Axial anomaly in $D=3+1$ light-cone QED

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We consider $(3+1)$-dimensional, Dirac electrons of arbitrary mass, propagating in the presence of electric and magnetic fields which are both parallel to the x^3 axis. The magnetic field is constant in space and time whereas the electric field depends arbitrarily upon the light-cone time parameter $x^+=(x^0+x^3)/\sqrt{2}$. We present an explicit solution to the Heisenberg equations for the electron field operator in this background. The electric field results in the creation of electron-positron pairs. We compute the expectation values of the vector and axial vector currents in the presence of a state which is free vacuum at $x^+=0$. Both current conservation and the standard result for the axial vector anomaly are verified for the first time ever in $(3+1)$-dimensional light-cone QED. An interesting feature of our operator solution is the fact that it depends in an essential way upon operators from the characteristic at $x^-=-L$, in addition to the usual dependence upon operators at $x^+=0$. This dependence survives even in the limit of infinite L. Ignoring the x^- operators leads to a progressive loss of unitarity, to the violation of current conservation, to the loss of renormalizability, and to an incorrect result for the axial vector anomaly.

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I. INTRODUCTION

It is customary in formulating a $(3+1)$-dimensional quantum field theory on the light cone to regard $x^+\equiv(x^0+x^3)/\sqrt{2}$ as the time coordinate. The complimentary null direction, $x^-\equiv(x^0-x^3)/\sqrt{2}$ is treated as a spatial coordinate, as are the transverse variables, $x^\perp=(x^1,x^2)$. In this view one is led to imagine that the Heisenberg field equations can be solved to express the operators at an arbitrary point $(x^+,x^-,x^\perp)$ in terms of the initial value operators on a surface of constant x^+.

However, it has been known for some time that solving the Klein-Gordon or Dirac equation on the light cone actually involves initial data on both characteristics [1]. In order to completely determine the operators in the wedge with $x^+>0$ and $x^->-L$ one must specify not only their values for $x^+=0$ with $x^-\geqslant -L$, but also for $x^-=-L$ with $x^+\geqslant 0$. This remains true even if L is taken to ∞ [2], although then the problem is segregated to the singularity at $p^+=0$. For free theories in trivial backgrounds, one can simply constrain this sector of the theory. Such a constraint is consistent because there is no mode mixing for these theories.

Interactions introduce mode mixing, and it is no longer obvious that the $p^+=0$ modes can be suppressed consistently. Nontrivial background fields can also result in mode mixing and recent results in this context seem to show conclusively that the $p^+=0$ modes cannot be ignored. We now have explicit and completely general solutions to the Heisenberg equations for Dirac electrons in the presence of an electric background field which points in the x^3 direction and is an arbitrary function of x^+ [3,4]. The homogeneous electric field results in e^+e^- pair production in an amazingly simple fashion. Each Fourier mode of fixed k^+ experiences pair production at the instant when its minimally coupled momentum, $p^+(x^+)\equiv k^+-eA_-(x^+)$, vanishes. At this instant the electron field operator suffers a drop in the amplitude proportional to the initial value data from the $x^+=0$ surface, with the missing amplitude being supplied by operators from the surface of constant x^-. Suppressing these other operators leads to a progressive loss of unitarity and to violation of current conservation. One also fails to produce the standard result for the axial vector anomaly in $1+1$ dimensions [4].

Although the first paper [3] applies to an arbitrary dimension, the operator solution was only valid in the limit $L\to\infty$. Since the limit could only be taken in the distributional sense, the solution was not sufficient to compute the expectation value of certain fermion bilinears. It is better to obtain a solution for arbitrary L, compute the expectation value of whatever operator is desired first, and *then* take the large L limit of the resulting $\mathbb{C}$-number. This was done in the second paper [4], but all the calculations were restricted to $1+1$ dimensions. In this paper we compute in $3+1$ dimensions. We have also extended the background to include a constant magnetic field which is co-linear with the electric field. This allows us to check the axial vector anomaly for the first time ever in $(3+1)$-dimensional light-cone QED.

