Gauged fermionic Q-balls

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We present a new model for a nontopological soliton (NTS) that contains fermions, scalar particles and a gauge field. Using a variational approach, we estimate the energy of the localized configuration, showing that it can be the lowest energy state of the system for a wide range of parameters.

I. INTRODUCTION

The study of solitons and nontopological solitons (NTSs) has a rich history. They have been proposed as building blocks for stars and black holes [1,2], and as dark matter candidates [3–5]. The first models for an NTS in $3+1$ dimensions were found by Lee and Wick, and by Friedberg, Lee and Sirlin [6,7]. The latter NTS contained one real scalar field to set up a false vacuum in which a second, complex scalar field was massless, allowing the NTS to be the lowest energy state for the system under certain conditions.

Coleman and collaborators extended this work to simpler objects dubbed Q-balls, which contained a single complex scalar field that possessed a conserved global symmetry [8,9]. More recent work has extended NTSs to contain fermions [10] and gauge fields [11]. Finally, work has been done by Kusenko extending NTSs to some supersymmetric field theories, where the corresponding false vacuum is set up in the superpotential [4,5].

In this paper we present a new model for an NTS containing mutually interacting fermions, scalar particles, and a $U(1)$ gauge field. The fermions have a Yukawa coupling to the scalars, and both carry a conserved global charge. This model is thus closer to more realistic theories of particle physics that fit in or are inspired by extensions of the standard model. We note that recently a fermionic Q-ball model was proposed by Anagnostopoulos et al. [12]. However, the approach adopted in the present manuscript is quite different, as we explicitly solve the Dirac equation, as opposed to modeling the fermions as a relativistic gas from the outset, and include the Yukawa coupling to the scalar field. Our results, nevertheless, agree with the general conclusions of Ref. [12], as we show that, indeed, it is possible to obtain NTSs which are the lowest energy state in the system for a wide range of parameters. In addition, there is an opportunity to extend our work to bring it in line with the nuclear bag model, as the expression derived below for the energy approximates some models used in nuclear physics. Throughout we work in natural units where $\hbar = c = 1$.

II. THE NTS

Consider the Lagrangian

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - U(|\phi|) + i\bar{\psi}\gamma^\mu D_\mu \psi$$

$$- m\left(1 - \frac{|\phi|}{F_-}\right)\bar{\psi}\psi, \tag{1}$$

where ϕ is the complex scalar field, $D_\mu = \partial_\mu - ieA_\mu$ is the $U(1)$ covariant derivative, $F_{\mu\nu}$ is the field tensor, and $U(|\phi|)$ is the potential for the scalar field. ψ is the 4-component spinor, $\bar{\psi}$ is the Dirac adjoint spinor, γ^μ are the four covariant Dirac matrices, m is a positive constant, and F_- is a constant chosen such that when $|\phi| = F_-$ the fermions are massless within the NTS. We can express the complex scalar field as two real fields using $\phi = (f/\sqrt{2})\exp(i\theta)$, to get

$$\mathcal{L} = \frac{1}{2}\partial_\mu f \partial^\mu f + \frac{1}{2}f^2(\partial_\mu \theta - eA_\mu)^2 - U(f) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+ i\bar{\psi}\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\left(1 - \frac{f}{F_-}\right)\bar{\psi}\psi. \tag{2}$$

