Order-v^4 corrections to S-wave quarkonium decay

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We compute corrections of relative order v^4 to the rates for the decays of $^1S_0$ heavy quarkonium into two photons and into light hadrons and for the decays of $^3S_1$ heavy quarkonium into a lepton pair and into light hadrons. In particular, we compute the coefficients of the decay operators that have the same quantum numbers as the heavy quarkonium. We also confirm previous calculations of the order-v^2 corrections to these rates. We find that the v expansion converges well for the decays of $^1S_0$ heavy quarkonium and for the decay of $^3S_1$ heavy quarkonium into a lepton pair. Large higher-order-in-v corrections appear in the decay of $^3S_1$ heavy quarkonium into light hadrons. However, we find that the series of coefficients of operators with $^3S_1$ quantum numbers, which yields a large correction in order v^2, yields a smaller correction in order v^4.

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I. INTRODUCTION

A formalism for the first-principles calculation of heavy-quarkonium decay rates in quantum chromodynamics (QCD) has been given in Ref. [1]. This formalism is based on the effective field theory nonrelativistic quantum chromodynamics (NRQCD). In it, one can write the decay rate of a quarkonium state H as

$$\Gamma(H)=\sum_n \frac{F_n}{m^{d_n-4}}\langle H|\mathcal{O}|H\rangle, \tag{1.1}$$

where F_n is a perturbatively calculable short-distance coefficient, m is the heavy-quark mass, the $\mathcal{O}_n$ are four-fermion NRQCD operators, and d_n is the mass dimension of $\mathcal{O}_n$. The terms in the sum over n may be classified according to their orders in v (Ref. [1]), where v is the heavy-quark–antiquark relative velocity. For charmonium, $v^2\approx 0.3$; for bottomonium, $v^2\approx 0.1$.

We concern ourselves in this paper with the decay of $^1S_0$ quarkonium into light hadrons and the decays of $^3S_1$ quarkonium into lepton pairs and into light hadrons. The coefficients of the operators of leading order in v and of relative order v^2 have been computed previously [1–9]. Some of the coefficients of the order-v^2 operators are sufficiently large as to cast doubt on the convergence of the v expansion for charmonium and bottomonium. In particular, the order-v^2 correction to the rate for the decay of $^3S_1$ quarkonium into light hadrons is $-5.32\langle v^2\rangle$, where $\langle v^2\rangle$ is the ratio of the expectation values of the order-v^2 and order-v^0 operators in the quarkonium state. Hence, in the case of charmonium, the order-v^2 correction is more than 100%.

In this paper, we compute the short-distance coefficients of the decay operators, through order v^4, that have the same quantum numbers as the quarkonium. Our calculations confirm previous results for the short-distance coefficients of the order-v^2 operators. We find that the v expansion is well behaved for the decays of $^1S_0$ quarkonium and for the decay of $^3S_1$ quarkonium into lepton pairs. In the case of the decay of $^3S_1$ quarkonium into light hadrons, large coefficients are associated with some of the operators of higher order in v. For the operators with $^3S_1$ quantum numbers, a large correction to the decay rate appears in order v^2, but the correction in order v^4 is considerably smaller. This suggests that the v expansion for operators with a given quantum number may converge well once one goes beyond the first nontrivial order.

II. NRQCD DECAY RATES

In this section, we present the NRQCD factorization expressions for the rates of $^1S_0$ quarkonium (e.g. η_c or η_b) decay to light hadrons (LH), $^3S_1$ quarkonium (e.g. J/ψ or Υ) decay to light hadrons, $^1S_0$ quarkonium decay to two photons, and $^3S_1$ quarkonium decay to e^+e^-.

Through relative order v^4, the rate for the decay of a $^1S_0$ state into light hadrons is given by

$$\begin{aligned}\Gamma(^1S_0\to\mathrm{LH})=&\frac{F_1(^1S_0)}{m^2}\langle ^1S_0|\mathcal{O}_1(^1S_0)|^1S_0\rangle\\&+\frac{G_1(^1S_0)}{m^4}\langle ^1S_0|\mathcal{P}_1(^1S_0)|^1S_0\rangle\\&+\frac{F_8(^3S_1)}{m^2}\langle ^1S_0|\mathcal{O}_8(^3S_1)|^1S_0\rangle\\&+\frac{F_8(^1S_0)}{m^2}\langle ^1S_0|\mathcal{O}_8(^1S_0)|^1S_0\rangle\\&+\frac{F_8(^1P_1)}{m^4}\langle ^1S_0|\mathcal{O}_8(^1P_1)|^1S_0\rangle\\&+\frac{H_1^1(^1S_0)}{m^6}\langle ^1S_0|\mathcal{Q}_1^1(^1S_0)|^1S_0\rangle\\&+\frac{H_1^2(^1S_0)}{m^6}\langle ^1S_0|\mathcal{Q}_1^2(^1S_0)|^1S_0\rangle.\end{aligned} \tag{2.1}$$

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The operators appearing in Eq. (2.1) are defined by

$$\mathcal{O}_1({}^1S_0)=\psi^\dagger\chi\chi^\dagger\psi, \tag{2.2a}$$

$$\mathcal{P}_1({}^1S_0)=\frac{1}{2}\left[\psi^\dagger\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi+\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi\chi^\dagger\psi\right], \tag{2.2b}$$

$$\mathcal{O}_8({}^3S_1)=\psi^\dagger\boldsymbol{\sigma}T_a\chi\cdot\chi^\dagger\boldsymbol{\sigma}T_a\psi, \tag{2.2c}$$

$$\mathcal{O}_8({}^1S_0)=\psi^\dagger T_a\chi\chi^\dagger T_a\psi, \tag{2.2d}$$

$$\mathcal{O}_8({}^1P_1)=\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)T_a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)T_a\psi, \tag{2.2e}$$

$$\mathcal{Q}_1^1({}^1S_0)=\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi, \tag{2.2f}$$

$$\mathcal{Q}_1^2({}^1S_0)=\frac{1}{2}\left[\psi^\dagger\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi+\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\chi\chi^\dagger\psi\right], \tag{2.2g}$$

$$\mathcal{Q}_1^3({}^1S_0)=\frac{1}{2}[\psi^\dagger\chi\chi^\dagger(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi-\psi^\dagger(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\chi\chi^\dagger\psi], \tag{2.2h}$$

where the subscript 1 or 8 indicates that the operator is a color singlet or a color octet, the superscript labels the three dimension-10 operators, ψ is the Pauli-spinor field that annihilates a heavy quark, $\chi^\dagger$ is the Pauli-spinor field that annihilates a heavy antiquark, $D^\mu=\partial^\mu+igA^\mu$ is the gauge-covariant derivative, A is the SU(3)-matrix-valued gauge field, g is the QCD coupling constant, $E^i=G^{0i}$, where $G^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+ig[A^\mu,A^\nu]$ is the gluon field strength, and the σ^i are Pauli matrices. The operator $\overleftrightarrow{\mathbf{D}}$ is defined by $\chi^\dagger\overleftrightarrow{\mathbf{D}}\psi=\chi^\dagger(\mathbf{D}\psi)-(\mathbf{D}\chi)^\dagger\psi$. The relative signs of the terms in each of these operators (and, in particular, $\mathcal{Q}_1^3$) are fixed by the requirements of Hermiticity and charge-conjugation invariance.

The matrix element of $\mathcal{Q}_1^3$ does not appear in Eq. (2.1) because, as we show in Appendix A, it can be eliminated in favor of $\mathcal{Q}_1^1$ and $\mathcal{Q}_1^2$ through the use of the equations of motion. From the velocity scaling rules in Ref. [1], we find that, in the ${}^1S_0$ state, the operator $\mathcal{O}_1({}^1S_0)$ has a matrix element of relative order v^0, the operator $\mathcal{P}_1({}^1S_0)$ has a matrix element of relative order v^2, the operator $\mathcal{O}_8({}^3S_1)$ has a matrix element of relative order v^3, and the operators $\mathcal{O}_8({}^1S_0)$, $\mathcal{O}_8({}^1P_1)$, $\mathcal{Q}_1^1({}^1S_0)$, $\mathcal{Q}_1^2({}^1S_0)$, and $\mathcal{Q}_1^3({}^1S_0)$ have matrix elements of relative order v^4.

The contributions of order α_s^2 and order α_s^3 to the short-distance coefficient $F_1({}^1S_0)$ have been computed in Refs. [2,3] and are given in Ref. [1]:

$$F_1({}^1S_0)=\frac{\pi C_F}{N_c}\alpha_s^2(2m)\left\{1+\left[\left(\frac{\pi^2}{4}-5\right)C_F+\left(\frac{199}{18}-\frac{13\pi^2}{24}\right)C_A-\frac{8}{9}n_f\right]\frac{\alpha_s}{\pi}\right\}, \tag{2.3}$$

where $N_c=3$ is the number of colors, $C_F=(N_c^2-1)/(2N_c)=4/3$, and $C_A=N_c$. The contribution of order α_s^2 to $G_1({}^1S_0)$ has been computed in Refs. [1,5].[1] It is

$$G_1({}^1S_0)=-\frac{4\pi C_F}{3N_c}\alpha_s^2. \tag{2.4}$$

We note that, to leading order in α_s, $[G_1({}^1S_0)]/[F_1({}^1S_0)]=-4/3$. Hence, the first relativistic correction is sizable in the case of the η_c. The contributions of order α_s^2 and order α_s^3 to $F_8({}^3S_1)$ and $F_8({}^1S_0)$ have been computed by Petrelli *et al.* [10]:

$$F_8({}^3S_1)=\frac{\pi n_f}{3}\alpha_s^2(\mu)\left\{1+\frac{\alpha_s}{\pi}\left[-\frac{13}{4}C_F+\left(\frac{133}{18}+\frac{2}{3}\log 2-\frac{\pi^2}{4}\right)C_A-\frac{10}{9}n_fT_F+2b_0\log\frac{\mu}{2m}\right]\right\}+5\alpha_s^3\left(-\frac{73}{4}+\frac{67}{36}\pi^2\right), \tag{2.5a}$$

$$F_8({}^1S_0)=2\pi B_F\alpha_s^2(\mu)\left\{1+\frac{\alpha_s}{\pi}\left[\left(-5+\frac{\pi^2}{4}\right)C_F+\left(\frac{122}{9}-\frac{17}{24}\pi^2\right)C_A-\frac{16}{9}n_fT_F+2b_0\log\frac{\mu}{2m}\right]\right\}, \tag{2.5b}$$

where μ is the QCD renormalization scale, n_f is the number

[1]Short-distance coefficients can be extracted from the results in Ref. [5] by first making the substitution $1/M_{\rm meson}^2\to(1/4m^2)(1-\varepsilon/m)$, where $-\varepsilon$ is the binding energy, and then making the identification $\varepsilon/m\to\langle{}^1S_0|\mathcal{P}_1({}^1S_0)|{}^1S_0\rangle/[m^2\langle{}^1S_0|\mathcal{O}_1({}^1S_0)|{}^1S_0\rangle]\approx\langle{}^3S_1|\mathcal{P}_1({}^3S_1)|{}^3S_1\rangle/[m^2\langle{}^3S_1|\mathcal{O}_1({}^3S_1)|{}^3S_1\rangle]$.

of light-quark flavors, $B_F=(N_c^2-4)/(4N_c)=5/12$, $T_F=1/2$, and $b_0=(11/6)C_A-(2/3)T_F n_f$. The contribution of order α_s^2 to $F_8(^1P_1)$ can be deduced from the results in Appendix A 2 of Ref. [1]:

$$F_8(^1P_1)=\frac{\pi N_c}{6}\alpha_s^2. \tag{2.6}$$

Owing to energy conservation, the operators associated with the short-distance coefficients $H_1^1(^1S_0)$ and $H_1^2(^1S_0)$ cannot be distinguished from each other in the Born-level decay of on-shell quarks. Consequently, if one uses on-shell matching between NRQCD and full QCD to compute the short-distance coefficients in Born-level decay processes, one can compute only $H_1^1(^1S_0)+H_1^2(^1S_0)$, not the individual coefficients. It is the quantity $H_1^1(^1S_0)+H_1^2(^1S_0)$ that we compute in this paper.

