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Duality for symmetric second rank tensors: The massive case

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A family of theories which are dual to the massive spin two Fierz-Pauli field $h_{\mu\nu}$, both free and coupled to external sources, is constructed in terms of a $T_{(\mu\nu)\sigma}$ tensor. The dualization method, a purely Lagrangian approach, is based on a first order parent Lagrangian, from which the dual partners are generated.

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I. INTRODUCTION

There usually is a great deal of freedom in the choice of variables for the description of a physical system. Different choices are considered equivalent when they are able to describe the same system. However, there might be practical reasons to prefer a given description to others. For example, in some cases it might be desirable to have a formulation where some symmetries are made explicit in the Lagrangian. This usually requires the use of a redundant set of variables to describe the system configurations, as in the case of gauge theories. Conversely, in other situations it is more convenient to choose a minimal, non-redundant, set of variables.

Duality, in its wider meaning, refers to two equivalent descriptions for a physical system using different fields [1]. One of the simplest cases is the scalar-tensor duality. It corresponds to the equivalence between a free massless scalar field φ, with field strength $f_\mu = \partial_\mu \varphi$, and a massless antisymmetric field $B_{\mu\nu}$, the Kalb-Ramond field, with field strength $H_{\mu\nu\sigma} = \partial_\mu B_{\nu\sigma} + \partial_\nu B_{\sigma\mu} + \partial_\sigma B_{\mu\nu}$ [2–4]. Another example is in fact a predecessor of the modern approach to duality, the electric-magnetic symmetry $(\vec{E}+i\vec{B}) \rightarrow e^{i\phi}(\vec{E}+i\vec{B})$ of the free Maxwell equations. When there are charged sources this symmetry can be maintained by introducing magnetic monopoles [5]. This transformation provides a connection between weak and strong couplings via the Dirac quantization condition. At the level of Yang-Mills theories with spontaneous symmetry breaking this kind of duality is expected, due to the existence of topological dyon-type solitons [6]. The extension of electromagnetic duality to $SL(2,R)$ and $SL(2,Z)$ plays an important role in the non-perturbative study of field and string theories [7] and has been extended to Born-Infeld theory [8].

These basic ideas have been subsequently generalized to arbitrary forms in arbitrary dimensions. Well known dualities are the ones between massless p-form and $(d-p-2)$-form fields and between massive p and $(d-p-1)$ forms in d-dimensional space-time [9]. These dualities among free fields have been proved by using the method of parent Lagrangians [10] as well as the canonical formalism [11]. They can be extended to include source interactions [12].

The above duality among forms can be understood as a relation between fields in different representations of the Lorentz group. The origin of this equivalence can be traced using the little group technique for constructing the representations of the Poincaré group in d dimensions. A detailed discussion of this observation suggests the possibility of generalizing the duality transformations among p forms to tensorial fields with arbitrary Young symmetry types. Consistent massless free [13], interacting [14], and massive [15] theories of mixed Young symmetry tensors were constructed in the past, but the attempts to prove a dual relation between these descriptions did not lead to a positive answer [15]. Additional interest in this type of theories arises from the recent formulation of $d=11$ dimensional supergravity as a gauge theory for the osp(32|1) superalgebra [16].

In an earlier paper [17] we have sketched a scheme to construct dual theories originally motivated by the relationship between field representations corresponding to associated Young diagrams. Here we fully develop this approach on a purely Lagrangian basis for the case of a massive spin-2 theory.

Let us consider the scalar field φ in order to illustrate our procedure for constructing dual theories. The starting point is the second order Lagrangian

$$L(\varphi) = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2}m^2\varphi^2 + J\varphi. \tag{1}$$

As the first step, we construct a first order Lagrangian, using a generalization of a procedure used in Ref. [18]. We are interested in a particular Lagrangian structure, which we will call the standard form

$$L(\varphi, L^\mu) = L^\mu \partial_\mu \varphi - \frac{1}{2}L^\mu L_\mu - \frac{1}{2}m^2\varphi^2 + J\varphi. \tag{2}$$

This standard form is defined by the kinetic term. It contains the derivative of the original field times a new auxiliary variable, which we call, in a rather loose way, the field strength of the original theory.

The key recipe to construct the dual theory is to introduce a point transformation in the configuration space for the auxiliary variable $L^{\mu}=\epsilon^{\mu\nu\sigma\tau}H_{\nu\sigma\tau}$, which leads to a new first order Lagrangian

$$L(\varphi,H_{\nu\sigma\tau})=H_{\nu\sigma\tau}\epsilon^{\mu\nu\sigma\tau}\partial_{\mu}\varphi+3H_{\nu\sigma\tau}H^{\nu\sigma\tau}-\frac{1}{2}m^{2}\varphi^{2}+J\varphi. \tag{3}$$

This turns out to be the parent Lagrangian from which both dual theories can be obtained. In fact, using the equation of motion for $H_{\nu\sigma\tau}$ we obtain $H_{\nu\sigma\tau}(\varphi)$ which takes us back to our starting action (1) after it is substituted in Eq. (3). On the other hand, we can also eliminate the field φ from the Lagrangian using its equation of motion

$$m^{2}\varphi=-\partial_{\mu}\epsilon^{\mu\nu\sigma\tau}H_{\nu\sigma\tau}+J. \tag{4}$$

In such a way we obtain the new theory

$$L(H_{\nu\sigma\tau})=\frac{1}{2}(\epsilon^{\mu\nu\sigma\tau}\partial_{\mu}H_{\nu\sigma\tau})^{2}+3m^{2}H_{\nu\sigma\tau}H^{\nu\sigma\tau}-J\epsilon^{\mu\nu\sigma\tau}\partial_{\mu}H_{\nu\sigma\tau}+\frac{1}{2}J^{2}, \tag{5}$$

which is equivalent to the original one through the transformation (4). In this form we have obtained a Lagrangian dual to Eq. (1).

For a massless theory, $m=0$, we lose the connection between the original field φ and the new one $H_{\nu\sigma\tau}$. In this case Eq. (4) becomes the constraint $\partial_{\mu}\epsilon^{\mu\nu\sigma\tau}H_{\nu\sigma\tau}=J$, which tells us that the field $H_{\nu\sigma\tau}$ can be considered as a field strength with an associated potential out of the sources.

Another paradigmatic example of dualization is the standard S duality for electrodynamics with a θ term. Let us consider the Euclidean Lagrangian

$$L=\frac{1}{8\pi}\left(aF_{\mu\nu}F^{\mu\nu}+ib\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right). \tag{6}$$

The standard Euclidean S-dualization recipe $F\to\tilde{F}$, $\tilde{\tilde{F}}\to+F$, $(a+ib)\to(a+ib)^{-1}$ leads to the new Lagrangian

$$\tilde{L}=-\frac{1}{8\pi}\left(\frac{a}{a^{2}+b^{2}}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}-\frac{ib}{a^{2}+b^{2}}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\tilde{F}^{\mu\nu}\tilde{F}^{\rho\sigma}\right). \tag{7}$$

Now we will show how to go from the original Lagrangian to the dual one, using the basic ideas of our approach. To begin with, we construct a first order Lagrangian for Eq. (6), introducing the Lagrange multiplier $G^{\mu\nu}$:

$$L(F,A,G)=\frac{1}{8\pi}\left(aF_{\mu\nu}F^{\mu\nu}+ib\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right)-\frac{1}{4\pi}[G^{\mu\nu}F_{\mu\nu}-G^{\mu\nu}(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu})]. \tag{8}$$

The Euler-Lagrange equation for $F_{\mu\nu}$ leads to

$$F^{\alpha\beta}=\frac{1}{(a^{2}+b^{2})}\left(aG^{\alpha\beta}-ib\frac{1}{2}G_{\mu\nu}\epsilon^{\mu\nu\alpha\beta}\right) \tag{9}$$

by a purely algebraic manipulation. This allows us to eliminate this field from Lagrangian (8), obtaining

$$L(A,G)=-\frac{1}{8\pi}\frac{1}{(a^{2}+b^{2})}\left(aG^{\alpha\beta}-ib\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}G_{\mu\nu}\right)G_{\alpha\beta}+\frac{1}{4\pi}G^{\mu\nu}(\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}) \tag{10}$$

which identifies $G^{\mu\nu}$ as the field strength of A^{μ}. The above first order Lagrangian is equivalent to the second order Lagrangian (6). This can be verified via the solution

$$G^{\alpha\beta}=a(\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha})+ib\frac{1}{2}(\partial_{\rho}A_{\sigma}-\partial_{\sigma}A_{\rho})\epsilon^{\rho\sigma\alpha\beta} \tag{11}$$

of the equation of motion for $G_{\mu\nu}$, together with the definition (9). The variation of A_{μ} in Lagrangian (10) produces the remaining equation $\partial_{\mu}G^{\mu\nu}=0$. Let us define the dual field $H_{\alpha\beta}$:

$$G^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}H_{\rho\sigma}. \tag{12}$$

By substitution in Eq. (10) we obtain

$$L=-\frac{1}{8\pi}\frac{1}{(a^{2}+b^{2})}\left(aH_{\kappa\lambda}-ib\frac{1}{2}\epsilon_{\kappa\lambda\rho\sigma}H^{\rho\sigma}\right)H^{\kappa\lambda}+\frac{1}{4\pi}\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}H^{\mu\nu}(\partial^{\rho}A^{\sigma}-\partial^{\sigma}A^{\rho}), \tag{13}$$

which is the correspondent parent Lagrangian. The variation of this last Lagrangian with respect to A^{μ} yields a Bianchi identity for $H_{\rho\sigma}$, $\epsilon^{\nu\mu\rho\sigma}\partial_{\mu}H_{\rho\sigma}=0$, which implies

$$H_{\rho\sigma}(B)=\partial_{\rho}B_{\sigma}-\partial_{\sigma}B_{\rho}, \tag{14}$$

where $H_{\rho\sigma}$ is identified as the dual field strength. Using this property in Eq. (13) leads to the second order Lagrangian

$$L(B)=-\frac{1}{8\pi}\frac{1}{(a^{2}+b^{2})}\left(aH_{\kappa\lambda}-ib\frac{1}{2}\epsilon_{\kappa\lambda\rho\sigma}H^{\rho\sigma}\right)H^{\kappa\lambda}, \tag{15}$$

the dual version of the original one. The above Lagrangian is precisely (7) with the notation $H^{\mu\nu}=\tilde{F}^{\mu\nu}$. The relation with the original theory appears at the level of the potentials and is given by Eqs. (11), (12) and (14). A similar method has been implemented for non-Abelian gauge theories in the context of the path integral formulation [19].