We assume the ground state will be spherically symmetric, and will have no magnetic field and hence no electric currents. Therefore, we may choose a gauge where $A_\mu = A_0(r)$. The boundary condition is that $A_0(r) \to 0$ as $r \to \infty$. In addition, we make the assumption that the scalar field oscillates in time with a regular frequency and hence $\theta = \omega t$, where ω is a positive constant [11]. The Lagrangian then becomes

$$L = 4\pi \int r^2 dr\left[-\frac{1}{2}f'^2 + \frac{1}{2e^2}g'^2 + \frac{1}{2}f^2 g^2 - U(f)\right.$$

$$\left. + \bar{\psi}\gamma^0(\omega - g)\psi - m\left(1 - \frac{f}{F_-}\right)\bar{\psi}\psi + i\bar{\psi}\gamma^\mu\partial_\mu\psi\right], \tag{3}$$

where $g \equiv \omega - eA_0(r)$. The Euler-Lagrange equations for g, f and ψ are

$$g'' + \frac{2}{r}g' + [e^2 \bar{\psi}\gamma^0\psi - e^2 f^2 g] = 0, \tag{4}$$

$$f'' + \frac{2}{r}f' + fg^2 - \frac{dU(f)}{df} + \frac{1}{F_-}m\bar{\psi}\psi = 0, \tag{5}$$

$$i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\left(1 - \frac{f}{F_-}\right)\psi = 0, \tag{6}$$

where it is understood that the only nonvanishing component of A_μ is A_0. The conserved currents and charges are given by

$$\mathcal{J}_\mu^{scalar} = -i(\phi^* D_\mu \phi - \phi D_\mu \phi^*), \tag{7}$$

$$Q = \int \mathcal{J}_0^{scalar} d^3x = 4\pi \int r^2 dr f^2 g, \tag{8}$$

$$\mathcal{J}_\mu^{fermion} = \bar{\psi}\gamma_\mu\psi, \tag{9}$$

$$N = \int \mathcal{J}_0^{fermion} d^3x = 4\pi \int r^2 dr \psi^\dagger \psi. \tag{10}$$

III. TAKING CARE OF THE FERMIONS

To proceed, we could take one of two approaches. We could use the Fermi gas approach as in [10,12]. Or we could attempt to solve the equations of motion directly. We attempt the second method here. We begin with the Dirac equation for the fermion field. We can write ψ in terms of two, 2-component, spin-1/2 spinors in the chiral representation as

$$\psi = \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}, \tag{11}$$

where $\phi_{R(L)}$ is a right (left)-handed spinor. It is easier to proceed if we switch to the non-covariant representation of the Dirac equation. Multiplying by $\beta^{-1} \equiv (\gamma^0)^{-1}$ on the left and using that $\gamma^i \equiv \beta\alpha^i$, we get the equation

$$i\frac{\partial\psi}{\partial t} = (-i\boldsymbol{\alpha}\cdot\nabla + \beta M + V_f(r))\psi, \tag{12}$$

where $V_f(r) = eA_0(r)$ is the potential for the fermions, and $M = (1 - f/F_-)m$ is the mass of the fermion. Following Lee et al. we will assume that $f(r) = const = F_- \equiv F$ inside the NTS [11]. We will also assume that ψ is of the simple form $\psi(r,t) = \psi(r)\exp(-iEt)$ where E is the energy of a single fermion. We therefore get the new equation

$$E\psi = (-i\boldsymbol{\alpha}\cdot\nabla + V_f(r))\psi. \tag{13}$$

We desire ψ to be spherically symmetric in the ground state. Since ϕ_R and ϕ_L must have opposite parity under spatial reflection, they cannot both be symmetric. Therefore, we must choose one of them to equal zero, and conventions of a right-handed coordinate system dictate that it is the left-handed component that must be zero. We can now expand in terms of an angular part and a radial part by writing [13]

$$\phi_R = ih(r)\Omega_{jlm}, \tag{14}$$

where Ω_{jlm} is a spherical spinor and the indices j,l,m are the quantum numbers of total angular momentum, orbital angular momentum, and the z component of angular momentum, respectively. In the spherically-symmetric ground state we have $l=0$ and so $j = l + 1/2 = 1/2$, and the spherical spinor for this case simplifies to [13]

$$\Omega_{(1/2)0(1/2)} = \begin{pmatrix} Y_{00} \\ 0 \end{pmatrix} = \sqrt{\frac{1}{4\pi}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{15}$$