Through relative order v^4, the decay rate for a $^3S_1$ state into light hadrons is

$$\begin{aligned}\Gamma(^3S_1\to\mathrm{LH})=&\frac{F_1(^3S_1)}{m^2}\langle ^3S_1|\mathcal{O}_1(^3S_1)|^3S_1\rangle+\frac{G_1(^3S_1)}{m^4}\langle ^3S_1|\mathcal{P}_1(^3S_1)|^3S_1\rangle+\frac{F_8(^1S_0)}{m^2}\langle ^3S_1|\mathcal{O}_8(^1S_0)|^3S_1\rangle\\&+\frac{F_8(^3S_1)}{m^2}\langle ^3S_1|\mathcal{O}_8(^3S_1)|^3S_1\rangle+\sum_{J=0,1,2}\frac{F_8(^3P_J)}{m^4}\langle ^3S_1|\mathcal{O}_8(^3P_J)|^3S_1\rangle+\frac{H_1^1(^3S_1)}{m^6}\langle ^3S_1|\mathcal{Q}_1^1(^3S_1)|^3S_1\rangle\\&+\frac{H_1^2(^3S_1)}{m^6}\langle ^3S_1|\mathcal{Q}_1^2(^3S_1)|^3S_1\rangle.\end{aligned} \tag{2.7}$$

The operator $\mathcal{O}_8(^1S_0)$ is defined in Eq. (2.2d), and the operator $\mathcal{O}_8(^3S_1)$ is defined in Eq. (2.2c). The remaining operators in Eq. (2.7) are defined by

$$\mathcal{O}_1(^3S_1)=\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi, \tag{2.8a}$$

$$\mathcal{P}_1(^3S_1)=\frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi+\psi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi\right], \tag{2.8b}$$

$$\mathcal{O}_8(^3P_0)=\frac{1}{3}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T_a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)T_a\psi, \tag{2.8c}$$

$$\mathcal{O}_8(^3P_1)=\frac{1}{2}\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma}\right)T_a\chi\cdot\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma}\right)T_a\psi, \tag{2.8d}$$

$$\mathcal{O}_8(^3P_2)=\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)T_a\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{D}^{(i}\sigma^{j)}\right)T_a\psi, \tag{2.8e}$$

$$\mathcal{Q}_1^1(^3S_1)=\psi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi\cdot\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi, \tag{2.8f}$$

$$\mathcal{Q}_1^2(^3S_1)=\frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\chi\cdot\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi+\psi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi\right], \tag{2.8g}$$

$$\mathcal{Q}_1^3(^3S_1)=\frac{1}{2}[\psi^\dagger\boldsymbol{\sigma}\chi\chi^\dagger\cdot\boldsymbol{\sigma}(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi-\psi^\dagger\boldsymbol{\sigma}(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\chi\cdot\chi^\dagger\boldsymbol{\sigma}\psi]. \tag{2.8h}$$

The operator $\mathcal{Q}_1^3(^3S_1)$ does not appear in Eq. (2.7) because, as we show in Appendix A, it can be eliminated in favor of $\mathcal{Q}_1^1(^3S_1)$ and $\mathcal{Q}_1^2(^3S_1)$ through the use of the equations of motion. From the velocity-scaling rules in Ref. [1], we find that, in the $^3S_1$ state, the operator $\mathcal{O}_1(^3S_1)$ has a matrix element of relative order v^0, the operator $\mathcal{P}_1(^3S_1)$ has a matrix element of relative order v^2, the operator $\mathcal{O}_8(^1S_0)$ has a matrix element of relative order v^3, and the operators $\mathcal{O}_8(^3S_1)$, $\mathcal{O}_8(^3P_0)$, $\mathcal{O}_8(^3P_1)$, $\mathcal{O}_8(^3P_2)$, $\mathcal{Q}_1^1(^3S_1)$, $\mathcal{Q}_1^2(^3S_1)$, and $\mathcal{Q}_1^3(^3S_1)$ have matrix elements of relative order v^4.

The order-α_s^3 and order-α_s^4 contributions to the short-distance coefficient $F_1(^3S_1)$ were computed by Mackenzie and Lepage [4] and can be found in Ref. [1], as can the order-α^2 contribution:

$$F_1({}^3S_1)=\frac{(N_c^2-1)(N_c^2-4)}{N_c^3}\,\frac{(\pi^2-9)}{18}\,\alpha_s^3(m)\left\{1+[-9.46(2)C_F+4.13(17)C_A-1.161(2)n_f]\frac{\alpha_s}{\pi}\right\}$$
$$+2\pi Q^2\left(\sum_{i=1}^{n_f}Q_i^2\right)\alpha^2\left[1-\frac{13}{4}C_F\frac{\alpha_s}{\pi}\right], \tag{2.9}$$

where Q is the electric charge of the heavy quark, and the Q_i are the electric charges of the light quarks. The order-α_s^3 contribution to the short-distance coefficient $G_1({}^3S_1)$ is computed in Ref. [5]:

$$G_1({}^3S_1)=-\frac{5(19\pi^2-132)}{729}\alpha_s^3. \tag{2.10}$$

To leading order in α_s^2, $G_1({}^3S_1)/[m^2F_1({}^3S_1)]=-(19\pi^2-132)/[12(\pi^2-9)]\approx-5.32$. Hence, the relativistic correction to J/ψ decay is greater in magnitude than the leading contribution. This situation casts some doubt on the validity of the v expansion. We investigate this issue further in this paper by calculating corrections of relative order v^4. The order-α_s^2 and order-α_s^3 contributions to the short-distance coefficients $F_8({}^1S_0)$ and $F_8({}^3S_1)$ are given in Eqs. (2.5b) and (2.5a), respectively. The order-α_s^2 and order-α_s^3 contributions to the short-distance coefficients $F_8({}^3P_J)$ have been computed by Petrelli *et al.* [10]:

$$F_8({}^3P_0)=6B_F\pi\alpha_s^2(\mu)\left\{1+\frac{\alpha_s}{\pi}\left[\left(-\frac{7}{3}+\frac{\pi^2}{4}\right)C_F+\left(\frac{463}{81}+\frac{35}{27}\log 2-\frac{17}{216}\pi^2\right)C_A+2b_0\log\frac{\mu}{2m}\right]\right\}+\frac{8}{9}n_fB_F\alpha_s^3\left(-\frac{29}{6}+\log\frac{2m}{\mu_\Lambda}\right), \tag{2.11a}$$

$$F_8({}^3P_1)=C_AB_F\alpha_s^3\left(\frac{1369}{54}-\frac{23}{9}\pi^2\right)+\frac{8}{9}n_fB_F\alpha_s^3\left(-\frac{4}{3}+\log\frac{2m}{\mu_\Lambda}\right), \tag{2.11b}$$

$$F_8({}^3P_2)=\frac{8B_F\pi}{5}\alpha_s^2(\mu)\left\{1+\frac{\alpha_s}{\pi}\left[-4C_F+\left(\frac{4955}{431}+\frac{7}{9}\log 2-\frac{43}{72}\pi^2\right)C_A+2b_0\log\frac{\mu}{2m}\right]\right\}+\frac{8}{9}n_fB_F\alpha_s^3\left(-\frac{29}{15}+\log\frac{2m}{\mu_\Lambda}\right), \tag{2.11c}$$

where μ_Λ is the NRQCD renormalization scale. The contribution to $F_8({}^1P_1)$ of order α_s^2 vanishes because Yang's theorem [11] forbids the decay of a spin-one particle into two equivalent massless vector particles (gluons). The contributions from decay into a light quark-antiquark pair vanish because the ${}^3P_J$ states are even under charge conjugation. Again, the individual quantities $H_1^1({}^3S_1)$ and $H_1^2({}^3S_1)$ cannot be distinguished in processes in which the heavy quark and antiquark decay on shell. We compute the quantity $H_1^1({}^3S_1)+H_1^2({}^3S_1)$ in this paper.

Through relative-order v^4, the decay of a ${}^1S_0$ state into two photons is given by

$$\Gamma({}^1S_0\to\gamma\gamma)=\frac{F_{\gamma\gamma}({}^1S_0)}{m^2}|\langle 0|\chi^\dagger\psi|{}^1S_0\rangle|^2+\frac{G_{\gamma\gamma}({}^1S_0)}{m^4}\mathrm{Re}\left[\langle{}^1S_0|\psi^\dagger\chi|0\rangle\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^1S_0\rangle\right]+\frac{H^1_{\gamma\gamma}({}^1S_0)}{m^6}$$
$$\times\langle{}^1S_0|\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi|0\rangle\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^1S_0\rangle+\frac{H^2_{\gamma\gamma}({}^1S_0)}{m^6}\mathrm{Re}\left[\langle{}^1S_0|\psi^\dagger\chi|0\rangle\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi|{}^1S_0\rangle\right]. \tag{2.12}$$

The product of matrix elements $\mathrm{Re}[\langle{}^1S_0|\psi^\dagger\chi|0\rangle\langle 0|\chi^\dagger(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi|{}^1S_0\rangle]$, which is of relative order v^4, does not appear in Eq. (2.12) because, as we show in Appendix A, it can be eliminated in favor of the products of matrix elements in the last two terms of Eq. (2.12) through the use of the equations of motion. From the velocity-scaling rules in Ref. [1], we find that, in Eq. (2.12), the product of matrix elements in the first line is of relative order v^0, the product of matrix elements in the second line is of relative order v^2, and the products of matrix elements in the third and fourth lines are of relative order v^4.