This paper focuses on the construction of a dual theory for a massive spin-2 field in four dimensions. It is organized as follows. In Sec. II we formulate the scheme for dualization pursued here. Section III contains the construction of an auxiliary first order Lagrangian which is equivalent to the usual one in terms of the standard Fierz-Pauli field $h_{\mu\nu}$ for a massive spin-2 particle. The general method for constructing such an auxiliary Lagrangian is briefly reviewed in Appendix A. An explicit proof of the equivalence between this auxiliary Lagrangian and the massive Fierz-Pauli Lagrangian is given in Appendix B. Section IV contains the definition of the dual field $T_{(\mu\nu)\sigma}$ together with the construction of the parent Lagrangian. In Sec. V the duality transformations arising from the parent Lagrangian are derived. The dual Lagrangian, in terms of $T_{(\mu\nu)\sigma}$, is obtained in Sec. VI together with the corresponding equations of motion and the set of Lagrangian contraints. These contraints allow us to make sure that we have obtained the correct number of degrees of freedom. Most of the calculations in this section are relegated to Appendix C. In Sec. VII we discuss the example of an external point mass source *m*. The massive spin two fields generated by this source are calculated in each of the dual theories, thus allowing the explicit verification of the duality trasformations. Finally we close with Sec. VIII which contains a summary of the work together with some comments regarding preliminary work in the zero mass limit of the present approach. A complete discussion of the massless case is deferred to a forthcoming publication.

II. THE DUALIZATION PROCEDURE

In general terms, the method applied to the previous examples can be summarized as follows, assuming that there is no external source for simplicity. We start from a second order theory for the free field Φ of a given tensorial character, which can be schematically presented as

$$L(\Phi)=\frac{1}{2}\partial\Phi\partial\Phi-\frac{M^2}{2}\Phi\Phi. \tag{16}$$

Next, we introduce an auxiliary field W to construct a first order formulation in the standard form

$$L(\Phi,W)=(\partial\Phi)W-\frac{1}{2}WW-\frac{M^2}{2}\Phi\Phi, \tag{17}$$

as explained in Appendix A. This identifies W as the field strength of Φ, with the equation of motion $\partial W+M^2\Phi=0$. Now, we introduce $\tilde{W}$ as the field strength dual to W via the change of variables $W=\epsilon\tilde{W}$ and substitute in the first order action (17) to obtain the Lagrangian

$$\tilde{L}(\Phi,\tilde{W})=(\partial\Phi)\epsilon\tilde{W}-\frac{1}{2}\epsilon\tilde{W}\epsilon\tilde{W}-\frac{M^2}{2}\Phi\Phi. \tag{18}$$

In this way we obtain the parent Lagrangian (18) which generates the pair of dual theories. The field $\tilde{W}$ can always be eliminated from Lagrangian (18) to recover the initial Lagrangian (16).

The Euler-Lagrange equation for Φ is $\epsilon\partial\tilde{W}+M^2\Phi=0$. If $M\neq 0$, or more generally if it is a regular matrix, this equation allows the algebraic elimination of the field Φ in Lagrangian (18), yielding a second order Lagrangian for $\tilde{W}$

$$\tilde{L}\left(\Phi\equiv-\frac{1}{M^2}\epsilon\partial\tilde{W},\ \ \tilde{W}\right)\propto\frac{1}{2}\epsilon\partial\tilde{W}\epsilon\partial\tilde{W}-\frac{M^2}{2}\epsilon\tilde{W}\epsilon\tilde{W}, \tag{19}$$

which is the dual to the original $L(\Phi)$.

If $M=0$, the parent Lagrangian reduces to

$$\tilde{L}(\Phi,\tilde{W})=(\partial\Phi)\epsilon\tilde{W}-\frac{1}{2}\epsilon\tilde{W}\epsilon\tilde{W}, \tag{20}$$

with the equations of motion

$$\epsilon\partial\tilde{W}=0, \tag{21}$$

$$\epsilon\tilde{W}\epsilon-\epsilon(\partial\Phi)=0, \tag{22}$$

preventing the algebraic solution for Φ. Nevertheless, Eq. (21) is a Bianchi identity for $\tilde{W}$ whose solution can be written symbolically as $\tilde{W}=\partial B$. That is to say, the dual field $\tilde{W}$ is a field strength and can be written in terms of a new potential B. Substituting $\tilde{W}$ in terms of B in Eq. (20) we arrive at the dual Lagrangian

$$L(B)=\frac{1}{2}\epsilon\partial B\epsilon\partial B. \tag{23}$$

Finally, the relation $\epsilon\partial B\epsilon-\epsilon\partial\Phi=0$, obtained from Eq. (22), provides the connection between the dual theories.

III. MASSIVE FIERZ-PAULI LAGRANGIAN

The Lagrangian for the massive Fierz-Pauli field is

$$\mathcal{L}=-\partial_\mu h^{\mu\nu}\partial_\alpha h^\alpha_\nu+\frac{1}{2}\partial_\alpha h^{\mu\nu}\partial^\alpha h_{\mu\nu}+\partial_\mu h^{\mu\nu}\partial_\nu h^\alpha_\alpha -\frac{1}{2}\partial_\alpha h^\mu_\mu\partial^\alpha h^\nu_\nu-\frac{M^2}{2}(h_{\mu\nu}h^{\mu\nu}-h^\mu_\mu h^\nu_\nu)+h_{\mu\nu}\Theta^{\mu\nu}, \tag{24}$$

where we are considering the source $\Theta^{\mu\nu}$, described by a symmetric tensor not necessarily conserved in contrast to the massless case. The kinetic part of the Lagrangian (24) is just the linearized Einstein Lagrangian. The equations of motion for $h_{\mu\nu}$ provide the following Lagragian constraints:

$$h^\alpha_\alpha=-\frac{1}{3M^2}\left(\Theta^\alpha_\alpha-\frac{2}{M^2}\partial_\alpha\partial_\beta\Theta^{\alpha\beta}\right), \tag{25}$$

$$\partial^\mu h_{\mu\nu} = \frac{1}{M^2}\left(\partial_\mu \Theta^\mu_\nu - \frac{1}{3}\partial_\nu \Theta^\alpha_\alpha + \frac{2}{3M^2}\partial_\nu\partial_\alpha\partial_\beta \Theta^{\alpha\beta}\right), \tag{26}$$

which show that the trace and the divergence of $h_{\mu\nu}$ do not propagate, vanishing outside the sources, as expected for a pure spin-2 theory. The resulting equation for $h_{\mu\nu}$ is

$$(\partial^\alpha\partial_\alpha + M^2)h_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3}\left(g_{\mu\nu} + \frac{1}{M^2}\partial_\mu\partial_\nu\right)\Theta^\alpha_\alpha. \tag{27}$$

Following the procedure sketched in Appendix A, we can construct an equivalent first order Lagrangian in the standard form. This Lagrangian is not unique because of the freedom in the choice of the auxiliary fields. Alternatively, we can construct a Lagrangian having the standard form with arbitrary coefficients, which are subsequently adjusted to obtain the original Lagrangian when the auxiliary fields are eliminated. In the present case the last approach is simpler, and we will follow it. Therefore, we start by proposing a field strength $K^{\alpha\{\beta\sigma\}}$ satisfying the following symmetry properties:

$$K^{\alpha\{\beta\sigma\}} = K^{\alpha\{\sigma\beta\}}, \tag{28}$$

$$K^{\alpha\{\beta\sigma\}} + K^{\beta\{\sigma\alpha\}} + K^{\sigma\{\alpha\beta\}} = 0. \tag{29}$$

These symmetry properties greatly simplify the manipulations and, as it will become evident in the following, they are consistent with the degrees of freedom of the spin-2 massive field. With this auxiliary field we construct the first order Lagrangian

$$\begin{aligned} L = &-\frac{1}{6}aK^{\alpha\{\beta\sigma\}}K_{\alpha\{\beta\sigma\}} + \frac{1}{8}qK^\beta K_\beta - \frac{2}{9}r\epsilon^{\gamma\delta\kappa\lambda}K_{\kappa\{\lambda}{}^{\sigma\}}K_{\gamma\{\delta\sigma\}} \\ &- \frac{e}{\sqrt{2}}K^{\alpha\{\beta\sigma\}}\partial_\alpha h_{\sigma\beta} - \frac{M^2}{2}(h_{\mu\nu}h^{\mu\nu} - h_\mu{}^\mu h_\nu{}^\nu) + \Theta_{\mu\nu}h^{\mu\nu} \\ &+ \Lambda_{\alpha\{\beta\sigma}(K^{\alpha\{\beta\sigma\}} + K^{\beta\{\sigma\alpha\}} + K^{\sigma\{\alpha\beta\}}), \end{aligned} \tag{30}$$

where $K_\alpha = K_{\alpha\{\lambda}{}^{\lambda\}}$. This Lagrangian has the most general mass term for the field $K^{\alpha\{\beta\sigma\}}$ with the symmetry properties (28), (29). Here $K^{\alpha\{\beta\sigma\}}$ is identified as the field strength of $h_{\sigma\beta}$. The constraint (29) is enforced by the Lagrange multiplier $\Lambda_{\alpha\{\beta\sigma\}} = \Lambda_{\alpha\{\sigma\beta\}}$. In Appendix B we show that the elimination of $K^{\alpha\{\beta\sigma\}}$ and $\Lambda_{\alpha\{\sigma\beta\}}$ in (30) leads effectively to the Fierz-Pauli Lagrangian when the coefficients satisfy

$$4r^2 = a(e^2 - a), \qquad 3q = 2a + e^2. \tag{31}$$

In such a case both theories are equivalent and Lagrangian (30) is the first order standard Lagrangian for the Fierz-Pauli massive field. From conditions (31) only two independent coefficients in Lagrangian (30) remain, one of them being the normalization of the auxiliary field.

