$O(\alpha^2 \ln(m_\mu/m_e))$ corrections to the electron energy spectrum in muon decay

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The $O(\alpha^2 \ln(m_\mu/m_e))$ corrections to the electron energy spectrum in muon decay are computed using the perturbative fragmentation function approach. The magnitude of these corrections is comparable to the anticipated precision of the TWIST experiment where the Michel parameters will be extracted from the measurement of the electron energy spectrum in muon decay.

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I. INTRODUCTION

Muon decay into an electron and a pair of neutrinos, $\mu \to e \nu_\mu \overline{\nu}_e$, is a classic process in particle physics. Although the high energy frontier has moved up from the energy scales comparable to the muon mass, precision physics of muons remains an interesting and inspiring source of information about the standard model (SM) and its possible extensions [1].

Among very different experiments with the muons that include the measurements of the muon anomalous magnetic moment, the muon lifetime, the $\mu \to e\gamma$ branching ratio, and the muon to electron conversion rate in muonic atoms, we focus here on the TWIST experiment [2,3], where the electron energy spectrum in muon decays will be measured to determine the Michel parameters [4,5] with a precision of $\sim 10^{-4}$. To confront these measurements with the SM predictions and to look for the signs of new physics, one needs an adequately accurate calculation of the electron energy spectrum within the SM.

Calculations of the electron energy spectrum in muon decay have a long and interesting history that dates back to the very early days of QED and the physics of weak interactions (see e.g. Ref. [6] for a historical recollection). In spite of the tremendous progress in precision calculations, the $O(\alpha^2)$ radiative corrections to the muon lifetime have been computed only recently [7], and the calculation of similar corrections to the electron energy spectrum has not even been attempted. One reason for this is that, in contrast to the total lifetime, the electron energy spectrum cannot be computed for vanishing electron mass since terms enhanced by the large logarithm $\ln(m_\mu/m_e)$ are present. These terms, excluding the ones that are related to the on-shell definition of the fine-structure constant commonly used in QED, cancel in the total rate rendering this calculation somewhat simpler.

At order $O(\alpha^2)$, corrections to the electron energy spectrum contain double-logarithmic $O(\ln^2(m_\mu^2/m_e^2))$ and single-logarithmic $O(\ln(m_\mu^2/m_e^2))$ terms and it is the purpose of this paper to present the calculation of those. The double-logarithmic terms were computed recently in Ref. [8]. It was pointed out there that the single-logarithmic $O(\ln(m_\mu^2/m_e^2))$ terms are required to match the precision of the TWIST experiment. Motivated by these considerations, we decided to perform this calculation. To accomplish this, we make use of the perturbative fragmentation function approach borrowed from QCD studies of heavy quark fragmentation in e^+e^- collisions.

II. PRELIMINARIES

According to the QCD factorization theorems [9], the differential cross section for producing a particle of a given type with a certain fraction of the initial energy can be written as a convolution of the hard scattering cross section computed with massless partons and the fragmentation function that describes the probability that a massless parton of a given type fragments to the observed physical particle in the final state. If we consider the process in which an energetic heavy quark (i.e. $m_Q \gg \Lambda_{\mathrm{QCD}}$) is produced and its energy is measured, we can identify the *massive* quark with the observed physical particle in the final state. It has been shown in QCD that in this case the perturbative fragmentation function can be defined and that this function absorbs all the terms that are singular in the limit of the vanishing heavy quark mass [10–12]. It is clear that these considerations should be applicable to QED as well.