and we see that indeed our wave function is spherically symmetric. The equations for $h(r)$ become [13]

$$\frac{dh}{dr} + (1+\kappa)\frac{h}{r} = 0, \tag{16}$$

$$[E - V_f(r)]h = 0, \tag{17}$$

where $\kappa \equiv -(l+1)$ so here $\kappa = -1$ [13]. The solution to Eq. (16) is then simply $h = const \equiv B$. However, we notice a problem with Eq. (17); namely, if $V_f(r)$ is nonconstant then it demands that $h=0$ since E must be a constant, and the equation must be satisfied for all r. How are we to get around this problem? We can approximate the fermionic energy inside the NTS as constant by taking the expectation value of the fermionic potential inside it. Then we can write $E \approx \langle E \rangle = \langle V_f(r) \rangle$ and Eq. (17) may be approximately satisfied inside the NTS. Before doing this we note that if h is constant, the expression for the conserved charge Eq. (10) becomes simply

$$N = \frac{B^2 R^3}{3}, \tag{18}$$

where R is the radius of the ball. We then get that

$$\langle V_f \rangle = 4\pi \int_0^R \psi^\dagger \sigma_3 V(r)\psi r^2 dr = \frac{3N}{R^3}\int_0^R r^2[\omega - g(r)]dr, \tag{19}$$

where σ_3 is the third Pauli spin matrix, and our approximate solution is

$$\psi(r,t) = \frac{1}{\sqrt{8\pi}}\left(\frac{3N}{R^3}\right)^{1/2} \exp(-it\langle V_f \rangle)\theta(R-r)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{20}$$

where $\theta(R-r)$ is the step function.

We can now use Eq. (20) in Eq. (4). The solution for $g(r)$ is

$$g(r) = \begin{cases} [\omega - e^2 Q/4\pi R]R \sinh(eFr)/r \sinh(eFR) \\ + \psi^\dagger \psi/F^2, \quad r \le R, \\ \omega - e^2 Q/4\pi r, \quad r > R, \end{cases}$$

(21)

where R is the radius of the soliton. We can now substitute for $f^2 g$ in Eq. (8) to get

$$e^2 Q = \int_0^R 4\pi dr [(g' r^2)' + e^2 r^2 \psi^\dagger \psi].$$

(22)

Using our solution for $g(r)$, Eq. (21), we get

$$\omega = \frac{e^2 Q}{4\pi R}\left[\frac{x}{x - \tanh(x)}\right] - \frac{e^2 N \tanh(x)}{4\pi R(x - \tanh(x))},$$

(23)

where $x \equiv eFR$.

IV. ESTIMATING THE GAUGED FERMIONIC Q-BALL ENERGY

The gauge invariant energy can be written as

$$E = 4\pi \int r^2 dr \left[\frac{1}{2}f'^2 + \frac{1}{2e^2}g'^2 + \frac{1}{2}f^2 g^2 + U(f) + i\bar{\psi}\gamma^i \partial_i \psi\right] + E_F,$$

(24)

where E_F is the relativistic Fermi energy given by $E_F = (3\pi/4)(3/2\pi)^{2/3}(N^{4/3}/R)$. Ignoring surface terms $\mathcal{O}(R^2)$, performing some partial integrations, and using our solution for ψ we get, for $r \le R$,

$$E \le \frac{1}{2}\omega Q + \frac{4}{3}\pi R^3 U(F) + E_F.$$

(25)

We can use Eq. (23) to get

$$E \le \frac{e^2 Q^2}{8\pi R}\left[\frac{x}{x - \tanh x}\right] + \frac{4}{3}\pi R^3 U(F) + \frac{C_1 N^{4/3}}{R} - \frac{e^2 QN}{8\pi R}\left[\frac{\tanh x}{x - \tanh x}\right],$$