IV. DUAL FIELD AND PARENT LAGRANGIAN

Now that we have identified the field strength $K^{\alpha\{\beta\sigma\}}$ for $h_{\mu\nu}$ and the corresponding first order theory, we can implement the transformation

$$K^{\alpha\{\beta\sigma\}} \to \Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)}, \qquad K^{\alpha\{\beta\sigma\}} = \epsilon^{\alpha\mu\nu\xi}\Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)} \tag{32}$$

that leads to the dual theory. Substituting this transformation in Eq. (30), we obtain the parent Lagrangian

$$\begin{aligned} L = &\, a\Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)}\Omega^{(\mu\nu\xi)}_{\{\beta\sigma\}} - \frac{1}{4}(2a + e^2)\Omega_{(\mu\nu\xi)}\Omega^{(\mu\nu\xi)} \\ &+ \frac{2}{3}\sqrt{a(e^2 - a)}\,\epsilon^\mu{}_{\rho\tau\chi}\Omega^{\{\lambda\sigma\}}_{(\mu\nu\lambda)}\Omega^{(\rho\tau\chi)\{\nu}{}_{\sigma\}} \\ &- \frac{e}{\sqrt{2}}\epsilon^{\alpha\mu\nu\xi}\Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)}\partial_\alpha h_{\sigma\beta} - \frac{M^2}{2}(h_{\mu\nu}h^{\mu\nu} - h_\mu{}^\mu h_\nu{}^\nu) \\ &+ \Theta_{\mu\nu}h^{\mu\nu} + \Lambda_{\alpha\beta\sigma}(\epsilon^{\alpha\mu\nu\xi}\Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)} + \epsilon^{\beta\mu\nu\xi}\Omega^{\{\sigma\alpha\}}_{(\mu\nu\xi)} \\ &+ \epsilon^{\sigma\mu\nu\xi}\Omega^{\{\alpha\beta\}}_{(\mu\nu\xi)}), \end{aligned} \tag{33}$$

where $\Omega_{(\mu\nu\xi)} = g_{\alpha\beta}\Omega^{\{\alpha\beta\}}_{(\mu\nu\xi)}$.

The dual theory is derived by eliminating $h_{\sigma\beta}$. Alternatively, by eliminating $\Omega^{(\mu\nu\xi)}_{\{\beta\sigma\}}$ from Eq. (33) we recover the Fierz-Pauli theory. The field $\Omega^{(\mu\nu\xi)}_{\{\beta\sigma\}}$ satisfies the constraint

$$\epsilon^{\alpha\mu\nu\xi}\Omega^{\{\beta\sigma\}}_{(\mu\nu\xi)} + \epsilon^{\beta\mu\nu\xi}\Omega^{\{\sigma\alpha\}}_{(\mu\nu\xi)} + \epsilon^{\sigma\mu\nu\xi}\Omega^{\{\alpha\beta\}}_{(\mu\nu\xi)} = 0, \tag{34}$$

as a consequence of Eq. (29). A simple way to warrant this constraint is to express $\Omega^{\{\beta\gamma\}}_{(\rho\sigma\tau)}$ in terms of a tensor $T^\gamma_{(\rho\sigma)} = -T^\gamma_{(\sigma\rho)}$, as follows:

$$\begin{aligned} \Omega^{\{\beta\gamma\}}_{(\rho\sigma\tau)} = &-\frac{1}{3\sqrt{2}}(g^\beta_\tau T_{(\rho\sigma)}{}^\gamma + g^\beta_\rho T_{(\sigma\tau)}{}^\gamma + g^\beta_\sigma T_{(\tau\rho)}{}^\gamma + g^\gamma_\tau T_{(\rho\sigma)}{}^\beta \\ &+ g^\gamma_\rho T_{(\sigma\tau)}{}^\beta + g^\gamma_\sigma T_{(\tau\rho)}{}^\beta). \end{aligned} \tag{35}$$

This expression identically satisfies the constraint, and avoids the necessity of its explicit use throughout the remaining manipulations. The duality transformation (32) now reads

$$K^{\alpha\{\beta\sigma\}} = -\frac{1}{\sqrt{2}}(T_{(\mu\nu)}{}^\sigma\epsilon^{\mu\nu\alpha\beta} + T_{(\mu\nu)}{}^\beta\epsilon^{\mu\nu\alpha\sigma}) \tag{36}$$

with $K^\alpha = -\frac{1}{2}K^{\alpha\{\beta}{}_{\beta\}} = -\sqrt{2}\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)\beta}$. The trace of $T_{(\mu\nu)\beta}$ does not contribute to the expression (35). Thus, we will take $T_{(\mu\nu)\beta}$ to be traceless and impose this constraint by means of a Lagrange multiplier. The analysis in Sec. VI will show that this choice is indeed compatible with the dynamics of the Fierz-Pauli field.

Using the identities

$$\begin{aligned} &\epsilon^{\mu\nu}{}_{\alpha\beta}T_{(\mu\nu)\sigma}(T^{(\alpha\beta)\sigma} + 2T^{(\sigma\alpha)\beta}) \\ &\quad = -\frac{2}{3}T^\nu_{(\mu\nu)}\epsilon^{\alpha\beta\gamma\mu}(T_{(\alpha\beta)\gamma} + T_{(\gamma\alpha)\beta} + T_{(\beta\gamma)\alpha},) \end{aligned} \tag{37}$$

$$\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)\sigma}T_{(\alpha\beta)}{}^{\sigma}=-2\epsilon_{\alpha}{}^{\mu\nu\beta}T_{(\mu\nu)\sigma}T^{(\sigma\alpha)}{}_{\beta}=-4\epsilon_{\alpha}{}^{\mu\nu\beta}T_{(\mu\sigma)\nu}T^{(\sigma\alpha)}{}_{\beta}, \qquad (38)$$

which follows from the antisymmetry of $T^{(\mu\nu)\sigma}$ and the null trace property $T^{(\mu\nu)}{}_{\nu}=0$, we rewrite the parent Lagrangian (33) as

$$L=\frac{1}{3}\left(2a-\frac{1}{2}e^{2}\right)T_{(\mu\nu)}{}^{\sigma}T^{(\mu\nu)}{}_{\sigma}+\frac{1}{3}(2a+e^{2})T_{(\mu\nu)\beta}T^{(\mu\beta)\nu}+\frac{1}{2}\sqrt{a(e^{2}-a)}\epsilon^{\mu\nu\kappa\lambda}T_{(\mu\nu)}{}^{\sigma}T_{(\kappa\lambda)\sigma}+eT_{(\mu\nu)}{}^{\sigma}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}h_{\sigma\beta}-\frac{M^{2}}{2}(h_{\mu\nu}h^{\mu\nu}-h_{\mu}{}^{\mu}h_{\nu}{}^{\nu})+\Theta_{\mu\nu}h^{\mu\nu}+\lambda_{\beta}T^{(\beta\alpha)}{}_{\alpha}. \qquad (39)$$

The above Lagrangian is equivalent to Eq. (12) in Ref. [17] when $\Theta_{\mu\nu}=0$.

V. DUALITY TRANSFORMATIONS

From the parent Lagrangian (39), the equation of motion for λ_{μ} yields

$$T^{(\mu\nu)}{}_{\nu}=0. \qquad (40)$$

Varying $h^{\mu\nu}$ in Eq. (39) we obtain the Euler-Lagrange equation

$$M^{2}(h_{\alpha}{}^{\alpha}g^{\mu\nu}-h^{\mu\nu})+\Theta^{\mu\nu}-\frac{e}{2}(\partial_{\sigma}T^{\mu}_{(\alpha\beta)}\epsilon^{\alpha\beta\sigma\nu}+\partial_{\sigma}T^{\nu}_{(\alpha\beta)}\epsilon^{\alpha\beta\sigma\mu})=0 \qquad (41)$$

which allows us to solve for $h_{\mu\nu}$

$$h^{\mu\nu}=-\frac{e}{2M^{2}}(\epsilon^{\alpha\beta\sigma\nu}\partial_{\sigma}T_{(\alpha\beta)}{}^{\mu}+\epsilon^{\alpha\beta\sigma\mu}\partial_{\sigma}T_{(\alpha\beta)}{}^{\nu})+\frac{1}{3M^{2}}g^{\mu\nu}(e\epsilon^{\alpha\beta\rho\kappa}\partial_{\rho}T_{(\alpha\beta)\kappa}-\Theta^{\alpha}_{\alpha})+\frac{1}{M^{2}}\Theta^{\mu\nu}, \qquad (42)$$

giving the first duality relation $h^{\mu\nu}=h^{\mu\nu}(T)$.

Varying $T^{(\mu\nu)\sigma}$ and contracting with the metric and the Levi-Civita tensors we have

$$T^{(\mu\nu)}{}_{\nu}=-\frac{1}{2a-e^{2}}\frac{e}{a}\left[\sqrt{a(e^{2}-a)}(\partial_{\sigma}h^{\sigma\mu}-\partial^{\mu}h)+\frac{3}{4}e\lambda^{\mu}\right]=0, \qquad (43)$$

$$T^{(\mu\nu)\sigma}+T^{(\nu\sigma)\mu}+T^{(\sigma\mu)\nu}=\frac{1}{2a-e^{2}}\epsilon^{\mu\nu\sigma\lambda}\left[e(\partial_{\sigma}h^{\sigma}{}_{\lambda}-\partial_{\lambda}h)+\frac{3}{2a}\sqrt{a(e^{2}-a)}\lambda_{\lambda}\right], \qquad (44)$$

$$a\epsilon_{\mu\nu\kappa\lambda}T^{(\mu\nu)\sigma}=2\sqrt{a(e^{2}-a)}\;T_{(\kappa\lambda)}{}^{\sigma}+e(\partial_{\kappa}h^{\sigma}{}_{\lambda}-\partial_{\lambda}h^{\sigma}{}_{\kappa})+\frac{1}{2}\epsilon^{\sigma}{}_{\kappa\lambda\mu}\lambda^{\mu}-\frac{1}{3}\frac{2a+e^{2}}{2a-e^{2}}(g^{\sigma}_{\kappa}g^{\rho}_{\lambda}-g^{\sigma}_{\lambda}g^{\rho}_{\kappa})\times\left[e(\partial_{\sigma}h^{\sigma}{}_{\rho}-\partial_{\rho}h)+\frac{3}{2a}\sqrt{a(e^{2}-a)}\lambda_{\rho}\right], \qquad (45)$$

after some algebraic manipulations. Using the above constraints, the equations of motion for $T^{(\mu\nu)\sigma}$ are

$$T^{(\mu\nu)\sigma}=-\frac{1}{2e}\epsilon^{\mu\nu\alpha\beta}\partial_{\alpha}h^{\sigma}_{\beta}+\frac{1}{2e}\epsilon^{\mu\nu\sigma\lambda}(\partial_{\sigma}h^{\sigma}{}_{\lambda}-\partial_{\lambda}h)-\frac{1}{2ae}\sqrt{a(e^{2}-a)}(\partial^{\mu}h^{\sigma\nu}-\partial^{\nu}h^{\sigma\mu})+\frac{1}{6ae}\sqrt{a(e^{2}-a)}[g^{\sigma\mu}(\partial_{\kappa}h^{\kappa\nu}-\partial^{\nu}h)-g^{\sigma\nu}(\partial_{\kappa}h^{\kappa\mu}-\partial^{\mu}h)], \qquad (46)$$

which is the final expression for the duality transformation $T^{(\mu\nu)\sigma}=T^{(\mu\nu)\sigma}(h)$. Summarizing, using only algebraic manipulations and without any mixing between the results of different variations, the Lagrangian equations of motions are Eqs. (42) and (46) together with the null trace condition. These equations are used in eliminating either $h^{\sigma\mu}$ or $T^{(\mu\nu)\sigma}$ from the parent Lagrangian, to obtain the corresponding Lagrangians for $T^{(\mu\nu)\sigma}$ or $h^{\sigma\mu}$, respectively. In fact, the degrees of freedom of $h^{\sigma\mu}$ are mapped into the traceless part of $T^{(\mu\nu)\sigma}$.