Applying this idea to muon decay, we can write the formula for the electron energy spectrum in the following way:

$$\frac{d\Gamma}{dx}(x,m_\mu,m_e) = \sum_{j=e,\gamma} \int_x^1 \frac{dz}{z} \frac{d\hat{\Gamma}_j}{dz}(z,m_\mu,\mu_f)\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right). \tag{1}$$

Here $z=2E/m_\mu$ is the fraction of energy carried away by a parton j, x is the same quantity for the observed physical massive electron, $d\hat{\Gamma}_j/dz$ is the energy distribution of the

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massless parton of type j computed in the modified minimal subtraction ($\overline{\rm MS}$) scheme, $\mathcal{D}_j$ is the fragmentation function of the parton j fragmenting into the massive electron, and μ_f is the factorization scale. Note that terms suppressed by the ratio of the electron mass to the muon mass, m_e^2/m_μ^2, cannot be described by Eq. (1). However, since these terms are known, for both the Born and the $O(\alpha)$ corrected electron energy spectrum [13,14], Eq. (1) is quite adequate for the anticipated level of experimental precision.

As mentioned above, the partonic decay rate $d\hat{\Gamma}_j/dz$ has to be computed in the $\overline{\rm MS}$ scheme. This requirement goes beyond the standard ultraviolet renormalization, since $d\hat{\Gamma}_j/dz$ contains collinear singularities. These singularities are removed from $d\hat{\Gamma}_j/dz$ by conventional renormalization in the $\overline{\rm MS}$ scheme, and the associated large collinear logarithms are absorbed into the fragmentation function $\mathcal{D}_j$.

The perturbative expansion for the energy distribution of the massless partons is

$$\frac{1}{\Gamma_0}\frac{d\hat{\Gamma}_j}{dz}(z,m_\mu,\mu_f)=A_j^{(0)}(z)+\frac{\overline{\alpha}(\mu_f)}{2\pi}\hat{A}_j^{(1)}(m_\mu,\mu_f,z)+\left(\frac{\overline{\alpha}(\mu_f)}{2\pi}\right)^2\hat{A}_j^{(2)}(m_\mu,\mu_f,z), \quad (2)$$

where $\Gamma_0=G_F^2m_\mu^5/(96\pi^3)$, $A_j^{(0)}(z)=z^2(3-2z)\delta_{je}$, $\overline{\alpha}(\mu_f)$ is the $\overline{\rm MS}$ renormalized fine structure constant, and terms of order $O(\alpha^3)$ and higher have been neglected. The $\overline{\rm MS}$ fine structure constant will later be converted into the on-shell fine structure constant $\alpha\approx 1/137.036$.

Before giving the explicit expressions for the coefficients $\hat{A}_j^{(1)}$, we would like to describe a simple idea, previously used in a number of QCD studies, that allows us to compute the $\alpha^2\ln(m_\mu/m_e)$ enhanced terms without an explicit two-loop calculation. Since $d\hat{\Gamma}_j/dz$ is computed for massless partons, it contains two energy scales, the muon mass m_μ and the factorization scale μ_f. Therefore, the only logarithms that arise are of the form $\ln(m_\mu/\mu_f)$. By choosing $\mu_f\sim m_\mu$, we effectively eliminate the large logarithms from the coefficients $\hat{A}_j^{(1)}$, and move all the large logarithms to the fragmentation function $\mathcal{D}_j$. Since the fragmentation function is process-independent and satisfies the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation, results from previous QCD studies can be employed to compute the large logarithmic corrections to the electron energy spectrum at order $O(\alpha^2)$.

Consider the fragmentation function $\mathcal{D}_j(x,\mu_f,m_e)$ which describes the probability that a massless parton j converts into a physical electron of mass m_e. This function satisfies the DGLAP evolution equation:

$$\frac{d\mathcal{D}_i(x,\mu_f,m_e)}{d\ln\mu_f^2}=\sum_j\int_x^1\frac{dz}{z}P_{ji}(z,\overline{\alpha}(\mu_f))\mathcal{D}_j\left(\frac{x}{z},\mu_f,m_e\right). \quad (3)$$