(26)

where $C_1 \equiv 3\pi/4(3/2\pi)^{2/3}$. The next step is to minimize this expression with respect to the various parameters at fixed N and Q. We examine the case of small e and expand to order e^3, where the Laurent series are given by $x/(x - \tanh x) \approx 3/x^2 + 6/5$ and $\tanh x/(x - \tanh x) \approx 3/x^2 + 1/5$. We then get an approximate form for E to $\mathcal{O}(e^3)$ of

$$E \le \frac{3Q^2}{8\pi F^2 R^3} + \frac{4}{3}\pi R^3 U(F) + \frac{C_1 N^{4/3}}{R} + \frac{3e^2 Q^2}{20\pi R} - \frac{3QN}{8\pi F^2 R^3} - \frac{e^2 QN}{40\pi R}.$$

(27)

These terms now have an easy physical interpretation. The first term is the zero-point energy of the scalar particles. The second term is the vacuum volume energy of the bag. The third term is the Fermi energy. The fourth term is the Coulomb repulsion of the scalar particles. The fifth and six terms represent the interactions of the fermions with the scalars and gauge fields, which may significantly alter the NTS's energy.

If we minimize this expression with respect to R we obtain, writing $R^2 \equiv y$,

$$4\pi U(F)y^3 - y\left(\frac{3e^2 Q^2}{20\pi} + C_1 N^{4/3} - \frac{e^2 QN}{40\pi}\right)$$
$$- \frac{9Q}{8\pi F^2}(Q - N) = 0.$$

(28)

The formal solution to this equation is given by

$$z = \cos\left[\frac{1}{3}\arccos\left(\frac{3B}{2A}\sqrt{\frac{3}{A}}\right) + \frac{2\pi n}{3}\right],$$

(29)

$$n = 0, \pm 1, \pm 2 \ldots$$

$$z \equiv y/\alpha, \quad \alpha \equiv \sqrt{(4A/3)},$$

$$A = \frac{C_1 N^{4/3}}{4\pi U(F)} + \frac{e^2 Q}{160\pi^2 U(F)}(6Q - N),$$

(30)

$$B = \frac{9Q}{32\pi^2 F^2 U(F)}(Q - N).$$

(31)

Unfortunately, since we must minimize the energy with respect to both R and F the details of selecting the correct root (or even if a positive real root exists) depend on the nature of $U(F)$. Hence, it is not possible to write down a general procedure for selecting a root. Once a suitable potential has been chosen however, the solution follows in a straightforward manner. To illustrate this, we solve for the potential

$$U(f) = \frac{\lambda^2 f^6}{6\mu^2} - \frac{f^4}{4} + \frac{\mu^2 f^2}{2}.$$

(32)