All previous relations greatly simplify on shell. Under this circumstance the condition

$$(\partial^{\kappa}h^{\lambda}_{\kappa}-\partial^{\lambda}h^{\kappa}_{\kappa})=\frac{1}{M^{2}}\partial_{\eta}\Theta^{\eta\lambda} \qquad (47)$$

is obtained, using Eqs. (25) and (26). The remaining on-shell constraints are

$$T_{(\mu\nu)}{}^{\mu}=0, \qquad (48)$$

$$\partial_{\beta}T_{(\mu\nu)}{}^{\beta}=-\frac{\sqrt{a(e^{2}-a)}}{3aeM^{2}}(\partial_{\mu}\partial_{\alpha}\Theta^{\alpha}_{\nu}-\partial_{\nu}\partial_{\alpha}\Theta^{\alpha}_{\mu}), \qquad (49)$$

$$\epsilon^{\lambda\mu\nu\beta}T_{(\mu\nu)\beta}=\frac{2}{eM^{2}}\partial_{\mu}\Theta^{\mu\lambda}, \qquad (50)$$

$$\partial_{\kappa}T^{(\kappa\lambda)\sigma}+\frac{1}{2a}\sqrt{a(e^{2}-a)}\epsilon_{\mu\nu}{}^{\kappa\lambda}\partial_{\kappa}T^{(\mu\nu)\sigma}=-\frac{(2a+e^{2})}{6aeM^{2}}\epsilon_{\nu\mu}{}^{\lambda\sigma}\partial^{\mu}\partial_{\beta}\Theta^{\nu\beta}+\frac{\sqrt{a(e^{2}-a)}}{3eaM^{2}}\times(-\partial^{\sigma}\partial_{\mu}\Theta^{\mu\lambda}+g^{\lambda\sigma}\partial_{\beta}\partial_{\eta}\Theta^{\eta\beta}). \qquad (51)$$

Using the constraint equations for both fields the on-shell duality relations become

$$h^{\mu\nu}=-\frac{e}{2M^2}(\epsilon^{\alpha\beta\sigma\nu}\partial_\sigma T_{(\alpha\beta)}{}^{\mu}+\epsilon^{\alpha\beta\sigma\mu}\partial_\sigma T_{(\alpha\beta)}{}^{\nu})+\frac{1}{3M^2}g^{\mu\nu}\left(\frac{2}{M^2}\partial_\gamma\partial_\lambda\Theta^{\gamma\lambda}-\Theta^{\alpha}_{\alpha}\right)+\frac{1}{M^2}\Theta^{\mu\nu}, \tag{52}$$

$$T_{(\mu\nu)\beta}=-\frac{1}{2e}\epsilon^{\alpha\sigma}{}_{\mu\nu}\partial_\alpha h_{\sigma\beta}-\frac{\sqrt{a(e^2-a)}}{2ae}(\partial_\mu h_{\nu\beta}-\partial_\nu h_{\mu\beta})-\frac{1}{2eM^2}\epsilon^{\sigma}{}_{\mu\nu\beta}\partial_\eta\Theta^{\eta}_{\sigma}-\frac{\sqrt{a(e^2-a)}}{6aeM^2}(g_{\nu\beta}\partial_\eta\Theta^{\eta}_{\mu}-g_{\mu\beta}\partial_\eta\Theta^{\eta}_{\nu}). \tag{53}$$

In the particular case where $a=e^2$ the duality transformations acquire the usual form

$$T_{(\mu\nu)\beta}=-\frac{1}{2e}\epsilon^{\alpha\sigma}{}_{\mu\nu}\partial_\alpha h_{\sigma\beta}-\frac{1}{2eM^2}\epsilon^{\sigma}{}_{\mu\nu\beta}\partial_\eta\Theta^{\eta}_{\sigma}, \tag{54}$$

involving only the Levi-Civita tensor. The constraint (51) reduces to $\partial_\kappa T^{(\kappa\lambda)\sigma}=0$ out of sources, which means that the field $T^{(\kappa\lambda)\sigma}$ contains purely transversal degrees of freedom. Otherwise, if $a\neq e^2$, the degrees of freedom of $h^{\sigma\beta}$ are also mapped in the longitudinal components $(\partial_\kappa T^{(\kappa\sigma)\beta}+\partial_\kappa T^{(\kappa\beta)\sigma})$.

VI. DUAL THEORY

The substitution of $h_{\sigma\beta}(T)$, given by Eq. (42), in the parent Lagrangian (39) leads to the following Lagrangian for $T_{(\mu\nu)\sigma}$:

$$L=\frac{1}{3}\left(2a-\frac{1}{2}e^2\right)T_{(\mu\nu)\sigma}T^{(\mu\nu)\sigma}+\frac{1}{3}(2a+e^2)T_{(\mu\nu)\beta}T^{(\mu\beta)\nu}+\frac{1}{2}\sqrt{a(e^2-a)}\epsilon^{\mu\nu\kappa\lambda}T_{(\mu\nu)}{}^{\sigma}T_{(\kappa\lambda)\sigma}+\frac{1}{2}h_{\sigma\beta}(T)\times(-e\,\epsilon^{\mu\nu\alpha\beta}\partial_\alpha T_{(\mu\nu)}{}^{\sigma}+\Theta^{\sigma\beta})+\lambda_\beta T^{\beta}. \tag{55}$$

After some algebra and dropping a global $-e^2/(2M^2)$ factor we obtain

$$L=\frac{4}{9}F_{(\alpha\beta\gamma)\nu}F^{(\alpha\beta\gamma)\nu}+\frac{2}{3}F_{(\alpha\beta\gamma)\nu}F^{(\alpha\beta\nu)\gamma}-F_{(\alpha\beta\mu)}{}^{\mu}F^{(\alpha\beta\nu)}{}_{\nu}-\frac{2M^2}{3e^2}\left[\left(2a-\frac{1}{2}e^2\right)T_{(\mu\nu)\sigma}T^{(\mu\nu)\sigma}+(2a+e^2)\times T_{(\mu\nu)\sigma}T^{(\mu\sigma)\nu}+\frac{3}{2}\sqrt{a(e^2-a)}\epsilon^{\mu\nu\alpha\beta}T_{(\mu\nu)\sigma}T_{(\alpha\beta)}{}^{\sigma}\right]+\lambda_\beta T^{\beta}+T_{(\alpha\beta)\mu}J^{(\alpha\beta)\mu}. \tag{56}$$

Here we have introduced the field strength

$$F_{(\alpha\beta\gamma)\nu}=\partial_\alpha T_{(\beta\gamma)\nu}+\partial_\beta T_{(\gamma\alpha)\nu}+\partial_\gamma T_{(\alpha\beta)\nu}, \tag{57}$$

and the source term is given as a function of the source $\Theta_{\mu\nu}$ by

$$J^{(\alpha\beta)\mu}=\frac{2}{e}\left(\frac{1}{3}\epsilon^{\alpha\beta\rho\mu}\partial_\rho\Theta^{\alpha}_{\alpha}-\epsilon^{\alpha\beta\sigma\nu}\partial_\sigma\Theta_{\nu}{}^{\mu}\right). \tag{58}$$

Note that the new source $J^{(\alpha\beta)\mu}$ is traceless and also satisfies

$$\epsilon_{\kappa\alpha\beta\mu}J^{(\alpha\beta)\mu}=-\frac{4}{e}\partial_\mu\Theta^{\mu}{}_{\kappa},\quad \partial_\alpha J^{(\alpha\beta)\mu}=0,$$

$$\partial_\mu J^{(\alpha\beta)\mu}=-\frac{2}{e}\epsilon^{\alpha\beta\sigma\nu}\partial_\sigma\partial^\mu\Theta_{\mu\nu}. \tag{59}$$

As stated previously $T_{(\mu\nu)\sigma}=-T_{(\nu\mu)\sigma}$ and therefore the field $T_{(\nu\mu)\sigma}$ has 24 components. But not all of them are true degrees of freedom, because there are cyclic variables. This becomes clear by defining

$$T_{(\mu\nu)\sigma}=\hat{T}_{(\mu\nu)\sigma}-\frac{1}{3}(g_{\mu\sigma}T_\nu-g_{\nu\sigma}T_\mu) \tag{60}$$

where $T_\mu\equiv T_{(\mu\beta)}{}^{\beta}$, and $\hat{T}_{(\mu\nu)\sigma}$ is a traceless field, $\hat{T}_{(\mu\nu)}{}^{\nu}=0$. Next, we rewrite Lagrangian (56) in terms of $\hat{T}^{(\chi\psi)\sigma}$ and T_μ and we further use the Euler-Lagrange equation for T_μ to eliminate this variable. The resulting Lagrangian contains linear and bilinear terms in λ_β. Finally, using the corresponding equation of motion for λ_β we can also eliminate this variable. In such a way we obtain an alternative version of the dual Lagrangian in terms of $\hat{T}_{(\mu\nu)\sigma}$

$$L=\frac{4}{9}\hat{F}_{(\alpha\beta\gamma)\nu}\hat{F}^{(\alpha\beta\gamma)\nu}+\frac{2}{3}\hat{F}_{(\alpha\beta\gamma)\nu}\hat{F}^{(\alpha\beta\nu)\gamma}-\hat{F}_{(\alpha\beta\mu)}{}^{\mu}\hat{F}^{(\alpha\beta\nu)}{}_{\nu}+\hat{T}_{(\alpha\beta)}{}^{\mu}J^{(\alpha\beta)}_{\mu}-\frac{2M^2}{3e^2}\left[\left(2a-\frac{1}{2}e^2\right)\hat{T}_{(\mu\nu)\sigma}\hat{T}^{(\mu\nu)\sigma}+(2a+e^2)\hat{T}_{(\mu\nu)\sigma}\hat{T}^{(\mu\sigma)\nu}+\frac{1}{2}(e^2-a)\hat{T}_{(\mu\nu)\sigma}(\hat{T}^{(\mu\nu)\sigma}+\hat{T}^{(\sigma\mu)\nu}+\hat{T}^{(\nu\sigma)\mu})+\frac{3}{2}\sqrt{a(e^2-a)}\epsilon^{\mu\nu\alpha\beta}\hat{T}_{(\mu\nu)\sigma}\hat{T}_{(\alpha\beta)}{}^{\sigma}\right]. \tag{61}$$

This clearly shows that the degrees of freedom are in the traceless field $\hat{T}^{(\mu\nu)\sigma}$, as has already been assumed in Sec. IV. When varying Lagrangian (61) it is necessary to take into account that not all the components of $\hat{T}^{(\mu\nu)\sigma}$ are independent, because of the traceless condition, and this is rather cumbersome.