The order-α^2 and order-$\alpha^2\alpha_s$ contributions to the short-distance coefficient $F_{\gamma\gamma}({}^1S_0)$ are calculated in Refs. [2,3,6] and are given in Ref. [1]:

$$F_{\gamma\gamma}({}^1S_0)=2\pi Q^4\alpha^2\left[1+\left(\frac{\pi^2}{4}-5\right)C_F\frac{\alpha_s}{\pi}\right]. \tag{2.13}$$

The order-α^2 contribution to $G_{\gamma\gamma}({}^1S_0)$ is computed in Refs. [1,5]:

$$G_{\gamma\gamma}({}^1S_0)=-\frac{8\pi Q^4}{3}\alpha^2. \tag{2.14}$$

To leading order in α_s, $G_{\gamma\gamma}({}^1S_0)/[m^2F_{\gamma\gamma}({}^1S_0)]=-4/3$. Hence, the first relativistic correction to this process is substantial for the η_c. In this paper, we compute the combination of short-distance coefficients $H^1_{\gamma\gamma}({}^1S_0)+H^2_{\gamma\gamma}({}^1S_0)$.

Through relative order v^4, the rate for a ${}^3S_1$ state to decay into an e^+e^- pair is

$$\begin{aligned}\Gamma({}^3S_1\to e^+e^-)=&\frac{F_{ee}({}^3S_1)}{m^2}|\langle 0|\chi^\dagger\boldsymbol{\sigma}\psi|{}^3S_1\rangle|^2+\frac{G_{ee}({}^3S_1)}{m^4}\mathrm{Re}\left[\langle {}^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle 0|\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^3S_1\rangle\right]\\&+\frac{H^1_{ee}({}^3S_1)}{m^6}\langle {}^3S_1|\psi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi|0\rangle\cdot\langle 0|\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^3S_1\rangle+\frac{H^2_{ee}({}^3S_1)}{m^6}\\&\times\mathrm{Re}\left[\langle {}^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle 0|\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi|{}^3S_1\rangle\right].\end{aligned} \tag{2.15}$$

The product of matrix elements $\mathrm{Re}[\langle {}^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle 0|\chi^\dagger\boldsymbol{\sigma}(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi|{}^3S_1\rangle]$, which is of relative order v^4, does not appear in Eq. (2.15) because, as we show in Appendix A, it can be eliminated in favor of the products of matrix elements in the last two terms of Eq. (2.15) through the use of the equations of motion. In Eq. (2.15), the product of matrix elements in the first line is of relative order v^0, the product of matrix elements in the second line is of relative order v^2, and the products of matrix elements in the third and fourth lines are of relative order v^4.

The order-α^2 and order-$\alpha^2\alpha_s$ contributions to the short-distance coefficient $F_{ee}({}^3S_1)$ are calculated in Refs. [7,8] and are given in Ref. [1]. The order-$\alpha^2\alpha_s^2$ contribution is calculated in Ref. [9]. Altogether, these contributions give

$$\begin{aligned}F_{ee}({}^3S_1)=&\frac{2\pi Q^2\alpha^2}{3}\left\{1-4C_F\frac{\alpha_s(m)}{\pi}+\left[-117.46\right.\right.\\&\left.\left.+0.82n_f+\frac{140\pi^2}{27}\ln\left(\frac{2m}{\mu_\Lambda}\right)\right]\left(\frac{\alpha_s}{\pi}\right)^2\right\}.\end{aligned} \tag{2.16}$$

The order-α^2 contribution to $G_{\gamma\gamma}({}^1S_0)$ is computed in Refs. [1,5]:

$$G_{ee}({}^3S_1)=-\frac{8\pi Q^2}{9}\alpha^2. \tag{2.17}$$

To leading order in α_s, $G_{ee}({}^3S_1)/[m^2F_{\mathrm{ee}}({}^3S_1)]=-4/3$. Hence, the first relativistic correction to this process is substantial for the J/ψ. In this paper, we compute the combination of short-distance coefficients $H^1_{ee}({}^3S_1)+H^1_{ee}({}^3S_1)$.

III. SPIN PROJECTORS

In computing the quarkonium decay rates, we use the covariant spin-projector method [12,13] to identify spin-singlet and spin-triplet amplitudes. For purposes of the computations in this paper, we need projection operators accurate at least through relative order v^4. In this section, we compute the required projectors to all orders in v.

The Dirac spinors, with the standard nonrelativistic normalization, may be written as

$$u(\mathbf{p})=\sqrt{\frac{E+m}{2E}}\begin{pmatrix}\xi\\ \dfrac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\xi\end{pmatrix}, \tag{3.1a}$$

$$v(-\mathbf{p})=\sqrt{\frac{E+m}{2E}}\begin{pmatrix}\dfrac{(-\mathbf{p})\cdot\boldsymbol{\sigma}}{E+m}\eta\\ \eta\end{pmatrix}, \tag{3.1b}$$

where ξ and η are two-component Pauli spinors, and $E(p)=\sqrt{m^2+\mathbf{p}^2}$. We take the heavy quark and antiquark momenta to be

$$p_Q=(1/2)P+p, \tag{3.2a}$$

$$p_{\bar{Q}}=(1/2)P-p, \tag{3.2b}$$

respectively, where in the quarkonium rest frame,

$$P=(2E(p),\mathbf{0}), \tag{3.3a}$$

$$p=(0,\mathbf{p}). \tag{3.3b}$$

Using Eq. (3.1), it is straightforward to express spin-singlet and spin-triplet combinations of spinor bilinears in terms of Dirac matrices and to write them in a covariant form. In the spin-singlet case, we have

$$\begin{aligned}\Pi_0(P,p)&=-\sum_{\lambda_1,\lambda_2} u(\mathbf{p},\lambda_1)\overline{v(-\mathbf{p},\lambda_2)}\langle \tfrac{1}{2}\,\lambda_1\,\tfrac{1}{2}\,\lambda_2|0\;0\rangle\\ &=\frac{1}{\sqrt{2}}\,\frac{E+m}{2E}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\frac{1+\gamma_0}{2}\,\gamma_5\left(1-\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\gamma_0\\ &=\frac{1}{2\sqrt{2}E(E+m)}\left(\frac{1}{2}\not{P}+m+\not{p}\right)\frac{\not{P}+2E}{4E}\\ &\quad\times\gamma_5\left(\frac{1}{2}\not{P}-m-\not{p}\right),\end{aligned} \tag{3.4}$$

where α_i and γ_μ are Dirac matrices in the Dirac representation, $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$, and we have chosen the normalization so that the projector (3.4) corresponds in NRQCD to the projector $I/\sqrt{2}$, where I is a unit Pauli matrix.[2] We note that $E(p)$ may be written in a Lorentz invariant fashion as

$$E(p)=(1/2)\sqrt{P^2}. \tag{3.5}$$

In the case of a spin-triplet state with polarization ϵ, we have

$$\begin{aligned}\Pi_1(P,p,\boldsymbol{\epsilon})&=\sum_{\lambda_1,\lambda_2} u(\mathbf{p},\lambda_1)\overline{v(-\mathbf{p},\lambda_2)}\langle \tfrac{1}{2}\,\lambda_1\,\tfrac{1}{2}\,\lambda_2|1\;\boldsymbol{\epsilon}\rangle\\ &=\frac{1}{\sqrt{2}}\,\frac{E+m}{2E}\left(1+\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\frac{1+\gamma_0}{2}\\ &\quad\times\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}\left(1-\frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{E+m}\right)\gamma_0\\ &=\frac{-1}{2\sqrt{2}E(E+m)}\left(\frac{1}{2}\not{P}+m+\not{p}\right)\\ &\quad\times\frac{\not{P}+2E}{4E}\not{\epsilon}\left(\frac{1}{2}\not{P}-m-\not{p}\right).\end{aligned} \tag{3.6}$$

Here, $|1\;\epsilon\rangle$ is the rotationally invariant linear combination $|1\;\epsilon\rangle=\epsilon^-|1\;1\rangle-\epsilon^+|1\;-1\rangle-\epsilon_3|1\;0\rangle$, with $\epsilon^\pm=(1/\sqrt{2})(\epsilon_1\pm i\epsilon_2)$. We have chosen the normalization so that the projector (3.6) corresponds in NRQCD to the projector $\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}/\sqrt{2}$. The expressions (3.4) and (3.6) are valid to all orders in v.

IV. RELATIVISTIC CORRECTIONS TO $^1S_0$ DECAYS

In this section we compute the short-distance coefficients that appear in the corrections through relative order v^4 to $^1S_0$ quarkonium decays into two photons and into light hadrons (two gluons).

We begin with the case of decay into two photons. We take the definitions of the heavy quark and antiquark momenta given in Eq. (3.2) and work in the quarkonium rest frame, as defined in Eq. (3.3). We take the outgoing photon momenta to be k and q, with polarization indices μ and ν, respectively. Consider first the diagram in which the quark emits the photon with momentum k. The spin-singlet amplitude corresponding to this diagram is

$$\begin{aligned}A_1(\mathrm{sing}\to\gamma\gamma)&=-ie^2Q^2\mathrm{Tr}\left[\Pi_0(P,p)\,\gamma^\nu\frac{\not{p}_Q-\not{k}+m}{-2p_Q\cdot k}\,\gamma^\mu\right]\\ &=\frac{-ie^2Q^2}{2\sqrt{2}E(E+m)}\,\frac{1}{2p_Q\cdot k}\mathrm{Tr}\left[\gamma^\nu\not{k}\gamma^\mu\left(\frac{1}{2}\not{P}+m+\not{p}\right)\frac{1+\gamma_0}{2}\,\gamma_5\left(\frac{1}{2}\not{P}-m-\not{p}\right)\right]\\ &=\frac{-ie^2Q^2}{2\sqrt{2}E(E+m)}\,\frac{1}{2p_Q\cdot k}\mathrm{Tr}\left[\gamma^\nu\not{k}\gamma^\mu(E+m+\not{p})\,\frac{1+\gamma_0}{2}\,\gamma_5(-E-m-\not{p})\right],\end{aligned} \tag{4.1}$$

[2] In Eq. (3.4), the standard Clebsch-Gordan coefficients are appropriate if the spinors in Eq. (3.1) are related to each other through a unitary transformation, which preserves the SU(2) algebra, such as the charge-conjugation transformation $\eta=-i\sigma_2\xi$. One such choice of spinors is $\xi=\binom{1}{0}$ or $\binom{0}{1}$ and $\eta=\binom{0}{1}$ or $\binom{-1}{0}$, for $\lambda_1=\pm 1/2$ and $\lambda_2=\pm 1/2$, respectively. On the other hand, a popular convention is $\xi=\binom{1}{0}$ or $\binom{0}{1}$ and $\eta=\binom{0}{1}$ or $\binom{1}{0}$, for $\lambda_1=\pm 1/2$ and $\lambda_2=\pm 1/2$, respectively. With this convention, the Clebsch-Gordan coefficients in Eq. (3.4) must be multiplied by an additional factor $(-1)^{(-1/2+\lambda_2)}$.