This Introduction is the first of seven sections. Section II explains light-cone notation and gauge choices. It also presents our solution of the Dirac equation in the previously described background. Section III describes quantization and also explains how to work in the presence of a state which is empty on the initial value surface. In Sec. IV we calculate the probability of pair creation. Section V is devoted to computing the expectation values of the vector currents. In Sec. VI, we show that the expectation values of the axial vector currents J_5^+, J_5^-, and the pseudoscalar J_5 obey the Adler-Bell-Jackiw anomaly to all orders in the magnetic field. Section VII gives concluding remarks.

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II. THE MODEL AND ITS SOLUTION

The Lagrangian density for QED is

$$\mathcal{L}=\overline{\Psi}\gamma^{\mu}(i\partial_{\mu}-eA_{\mu}-m)\Psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1}$$

In four dimensions μ and ν run from 0 to 3. A_{μ} is the gauge potential, Ψ is the Dirac bispinor, and $F_{\mu\nu}\equiv\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the Maxwell field strength tensor. We employ the conventions of Bjorken and Drell [5], who give $\eta^{\mu\nu}$ timelike signature and $\{\gamma^{\mu},\gamma^{\nu}\}=2\eta^{\mu\nu}$.

The coordinates of light-cone quantum field theory are [6]

$$x^{\pm}\equiv\frac{1}{\sqrt{2}}(x^{0}\pm x^{3}),\quad x^{\perp}\equiv(x^{1},x^{2}). \tag{2}$$

Any vector can be expressed in this basis. For example, the inner product of two Lorentz vectors is

$$a^{\mu}b_{\mu}=a^{+}b^{-}+a^{-}b^{+}-a^{\perp}\cdot b^{\perp}. \tag{3}$$

From Eq. (3) we are able to extract the nonvanishing components of the light-cone metric as $\eta^{+-}=\eta^{-+}=-\eta^{11}=-\eta^{22}=1$. Therefore, raising and lowering are accomplished thusly: $a_{+}=a^{-}$, $a_{-}=a^{+}, a_{1}=-a^{1}, a_{2}=-a^{2}$. Further, the divergence of any 4-vector is $\partial_{\mu}V^{\mu}=\partial_{+}V^{+}+\partial_{-}V^{-}+\nabla_{\perp}\cdot V^{\perp}$.

Light-cone gamma matrices satisfy

$$(\gamma^{\pm})^{2}=0,\ \ \{\gamma^{+},\gamma^{-}\}=2,\ \ \{\gamma^{i},\gamma^{j}\}=-2\delta^{ij}. \tag{4}$$

Dirac spinors on the light-cone are decomposed by the projectors

$$P_{\pm}\equiv\frac{1}{2}\gamma^{\mp}\gamma^{\pm}=\frac{1}{2}(I\pm\gamma^{0}\gamma^{3}). \tag{5}$$

Acting these on the full bispinor gives its + and − components:

$$\psi_{\pm}=P_{\pm}\Psi,\qquad \Psi=\psi_{+}+\psi_{-}. \tag{6}$$

Our electric and magnetic backgrounds are $\vec{E}(x^{+},x^{-},x^{\perp})=E(x^{+})\hat{x}_{3}$ and $\vec{B}(x^{+},x^{-},x^{\perp})=B\hat{x}_{3}$, respectively. We fix the gauge with

$$A_{+}(x^{+},x^{-},x^{\perp})=0. \tag{7}$$

We can also impose the surface conditions

$$A_{-}(0,x^{-},x^{\perp})=0,$$
$$A_{1}(0,0,x^{\perp})=-A_{2}(0,0,x^{\perp}). \tag{8}$$

In this gauge the nonzero components of the vector potential are

$$A_{-}(x^{+})=-\int_{0}^{x^{+}}dyE(y),\quad A_{\perp}(x^{\perp})=\frac{B}{2}(x^{2}\hat{x}_{1}-x^{1}\hat{x}_{2}). \tag{9}$$