Consequently, in order to study the properties of the dual field $T^{(\mu\nu)\sigma}$ it is more convenient to go back to Lagrangian

(56), because there we have to impose only the antisymmetry constraint. Starting from this Lagrangian we obtain the equations of motion

$$
\begin{aligned}
E^{(\beta\gamma)\nu} := & \frac{2}{3}\partial_\alpha[2\,F^{(\alpha\beta\gamma)\nu}+(F^{(\alpha\beta\nu)\gamma}+F^{(\gamma\alpha\nu)\beta}+F^{(\beta\gamma\nu)\alpha})] \\
& -\partial^\nu F^{(\beta\gamma\kappa)}{}_\kappa-\partial_\alpha(g^{\gamma\nu}F^{(\alpha\beta\kappa)}{}_\kappa+g^{\beta\nu}F^{(\gamma\alpha\kappa)}{}_\kappa) \\
& +\frac{M^2}{e^2}\sqrt{a(e^2-a)}\,\epsilon^{\beta\gamma\kappa\lambda}T_{(\kappa\lambda)}{}^\nu+\frac{2M^2}{3e^2} \\
& \times\left[\left(2a-\frac{1}{2}e^2\right)T^{(\beta\gamma)\nu}+\frac{1}{2}(2a+e^2)\right. \\
& \left.\times(T^{(\beta\nu)\gamma}-T^{(\gamma\nu)\beta})\right]-\frac{1}{4}(\lambda^\beta g^{\gamma\nu}-\lambda^\gamma g^{\beta\nu}) \\
& -\frac{1}{2}J^{(\beta\gamma)\nu}=0,
\end{aligned}
\tag{62}
$$

$$
T^{(\mu\nu)}{}_\nu=0. \tag{63}
$$

In order to make explicit the Lagrangian constraints arising from Eq. (62), which determine the number of propagating degrees of freedom associated with the massive field $T^{(\mu\nu)\rho}$, it is sufficient to consider the equations of motion out of sources. In Appendix C we provide some details of the derivation and the results are summarized here. The constraints are

$$
T^{(\mu\alpha)}{}_\alpha=0,\quad T^{(\mu\alpha)\beta}+T^{(\alpha\beta)\mu}+T^{(\beta\mu)\alpha}=0,\quad \partial_\theta T^{(\alpha\beta)\theta}=0, \tag{64}
$$

$$
\begin{aligned}
0=&(\partial_\beta T^{(\beta\gamma)\nu}+\partial_\beta T^{(\beta\nu)\gamma})+\sqrt{\frac{(e^2-a)}{a}}\frac{1}{2}(\epsilon^{\beta\gamma\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^\nu \\
&+\epsilon^{\beta\nu\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^\gamma):=S^{\{\gamma\nu\}},
\end{aligned}
\tag{65}
$$

where we observe that the antisymmetric part of $\partial_\beta T^{(\beta\gamma)\nu}$ turns out to be zero, as shown in the sourceless version of Eq. (C18) of Appendix C. The count of the number of independent degrees of freedom goes as follows. The field $T^{(\mu\nu)\rho}$ has 24 independent components. Equations (64) provide $4+4+6=14$ constraints, respectively, thus leaving 24-14=10 independent variables up to this level. Because of the symmetry $S^{\gamma\nu}=S^{\nu\gamma}$, the remaining Eq. (65) provides only 10 relations. Nevertheless, among them we find 5 additional identities: 4 arising from $\partial_\nu S^{\{\gamma\nu\}}=0$ and 1 arising from $g_{\gamma\nu}S^{\{\gamma\nu\}}=0$, leaving only 5 additional independent constraints. Thus, Eq. (65) reduces to $10-5=5$ the previous 10 independent degrees of freedom, as appropriate for a massive spin-2 system.

Taking into account the constraints (64)–(65), the equation of motion in the source free region becomes

$$
\begin{aligned}
&\partial^2 T^{(\beta\gamma)\nu}+\partial_\alpha\partial^\gamma T^{(\alpha\beta)\nu}-\partial_\alpha\partial^\beta T^{(\alpha\gamma)\nu} \\
&\quad+\frac{M^2}{2e^2}\sqrt{a(e^2-a)}\,\epsilon^{\beta\gamma\kappa\lambda}T_{(\kappa\lambda)}{}^\nu+\frac{M^2a}{e^2}T^{(\beta\gamma)\nu}=0.
\end{aligned}
\tag{66}
$$

Furthermore, from Eq. (65) we can write

$$
\begin{aligned}
&\partial_\alpha\partial_\gamma T^{(\alpha\beta)\nu}-\partial_\alpha\partial_\beta T^{(\alpha\gamma)\nu} \\
&\quad=-\frac{D}{4a}[\partial_\gamma(\epsilon^{\alpha\beta\kappa\lambda}\partial_\alpha T_{(\kappa\lambda)}{}^\nu+\epsilon^{\alpha\nu\kappa\lambda}\partial_\alpha T_{(\kappa\lambda)}{}^\beta) \\
&\quad\quad-\partial_\beta(\epsilon^{\alpha\gamma\kappa\lambda}\partial_\alpha T_{(\kappa\lambda)}{}^\nu+\epsilon^{\alpha\nu\kappa\lambda}\partial_\alpha T_{(\kappa\lambda)}{}^\gamma)].
\end{aligned}
\tag{67}
$$

The above equation and its dual imply

$$
\begin{aligned}
&(\partial^\gamma\partial_\alpha T^{(\alpha\nu)\beta}-\partial^\beta\partial_\alpha T^{(\alpha\nu)\gamma}) \\
&\quad=-\frac{D^2}{ae^2}\partial^2 T^{(\beta\gamma)\nu}+\frac{D}{2e^2}\partial^2\epsilon^{\beta\gamma}{}_{\kappa\lambda}T^{(\kappa\lambda)\nu}.
\end{aligned}
\tag{68}
$$

Hence, the equation of motion (66) and its dual can be written as

$$
\frac{a}{e^2}\partial^2 T+\frac{D}{e^2}\partial^2 T^*+\frac{aM^2}{e^2}T+\frac{DM^2}{e^2}T^*=0, \tag{69}
$$

$$
\frac{a}{e^2}\partial^2 T^*-\frac{D}{e^2}\partial^2 T+\frac{aM^2}{e^2}T^*-\frac{DM^2}{e^2}T=0, \tag{70}
$$

where we have omitted the indices of $T^{(\beta\gamma)\nu}$ and T^* means the dual of T. Finally, solving for $T^{(\beta\gamma)\nu}$ we get

$$
(\partial^2+M^2)T^{(\beta\gamma)\nu}=0. \tag{71}
$$

The simpler case $e^2=a$ is reminiscent of the standard duality transformations and the constraint equations simplify to

$$
T^{(\mu\theta)}{}_\theta=0,\qquad T^{(\mu\alpha\beta)}=0,\qquad \partial_\theta T^{(\mu\nu)\theta}=0,\qquad \partial_\theta T^{(\theta\mu)\nu}=0. \tag{72}
$$

VII. POINT MASS SOURCE

As an illustration let us discuss the field generated by a point mass m at rest. The corresponding source is the energy-momentum tensor of a point mass at rest and has the components $\Theta_{00}=16\pi m\,\delta(\mathbf{r}),\Theta_{0i}=\Theta_{ji}=0$. From Eq. (27), the resulting field configuration is

$$
h_{00}=\frac{8}{3}\frac{m}{r}e^{-Mr},\quad h_{ij}=\frac{1}{2}\delta_{ij}h_{00}-\frac{1}{2M^2}\partial_j\partial_i h_{00},\quad h_{0i}=0. \tag{73}
$$

The last term in h_{ij} is irrelevant when the field is coupled to a conserved source.

It is interesting to observe how the zero mass limit discontinuity (the van Dam–Veltman–Zakharov discontinuity

[20,21]) manifests itself here. In the limit $M\rightarrow 0$, h_{00} converges to $\frac{4}{3}$ of the Newtonian potential m/r. Besides, the component h_{ij} has a divergent term plus $\frac{1}{2}h_{00}\delta_{ij}$. This has to be contrasted with the massless spin-2 theory in the Lorentz gauge, where the non-zero fields are

$$\tilde{h}_{00}=\frac{2m}{r}, \qquad \tilde{h}_{ij}=\tilde{h}_{00}\delta_{ij}+\partial_i\partial_j f(r), \tag{74}$$

where the last term in $\tilde{h}_{ij}$ accounts for a remaining gauge freedom associated with a time-independent spatial rotation.