where e is the electromagnetic coupling constant.[3] In the projector $(1+\gamma_0)/2$ in the last line, the term proportional to 1 gives a vanishing trace, while the term proportional to γ_0 gives

$$A_1(\text{sing}\to\gamma\gamma)=-e^2Q^2\frac{m}{\sqrt{2}E}\epsilon^{\nu\rho\mu 0}k_\rho\frac{1}{p_Q\cdot k}. \quad (4.2)$$

Similarly, the diagram in which the antiquark emits the photon with momentum k yields an amplitude

$$A_2(\text{sing}\to\gamma\gamma)=e^2Q^2\frac{m}{\sqrt{2}E}\epsilon^{\mu\rho\nu 0}k_\rho\frac{1}{p_{\bar{Q}}\cdot k}. \quad (4.3)$$

Adding $A_1(\text{sing}\to\gamma\gamma)$ and $A_2(\text{sing}\to\gamma\gamma)$, we obtain the complete amplitude for $^1S_0$ charmonium decay into two photons:

$$A(\text{sing}\to\gamma\gamma)=-e^2Q^2\frac{m}{\sqrt{2}E}\epsilon^{\nu\rho\mu 0}k_\rho \times\left(\frac{1}{E^2-\mathbf{p}\cdot\mathbf{k}}+\frac{1}{E^2+\mathbf{p}\cdot\mathbf{k}}\right). \quad (4.4)$$

We project out the S-wave part of the amplitude by averaging over the angles of $\mathbf{p}$:

$$\begin{aligned}A(^1S_0\to\gamma\gamma)&=\frac{1}{2}\int_{-1}^{1}d(\cos\theta)(-e^2Q^2)\frac{m}{\sqrt{2}E}\epsilon^{\nu\rho\mu 0}k_\rho \times\left(\frac{1}{E^2-|\mathbf{p}||\mathbf{k}|\cos\theta}+\frac{1}{E^2+|\mathbf{p}||\mathbf{k}|\cos\theta}\right)\\&=-e^2Q^2\epsilon^{\nu\rho\mu 0}k_\rho\frac{m}{\sqrt{2}E^2|\mathbf{p}|}\ln\frac{E+|\mathbf{p}|}{E-|\mathbf{p}|}, \end{aligned}\quad (4.5)$$

where we have used $\mathbf{k}^2=E^2$. Multiplying the expression (4.5) by its complex conjugate, by the two-body phase space $1/(8\pi)$, and by a factor 1/2! for two identical particles in the final state, we obtain the decay width for a $^1S_0$ $Q\bar{Q}$ state into two photons:

$$\Gamma(^1S_0\to\gamma\gamma)=\frac{\pi m^2Q^4\alpha^2}{E^2\mathbf{p}^2}\ln^2\frac{E+|\mathbf{p}|}{E-|\mathbf{p}|}. \quad (4.6)$$

Here, and in succeeding computations of the decay widths of two-particle states, we suppress a factor of the inverse volume that is associated with the normalization of the initial state.

[3] In computing the short-distance coefficients for electromagnetic decay processes, we suppress the trivial color factors, which ultimately cancel when one matches decay rates in full QCD and NRQCD.

From Eq. (2.12), we find that the decay width for a $^1S_0$ $Q\bar{Q}$ state into two photons in NRQCD in order α_s^0 and through relative order v^4 is

$$\begin{aligned}\Gamma_{\text{NRQCD}}(^1S_0\to\gamma\gamma)=2[&(1/m^2)F_{\gamma\gamma}(^1S_0)\\&+(\mathbf{p}^2/m^4)G_{\gamma\gamma}(^1S_0)+(\mathbf{p}^4/m^6)H^1_{\gamma\gamma}(^1S_0)\\&+(\mathbf{p}^4/m^6)H^2_{\gamma\gamma}(^1S_0)],\end{aligned} \quad (4.7)$$

where the factor two on the right side of Eq. (4.7) comes from the spin factor for normalized heavy-quark states.

Comparing powers of $\mathbf{p}^2/m^2$ in Eqs. (4.6) and (4.7), we obtain the short-distance coefficients at leading order in α_s:

$$F_{\gamma\gamma}(^1S_0)=2\pi Q^4\alpha^2, \quad (4.8a)$$

$$G_{\gamma\gamma}(^1S_0)=-\frac{8\pi}{3}Q^4\alpha^2, \quad (4.8b)$$

$$H^1_{\gamma\gamma}(^1S_0)+H^2_{\gamma\gamma}(^1S_0)=\frac{136\pi}{45}Q^4\alpha^2. \quad (4.8c)$$

Our results for $F_{\gamma\gamma}(^1S_0)$ and $G_{\gamma\gamma}(^1S_0)$ confirm those given in Refs. [1–3,6] and [1], respectively. Our result for $H^1_{\gamma\gamma}(^1S_0)+H^2_{\gamma\gamma}(^1S_0)$ is new.

At leading order in α_s, the decay of a $^1S_0$ $Q\bar{Q}$ state to light hadrons proceeds through an annihilation into two gluons. Hence, we may obtain the decay width for a $^1S_0$ $Q\bar{Q}$ state into light hadrons by multiplying the width into two photons [Eq. (4.6)] by a color factor $C_F/2$ times $\alpha_s^2/(\alpha^2Q^4)$:

$$\Gamma(^1S_0\to\text{LH})=\frac{\pi C_Fm^2\alpha_s^2}{2E^2\mathbf{p}^2}\ln^2\frac{E+|\mathbf{p}|}{E-|\mathbf{p}|}. \quad (4.9)$$

From Eq. (2.1), we find that the decay width for a $^1S_0$ $Q\bar{Q}$ state into two photons in NRQCD in order α_s^2 and through relative order v^4 is

$$\begin{aligned}\Gamma_{\text{NRQCD}}(^1S_0\to\text{LH})=2N_c[&(1/m^2)F_1(^1S_0)\\&+(\mathbf{p}^2/m^4)G_1(^1S_0)+(\mathbf{p}^4/m^6)H^1_1(^1S_0)\\&+(\mathbf{p}^4/m^6)H^2_1(^1S_0)],\end{aligned} \quad (4.10)$$

where the factor $2N_c$ on the right side of Eq. (4.10) comes from the spin and color factors for normalized heavy-quark states. The matrix elements of the color-octet operators do not contribute to Eq. (4.10) in order α_s^2. Comparing Eqs. (4.9) and (4.10), we obtain the short-distance coefficients at leading order in α_s:

$$F_1(^1S_0)=\frac{\pi C_F}{N_c}\alpha_s^2, \quad (4.11a)$$

$$G_1(^1S_0)=-\frac{4\pi C_F}{3N_c}\alpha_s^2, \quad (4.11b)$$

$$H^1_1(^1S_0)+H^2_1(^1S_0)=\frac{68\pi C_F}{45N_c}\alpha_s^2. \quad (4.11c)$$

Our result for $F_1(^1S_0)$ is in agreement with that given in Refs. [1–3], and our result for $G_1(^1S_0)$ is in agreement with that given in Ref. [1]. Our result for $H_1^1(^1S_0)+H_1^2(^1S_0)$ is new.

V. RELATIVISTIC CORRECTIONS TO $^3S_1$ DECAY TO e^+e^-

Next we turn to the case of the decay of a $^3S_1$ quarkonium state into an e^+e^- pair. Again, we work in the quarkonium rest frame defined in Eq. (3.3). The amplitude for a quark and antiquark in a spin-triplet state with the momenta given in Eq. (3.2) to decay into a virtual photon with polarization index μ is given by

$$A(\text{trip}\rightarrow\gamma^*)=ieQ\,\text{Tr}[\Pi_1(P,p,\epsilon)\gamma_\mu]$$
$$=ieQ\sqrt{2}\left[\frac{p_\mu p\cdot\epsilon}{E(E+m)}+\epsilon_\mu\right]. \tag{5.1}$$

We can project out the S-wave part of the amplitude by averaging over the angles of $\mathbf{p}$:

$$A(^3S_1\rightarrow\gamma^*)=\frac{1}{2}\int_{-1}^{1}d(\cos\theta)\,ieQ\sqrt{2}\left[\frac{p_\mu p\cdot\epsilon}{E(E+m)}+\epsilon_\mu\right]$$
$$=ieQ\sqrt{2}\left(\frac{2}{3}+\frac{m}{3E}\right)\epsilon_\mu. \tag{5.2}$$

In order to obtain the decay rate into an e^+e^- pair, we multiply the expression (5.2) by its complex conjugate with index ν, by a photon-propagator factor $-ig_{\mu\rho}/k^2$, by a complex-conjugated photon-propagator factor $ig_{\nu\sigma}/k^2$, and by twice the imaginary part of the e^+e^--pair contribution to the photon's vacuum polarization, namely, $(g_{\rho\sigma}k^2-k_\rho k_\sigma)\times(-2/3)\alpha$. Here k is the virtual photon's momentum. The result is

$$\Gamma(^3S_1\rightarrow e^+e^-)=\frac{4\pi Q^2\alpha^2}{3E^2}\left(\frac{2}{3}+\frac{m}{3E}\right)^2, \tag{5.3}$$

where we have used $k\cdot\epsilon=0$, $\epsilon\cdot\epsilon^*=-1$, and $k^2=4E^2$.

From Eq. (2.15), we see that, in NRQCD through relative order v^4, the decay width for a $^3S_1\ Q\bar{Q}$ state into an e^+e^- pair is

$$\Gamma_{\text{NRQCD}}(^3S_1\rightarrow e^+e^-)=2[(1/m^2)F_{ee}(^3S_1)+(\mathbf{p}^2/m^4)G_{ee}(^3S_1)+(\mathbf{p}^4/m^6)H_{ee}^1(^3S_1)+(\mathbf{p}^4/m^6)H_{ee}^2(^3S_1)]. \tag{5.4}$$

The factor two on the right side of Eq. (5.4) comes from the spin factor for normalized heavy-quark states.

Comparing powers of $\mathbf{p}^2/m^2$ in Eqs. (5.3) and (5.4), we obtain the short-distance coefficients at leading order in α_s:

$$F_{ee}(^3S_1)=\frac{2\pi}{3}Q^2\alpha^2, \tag{5.5a}$$

$$G_{ee}(^3S_1)=-\frac{8\pi}{9}Q^2\alpha^2, \tag{5.5b}$$

$$H_{ee}^1(^3S_1)+H_{ee}^2(^3S_1)=\frac{58\pi}{54}Q^2\alpha^2. \tag{5.5c}$$

Our result for $F_{ee}(^3S_1)$ agrees with that given in Refs. [1,7,8], and our result for $G_{ee}(^3S_1)$ agrees with that given in Ref. [1]. Our result for $H_{ee}^1(^3S_1)+H_{ee}^2(^3S_1)$ is new.