With these conventions the Dirac equation is

$$[i\gamma^{+}\partial_{+}+i\gamma^{-}(\partial_{-}+ieA_{-})+i\gamma^{\perp}\cdot\mathcal{D}_{\perp}-m]\Psi(x)=0, \tag{10}$$

where $\mathcal{D}_{\perp}\equiv\nabla_{\perp}+ieA_{\perp}$ is the transverse covariant derivative of QED. Alternately multiplying this equation by $\frac{1}{2}\gamma^{-}$ and $\frac{1}{2}\gamma^{+}$ gives two coupled equations involving the light-cone spinors:

$$i\partial_{+}\psi_{+}(x)=\tfrac{1}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\gamma^{-}\psi_{-}, \tag{11}$$

$$(i\partial_{-}-eA_{-})\psi_{-}(x)=\tfrac{1}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\gamma^{+}\psi+. \tag{12}$$

One solves this system by integrating Eq. (11) with respect to x^{+} and Eq. (12) with respect to x^{-},

$$\psi_{+}(x^{+},x^{-},x^{\perp})=\psi_{+}(0,x^{-},x^{\perp})-\frac{i}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\times\int_{0}^{x^{+}}du\,\gamma^{-}\psi_{-}(u,x^{-},x^{\perp}), \tag{13}$$

$$\psi_{-}(x^{+},x^{-},x^{\perp})=e^{-ieA_{-}(x^{+})(x^{-}+L)}\psi_{-}(x^{+},-L,x^{\perp})-\frac{i}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\int_{-L}^{x^{-}}dv\times e^{-ieA_{-}(x^{+})(x^{-}-v)}\gamma^{+}\psi_{+}(x^{+},v,x^{\perp}). \tag{14}$$

These equations implicitly express $\psi_{\pm}(x^{+},x^{-},x^{\perp})$ in terms of ψ_{+}, for $x^{+}=0$ and $x^{-}>-L$, and ψ_{-}, for $x^{+}>0$ and $x^{-}=-L$. To make the relation explicit we substitute Eq. (14) into Eq. (13) and iterate. The result is an infinite series:

$$\psi_{+}(x^{+},x^{-},x^{\perp})=\sum_{n=0}^{\infty}\left[-\frac{1}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})(m-i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\right]^{n}\int_{0}^{x^{+}}du_{1}\int_{-L}^{x^{-}}dv_{1}e^{-ieA_{-}(u_{1})(x^{-}-v_{1})}\times\int_{0}^{u_{1}}du_{2}\int_{-L}^{v_{1}}dv_{2}e^{-ieA_{-}(u_{2})(v_{1}-v_{2})}\cdots\int_{0}^{u_{n-1}}du_{n}\int_{-L}^{v_{n-1}}dv_{n}e^{-ieA_{-}(u_{n})(v_{n-1}-v_{n})}\times\left\{\psi_{+}(0,v_{n},x^{\perp})-\frac{i}{2}(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\int_{0}^{u_{n}}du\,e^{-ieA_{-}(u_{n})(v_{n}+L)}\gamma^{-}\psi_{-}(u,-L,x^{\perp})\right\}. \tag{15}$$

This series can be summed as in [4]. The result is

$$\psi_+(x^+,x^-,x^\perp)=\int_{-L}^{\infty}dv\int_{-\infty}^{+\infty}\frac{dk^+}{2\pi}e^{i(k^++i/L)(v-x^-)}\times\Big\{\mathcal{U}(x^\perp,\tau(0,x^+;k^+))\psi_+(0,v,x^\perp)-\frac{i}{2}(m+i\gamma^\perp\cdot\mathcal{D}_\perp)$$
$$\times\int_0^{x^+}du\,e^{-ieA_-(u)(v+L)}\mathcal{U}(x^\perp,\tau(u,x^+;k^+))\gamma^-\psi_-(u,-L,x^\perp)\Big\}. \tag{16}$$

