\psi}_1 + \bar{\psi}_2) \tag{A20}$$

into Eq. (A18), then we obtain

$$\delta\phi = \epsilon\psi_+, \qquad \delta\phi^* = \epsilon\psi_-,$$
$$\delta\bar{\psi}_- = -\epsilon\partial_{\bar{z}}\phi - \epsilon W^{*\prime}, \quad \delta\bar{\psi}_+ = -\epsilon\partial_z\phi' - \epsilon W' \tag{A21}$$

and this coincides with $\delta_1 - \delta_2$ after eliminating auxiliary fields D and D^* by their equation of motion.

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