VI. RELATIVISTIC CORRECTIONS TO $^3S_1$ DECAY TO LIGHT HADRONS

In the decay of a heavy-quark–antiquark state, diagrams in which only two of the final-state gluons attach to the heavy-quark line have a common heavy-quark color factor. Hence, (Abelian) charge-conjugation symmetry forbids such diagrams in the decay of a $^3S_1$ state. Furthermore, color conservation forbids diagrams in which only one of the final-state gluons attaches to the heavy-quark line. Thus, in leading order in α_s, a $^3S_1$ heavy-quark–antiquark state decays into three gluons, and the decay proceeds through diagrams in which all three gluons attach to the heavy-quark line. (Since no triple-gluon vertices appear, there are no ghost contributions.)

In this decay process, in contrast with the decay processes that we have analyzed in the preceding sections, the kinematics allow one of the final-state gluons to have zero energy. Hence, the possibility arises that the decay rate contains an infrared (IR) divergence. Simple power counting arguments show that an IR divergence can arise only if the soft gluon attaches to an incoming (on-shell) heavy-quark or heavy-antiquark leg. Therefore, one can use NRQCD to analyze the interaction of this soft gluon with the heavy quark.

One can see from power-counting arguments, that, through relative order v^4, in the Coulomb gauge, a gluon that interacts with a quark or an antiquark can yield an IR divergence only if the interactions are of the type $\psi^\dagger\mathbf{D}\cdot\mathbf{A}\psi$ or $\chi^\dagger\mathbf{D}\cdot\mathbf{A}\chi$. (The $\psi^\dagger\mathbf{B}\cdot\boldsymbol{\sigma}\psi$ and $\chi^\dagger\mathbf{B}\cdot\boldsymbol{\sigma}\chi$ interactions have the correct dimensions to produce an IR divergence, but the factors of $\mathbf{B}$ bring in powers of the gluon momentum that protect against an IR divergence.) The factor of $\mathbf{D}$ translates into a factor of the incoming quark or antiquark momentum. Factors of the gluon momentum do not appear since they are orthogonal to the gluon propagator in the Coulomb gauge. Therefore, the interactions of the gluon yield two factors of the incoming quark or antiquark momentum in the squared amplitude. Two additional factors of the incoming quark or antiquark momentum are required in order to have a nonzero overlap with an incoming S-wave state. Hence, an IR divergence in the decay rate must be associated with at least four factors of the incoming quark or antiquark momentum. That is, an IR divergence can first appear in relative order v^4. Because the soft gluon in a $\mathbf{D}\cdot\mathbf{A}$ interaction changes the incoming S-wave color-singlet quark-antiquark state into a P-wave color-octet quark-antiquark state, we expect that the IR divergence will be absorbed into matrix elements of the P-wave color-octet operators in Eq. (2.7).

Now let us turn to the actual computation of the rate for a

${}^3S_1\,Q\bar{Q}$ state to decay into three gluons. We present only the outlines of that calculation here. We used the symbolic manipulation program MATHEMATICA and the package FEYNCALC [14] to handle the tedious, but straightforward, details of the algebra.

We regulate the anticipated IR divergence by computing in $D=4-2\epsilon$ dimensions. We work in the quarkonium rest frame, assign the incoming quark and antiquark momenta as in Eqs. (3.2) and (3.3), and take the outgoing gluon momenta to be k_1, k_2, and k_3. First we compute the sum of the six Feynman diagrams for this process, making use of the projector (3.6). Although we are working in $D=4-2\epsilon$ dimensions, we can follow the approach of Ref. [10] and simply use the D-dimensional version of the spin-1 projector (3.6). As explained in Ref. [10], we need not consider projectors for the higher-spin evanescent NRQCD operators that appear in D dimensions because the contributions that contain an IR pole in one loop do not mix the higher-spin operators with the spin-1 operators.

At this point, we could square the amplitude, integrate over the phase space, and expand in powers of $\mathbf{p}$ in order to obtain the desired result. However, the amount of algebra would be greatly reduced if we could make the expansion in powers of $\mathbf{p}$ before carrying out the phase-space integration. Such a strategy is complicated by the fact that the phase space depends on $\mathbf{p}$ through the total energy of the incoming $Q\bar{Q}$ state, but we can make that dependence explicit by introducing a rescaling of phase-space integration variables:

$$k_i \to k_i E(p)/m. \tag{6.1}$$

Then, the final-state phase space transforms as

$$\prod_i \left(\frac{d^{(D-1)}k_i}{2(k_i)_0}\right)\delta^D\!\left(P-\sum_i k_i\right) \to \prod_i \left(\frac{d^{(D-1)}k_i}{2(k_i)_0}\right)\delta^D\!\left(\frac{mP}{E(p)}-\sum_i k_i\right) f(p), \tag{6.2}$$

where

$$f(p)=\left[\frac{E(p)}{m}\right]^{(D-2)^3/D} = \left[\frac{E(p)}{m}\right]^2\left[1-\frac{5}{2}\epsilon\log\frac{E^2(p)}{m^2}\right]+O(\epsilon^2). \tag{6.3}$$

All of the dependence on $\mathbf{p}$ on the right side of Eq. (6.2) is contained in the explicit factor $f(p)$. The remaining factors correspond to the phase space evaluated at the $Q\bar{Q}$ threshold $p=0$. Therefore, after rescaling the k_i according to Eq. (6.1), we can obtain the necessary expansion in powers of $\mathbf{p}$ by expanding the amplitude, its complex conjugate, and $f(p)$ in powers of $\mathbf{p}$ *before* carrying out the phase-space integration. Note that an IR pole in ϵ first appears in the rate, excluding the factor $f(p)$, only in the relative-order v^4. Hence, we can drop the term proportional to ϵ on the right side of Eq. (6.3), which contributes an additional factor v^2.

We expand the amplitude in a power series in $\mathbf{p}$, through order $\mathbf{p}^4$. The terms containing no powers of $\mathbf{p}$ yield a pure S-wave contribution. For the terms containing two powers of p, we extract the S-wave contribution by making the replacement

$$p_\mu p_\nu \to \mathbf{p}^2 T_{\mu\nu}. \tag{6.4}$$

For the terms containing four powers of p, we extract the S-wave contribution by making the replacement

$$p_\mu p_\nu p_\rho p_\sigma \to \mathbf{p}^4 T_{\mu\nu\rho\sigma}. \tag{6.5}$$

Here,

$$T_{\mu\nu}=\frac{1}{D-1}\Pi_{\mu\nu}, \tag{6.6}$$

$$T_{\mu\nu\rho\sigma}=\frac{1}{(D-1)(D+1)}[\Pi_{\mu\nu}\Pi_{\rho\sigma}+\Pi_{\mu\rho}\Pi_{\nu\sigma}+\Pi_{\mu\sigma}\Pi_{\nu\rho}], \tag{6.7}$$

and

$$\Pi_{\mu\nu}=-g_{\mu\nu}+\frac{P_\mu P_\nu}{4E^2(p)}. \tag{6.8}$$

Next, we multiply the amplitude by its complex conjugate. We evaluate the gluon polarization sums using the Feynman-gauge expression

$$\epsilon_\mu \epsilon_\nu^* = -g_{\mu\nu}, \tag{6.9}$$

we evaluate the spin-triplet-state polarization sum using

$$\epsilon_\mu \epsilon_\nu^* = \Pi_{\mu\nu}, \tag{6.10}$$

and we divide by $D-1$ to obtain the average over the spin-triplet-state polarizations. Owing to the charge-conjugation invariance of the amplitude, only the part of the color factor that is symmetric in the color indices survives. It is given by

$$\frac{1}{16N_c}d^{abc}d^{abc}=\frac{(N_c^2-1)(N_c^2-4)}{16N_c^2}. \tag{6.11}$$

Multiplying by this color factor and by $f(p)$, we obtain the "squared matrix element" that must be integrated over the $p=0$ three-body phase space to obtain the decay rate. We write the coefficients of $\mathbf{p}^0$, $\mathbf{p}^2$, and $\mathbf{p}^4$ in terms of the invariants

$$s=2k_1\cdot k_2, \tag{6.12a}$$

$$t=2k_1\cdot k_3, \tag{6.12b}$$

$$u=2k_2\cdot k_3, \tag{6.12c}$$

where, since we have set $p=0$ in these coefficients, the energy-momentum conservation relation now reads

$$k_1+k_2+k_3=2m. \tag{6.13}$$

The expressions for these coefficients in four dimensions are given in Appendix B.

We re-write the coefficients of $\mathbf{p}^0$, $\mathbf{p}^2$, and $\mathbf{p}^4$ in terms of the invariants

$$x_i=\frac{2P\cdot k_i}{(2m)^2}\bigg|_{p=0}. \tag{6.14}$$

It follows that

$$s=2m^2(x_1+x_2-x_3), \tag{6.15a}$$

$$t=2m^2(x_1-x_2+x_3), \tag{6.15b}$$

$$u=2m^2(-x_1+x_2+x_3). \tag{6.15c}$$

The D-dimensional three-body phase space for decay of a particle of mass M is [10]

$$d\Phi_{(3)}=\frac{M^2}{2(4\pi)^3}\left(\frac{4\pi}{M^2}\right)^{2\epsilon}\frac{1}{\Gamma(2-2\epsilon)}\times\prod_{i=1}^{3}(1-x_i)^{-\epsilon}dx_i\,\delta\left(2-\sum_{i=1}^{3}x_i\right). \tag{6.16}$$

The phase space at $p=0$ is obtained by making the identification $M=2m$.

It is convenient to make a further change of variables, so that the limits of integration are independent of the integration variables. To this end, we write

$$x_1=x, \tag{6.17a}$$

$$x_2=1-xy, \tag{6.17b}$$

$$x_3=1-x(1-y). \tag{6.17c}$$

This change of variables is particularly useful in analyzing the infrared singularities, since it avoids the difficulty that, at the singular points $x_i=0$, the range of integration in one of the variables x_j $(i\neq j)$ vanishes. Now the phase space is

$$d\Phi_{(3)}=\frac{M^2}{2(4\pi)^3}\left(\frac{4\pi}{M^2}\right)^{2\epsilon}\frac{1}{\Gamma(2-2\epsilon)}\times[x^2(1-x)y(1-y)]^{-\epsilon}x\,dx\,dy, \tag{6.18}$$

where x and y range from 0 to 1.