Here P_{ji} is the time-like splitting function which, to the order we work to, can be written as

$$P_{ji}(x,\overline{\alpha}(\mu_f))=\frac{\overline{\alpha}(\mu_f)}{2\pi}P_{ji}^{(0)}(x)+\left(\frac{\overline{\alpha}(\mu_f)}{2\pi}\right)^2P_{ji}^{(1)}(x)+O(\overline{\alpha}^3). \quad (4)$$

Equation (3) can be solved as a power series in $\overline{\alpha}$ if the initial condition for the function $\mathcal{D}_j$ at the scale μ_0 is provided. This initial condition can be obtained from QCD studies of heavy quark fragmentation [10]; when generalized to QED, they imply that the fragmentation of a massless electron into a physical electron is described by

$$\mathcal{D}_e^{\rm ini}(x,\mu_0,m_e)=\delta(1-x)+\frac{\overline{\alpha}(\mu_0)}{2\pi}d_1(x,\mu_0,m_e)+O(\alpha^2),$$

$$d_1(x,\mu_0,m_e)\equiv d_1(x)=\left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_0^2}{m_e^2}-2\ln(1-x)-1\right)\right]_+. \quad (5)$$

Similarly, the function that describes photon fragmentation into physical electron is

$$\mathcal{D}_\gamma^{\rm ini}(x,\mu_0,m_e)=\frac{\overline{\alpha}(\mu_0)}{2\pi}(x^2+(1-x)^2)\ln\frac{\mu_0^2}{m_e^2}+O(\alpha^2). \quad (6)$$

As we will show in the next section, the $O(\alpha^2)$ terms in the initial conditions for the fragmentation functions are not needed for our purposes since our scale choice $\mu_0\sim m_e$ guarantees that no large logarithms appear in the initial condition for $\mathcal{D}_j$. Also, since the fragmentation function does not contain large logarithms at the lowest order $O(\alpha^0)$, the second order coefficient in $d\hat{\Gamma}_j/dz$ is not needed as well. On the other hand, the $O(\alpha)$ coefficients in $d\hat{\Gamma}_j/dz$ have to be known exactly. They are

$$\begin{aligned}\hat{A}_e^{(1)}(z)=&\left(2z^2(2z-3)\ln\left[\frac{z}{1-z}\right]+2z+\frac{8}{3}z^3+\frac{5}{6}-4z^2\right)\\&\times\ln\left(\frac{m_\mu^2}{\mu_f^2}\right)+2z^2(2z-3)(4\zeta_2-4{\rm Li}_2(z)\\&+2\ln^2z-3\ln z\ln(1-z)-\ln^2(1-z))\\&+\left(\frac{5}{3}-2z-13z^2+\frac{34}{3}z^3\right)\ln(1-z)\\&+\left(\frac{5}{3}+4z-2z^2-6z^3\right)\\&\times\ln z+\frac{5}{6}-\frac{23}{3}z-\frac{3}{2}z^2+\frac{7}{3}z^3,\end{aligned} \quad (7)$$

$$\hat{A}_\gamma^{(1)}(z)=\left(\ln\frac{m_\mu^2}{\mu_f^2}+\ln(1-z)\right)\left(\frac{1}{z}-\frac{5}{3}+2z-2z^2+\frac{2}{3}z^3\right)+\ln z\left(\frac{2}{z}-\frac{10}{3}+4z\right)-\frac{1}{z}+\frac{1}{3}+\frac{35}{12}z-2z^2-\frac{1}{4}z^3. \tag{8}$$

Having made these preliminary remarks, we can now derive the fragmentation function and use it to calculate the electron energy spectrum in muon decay.