We choose $\lambda = 0.444$, $\mu = .25$, $e = .1$, $Q = 10000$, and $M_\psi = 2$ [we must choose $\lambda^2 > 3/16$ to insure that $U(f) > 0$ for all $f \ne 0$], where μ and M_ψ have dimensions of mass (M_ψ is the mass of the fermions in the true vacuum). The energy scale is set by F, which has dimensions of mass. We first solve the problem with no fermions present, which is identical to the scenario considered in [11]. This gives us an idea of a possible range of values for R and F (based on this analysis, we set $F = 0.48$ throughout) to search for fermionic NTSs. Writing the energy of the free scalars and fermions as $E_{\text{free}} = \mu Q + M_\psi N$, the NTS is stable whenever $E_{NTS}/E_{\text{free}} < 1$. In Fig. 1, we show the ratio E_{NTS}/E_{free} as a function of increasing fermion number, for several values of the radius. From a more detailed analysis we can show that, for the parameters used in Fig. 1, the condition for the existence of NTSs is satisfied for $35 \lesssim R \lesssim 111$. Clearly, the same sort of range search can be performed for any set of parameters.
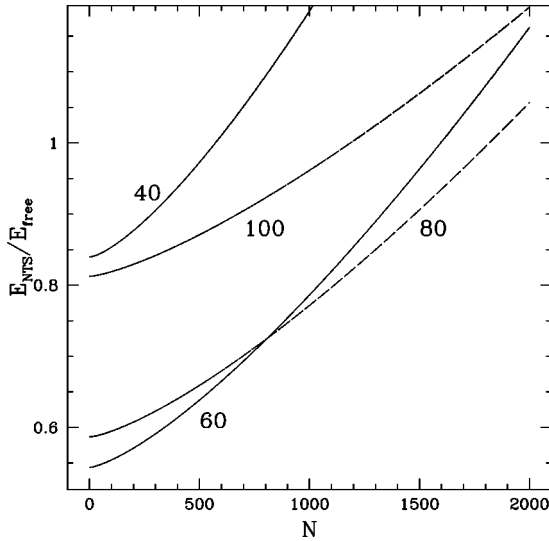
FIG. 1. Ratio of NTS energy to the energy of free fermions and scalars as a function of fermionic charge N. The labels indicate the different values of the NTS radius.
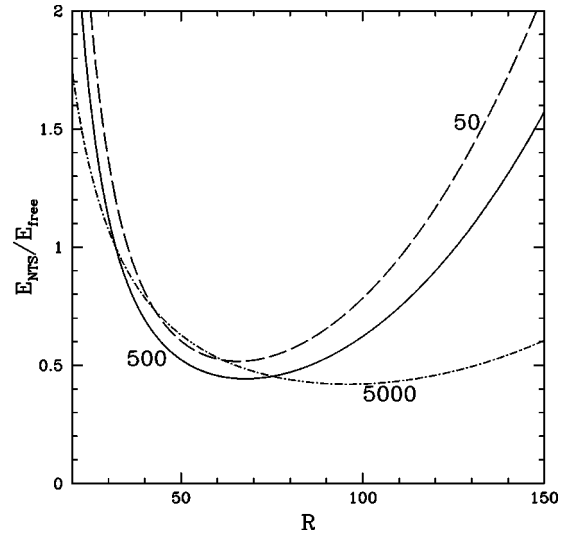

FIG. 2. Ratio of NTS energy to the energy of free fermions and scalars as a function of radius R. The labels indicate the different values of the fermionic charge N.

In Fig. 2 we show the ratio E_{NTS}/E_{free} as a function of the NTS radius for different values of the fermionic charge N. Clearly, there is a wide range of values for N wherein the NTS is the preferred energy configuration. For large radii (larger than the range of values displayed in the figure), the NTS energy is independent of N, as can be easily seen from Eq. (27).

We conclude with a few remarks about our NTS. First, we see that if we use a potential that satisfies Coleman's condition [8] $\min[2U/|\phi^2|] < \mu^2$ (μ is the mass of the free scalar particles), in order to set up a false vacuum where the fermions can be massless, then we can always find parameters where $E_{NTS} < E_{\text{free}}$. Second, we see that the presence of both fermions and the gauge field increases the energy and the radius of the NTS, while the attractive Yukawa coupling between the fermions and scalars decreases both in relation to the ungauged scalar Q-balls studied by Coleman and collaborators [8,9]. Third, we note that the asymptotic form of the energy is

$$\lim_{R \to \infty} E \to \frac{4}{3} \pi R^3 U(F). \tag{33}$$

Hence, if we scale $U(F)$ in such a way that $R^3 U(F) = \text{const} < \frac{3}{4} E_{\text{free}}$, our NTS can be stable at arbitrarily large radii. Physically, we have obtained a state of matter with scalar particles and fermions uniform throughout. We thus see that it is possible to form a NTS out of gauged fermions and scalar particles, which can be the preferred energy state of the system for a wide range of parameters. It would be interesting to investigate the solutions to the set of coupled equations numerically, obtaining a more detailed analysis of the allowed parameter space.

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