In the cases of the terms proportional to $\mathbf{p}^0$ and $\mathbf{p}^2$, the integrations over the phase space are IR finite, and we can carry out the integrations with $D=4$. Multiplying by 1/3! for three identical particles in the final state, we obtain

$$\Gamma^{(0)}=\frac{1}{m^2}\frac{(N_c^2-1)(N_c^2-4)}{9N_c^2}(\pi^2-9)\alpha_s^3, \tag{6.19}$$

$$\Gamma^{(2)}=-\frac{\mathbf{p}^2}{m^4}\frac{(N_c^2-1)(N_c^2-4)}{108N_c^2}(19\pi^2-132)\alpha_s^3. \tag{6.20}$$

In the case of the term proportional to $\mathbf{p}^4$, we must first separate the IR singular parts in the matrix element squared.[4] These are given by

$$\widetilde{\mathcal{M}}_{\rm IR}=\frac{\mathbf{p}^4}{m^8}\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\frac{128\pi^3\alpha_s^3}{(3-2\epsilon)^3}\times[(3-2\epsilon)(1-\epsilon)-2(2-\epsilon)y(1-y)]\mu^{6\epsilon}\times\left(\frac{1}{x^2}+\frac{1}{(1-xy)^2}+\frac{1}{[1-x(1-y)]^2}\right). \tag{6.21}$$

Integrating $\widetilde{\mathcal{M}}_{\rm IR}$ over the phase space (6.18) and multiplying by 1/3! for three identical particles in the final state, we obtain

$$\Gamma_{\rm IR}^{(4)}=-\frac{\mathbf{p}^4}{m^6}\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\alpha_s^3\frac{1}{\epsilon}\left(\frac{4\pi}{M^2}\right)^{2\epsilon}\times\mu^{6\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma^2(2-2\epsilon)}\frac{(1-\epsilon)^2(7-4\epsilon)}{(3-2\epsilon)^4}. \tag{6.22a}$$

Neglecting terms of order ϵ, we may write this expression as

$$\Gamma_{\rm IR}^{(4)}=-\frac{\mathbf{p}^4}{m^6}\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\alpha_s^3\left[\frac{7}{81\epsilon}\left(\frac{4\pi}{M^2}\right)^{2\epsilon}\times\mu^{6\epsilon}\frac{(1-\epsilon\gamma_E)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}+\frac{44}{243}\right], \tag{6.22b}$$

where γ_E is Euler's constant. After subtracting the IR singular terms (6.21) from the integrand, we can carry out the phase-space integration over the remainder with $D=4$. Multiplying by 1/3! for three identical particles in the final state, we obtain

$$\Gamma_{\rm finite}^{(4)}=\frac{\mathbf{p}^4}{m^6}\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\alpha_s^3\left[-\frac{3563}{2430}+\frac{1609}{6480}\pi^2\right]. \tag{6.23}$$

The complete decay width in full QCD of a ${}^3S_1$ $Q\overline{Q}$ state into light hadrons through order v^4 is then

$$\Gamma({}^3S_1\to{\rm LH})=\Gamma^{(0)}+\Gamma^{(2)}+\Gamma_{\rm IR}^{(4)}+\Gamma_{\rm finite}^{(4)}, \tag{6.24}$$

[4] In Ref. [10], an alternative method for dealing with the singular part was employed. The region of integration was partitioned into three regions that are related by interchange of the three gluon momenta. Only the region containing the singularity at $x=0$ was retained, and the contribution from this region was multiplied by three to obtain the complete result. The method that we present in this paper has the advantage that the limits of integration are simpler, and hence, the integrals are evaluated more easily. Also, certain terms that cancel between the singular and non-singular contributions in the method of Ref. [10] never appear in the present method.

where the quantities on the right side are given in Eqs. (6.19), (6.20), (6.22), and (6.23).

To determine the short-distance coefficients, we match these results with the NRQCD expression for the decay width (2.7), evaluated in the $^3S_1$ $Q\bar{Q}$ state. Since we have computed the full QCD decay rate in order α_s^3, we must evaluate each contributing term in Eq. (2.7) with accuracy α_s^3. The coefficients $F_8(^1S_0)$, $F_8(^3S_1)$, and $F_8(^3P_J)$ are of order α_s^2. Therefore, we must evaluate the corresponding matrix elements through order α_s. We evaluate the matrix elements corresponding to the unknown coefficients $F_1(^3S_1)$, $G_1(^3S_1)$, $H_1^1(^3S_1)$, and $H_1^2(^3S_1)$ at order α_s^0.

The color-octet matrix elements $\langle {}^3S_1|\mathcal{O}_8(^1S_0)|{}^3S_1\rangle$ and $\langle {}^3S_1|\mathcal{O}_8(^3S_1)|{}^3S_1\rangle$ have a vanishing contribution at order α_s^0 in the color-singlet $Q\bar{Q}$ state. The order-α_s^1 contribution comes from four diagrams in which a gluon connects an initial-state Q or $\bar{Q}$ with a final-state Q or $\bar{Q}$. The interaction of the gluon with the Q or $\bar{Q}$ cannot be of the $\mathbf{p}\cdot\mathbf{A}$ form, since that interaction changes the orbital angular momentum by one unit. Any other NRQCD interaction must involve at least one power of the gluon momentum. Hence, it is easy to see, by simple power counting arguments, that the integration over the gluon momentum is ultraviolet (UV) power divergent. It therefore vanishes in dimensional regularization.

The color-octet matrix elements $\langle {}^3S_1|\mathcal{O}_8(^3P_J)|{}^3S_1\rangle$ also have a vanishing contribution at order α_s^0 in the color-singlet $Q\bar{Q}$ state. Again, the order-α_s^1 contribution comes from four diagrams in which a gluon connects an initial-state Q or $\bar{Q}$ with a final-state Q or $\bar{Q}$. The contribution at leading order in v arises from $\mathbf{p}\cdot\mathbf{A}$ interactions between the gluon and the Q or $\bar{Q}$. A straightforward computation yields

$$\langle {}^3S_1|\mathcal{O}_8(^3P_J)|{}^3S_1\rangle = \frac{\mathbf{p}^4}{m^2}\,\frac{8(2J+1)C_F}{81\pi}\alpha_s\int_0^\infty \frac{dk}{k}. \tag{6.25}$$

This integral has logarithmic IR and UV divergences. Since it is scale invariant, it vanishes in dimensional regularization. It can be written as

$$\langle {}^3S_1|\mathcal{O}_8(^3P_J)|{}^3S_1\rangle = \frac{\mathbf{p}^4}{m^2}\,\frac{4(2J+1)C_F}{81\pi}\,\frac{\mu^{2\epsilon}}{\mu_\Lambda^{2\epsilon}}\alpha_s \times\left\{\left[\frac{1}{\epsilon_{\rm UV}}+\log(4\pi)-\gamma_E\right]-\left[\frac{1}{\epsilon_{\rm IR}}+\log(4\pi)-\gamma_E\right]\right\}, \tag{6.26}$$

where $\epsilon_{\rm UV}$ and $\epsilon_{\rm IR}$ are $(4-D)/2$, and μ_Λ is the NRQCD renormalization scale. We renormalize the expression (6.26) in the modified minimal subtraction ($\overline{\rm MS}$) scheme by subtracting the contribution proportional to $1/\epsilon_{\rm UV}+\log(4\pi)-\gamma_E$. The renormalized matrix element is

$$\langle {}^3S_1|\mathcal{O}_8(^3P_J)|{}^3S_1\rangle_{\overline{\rm MS}} = -\frac{\mathbf{p}^4}{m^2}\,\frac{4(2J+1)C_F}{81\pi}\,\frac{\mu^{2\epsilon}}{\mu_\Lambda^{2\epsilon}}\alpha_s \times\left[\frac{1}{\epsilon}+\log(4\pi)-\gamma_E\right], \tag{6.27}$$

where we have made the identification $\epsilon_{\rm IR}=\epsilon$.

Making use of these results for the matrix elements, we find that the decay width in NRQCD in order α_s^3 is

$$\begin{aligned}\Gamma_{\rm NRQCD}(^3S_1\to{\rm LH}) = 2N_c[&(1/m^2)F_1(^3S_1)\\ &+(\mathbf{p}^2/m^4)G_1(^3S_1)+(\mathbf{p}^4/m^6)H^1_{\gamma\gamma}(^1S_0)\\ &+(\mathbf{p}^4/m^6)H^2_{\gamma\gamma}(^1S_0)]\\ &+(\mathbf{p}^4/m^6)\sum_{J=0,1,2}c_JF_8(^3P_J),\end{aligned} \tag{6.28a}$$

where the factor $2N_c$ in front of the square brackets comes from the color and spin factors for normalized heavy-quark states and

$$c_J = -\frac{2(N_c^2-1)}{81\pi N_c}(2J+1)\frac{\mu^{2\epsilon}}{\mu_\Lambda^{2\epsilon}}\alpha_s\left[\frac{1}{\epsilon}+\log(4\pi)-\gamma_E\right]. \tag{6.28b}$$

The short-distance coefficients $F_8(^3P_J)$ have been computed in $D=4-2\epsilon$ dimensions in order α_s^2 by Petrelli *et al.* [10]:

$$F_8(^3P_0) = 18\pi B_F\alpha_s^2\left(\frac{4\pi}{M^2}\right)^\epsilon \mu^{4\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,\frac{1-\epsilon}{3-2\epsilon}, \tag{6.29a}$$

$$F_8(^3P_1) = 0, \tag{6.29b}$$

$$F_8(^3P_2) = 4\pi B_F\alpha_s^2\left(\frac{4\pi}{M^2}\right)^\epsilon \mu^{4\epsilon} \times\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,\frac{6-13\epsilon+4\epsilon^2}{(3-2\epsilon)(5-2\epsilon)}. \tag{6.29c}$$

It follows that

$$\sum_{J=0,1,2}c_JF_8(^3P_J) = -\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\alpha_s^3\left[\frac{7}{81\epsilon}\left(\frac{4\pi\mu^3}{M\mu_\Lambda}\right)^{2\epsilon} \times\frac{(1-\epsilon\gamma_E)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{1}{15}\right], \tag{6.30}$$

where we have neglected terms of order ϵ.

Using Eq. (6.30), we can compare the width in full QCD [Eq. (6.24)] with the width in NRQCD [Eq. (6.28)] to compute the short-distance coefficients. As expected, the IR poles in ϵ cancel, and we obtain

TABLE I. Short-distance coefficients and estimates of sizes of corresponding matrix elements for the decay of a $^1S_0$ quarkonium state to two photons.