III. THE FRAGMENTATION FUNCTION

In this section we compute the fragmentation function. For this purpose, we have to solve the DGLAP equation (3) in a way consistent with the initial conditions. When solving this equation perturbatively, we express the running fine structure constant in the $\overline{\mathrm{MS}}$ scheme in terms of the fine structure constant defined in the on-shell scheme, the standard renormalization scheme for QED. To order $O(\alpha^2)$, the well known relation between the $\overline{\mathrm{MS}}$ and on-shell coupling constants is

$$\overline{\alpha}(\mu_f)=\alpha+\frac{\alpha^2}{3\pi}\ln\frac{\mu_f^2}{m_e^2}. \tag{9}$$

Solving Eq. (3) iteratively, we obtain

$$\begin{aligned}\mathcal{D}_e(x,\mu_f,m_e)&=\delta(1-x)+\frac{\alpha}{2\pi}(LP_{ee}^{(0)}(x)+d_1(x,\mu_0,m_e))\\&+\left(\frac{\alpha}{2\pi}\right)^2\Bigg(L^2\left[\frac{1}{2}P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x)+\frac{1}{3}P_{ee}^{(0)}(x)\right.\\&\left.+\frac{1}{2}P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)}(x)\right]+L[P_{ee}^{(0)}\otimes d_1(x)\\&+P_{ee}^{(1)}(x)]\Bigg)+O(\alpha^2L^0,\alpha^3),\end{aligned} \tag{10}$$

$$\mathcal{D}_\gamma(x,\mu_f,m_e)=\frac{\alpha}{2\pi}LP_{e\gamma}^{(0)}(x)+O(\alpha^2), \tag{11}$$

where $L=\ln(\mu_f^2/\mu_0^2)$ and the convolution operation is defined in a standard way:

$$A\otimes B(x)=\int_0^1 dz\int_0^1 dz'\,\delta(x-zz')A(z)B(z')=\int_x^1\frac{dz}{z}A(z)B\left(\frac{x}{z}\right). \tag{12}$$

The leading order expressions for the splitting functions used in Eqs. (10),(11) are

$$P_{ee}^{(0)}(x)=\left[\frac{1+x^2}{1-x}\right]_+,\quad P_{\gamma e}^{(0)}(x)=\frac{1+(1-x)^2}{x},$$

$$P_{e\gamma}^{(0)}(x)=x^2+(1-x)^2. \tag{13}$$

At the next-to-leading order the time-like splitting functions have been derived for QCD in Refs. [15–18]; by choosing appropriate color structures, they can be translated to QED in a straightforward way. Since, experimentally, one will probably distinguish between events with one or more electrons in the final state, we decided to split the corresponding second order function $P_{ee}^{(1)}(x)$ into four parts, in the same way as in Ref. [19]:

$$P_{ee}^{(1)}(x)=P_{ee}^{(1,\gamma)}(x)+P_{ee}^{(1,\mathrm{NS})}(x)+P_{ee}^{(1,\mathrm{S})}(x)+P_{ee}^{(1,\mathrm{int})}(x). \tag{14}$$

Here $P_{ee}^{(1,\gamma)}(x)$ is determined by the set of Feynman diagrams with only photonic corrections (i.e. no additional electrons in the final state or closed electron loops in virtual corrections); $P_{ee}^{(1,\mathrm{NS})}(x)$ describes corrections due to non-singlet real and virtual e^+e^- pairs; $P_{ee}^{(1,\mathrm{S})}(x)$ contains the contribution of the singlet e^+e^- pair and $P_{ee}^{(1,\mathrm{int})}(x)$ describes the interference of the singlet and the non-singlet pairs. These functions can be written as

$$\begin{aligned}P_{ee}^{(1,\gamma)}(x)&=\delta(1-x)\left(\frac{3}{8}-3\zeta_2+6\zeta_3\right)+\frac{1+x^2}{1-x}\\&\times(2\ln x\ln(1-x)-2\ln^2x-2\mathrm{Li}_2(1-x))\\&+\frac{1}{2}(1+x)\ln^2x+2x\ln x-3x+2,\end{aligned} \tag{15}$$