The various hitherto undefined functions are

$$\mathcal{U}(x^\perp,\tau)\equiv e^{-i\mathcal{H}[eA_\perp(x_\perp)]\tau}, \tag{17}$$

$$\mathcal{H}[eA_\perp(x_\perp)]\equiv\tfrac{1}{2}(m+i\gamma^\perp\cdot\mathcal{D}_\perp)(m-i\gamma^\perp\cdot\mathcal{D}_\perp), \tag{18}$$

$$\tau(u,x^+;k^+)\equiv\int_u^{x^+}\frac{du'}{k^+-eA_-(u')+i/L}. \tag{19}$$

To shorten expressions in later sections we make the definitions

$$\tau_+\equiv\tau(0,x^+;k^+),\qquad \tau_-\equiv\tau(u,x^+;k^+) \tag{20}$$

$$\tau_+^*\equiv\tau^*(0,x^+;q^+),\qquad \tau_-^*\equiv\tau(y,x^+;q^+) \tag{21}$$

$$\tau_{++}\equiv\tau_+^*-\tau_+,\qquad \tau_{--}\equiv\tau_-^*-\tau_-. \tag{22}$$

At this stage our solution (16) is still valid for *any* $A_\perp(x^\perp)$; however, its dependence upon the initial value operators is complicated by the transverse covariant derivative operator. To exhibit this dependence we express $\mathcal{U}$ as a kernel:

$$\mathcal{U}(x^\perp,\tau)f(x^\perp)\equiv\int d^2y^\perp K(x^\perp,y^\perp;\tau)f(y^\perp). \tag{23}$$

One obtains the kernel by treating $\mathcal{H}[eA_\perp(x^\perp)]$ as a first quantized Hamiltonian. The spinor structure factors out through the reduction

$$\mathcal{H}[eA_\perp(x^\perp)]=\frac{1}{2}[m^2-\mathcal{D}_\perp\cdot\mathcal{D}_\perp]+\beta\Sigma^3, \tag{24}$$

where $\Sigma^3\equiv(i/2)[\gamma^1,\gamma^2]$ and $\beta\equiv|e|B/2=-eB/2$. For our linear $A_\perp(x^\perp)$, Eq. (9), the Hamiltonian is that of a rotated, 2-dimensional harmonic oscillator. Identifying its kernel is straightforward:

$$\mathcal{K}(x^\perp,y^\perp;\tau)\equiv e^{-i\beta\Sigma^3\tau}\mathcal{G}(x^\perp,y^\perp;\tau). \tag{25}$$

The function $\mathcal{G}(x^\perp,y^\perp;\tau)$ is

$$-\frac{i\beta e^{-(i/2)m^2\tau}}{2\pi\sin(\beta\tau)}\exp\left[\frac{i}{2}\beta\cot(\beta\tau)(x^\perp-y^\perp)^2-iex^\perp\cdot A_\perp(y^\perp)\right]. \tag{26}$$

We will often use its Fourier transform on $y^\perp$:

$$\tilde{\mathcal{G}}(x^\perp,k^\perp,s)=\frac{e^{-(i/2)m^2s}e^{-ik^\perp\cdot x^\perp}}{\cos(\beta s)}\times\exp\left[\frac{i}{2\beta}\tan(\beta s)(k^\perp-eA_\perp(x^\perp))^2\right]. \tag{27}$$

In terms of the kernel our solution for ψ_+ is

$$\psi_+(x^+,x^-,x^\perp)=\int_{-L}^{\infty}dv\int_{-\infty}^{+\infty}\frac{dk^+}{2\pi}e^{i(k^++i/L)(v-x^-)}\int d^2y^\perp\Big\{\mathcal{K}(x^\perp,y^\perp;\tau(0,x^+;k^+))\psi_+(0,v,y^\perp)-\frac{i}{2}(m+i\gamma^\perp\cdot\mathcal{D}_\perp)$$
$$\times\int_0^{x^+}du\,e^{-ieA_-(u)(v+L)}\mathcal{K}(x^\perp,y^\perp;\tau(u,x^+;k^+))\gamma^-\psi_-(u,-L,y^\perp)\Big\}. \tag{28}$$