Coefficient	Value	Matrix element
$F_{\gamma\gamma}(^1S_0)$	1	1
$G_{\gamma\gamma}(^1S_0)$	-1.33	v^2
$H^1_{\gamma\gamma}(^1S_0)+H^2_{\gamma\gamma}(^1S_0)$	1.51	v^4

$$F_1(^3S_1)=\frac{(N_c^2-1)(N_c^2-4)}{N_c^3}\,\frac{(\pi^2-9)}{18}\alpha_s^3, \tag{6.31a}$$

$$G_1(^3S_1)=\frac{(N_c^2-1)(N_c^2-4)}{N_c^3}\left(\frac{11}{18}-\frac{19}{216}\pi^2\right)\alpha_s^3, \tag{6.31b}$$

$$H_1^1(^3S_1)+H_1^2(^3S_1)=\frac{(N_c^2-1)(N_c^2-4)}{N_c^3}\times\left(-\frac{833}{972}+\frac{1609}{12960}\pi^2+\frac{7}{81}\log\frac{2m}{\mu_\Lambda}\right)\alpha_s^3. \tag{6.31c}$$

Our result for $F_1(^3S_1)$ agrees with that given in Ref. [4], and our result for $G_1(^3S_1)$ agrees with that given in Ref. [5]. Our result for $H_1^1(^3S_1)+H_1^2(^3S_1)$ is new.

VII. DISCUSSION

In this paper, we have computed short-distance coefficients for the decays of a $^1S_0$ heavy-quarkonium state to two photons and to light hadrons and the decays of a $^3S_1$ heavy-quarkonium state to a lepton pair and to light hadrons. Specifically, we have computed the coefficients of the operators whose matrix elements are of order v^4 and whose quantum numbers are those of the quarkonium state.

In our computation, we are able to obtain only the combinations $H^1(^{2S+1}L_J)+H^2(^{2S+1}L_J)$, rather than the individual coefficients $H^1(^{2S+1}L_J)$ and $H^2(^{2S+1}L_J)$, because the corresponding operators, $\mathcal{Q}^1(^{2S+1}L_J)$ and $\mathcal{Q}^2(^{2S+1}L_J)$, have identical matrix elements for on-shell heavy quarks in the center-of-momentum frame. In order to obtain the values of the individual coefficients, it would be necessary to consider matrix elements of the operators $\mathcal{Q}^1(^{2S+1}L_J)$ and $\mathcal{Q}^2(^{2S+1}L_J)$ in which the heavy $Q\bar{Q}$ interact with additional quanta before reaching the annihilation vertex. Alternatively, one could consider matrix elements of the operators $\mathcal{Q}^3(^{2S+1}L_J)$, which, as we have shown in Appendix A, are related to the operators $\mathcal{Q}^1(^{2S+1}L_J)$ and $\mathcal{Q}^2(^{2S+1}L_J)$ through the equations of motion.

In Tables I–IV, we show the numerical values of the short-distance coefficients that appear through order v^4 for the decays that we consider in this paper. For each coefficient, we take into account only the contribution that is leading in α_s. In each case, we normalize the short-distance

TABLE II. Short-distance coefficients and estimates of sizes of corresponding matrix elements for the decay of a $^1S_0$ quarkonium state to light hadrons.

Coefficient	Value	Matrix element
$F_1(^1S_0)$	1	1
$G_1(^1S_0)$	-1.33	v^2
$F_8(^3S_1)$	$0.75n_f$	$v^3/(2N_c)$
$F_8(^1S_0)$	1.88	$v^4/(2N_c)$
$F_8(^1P_1)$	1.13	$v^4/(2N_c)$
$H_1^1(^1S_0)+H_1^2(^1S_0)$	1.51	v^4

coefficients to the coefficient of the operator whose matrix element is of leading order in v. In the third column of each table, we use the velocity-scaling rules [1] to estimate the size of the matrix element of the operator that is associated with each coefficient, relative to the size of the matrix element of leading order in v. In the case of the color-octet operators, we adopt the approach of Ref. [10], multiplying the velocity-scaling estimate by a factor $1/(2N_c)$ to account for the relative spin and color normalizations of the color-singlet and color-octet operators as we have defined them in this paper.

In the case of charmonium, $v^2\approx 0.3$ and $\alpha_s(m_c)\approx 0.35$. Then, we see from Tables I–III that the convergence of the v expansion is reasonable for the $^1S_0$ decays into two photons and into light hadrons and for the $^3S_1$ decay into light hadrons.

On the other hand, the coefficients in Table IV cast some doubt on the convergence of the v expansion in the case of the $^3S_1$ decay into light hadrons. In the case of charmonium, all of the contributions of higher order in v are larger in magnitude than the order-v^0 contribution, with the exception of the $H_1^1(^3S_1)+H_1^2(^3S_1)$ contribution. The color-octet coefficients, other than $F_8(^3P_1)$, are enhanced by π/α_s, relative to $F_1(^3S_1)$, since the corresponding color-octet Fock states can decay into two gluons or into light-quark pairs, rather than into three gluons. In addition to this enhancement, some of the coefficients of π/α_s are quite large. However, one can, through a redefinition of the color-singlet operators, incorporate the factors $1/(2N_c)$, which we have associated with the matrix elements, into the short-distance coefficients [10]. Then, aside from the π/α_s enhancement, only $F_8(^3P_0)$ is especially large. In the case of the color-singlet coefficients, $G_1(^3S_1)$ is quite large in magnitude rela-

TABLE III. Short-distance coefficients and estimates of sizes of corresponding matrix elements for the decay of a $^3S_1$ quarkonium state to a lepton pair.

Coefficient	Value	Matrix element
$F_{ee}(^3S_1)$	1	1
$G_{ee}(^3S_1)$	-1.33	v^2
$H^1_{ee}(^3S_1)+H^2_{ee}(^3S_1)$	1.61	v^4

TABLE IV. Short-distance coefficients and estimates of sizes of corresponding matrix elements for the decay of a $^3S_1$ quarkonium state to light hadrons.

Coefficient	Value	Matrix element
$F_1(^3S_1)$	1	1
$G_1(^3S_1)$	-5.32	v^2
$F_8(^1S_0)$	$11.64\pi/\alpha_s$	$v^3/(2N_c)$
$F_8(^3S_1)$	$4.66n_f\pi/\alpha_s$	$v^4/(2N_c)$
$F_8(^3P_0)$	$34.93\pi/\alpha_s$	$v^4/(2N_c)$
$F_8(^3P_1)$	$2.26-6.90n_f+5.17n_f\log(2m/\mu_\Lambda)$	$v^4/(2N_c)$
$F_8(^3P_2)$	$9.31\pi/\alpha_s$	$v^4/(2N_c)$
$H_1^1(^3S_1)+H_1^2(^3S_1)$	$7.62+1.79\log(2m/\mu_\Lambda)$	v^4

tive to $F_1(^3S_1)$. However, the quantity $H_1^1(^3S_1)+H_1^2(^3S_1)$ is not significantly larger in magnitude than $G_1(^3S_1)$, giving some hope that the v expansion may ultimately be well behaved.

The estimates of the sizes of the relativistic corrections strongly suggest that, in order to carry out a meaningful phenomenological analysis of *S*-wave quarkonium decays into light hadrons, one would need to take into account contributions beyond leading order in v. (For a further discussion of this point, see Ref. [15].) All of the contributions listed in Table IV, except for that of $F_8(^3P_1)$, would be needed to achieve a precision of better than 50%.

Unfortunately, most of the required matrix elements are unknown. However, the number of unknown quantities can be reduced drastically by making use of the heavy-quark spin symmetry and the vacuum-saturation approximation [1], although the accuracy of these approximations is not always sufficient to allow a calculation of the decay rates through relative order v^4. Owing to the heavy-quark spin symmetry, the matrix elements of $\mathcal{O}_1(^1S_0)$, $\mathcal{P}_1(^1S_0)$, $\mathcal{O}_8(^3S_1)$, $\mathcal{O}_8(^1S_0)$, $\mathcal{Q}_1^1(^1S_0)$, and $\mathcal{Q}_1^2(^1S_0)$ in a $^1S_0$ state are equal to the matrix elements of $\mathcal{O}_1(^3S_1)$, $\mathcal{P}_1(^3S_1)$, $\mathcal{O}_8(^1S_0)$, $\mathcal{O}_8(^3S_1)$, $\mathcal{Q}_1^1(^3S_1)$, and $\mathcal{Q}_1^2(^3S_1)$ in a $^3S_1$ state, respectively, up to corrections of relative order v^2. Also owing to the heavy-quark spin symmetry, the matrix elements of the operators $\mathcal{O}_8(^3P_J)$ in a $^3S_1$ state are equal to $(2J+1)/9$ times the matrix element of $\mathcal{O}_8(^1P_1)$ in a $^1S_0$ state, up to corrections of relative order v^2. According to the vacuum-saturation approximation, the matrix elements of the operators for the electromagnetic decays are equal to the matrix elements of the color-singlet hadronic-decay operators with the same quantum numbers, up to corrections of relative order v^4. It also follows from the vacuum-saturation approximation that the matrix element of $\mathcal{Q}_1^1(^{2S+1}S_J)$ is equal to the square of the matrix element of $\mathcal{P}_1(^{2S+1}S_J)$ divided by the matrix element of $\mathcal{O}_1(^{2S+1}S_J)$, up to corrections of order v^4. However, the matrix element of $\mathcal{Q}_1^2(^{2S+1}S_J)$ is not known to be related to the others.

The matrix elements of $\mathcal{O}_1(^3S_1)$ in the J/ψ and Υ states are known from phenomenology. The matrix elements of $\mathcal{O}_1(^3S_1)$ and $\mathcal{P}_1(^3S_1)$ in the J/ψ and Υ states have also been computed on the lattice [16], although the lattice determinations of the matrix elements of $\mathcal{P}_1(^3S_1)$ are rather imprecise, owing to large uncertainties in the perturbation series that relates the lattice and continuum matrix elements. According to the Gremm-Kapustin relation [17], for dimensionally regulated matrix elements, the matrix element of $\mathcal{P}_1(^3S_1)$ is equal to the matrix element of $\mathcal{O}_1(^3S_1)$ times $(M-2m_{\rm pole})/m$, up to corrections of relative order v^2. Here, M is the quarkonium mass, and $m_{\rm pole}$ is the heavy-quark pole mass. The remaining unknown operator matrix elements could, in principle, be determined in lattice numerical simulations.

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APPENDIX A: RELATION BETWEEN THE OPERATORS OF ORDER v^4

In this appendix, we demonstrate that the operators $\mathcal{Q}_1^i(^1S_0)$ [Eq. (2.2)] are related to each other by the equations of motion, as are the operators $\mathcal{Q}_1^i(^3S_1)$ [Eq. (2.8)], the vacuum-saturated versions of the $\mathcal{Q}_1^i(^1S_0)$, and the vacuum-saturated versions of the $\mathcal{Q}_1^i(^3S_1)$. We assume that these operators are integrated over all space-time, so that we can employ integration by parts in re-writing them.