$$\begin{aligned}P_{ee}^{(1,\mathrm{NS})}(x)&=\delta(1-x)\left(-\frac{4}{3}\zeta_2-\frac{1}{6}\right)-\frac{20}{9}\left[\frac{1}{1-x}\right]_+\\&-\frac{2}{3}\frac{1+x^2}{1-x}\ln x-\frac{2}{9}+\frac{22}{9}x,\end{aligned} \tag{16}$$

$$\begin{aligned}P_{ee}^{(1,\mathrm{S})}(x)&=(1+x)\ln^2x+\left(-5-9x-\frac{8}{3}x^2\right)\ln x-8-\frac{20}{9x}\\&+4x+\frac{56}{9}x^2,\end{aligned} \tag{17}$$

$$\begin{aligned}P_{ee}^{(1,\mathrm{int})}(x)&=\frac{1+x^2}{1-x}\left(2\mathrm{Li}_2(1-x)+\frac{3}{2}\ln x\right)-\frac{7}{2}(1+x)\ln x\\&-7+8x,\end{aligned} \tag{18}$$

where we have used

$$\zeta_n\equiv\sum_{k=1}^{\infty}\frac{1}{k^n},\qquad \zeta_2=\frac{\pi^2}{6},$$

$$\mathrm{Li}_2(x)\equiv-\int_0^x dz\frac{\ln(1-z)}{z}. \tag{19}$$

Finally, we give the explicit formulas for various convolutions which appear in Eq. (10):

$$P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x)=\delta(1-x)\left(\frac{9}{4}-4\zeta_2\right)+\left[\frac{1}{1-x}(6+8\times\ln(1-x))\right]_+-\frac{4}{1-x}\ln x+(1+x)[3\ln x-4\ln(1-x)]-x-5, \tag{20}$$

$$P_{\gamma e}^{(0)}\otimes P_{e\gamma}^{(0)}(x)=\frac{1-x}{3x}(4+7x+4x^2)+2(1+x)\ln x, \tag{21}$$

$$P_{ee}^{(0)}\otimes d_1(x)=P_{ee}^{(0)}\otimes P_{ee}^{(0)}(x)\left(\ln\frac{\mu_0^2}{m_e^2}-1\right)+\delta(1-x)\times\left(\frac{21}{4}-8\zeta_3\right)+\left[\frac{1}{1-x}(7+8\zeta_2-6\ln(1-x)-12\ln^2(1-x))\right]_++\frac{8}{1-x}\ln x\ln(1-x)+(1+x)[6\ln^2(1-x)-6\ln x\ln(1-x)-2\mathrm{Li}_2(1-x)-4\zeta_2]+2x\ln x+(7-x)\times\ln(1-x)-\frac{11}{2}-\frac{3}{2}x. \tag{22}$$

Using these results in Eqs. (10),(11), we obtain explicit expressions for the fragmentation functions $\mathcal{D}_{e,\gamma}(x,\mu_f,m_e)$.

IV. THE ELECTRON ENERGY SPECTRUM

To obtain the electron energy spectrum we have to convolute the fragmentation functions in Eqs. (10),(11) with $d\hat{\Gamma}_j/dz$. All the necessary ingredients to do that can be found in the previous sections. Before presenting our results for the electron energy spectrum, we note that the dependence on the factorization scale cancels explicitly in the final result, except for the terms that are not enhanced by any large logarithm and therefore beyond the scope of this paper.

We split the final result into four pieces: the pure photonic radiative corrections, the non-singlet pair radiative corrections, the singlet pair radiative corrections, and the corrections due to the interference of the singlet and the non-singlet pairs. With this separation, we can write the electron energy spectrum as

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dx}=\Delta^{(\gamma)}+\Delta^{(\mathrm{NS})}+\Delta^{(\mathrm{S})}+\Delta^{(\mathrm{int})}. \tag{23}$$