Next we consider the corresponding theory for $T_{(\alpha\beta)\gamma}$, in the simpler case of $a=e^2$. We refer the reader to Appendix C for the notation. Here the dual source is

$$J^{(\alpha\beta)\mu}=\frac{32\pi m}{e}\left(g^{\mu 0}\epsilon^{0\alpha\beta\rho}-\frac{1}{3}\epsilon^{\alpha\beta\mu\rho}\right)\partial_\rho\delta(\mathbf{r}), \tag{75}$$

$$J^{(\alpha\beta\mu)}=\frac{32\pi m}{e}(g^{\mu 0}\epsilon^{0\alpha\beta\rho}+g^{\alpha 0}\epsilon^{0\beta\mu\rho}+g^{\beta 0}\epsilon^{0\mu\alpha\rho}-\epsilon^{\alpha\beta\mu\rho})\times\partial_\rho\delta(\mathbf{r}). \tag{76}$$

Our conventions are $\epsilon^{0123}=\epsilon_{123}=+1$. The equation of motion, arising from Eq. (C26) of Appendix C, becomes

$$(\partial_\alpha\partial^\alpha+M^2)T^{(\beta\gamma)\nu}=\frac{1}{4}J^{(\beta\gamma)\nu}+\frac{1}{4}J^{(\gamma\beta\nu)}+\frac{1}{6M^2}\partial_\alpha\partial^\alpha J^{(\gamma\beta\nu)}, \tag{77}$$

with the constraints

$$M^2T^{(\beta\gamma\nu)}=-\frac{1}{2}J^{(\beta\gamma\nu)}, \qquad M^2\partial_\theta T^{(\mu\nu)\theta}=0,$$
$$M^2\partial_\beta T^{(\beta\gamma)\nu}=0. \tag{78}$$

From here, the non-zero components of $T_{(\alpha\beta)\gamma}$ are

$$T_{(0i)j}=\frac{2m}{3e}(1+Mr)\frac{e^{-Mr}}{r^3}\epsilon_{ijk}x_k, \tag{79}$$

$$T_{(ij)0}=-\frac{4m}{3e}(1+Mr)\frac{e^{-Mr}}{r^3}\epsilon_{ijk}x_k. \tag{80}$$

We can now compare both theories. In terms of the massive Fierz-Pauli solution, the solution for $T_{(\mu\nu)\sigma}$ can be written as

$$T_{(0i)j}=-\frac{1}{4e}\epsilon_{ijk}\partial_k h_{00}, \tag{81}$$

$$T_{(ij)0}=+\frac{1}{2e}\epsilon_{ijk}\partial_k h_{00}. \tag{82}$$

It is straightforward to verify that both solutions, $h_{\mu\nu}$ and $T_{(\alpha\beta)\gamma}$, are in fact related by the duality transformations. Out of the source the on-shell duality relations are

$$h_{\alpha\beta}=-\frac{e}{2M^2}(\epsilon_{\gamma\delta\rho\alpha}\partial^\rho T_\beta^{(\gamma\delta)}+\epsilon_{\gamma\delta\rho\beta}\partial^\rho T_\alpha^{(\gamma\delta)}), \tag{83}$$

$$T_{(\alpha\beta)\gamma}=-\frac{1}{2e}\epsilon^{\rho\delta}{}_{\alpha\beta}\partial_\rho h_{\delta\gamma}. \tag{84}$$

From Eq. (84) and Eqs. (79),(80) we obtain Eqs. (81) and (82). Conversely, applying Eq. (83) to the expressions (81) and (82) we recover Eqs. (79) and (80). It is interesting to observe that the term that diverges in the zero mass limit in Eq. (73) does not contribute to $T_{(\alpha\beta)\gamma}$, which remains non-divergent in this limit. Thus the description in terms of $T_{(\alpha\beta)\gamma}$ seems more suitable for studying the massless limit.

The analysis of how the massless limit and the van Dam–Veltman–Zakharov discontinuity appears in the dual theory requires the discussion of duality in the case of $M=0$. We postpone the discussion of this interesting point to a forthcoming paper.

VIII. SUMMARY AND FINAL REMARKS

The dualization scheme presented in this work is based on a first order parent Lagrangian from which either the original theory or the dual one can be obtained, by means of permissible substitutions arising from the algebraic solutions of the corresponding equations of motion. This procedure guarantees the equivalence of both theories.

Given a Lagrangian to be dualized, we first construct an equivalent auxiliary first order Lagrangian written in standard form, which can always be done by using the method of Lagrange multipliers in the manner described in Refs. [18] and [22]. This first order Lagrangian is not unique and provides the identification of what we have called the field strength of the original field. Dualization occurs at this level, through the introduction of the dual tensor defined by the contraction of the Levi-Civita tensor with the field strength. Different possibilities might arise at this level which will produce alternative dual theories.

Substitution of the field strength in terms of the dual tensor in the auxiliary first order Lagrangian produces the parent Lagrangian, which is a functional of the original field configuration together with the new dual field. On the one hand, the elimination of the dual field from this Lagrangian, via its equations of motion, always takes us back to the original second order theory. On the other hand, the elimination of the original field from the parent Lagrangian defines the dual theory.

This dual tensor plays different roles in the massive and the massless cases, because the duality transformation is singular in the limit $M\rightarrow 0$. The mapping between dual theories is also very different according to these cases. For $M\neq 0$ the dual tensor turns out to be the basic configuration variables of the dual theory and its definition in terms of the original field strength provides the relation among the resultant theories. Here the dual field is interpreted as a potential. For $M=0$ the equation of motion of the dual field becomes a constraint (a Bianchi identity) on the dual variable, which implies that it can be written in terms of a potential. Hence the dual field can be interpreted as a new field strength. The connection between both theories is now given by a relation between the original and dual potentials which usually involves derivatives. Summarizing, for massive theories duality relates field strengths and potentials, while for massless theories it relates the corresponding potentials.

We have applied this scheme to the massive spin-2 field coupled to external sources, obtaining a family of dual theories. The starting point is the symmetric massive Fierz-Pauli field $h_{\mu\nu}$ with its standard Lagrangian. The corresponding first order auxiliary Lagrangian, which has two independent parameters, is written in terms of $h_{\mu\nu}$ plus the field strength $K_{\alpha\{\beta\gamma\}}$. The latter satisfies additional symmetry properties. We have explicitly shown that the elimination of the auxiliary field leads to the original massive Fierz-Pauli Lagrangian. At this stage there is some freedom in the election of the dual field Ω and we have chosen the relation $K^{\alpha\{\beta\gamma\}} = \epsilon^{\alpha\mu\nu\rho}\,\Omega^{\{\beta\gamma\}}_{(\mu\nu\rho)}$. In order to partially fulfill the induced symmetry properties of $\Omega^{\{\beta\gamma\}}_{(\mu\nu\rho)}$ we have introduced the auxiliary tensor $T^{(\alpha\beta)\gamma}$ which is required to be antisymmetric in the first two indices, and which serves as the basic dual field in the sequel. This field is reminiscent of what is called the Fierz tensor in Ref. [23]. Our approach for the massive case is different from the latter reference because we take $T^{(\alpha\beta)\gamma}$ as the basic variable for the massive situation. The Lagrangian for this field is subsequently constructed by eliminating $h^{\mu\nu}$ from the parent Lagrangian. By construction this dual theory is equivalent to the initial Fierz-Pauli description, and the connection between both is established. The correct number of degrees of freedom in the dual theory is verified by identifying the Lagrange constraints arising from the equations of motion.

Finally, we have discussed the case of the massive field generated by a point mass m at rest, which is described using both the original and the dual theory. This simple example suggests that the description in terms of $T^{(\alpha\beta)\gamma}$ behaves continuously in the limit $M\to 0$, in contrast with that in terms of $h^{\mu\nu}$. The latter theory develops a singularity in the $M\to 0$ case , while the components of the dual field remain finite.

We postpone for a separate publication a detailed discussion of the $M=0$ case. This situation is directly related to the problem of dualizing linearized gravity, which has been the subject of recent investigations [24–26]. Our preliminary work on this subject shows some interesting features: (i) the zero mass limit of the dual Lagrangian for $T^{(\alpha\beta)\gamma}$, given in Eq. (56), has no arbitrary parameters so that one would expect it to be completely determined by a set of gauge symmetries. (ii) The Dirac analysis of the constraints leads to the count of two degrees of freedom per space point, in contrast with the results in Ref. [15]. The analysis of the gauge structure that arises in the approach pursued here should be of some interest, together with the discussion of the van Dam–Veltman–Zakharov discontinuity in the dual theory.

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APPENDIX A: FIRST ORDER LAGRANGIAN IN THE STANDARD FORM

Here we show how to construct a first order Lagrangian equivalent to a given second order Lagrangian using the method of Lagrange multipliers. This approach has been presented in the framework of classical mechanics for regular systems in Ref. [18], to construct a Hamiltonian formalism without the use of a Legendre transformation. Reference [22] deals with its application to singular systems. To give a general idea of the method, we simply sketch it for the case of regular field theories. Consider a given regular Lagrangian

$$L = L(\psi^a, \psi^a{}_{,\mu}). \tag{A1}$$

Let us assume that we want to introduce a set of new functions $f^a_\mu = f^a_\mu(\psi^b{}_{,\mu})$, and treat them as new independent fields. To do this it must be possible to solve $\psi^a{}_{,\mu}$ in terms of f^a_μ, i.e. $|\partial f^a_\mu/\partial\psi^b_{,\nu}|\neq 0$. Substituting this solution in Eq. (A1) and imposing the constraint $f^a_\mu - f^a_\mu(\psi^b_{,\mu})=0$ we obtain a new Lagrangian

$$\tilde{L} = L(\psi^a, f^a_\mu) + \lambda^\mu_a (f^a_\mu(\psi^b_{,\mu}) - f^a_\mu), \tag{A2}$$

which is clearly equivalent to the original one. The auxiliary functions f^a_μ exclusively appear as algebraic variables, without derivatives. Therefore, they can be eliminated by using their equations of motion

$$\frac{\partial\tilde{L}}{\partial f^a_\mu} = \frac{\partial L}{\partial f^a_\mu} - \lambda^\mu_a = 0. \tag{A3}$$

The solution of this equation is a set of functions $f^a_\mu = f^a_\mu(\psi^a, \lambda^\mu_a)$, and the resulting Lagrangian has the form

$$\tilde{L} = \lambda^\mu_a f^a_\mu(\psi^b_{,\mu}) - \lambda^\mu_a f^a_\mu(\psi^a, \lambda^\mu_a) + L(\psi^a, f^a_\mu(\psi^a, \lambda^\mu_a)). \tag{A4}$$

The first term of the above Lagrangian shows that λ^μ_a define the field strength of ψ^b. Their relationship with the configuration variables is given by their equation of motion in the first order theory

$$f^a_\mu(\psi^b_{,\mu}) - f^a_\mu(\psi^a, \lambda^\mu_a) - \lambda^\nu_b \frac{\partial f^b_\nu}{\partial\lambda^\mu_a} + \frac{\partial L}{\partial\lambda^\mu_a} = 0. \tag{A5}$$

This definition of the field strength is not unique, because it depends on the choice of the functions $f^a_\mu(\psi^b_{,\mu})$.