We begin by considering the operator

$$\mathcal{Q}_1^1(^1S_0)=\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi. \tag{A1}$$

Now,

$$\chi^\dagger(-i\overleftrightarrow{\mathbf{D}})^2\psi=\chi^\dagger(-i\overrightarrow{\boldsymbol{\partial}}-g\mathbf{A}+i\overleftarrow{\boldsymbol{\partial}}-g\mathbf{A})(-i\overrightarrow{\boldsymbol{\partial}}-g\mathbf{A}+i\overleftarrow{\boldsymbol{\partial}}-g\mathbf{A})\psi=\chi^\dagger[2(-i\overrightarrow{\boldsymbol{\partial}}-g\mathbf{A})^2+2(i\overleftarrow{\boldsymbol{\partial}}-g\mathbf{A})^2-(i\overrightarrow{\boldsymbol{\partial}}+i\overleftarrow{\boldsymbol{\partial}})^2]\psi]$$
$$=\chi^\dagger[4m(i\overrightarrow{\partial}_0-gA_0)+4m(i\overleftarrow{\partial}_0-gA_0)-(i\overrightarrow{\boldsymbol{\partial}}+i\overleftarrow{\boldsymbol{\partial}})^2]\psi=[4im\overrightarrow{\partial}_0-(-i\overrightarrow{\boldsymbol{\partial}})^2]\chi^\dagger\psi, \tag{A2}$$

where we have used the equations of motion at leading order in v in the third line. Furthermore, under integration by parts, which is equivalent to energy-momentum conservation in momentum space,

$$[\psi^\dagger(-i\overleftrightarrow{\mathbf{D}})^2\chi][4im\overrightarrow{\partial}_0-(-i\overrightarrow{\boldsymbol{\partial}})^2]\chi^\dagger\psi\rightarrow\{[-4mi\overrightarrow{\partial}_0-(i\overrightarrow{\boldsymbol{\partial}})^2][\psi^\dagger(-i\overleftrightarrow{\mathbf{D}})^2\chi]\}\chi^\dagger\psi. \tag{A3}$$

Let us focus on the first $Q\overline{Q}$ bilinear on the right of Eq. (A3). It is

$$[-4mi\overrightarrow{\partial}_0-(i\overrightarrow{\boldsymbol{\partial}})^2][\psi^\dagger(-i\overleftrightarrow{\mathbf{D}})^2\chi]=\psi^\dagger\{-4m(i\overleftarrow{\partial}_0+gA_0)(-i\overleftrightarrow{\mathbf{D}})^2-4m(-i\overleftrightarrow{\mathbf{D}})^2(i\overrightarrow{\partial}_0-gA_0)+4m[gA_0,(-i\overleftrightarrow{\mathbf{D}})^2]$$
$$-4m[i\overrightarrow{\partial}_0,(-i\overleftrightarrow{\mathbf{D}})^2]-(i\overleftarrow{\boldsymbol{\partial}}+i\overrightarrow{\boldsymbol{\partial}})^2(-i\overleftrightarrow{\mathbf{D}})^2\}\chi. \tag{A4}$$

Then, using the equations of motion, we have

$$[-4mi\overrightarrow{\partial}_0-(i\overrightarrow{\boldsymbol{\partial}})^2][\psi^\dagger(-i\overleftrightarrow{\mathbf{D}})^2\chi]=\psi^\dagger\{2(i\overleftarrow{\boldsymbol{\partial}}-g\mathbf{A})^2(-i\overleftrightarrow{\mathbf{D}})^2+2(-i\overleftrightarrow{\mathbf{D}})^2(-i\overrightarrow{\boldsymbol{\partial}}-g\mathbf{A})^2+4m[gA_0,(-i\overleftrightarrow{\mathbf{D}})^2]$$
$$-4m[i\overrightarrow{\partial}_0,(-i\overleftrightarrow{\mathbf{D}})^2]-(i\overleftarrow{\boldsymbol{\partial}}+i\overrightarrow{\boldsymbol{\partial}})^2(-i\overleftrightarrow{\mathbf{D}})^2\}\chi$$
$$=\psi^\dagger\{(-i\overleftrightarrow{\mathbf{D}})^4+4m[(-iD_0),(-i\overleftrightarrow{\mathbf{D}})^2]\}\chi=\psi^\dagger[(-i\overleftrightarrow{\mathbf{D}})^4-8m(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})]\chi, \tag{A5}$$

where, in arriving at the second equality, we have dropped some terms proportional to $[\overleftrightarrow{D}_i,\overleftrightarrow{D}_j]=-2ig\epsilon_{ijk}B_k$ that are order v^2 relative to the terms that we have retained.

Thus, taking into account both $Q\overline{Q}$ bilinears, we have

$$\mathcal{Q}_1^1({}^1S_0)\rightarrow\psi^\dagger\left[\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4-(m/2)(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\right]\chi\chi^\dagger\psi. \tag{A6}$$

Carrying out this procedure symmetrically on the left and right $Q\overline{Q}$ bilinears of $\mathcal{Q}_1^1({}^1S_0)$, we conclude that, under the equations of motion and integration by parts,

$$\mathcal{Q}_1^1({}^1S_0)\rightarrow\mathcal{Q}_1^2({}^1S_0)+(m/2)\mathcal{Q}_1^3({}^1S_0). \tag{A7}$$

A similar analysis in the spin-triplet case yields

$$\mathcal{Q}_1^1({}^3S_1)\rightarrow\mathcal{Q}_1^2({}^3S_1)+(m/2)\mathcal{Q}_1^3({}^3S_1). \tag{A8}$$

The vacuum-saturated versions of these relations, which are relevant to the electromagnetic decays are

$$\langle{}^1S_0|\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi|0\rangle\langle0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^1S_0\rangle$$
$$\rightarrow\mathrm{Re}\left[\langle{}^1S_0|\psi^\dagger\chi|0\rangle\langle0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi|{}^1S_0\rangle\right]$$
$$+(m/2)\mathrm{Re}[\langle{}^1S_0|\psi^\dagger\chi|0\rangle$$
$$\times\langle0|\chi^\dagger(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi|{}^1S_0\rangle] \tag{A9}$$

and

$$\langle{}^3S_1|\psi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\chi|0\rangle\cdot\langle0|\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^2\psi|{}^3S_1\rangle$$
$$\rightarrow\mathrm{Re}\left[\langle{}^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle0|\chi^\dagger\boldsymbol{\sigma}\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)^4\psi|{}^3S_1\rangle\right]$$
$$+(m/2)\mathrm{Re}[\langle{}^3S_1|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle\cdot\langle0|\chi^\dagger\boldsymbol{\sigma}(\overleftrightarrow{\mathbf{D}}\cdot g\mathbf{E}$$
$$+g\mathbf{E}\cdot\overleftrightarrow{\mathbf{D}})\psi|{}^3S_1\rangle]. \tag{A10}$$

APPENDIX B: SQUARED MATRIX ELEMENTS FOR ${}^3S_1$ DECAY INTO LIGHT HADRONS

In this appendix we give the expressions for the terms of order $\mathbf{p}^{(0)}$, $\mathbf{p}^{(2)}$, and $\mathbf{p}^{(4)}$ in the square of the matrix element for a ${}^3S_1$ $Q\overline{Q}$ state to decay into light hadrons (three gluons). These terms are denoted by $\widetilde{\mathcal{M}}^{(0)}$, $\widetilde{\mathcal{M}}^{(2)}$, and $\widetilde{\mathcal{M}}^{(4)}$, respectively. The quantity m is the heavy-quark mass. The invariants s, t, and u are defined in Eq. (6.12).

$$\tilde{\mathcal{M}}^{(0)}=\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\,\frac{2048\pi^3\alpha_s^3}{3}(16m^4s^2-8m^2s^3+s^4+16m^4st-12m^2s^2t+2s^3t+16m^4t^2-12m^2st^2+3s^2t^2-8m^2t^3$$
$$+2st^3+t^4)\frac{1}{(4m^2-s)^2(4m^2-t)^2(s+t)^2}; \tag{B1}$$

$$\tilde{\mathcal{M}}^{(2)}=-\mathbf{p}^2\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\,\frac{8192\pi^3\alpha_s^3}{9}(768m^{10}s^2-256m^8s^3-48m^6s^4+24m^4s^5-2m^2s^6+512m^{10}st-128m^8s^2t$$
$$-256m^6s^3t+120m^4s^4t-18m^2s^5t+s^6t+768m^{10}t^2-128m^8st^2-256m^6s^2t^2+176m^4s^3t^2-37m^2s^4t^2+3s^5t^2$$
$$-256m^8t^3-256m^6st^3+176m^4s^2t^3-44m^2s^3t^3+5s^4t^3-48m^6t^4+120m^4st^4-37m^2s^2t^4+5s^3t^4+24m^4t^5$$
$$-18m^2st^5+3s^2t^5-2m^2t^6+st^6)\,\frac{1}{m^2(4m^2-s)^3(4m^2-t)^3(s+t)^3}; \tag{B2}$$

$$\tilde{\mathcal{M}}^{(4)}=\mathbf{p}^4\frac{(N_c^2-1)(N_c^2-4)}{N_c^2}\,\frac{1024\pi^3\alpha_s^3}{135}(1966080m^{16}s^2+1409024m^{14}s^3-1445888m^{12}s^4+223232m^{10}s^5+47104m^8s^6$$
$$-14080m^6s^7+880m^4s^8+1310720m^{16}st+2588672m^{14}s^2t-4034560m^{12}s^3t+1176576m^{10}s^4t+97536m^8s^5t$$
$$-86848m^6s^6t+12352m^4s^7t-552m^2s^8t+1966080m^{16}t^2+2588672m^{14}st^2-3047424m^{12}s^2t^2+984064m^{10}s^3t^2$$
$$+371456m^8s^4t^2-275264m^6s^5t^2+56032m^4s^6t^2-4688m^2s^7t^2+155s^8t^2+1409024m^{14}t^3-4034560m^{12}st^3$$
$$+984064m^{10}s^2t^3+525312m^8s^3t^3-408960m^6s^4t^3+108304m^4s^5t^3-12452m^2s^6t^3+620s^7t^3-1445888m^{12}t^4$$
$$+1176576m^{10}st^4+371456m^8s^2t^4-408960m^6s^3t^4+128640m^4s^4t^4-18468m^2s^5t^4+1240s^6t^4+223232m^{10}t^5$$
$$+97536m^8st^5-275264m^6s^2t^5+108304m^4s^3t^5-18468m^2s^4t^5+1550s^5t^5+47104m^8t^6-86848m^6st^6$$
$$+56032m^4s^2t^6-12452m^2s^3t^6+1240s^4t^6-14080m^6t^7+12352m^4st^7-4688m^2s^2t^7+620s^3t^7+880m^4t^8$$
$$-552m^2st^8+155s^2t^8)\frac{1}{m^4(4m^2-s)^4(4m^2-t)^4(s+t)^4}. \tag{B3}$$

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