We begin with the photonic corrections. Computing the convolutions, we arrive at the following result:

$$\Delta^{(\gamma)}=f_0(x)+\frac{\alpha}{2\pi}f_1(x)+\left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{2}f_2^{(0,\gamma)}(x)\ln^2\left(\frac{m_\mu^2}{m_e^2}\right)+f_2^{(1,\gamma)}(x)\ln\left(\frac{m_\mu^2}{m_e^2}\right)+\cdots\right], \tag{24}$$

where ellipses represent both the $O(\alpha^2)$ terms without logarithmic enhancement and terms of higher order in the expansion in the fine structure constant. The $O(\alpha^0)$ energy spectrum is given by $f_0(x)=x^2(3-2x)$. The $O(\alpha)$ correction, $f_1(x)$, was calculated in Ref. [13]. The coefficient of the double-logarithmic term is

$$f_2^{(0,\gamma)}(x)=4x^2(3-2x)\left[\frac{1}{2}\ln^2x+\ln^2(1-x)-2\ln x\ln(1-x)-\mathrm{Li}_2(1-x)-\zeta_2\right]+\left(\frac{10}{3}+8x-16x^2+\frac{32}{3}x^3\right)\times\ln(1-x)+\left(-\frac{5}{6}-2x+8x^2-\frac{32}{3}x^3\right)\ln x+\frac{11}{36}+\frac{17}{6}x+\frac{8}{3}x^2-\frac{32}{9}x^3, \tag{25}$$

and is therefore in agreement with the recent results in Ref. [8]. The coefficient of the single-logarithmic term for the pure photonic corrections, one of the new results of this paper, is

$$f_2^{(1,\gamma)}(x)=2x^2(3-2x)(-2\mathrm{Li}_3(x)-2\mathrm{S}_{1,2}(x)+2\mathrm{Li}_2(x)\ln(1-x)+2\mathrm{Li}_2(x)\ln x+5\ln x\ln^2(1-x)-5\ln^2x\ln(1-x)+2\ln^3x-2\zeta_2\ln(1-x)-2\zeta_2\ln x+7\zeta_3)+\mathrm{Li}_2(x)\left(\frac{10}{3}+14x-40x^2+\frac{92}{3}x^3\right)+\ln x\ln(1-x)\left(\frac{25}{3}+32x-54x^2+\frac{92}{3}x^3\right)+\ln^2(1-x)(-12x-4x^2+8x^3)+\ln^2x\left(-\frac{25}{12}-5x+22x^2-\frac{70}{3}x^3\right)+\ln(1-x)\left(-\frac{17}{3}-\frac{53}{3}x+\frac{64}{3}x^2-12x^3\right)+\ln x\left(-\frac{3}{4}+\frac{37}{6}x+\frac{4}{3}x^2+\frac{44}{9}x^3\right)+\zeta_2\left(-\frac{10}{3}-2x+35x^2-\frac{98}{3}x^3\right)+\frac{211}{216}-\frac{287}{12}x+\frac{83}{3}x^2-\frac{559}{54}x^3, \tag{26}$$

where

$$\mathrm{Li}_3(x) \equiv \int_0^x dz \frac{\mathrm{Li}_2(z)}{z},$$

$$\mathrm{S}_{1,2}(x) \equiv \frac{1}{2}\int_0^x dz \frac{\ln^2(1-z)}{z}. \tag{27}$$

The correction arising from non-singlet electron-positron pairs, including the effects of the running coupling constant, is

$$\Delta^{(\mathrm{NS})} = \left(\frac{\alpha}{2\pi}\right)^2 \left[\frac{1}{3} f_2^{(0,\mathrm{NS})}(x) \ln^2\left(\frac{m_\mu^2}{m_e^2}\right) + f_2^{(1,\mathrm{NS})}(x) \ln\left(\frac{m_\mu^2}{m_e^2}\right) + \cdots\right], \tag{28}$$

with

$$f_2^{(0,\mathrm{NS})}(x) = 2x^2(3-2x)\ln\frac{1-x}{x} + \frac{5}{6} + 2x - 4x^2 + \frac{8}{3}x^3, \tag{29}$$