APPENDIX B: EQUIVALENCE BETWEEN SECOND ORDER AND FIRST ORDER LAGRANGIANS

In this appendix we show that Lagrangian (30), where the coupling constants satisfy Eqs. (31), is a first order Lagrangian for the massive Fierz-Pauli theory (24). From Lagrangian

(30) we obtain the equations of motion for $K_{\alpha\{\beta\sigma\}}$ and $\Lambda_{\alpha\{\beta\sigma\}}$

$$-\frac{1}{3}aK_{\alpha\{\beta\sigma\}}+\frac{1}{4}q\,g_{\beta\sigma}K_{\alpha}-\frac{2}{9}r(\epsilon^{\gamma\delta}{}_{\alpha\beta}K_{\gamma\{\delta\sigma\}}+\epsilon^{\gamma\delta}{}_{\alpha\sigma}K_{\gamma\{\delta\beta\}})$$
$$-\frac{e}{\sqrt{2}}\partial_{\alpha}h_{\beta\sigma}+\Phi_{\alpha\beta\sigma}=0, \tag{B1}$$

$$K^{\alpha\{\beta\sigma\}}+K^{\beta\{\sigma\alpha\}}+K^{\sigma\{\alpha\beta\}}=0, \tag{B2}$$

where $\Phi_{\alpha\beta\sigma}=\Lambda_{\alpha\{\beta\sigma\}}+\Lambda_{\beta\{\sigma\alpha\}}+\Lambda_{\sigma\{\alpha\beta\}}$ is a completely symmetric tensor. Equation (B2) leads to a constraint between the two possible contractions of the indices of $K^{\alpha\{\beta\sigma\}}$

$$K_{\alpha}+2K^{\beta}{}_{\{\beta\alpha\}}=0. \tag{B3}$$

First, we solve the field strength $K_{\alpha\{\beta\gamma\}}$ in terms of the potential field $h_{\sigma\beta}$. Taking the two possible traces in Eq. (B1) we obtain

$$K_{\alpha}=\frac{2\sqrt{2}e}{(2a-3q)}(\partial^{\beta}h_{\beta\alpha}-\partial_{\alpha}h_{\beta}{}^{\beta}). \tag{B4}$$

Performing a cyclic permutation of the indices in Eq. (B1) and adding the results we get

$$\Phi_{\alpha\beta\sigma}=\frac{e}{3\sqrt{2}}(\partial_{\alpha}h_{\beta\sigma}+\partial_{\beta}h_{\sigma\alpha}+\partial_{\sigma}h_{\alpha\beta})$$
$$-\frac{q}{12}(g_{\sigma\beta}K_{\alpha}+g_{\beta\alpha}K_{\sigma}+g_{\alpha\sigma}K_{\beta}). \tag{B5}$$

The contraction of Eq. (B1) with $\epsilon^{\alpha\beta}{}_{\mu\nu}$ gives

$$-\frac{1}{3}a\epsilon^{\gamma\delta}{}_{\mu\nu}K_{\gamma\{\delta\sigma\}}+\frac{1}{4}q\,\epsilon^{\alpha}_{\sigma\mu\nu}K_{\alpha}+\frac{2}{3}r(K_{\mu\{\nu\sigma\}}-K_{\nu\{\mu\sigma\}})$$
$$+\frac{1}{3}r(g_{\mu\sigma}K_{\nu}-g_{\nu\sigma}K_{\mu})-\frac{e}{\sqrt{2}}\epsilon^{\gamma\delta}{}_{\mu\nu}\partial_{\gamma}h_{\delta\sigma}=0. \tag{B6}$$

Combining this last equation with Eq. (B1) to eliminate terms proportional to $\epsilon^{\gamma\delta}{}_{\mu\nu}K_{\gamma\{\delta\sigma\}}$ we obtain the following expression for $K_{\alpha\{\beta\sigma\}}$ in terms of $h_{\alpha\beta}$:

$$K_{\alpha\{\beta\sigma\}}=-\frac{\sqrt{2}}{e}[A\,P_{\alpha\{\beta\sigma\}}+BQ_{\alpha\{\beta\sigma\}}+CR_{\alpha\{\beta\sigma\}}\}, \tag{B7}$$

where

$$P_{\alpha\{\beta\sigma\}}=g_{\alpha\sigma}(\partial^{\gamma}h_{\gamma\beta}-\partial_{\beta}h_{\gamma}{}^{\gamma})+g_{\alpha\beta}(\partial^{\gamma}h_{\gamma\sigma}-\partial_{\sigma}h_{\gamma}{}^{\gamma})$$
$$-2g_{\beta\sigma}(\partial^{\gamma}h_{\gamma\alpha}-\partial_{\alpha}h_{\gamma}{}^{\gamma}), \tag{B8}$$

$$Q_{\alpha\{\beta\sigma\}}=\partial_{\beta}h_{\sigma\alpha}+\partial_{\sigma}h_{\alpha\beta}-2\partial_{\alpha}h_{\beta\sigma}, \tag{B9}$$

$$R_{\alpha\{\beta\sigma\}}=\epsilon^{\gamma\delta}{}_{\alpha\beta}\partial_{\gamma}h_{\delta\sigma}+\epsilon^{\gamma\delta}{}_{\alpha\sigma}\partial_{\gamma}h_{\delta\beta}, \tag{B10}$$

together with the coefficients

$$A=e^{2}\left(\frac{2}{3}r^{2}+\frac{1}{4}aq\right)\left(a-\frac{3}{2}q\right)^{-1}(a^{2}+4r^{2})^{-1}, \tag{B11}$$

$$B=-\frac{1}{2}ae^{2}(a^{2}+4r^{2})^{-1}, \tag{B12}$$

$$C=-re^{2}(a^{2}+4r^{2})^{-1}. \tag{B13}$$

All three tensors appearing on the right-hand side of Eq. (B7) have a vanishing cyclic sum. The Lagrangian in terms of $h_{\alpha\beta}$ can be obtained by replacing expression (B7) in the first order Lagrangian (30), or more simply, noting that the contribution from the mass terms for $K^{\alpha\{\beta\sigma\}}$ is $(-1/2)$ of the contribution from the interaction term $-(e/\sqrt{2})K^{\alpha\{\beta\sigma\}}\partial_{\alpha}h_{\beta\sigma}$. Half of this last contribution gives the kinetic terms of the $h_{\alpha\beta}$ Lagrangian. The interaction term gives

$$(2A+2B)\partial_{\mu}h^{\mu\nu}\partial_{\alpha}h_{\nu}{}^{\alpha}+(-2B)\partial_{\alpha}h^{\mu\nu}\partial^{\alpha}h_{\mu\nu}$$
$$+(-4A)\partial_{\mu}h^{\mu\nu}\partial_{\nu}h_{\alpha}{}^{\alpha}+(2A)\partial_{\alpha}h_{\mu}{}^{\mu}\partial^{\alpha}h_{\nu}{}^{\nu}, \tag{B14}$$

after substituting Eq. (B7). Let us observe that the term proportional to C gives no contribution. Comparing this expression with the kinetic part of the original Lagrangian (24) we obtain $A=(-1/2)$, $B=(-1/2)$. From here the relations (31) follow.

APPENDIX C: LAGRANGIAN CONSTRAINTS FOR THE PROPAGATING FIELD $T^{(\mu\nu)\rho}$

We start from Eqs. (62) and (63). The zero trace condition implies $F^{(\alpha\beta\theta)}{}_{\theta}=\partial_{\theta}T^{(\alpha\beta)\theta}$. We will also use the property $\epsilon_{\alpha\beta\gamma\delta}F^{(\beta\gamma\delta)\psi}=3\epsilon_{\alpha\beta\gamma\delta}\partial^{\beta}T^{(\gamma\delta)\psi}$, together with the notation $T^{(\mu\nu\rho)}=T^{(\mu\nu)\rho}+T^{(\nu\rho)\mu}+T^{(\rho\mu)\nu}$ and $D=\sqrt{a(e^{2}-a)}$.

Useful relations to determine the Lagrangian constraints are obtained according to the following manipulations.

$g_{\gamma\nu}E^{(\beta\gamma)\nu}=0$ implies

$$\lambda^{\beta}=\frac{4DM^{2}}{3e^{2}}\epsilon^{\beta\sigma\kappa\tau}T_{(\kappa\tau)\sigma}. \tag{C1}$$

$\partial_{\beta}E^{(\beta\gamma)\nu}=0$ leads to

$$DM^{2}\epsilon^{\beta\gamma\kappa\lambda}\partial_{\beta}T_{(\kappa\lambda)}{}^{\nu}-\frac{e^{2}}{4}(g^{\gamma\nu}\partial_{\beta}\lambda^{\beta}-\partial^{\nu}\lambda^{\gamma})+\frac{2}{3}M^{2}\partial_{\beta}$$
$$\times\left[\left(2a-\frac{1}{2}e^{2}\right)T^{(\beta\gamma)\nu}+\frac{1}{2}(2a+e^{2})\right.$$
$$\left.\times(T^{(\beta\nu)\gamma}-T^{(\gamma\nu)\beta})\right]=0. \tag{C2}$$

This expression can be decomposed in the symmetric and antisymmetric part

$$M^2\partial_\beta(T^{(\beta\gamma)\nu}+T^{(\beta\nu)\gamma})$$
$$=-\frac{DM^2}{2a}(\epsilon^{\beta\gamma\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^{\nu}+\epsilon^{\beta\nu\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^{\gamma})$$
$$+\frac{e^2}{4a}\left(g^{\gamma\nu}\partial_\beta\lambda^\beta-\frac{1}{2}(\partial^\nu\lambda^\gamma+\partial^\gamma\lambda^\nu)\right) \qquad (C3)$$

$$M^2(a-e^2)\partial_\beta(T^{(\beta\gamma)\nu}-T^{(\beta\nu)\gamma})$$
$$=-\frac{3DM^2}{2}(\epsilon^{\beta\gamma\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^{\nu}-\epsilon^{\beta\nu\kappa\lambda}\partial_\beta T_{(\kappa\lambda)}{}^{\gamma})$$
$$+M^2\partial_\beta(2a+e^2)T^{(\gamma\nu)\beta}+\frac{3e^2}{8}(\partial^\gamma\lambda^\nu-\partial^\nu\lambda^\gamma). \qquad (C4)$$