$$\begin{aligned} f_2^{(1,\mathrm{NS})}(x) = {} & 2x^2(3-2x)\biggl(-2\mathrm{Li}_2(1-x) - \frac{2}{3}\ln x \ln(1-x) \\ & + \frac{2}{3}\ln^2(1-x) - \ln^2 x - \frac{2}{3}\zeta_2\biggr) + \ln(1-x) \\ & \times\left(\frac{10}{9} - \frac{4}{3}x - \frac{46}{3}x^2 + 12x^3\right) + \ln x\biggl(\frac{5}{9} + \frac{4}{3}x \\ & + 8x^2 - \frac{76}{9}x^3\biggr) - \frac{11}{6} - \frac{19}{3}x + \frac{100}{9}x^2 - \frac{64}{9}x^3. \end{aligned} \tag{30}$$

Next, we present the result for the singlet pair correction. Writing

$$\Delta^{(\mathrm{S})} = \left(\frac{\alpha}{2\pi}\right)^2 \left[\frac{1}{2} f_2^{(0,\mathrm{S})}(x) \ln^2\left(\frac{m_\mu^2}{m_e^2}\right) + f_2^{(1,\mathrm{S})}(x) \ln\left(\frac{m_\mu^2}{m_e^2}\right) + \cdots\right], \tag{31}$$

we obtain

$$f_2^{(0,\mathrm{S})}(x) = \frac{2}{3x} + \frac{17}{9} + 3x - \frac{14}{3}x^2 - \frac{8}{9}x^3 + \left(\frac{5}{3} + 4x + 4x^2\right)\ln x, \tag{32}$$

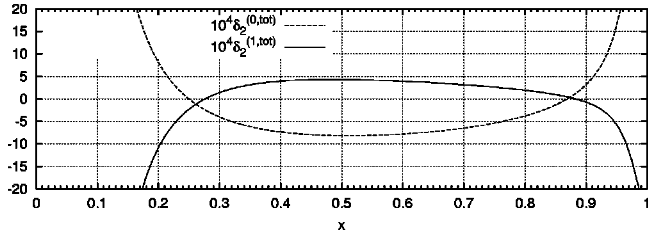

FIG. 1. Double and single logarithmic corrections as a function of x.

$$\begin{aligned} f_2^{(1,\mathrm{S})}(x) = {} & [\mathrm{Li}_2(1-x) + \ln x \ln(1-x)]\left(\frac{5}{3} + 4x + 4x^2\right) \\ & + \ln^2 x\left(\frac{5}{2} + 6x + 4x^2\right) + \ln(1-x)\biggl(\frac{17}{9} + \frac{2}{3x} + 3x \\ & - \frac{14}{3}x^2 - \frac{8}{9}x^3\biggr) + \ln x\left(\frac{8}{9} + \frac{4}{3x} - \frac{5}{6}x - \frac{19}{3}x^2\right) \\ & - \frac{1}{3x} - \frac{67}{9} + \frac{43}{18}x + \frac{77}{18}x^2 + \frac{10}{9}x^3. \end{aligned} \tag{33}$$

Finally, for the interference term we find

$$\Delta^{(\mathrm{int})} = \left(\frac{\alpha}{2\pi}\right)^2 \left[f_2^{(1,\mathrm{int})}(x) \ln\left(\frac{m_\mu^2}{m_e^2}\right) + \cdots\right], \tag{34}$$

where

$$\begin{aligned} f_2^{(1,\mathrm{int})}(x) = {} & 2x^2(3-2x)\bigl(2\mathrm{Li}_3(1-x) - 4\mathrm{S}_{1,2}(1-x) \\ & - 2\mathrm{Li}_2(1-x)\ln x\bigr) + \mathrm{Li}_2(1-x)\biggl(\frac{5}{3} + 4x \\ & - 26x^2 + \frac{52}{3}x^3\biggr) + \ln^2 x\left(-9x^2 + \frac{26}{3}x^3\right) \\ & + \ln x\left(-\frac{5}{3} - \frac{5}{3}x - \frac{28}{3}x^2\right) - \frac{62}{9} + \frac{41}{3}x \\ & - \frac{55}{3}x^2 + \frac{104}{9}x^3. \end{aligned} \tag{35}$$

V. CONCLUSIONS

By applying techniques of perturbative QCD to QED, we have computed the $O(\alpha^2)$ corrections to the electron energy spectrum in unpolarized muon decay, keeping all the terms enhanced by logarithms of the muon to electron mass ratio. The double logarithmic $O(\alpha^2(\ln^2 m_\mu^2/m_e^2))$ corrections are in agreement with the recent results of Ref. [8]. The single logarithmic $O(\alpha^2 \ln(m_\mu^2/m_e^2))$ corrections, the new result presented in this paper, are important to match the precision requirements of the TWIST experiment. To illustrate the significance of the single logarithmic terms, we plot in Fig. 1 both the double and the single $O(\alpha^2)$ logarithmic corrections, defined as

$$\delta_2^{(0,\mathrm{tot})}(x)=\frac{1}{f_0(x)}\left(\frac{\alpha}{2\pi}\right)^2\left[\frac{1}{2}f_2^{(0,\gamma)}(x)+\frac{1}{3}f_2^{(0,\mathrm{NS})}(x)\right.$$
$$\left.+\frac{1}{2}f_2^{(0,\mathrm{S})}(x)\right]\ln^2\left(\frac{m_\mu^2}{m_e^2}\right),$$

$$\delta_2^{(1,\mathrm{tot})}(x)=\frac{1}{f_0(x)}\left(\frac{\alpha}{2\pi}\right)^2[f_2^{(1,\gamma)}(x)+f_2^{(1,\mathrm{NS})}(x)$$
$$+f_2^{(1,\mathrm{S})}(x)+f_2^{(1,\mathrm{int})}(x)]\ln\left(\frac{m_\mu^2}{m_e^2}\right). \quad (36)$$

As follows from Fig. 1, the $O(\alpha^2\ln(m_\mu^2/m_e^2))$ corrections computed in this paper are required for the theoretical prediction at the precision level 10^{-4}. Moreover, within the acceptance region of the TWIST experiment, $0.3 \lesssim x \lesssim 0.98$, the magnitude of the $O(\alpha^2\ln(m_\mu/m_e))$ corrections is comparable to the magnitude of the $O(\alpha^2\ln^2(m_\mu/m_e))$ terms. It is interesting to note that the double-logaritihmic and the single-logarithmic corrections have opposite signs.

Since the leading and the sub-leading logarithmic corrections tend to interfere destructively and since the sub-leading corrections are larger than the precision of the TWIST experiment, the full calculation of the $O(\alpha^2)$ corrections to the electron energy spectrum becomes very desirable. It seems that without such a calculation the intrinsic theory uncertainty in the SM prediction for the electron energy spectrum cannot be pushed below a few$\times 10^{-4}$.

A complete calculation of the $O(\alpha^2)$ corrections to the electron energy spectrum in muon decay is a very difficult task and it is unclear if it is currently possible. However, it is possible to extend our analysis to further improve on the theory prediction. First of all, by using the techniques described in this paper, it is straightforward to compute the electron energy spectrum in polarized muon decay. Furthermore, the $O(\alpha^3\ln^3(m_\mu^2/m_e^2))$ corrections can be obtained from the DGLAP equation. The resummation of corrections that are singular in the limit $x\to 1$ can also be performed and its influence on the theoretical prediction for the spectrum can be studied. These analyses, as well as a detailed discussion of the present theoretical uncertainty in the electron energy spectrum in polarized muon decay, are presented in Ref. [20].

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