Applying ∂_ν to Eq. (C2) we have

$$4M^2[D\epsilon^{\beta\gamma\kappa\lambda}\partial_\nu\partial_\beta T_{(\kappa\lambda)}{}^{\nu}+2a\partial_\nu\partial_\beta T^{(\beta\gamma)\nu}]$$
$$=e^2\partial_\nu(\partial^\gamma\lambda^\nu-\partial^\nu\lambda^\gamma). \qquad (C5)$$

From $\epsilon_{\beta\gamma\nu\psi}E^{(\beta\gamma)\nu}=0$ we obtain

$$2\epsilon_{\beta\gamma\nu\psi}\partial_\alpha F^{(\beta\gamma\nu)\alpha}-6M^2\epsilon_{\beta\gamma\nu\psi}T^{(\beta\gamma)\nu}-3\epsilon_{\beta\gamma\nu\psi}J^{(\beta\gamma)\nu}=0. \qquad (C6)$$

Contracting the above equation with $\epsilon^{\rho\sigma\tau\psi}$ leads to

$$\partial_\alpha F^{(\beta\gamma\nu)\alpha}-M^2T^{(\beta\gamma\nu)}=\frac{1}{2}J^{(\beta\gamma\nu)}. \qquad (C7)$$

$\partial_\gamma E^{(\beta\nu)\gamma}=0$ implies

$$\Lambda^{\nu\mu}\equiv(\partial^\nu\lambda^\mu-\partial^\mu\lambda^\nu)$$
$$=\frac{4DM^2}{e^2}\epsilon^{\mu\nu\kappa\lambda}\partial_\theta T_{(\kappa\lambda)}{}^{\theta}$$
$$+\frac{8aM^2}{e^2}\partial_\theta T^{(\mu\nu)\theta}+\frac{8D^2M^2}{3ae^2}\partial_\theta T^{(\theta\mu\nu)}. \qquad (C8)$$

This equation contains $\partial_\theta T^{(\mu\nu)\theta}$ and its dual $\frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\partial_\theta T^{(\mu\nu)\theta}$. We now show that this relation leads to a solution of $\Lambda_{\theta\beta}$. Taking into account that the constraint (C1) gives $*\Lambda_{\theta\beta}=(8DM^2/3e^2)\partial_\theta T^{(\theta\mu\nu)}$ for the dual of $\Lambda_{\theta\beta}$, we can rewrite Eq. (C8) as

$$\Lambda^{\mu\nu}+\frac{D}{a}(*\Lambda^{\mu\nu})=-\frac{4DM^2}{e^2}\epsilon^{\mu\nu\kappa\lambda}\partial_\theta T^{\theta}_{(\kappa\lambda)}-\frac{8aM^2}{e^2}$$
$$\times\partial_\theta T^{(\mu\nu)\theta}. \qquad (C9)$$

The dual of the above equation together with the property$*(*\Lambda)=-\Lambda$ produce a second independent equation

$$(*\Lambda^{\mu\nu})-\frac{D}{a}\Lambda^{\mu\nu}=\frac{8DM^2}{e^2}\partial_\theta T^{(\mu\nu)\theta}-\frac{4aM^2}{e^2}\epsilon^{\mu\nu}_{\alpha\beta}\partial_\theta T^{(\alpha\beta)\theta}. \qquad (C10)$$

Solving the system we are left with

$$\Lambda^{\mu\nu}=-\frac{8M^2a}{e^2}\partial_\theta T^{(\mu\nu)\theta}, \qquad (C11)$$

$$*\Lambda^{\mu\nu}=-\frac{4M^2a}{e^2}\epsilon^{\mu\nu}_{\alpha\beta}\partial_\theta T^{(\alpha\beta)\theta}. \qquad (C12)$$

These expressions are consistent with the duality relationship. Equation (C11) directly gives

$$M^2\partial_\theta T^{(\mu\nu)\theta}=-\frac{e^2}{8a}\Lambda^{\mu\nu}. \qquad (C13)$$

Taking now the divergence of Eq. (C12) and comparing with Eq. (C5) yields

$$4M^2\epsilon^{\beta\nu\kappa\lambda}\partial_\sigma\partial_\beta T_{(\kappa\lambda)}{}^{\sigma}=0. \qquad (C14)$$

Contracting the free index of this last equation with a Levi-Civita tensor, together with Eq. (C7) implies

$$M^2T^{(\beta\gamma\nu)}=-\frac{1}{2}J^{(\beta\gamma\nu)}. \qquad (C15)$$

Taking the divergence of this equation respect to one of the antisymmetric indices we get

$$M^2\partial_\beta(T^{(\beta\gamma)\nu}-T^{(\beta\nu)\gamma})=-M^2\partial_\beta T^{(\gamma\nu)\beta}-\frac{1}{2}\partial_\beta J^{(\gamma\nu)\beta}. \qquad (C16)$$

Using Eq. (C15) in Eq. (C1) we obtain a relationship between the Lagrange multiplier λ_σ and the source:

$$\lambda^\beta=-\frac{2D}{9e^2}\epsilon^{\beta\kappa\tau\sigma}J_{(\kappa\tau\sigma)}. \qquad (C17)$$

Finally, Eqs. (C16) and (C13) imply

$$M^2\partial_\beta(T^{(\beta\gamma)\nu}-T^{(\beta\nu)\gamma})=\frac{e^2}{8a}\Lambda^{\gamma\nu}-\frac{1}{2}\partial_\alpha J^{(\gamma\nu)\alpha}. \qquad (C18)$$

From the above results the following independent Lagrangian constraints arise:

$$\lambda_\sigma = -\frac{2D}{9e^2}\epsilon_{\sigma\beta\gamma\nu}J^{(\beta\gamma\nu)}, \tag{C19}$$

$$M^2 T^{(\beta\gamma\nu)} = -\frac{1}{2}J^{(\beta\gamma\nu)}, \tag{C20}$$

$$M^2\partial_\theta T^{(\mu\nu)\theta} = -\frac{e^2}{8a}\Lambda^{\mu\nu}, \tag{C21}$$

$$M^2\partial_\beta T^{(\beta\gamma)\nu} = \frac{1}{16a}[e^2\Lambda^{\gamma\nu} - 4a\partial_\alpha J^{(\gamma\nu)\alpha}] + \frac{e^2}{8a}\bigg[g^{\gamma\nu}\partial_\rho\lambda^\rho - \frac{1}{2}(\partial^\nu\lambda^\gamma + \partial^\gamma\lambda^\nu) - 2\frac{DM^2}{e^2}\partial_\beta(\epsilon^{\beta\gamma\kappa\lambda}T_{(\kappa\lambda)}{}^\nu + \epsilon^{\beta\nu\kappa\lambda}T_{(\kappa\lambda)}{}^\gamma)\bigg], \tag{C22}$$

and the equation of motion is

$$2\partial_\alpha\partial^\alpha T^{(\beta\gamma)\nu} + \frac{D}{2a}\partial_\sigma[\partial^\beta(\epsilon^{\sigma\gamma\kappa\lambda}T_{(\kappa\lambda)}{}^\nu + \epsilon^{\sigma\nu\kappa\lambda}T_{(\kappa\lambda)}{}^\gamma) - \partial^\gamma(\epsilon^{\sigma\beta\kappa\lambda}T_{(\kappa\lambda)}{}^\nu + \epsilon^{\sigma\nu\kappa\lambda}T_{(\kappa\lambda)}{}^\beta)] + \frac{M^2}{e^2}\sqrt{a(e^2-a)}\epsilon^{\beta\gamma\kappa\lambda}T_{(\kappa\lambda)}{}^\nu$$
$$+ \frac{2aM^2}{e^2}T^{(\beta\gamma)\nu} = \frac{1}{2}J^{(\beta\gamma)\nu} + \frac{2a+e^2}{6e^2}J^{(\gamma\beta\nu)} + \frac{1}{6M^2}(2\partial_\alpha\partial^\alpha J^{(\gamma\beta\nu)} + 2\partial^\nu\partial_\alpha J^{(\beta\gamma)\alpha} + \partial^\beta\partial_\alpha J^{(\nu\gamma)\alpha} + \partial^\gamma\partial_\alpha J^{(\beta\nu)\alpha})$$
$$+ \frac{1}{4}(g^{\gamma\nu}\lambda^\beta - g^{\beta\nu}\lambda^\gamma) + \frac{e^2}{4M^2a}\partial^\beta\left(g^{\gamma\nu}\partial_\rho\lambda^\rho - \frac{1}{2}\partial^\nu\lambda^\gamma\right) - \frac{e^2}{4M^2a}(g^{\beta\nu}\partial^\gamma - g^{\gamma\nu}\partial^\beta)\partial_\rho\lambda^\rho$$
$$- \frac{e^2}{24aM^2}(8\partial^\nu\Lambda^{\beta\gamma} + \partial^\beta\Lambda^{\nu\gamma} + \partial^\gamma\Lambda^{\beta\nu}) - \frac{e^2}{8aM^2}\partial_\alpha(g^{\gamma\nu}\Lambda^{\alpha\beta} + g^{\beta\nu}\Lambda^{\gamma\alpha}). \tag{C23}$$

In the case $a=e^2$, we have

$$\lambda^\beta = 0, \quad M^2 T^{(\beta\gamma\nu)} = -\frac{1}{2}J^{(\beta\gamma\nu)}, \tag{C24}$$

$$M^2\partial_\theta T^{(\mu\nu)\theta} = 0, \quad M^2\partial_\beta T^{(\beta\gamma)\nu} = -\frac{1}{4}\partial_\alpha J^{(\gamma\nu)\alpha}, \tag{C25}$$

$$(\partial_\alpha\partial^\alpha + M^2)T^{(\beta\gamma)\nu} = \frac{1}{4}J^{(\beta\gamma)\nu} + \frac{1}{4}J^{(\gamma\beta\nu)} + \frac{1}{12M^2}(2\partial_\alpha\partial^\alpha J^{(\gamma\beta\nu)} + 2\partial^\nu\partial_\alpha J^{(\beta\gamma)\alpha} + \partial^\beta\partial_\alpha J^{(\nu\gamma)\alpha} + \partial^\gamma\partial_\alpha J^{(\beta\nu)\alpha}). \tag{C26}$$

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