The solution for ψ_- is obtained by inverting Eq. (11). That is,

$$\psi_-=i\gamma^+(m+i\gamma^\perp\cdot\mathcal{D}_\perp)^{-1}\partial_+\psi_+. \tag{29}$$

The inverse operator can be obtained by noting that $\mathcal{K}$ is the Green's function for a time dependent Schrödinger equation,

$$\left(\mathcal{H}-i\frac{\partial}{\partial s}\right)\mathcal{K}(x^\perp,y^\perp;s)=0. \tag{30}$$

Integrating Eq. (30) gives us the inverse operator,

$$(m+i\gamma^\perp\cdot\mathcal{D}_\perp)^{-1}\delta^2(x^\perp-y^\perp)=i\int_0^\infty ds(m-i\gamma^\perp\cdot\mathcal{D}_\perp(x_\perp))\mathcal{K}(x^\perp,y^\perp;s). \tag{31}$$

It is sometimes desirable to express dependence upon the transverse coordinates using the harmonic oscillator basis of $\mathcal{H}$. One begins by defining first quantized momenta:

$$p_x \equiv -i\frac{\partial}{\partial x}, \quad p_y \equiv -i\frac{\partial}{\partial y}. \tag{32}$$

With the position operators these are formed into lowering operators:

$$a_x \equiv \frac{1}{\sqrt{2\beta}}(\beta x + i p_x), \quad a_y \equiv \frac{1}{\sqrt{2\beta}}(\beta y + i p_y). \tag{33}$$

Finally, one defines complex raising and lowering operators:

$$a_\pm \equiv \frac{1}{\sqrt{2}}(a_x \pm i a_y), \quad a_\pm^\dagger \equiv \frac{1}{\sqrt{2}}(a_x^\dagger \mp i a_y^\dagger). \tag{34}$$

The original coordinates and derivatives have the following expressions:

$$x = \frac{1}{2\sqrt{\beta}}(a_+ + a_+^\dagger + a_- + a_-^\dagger),$$

$$y = \frac{i}{2\sqrt{\beta}}(-a_+ + a_+^\dagger + a_- - a_-^\dagger), \tag{35}$$

$$\partial_x = \frac{\sqrt{\beta}}{2}(a_+ - a_+^\dagger + a_- - a_-^\dagger),$$

$$\partial_y = i\frac{\sqrt{\beta}}{2}(-a_+ - a_+^\dagger + a_- + a_-^\dagger). \tag{36}$$

Hence the covariant derivative operators are

$$\mathcal{D}_x = \partial_x - i\beta y = \sqrt{\beta}(a_- - a_-^\dagger), \tag{37}$$

$$\mathcal{D}_y = \partial_y + i\beta x = \sqrt{\beta}(i a_- + i a_-^\dagger). \tag{38}$$

The point of this technology is to give a simple expression for the first quantized Hamiltonian,

$$\mathcal{H} = \frac{1}{2}m^2 + (2a_-^\dagger a_- + 1 + \Sigma^3)\beta. \tag{39}$$

Its normalized eigenstates are

$$W_{n_\pm}(x^\perp) \equiv \frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}\sqrt{\frac{\beta}{\pi}}e^{-(\beta/2)\|x^\perp\|^2}. \tag{40}$$

The fields can be expressed in this basis as follows:

$$\psi_+(x^+,x^-,n_\pm) \equiv \int d^2x^\perp W_{n_\pm}^*(x^\perp)\psi_+(x^+,x^-,x^\perp) \equiv \langle\langle W_{n_\pm}|\psi_+(x^+,x^-)\rangle\rangle. \tag{41}$$

Our solution (16) assumes the form

$$\psi_\pm(x^+,x^-,x^\perp) = \int_{-\infty}^{\infty}\frac{dk^+}{2\pi}e^{-i(k^+ + i/L)x^-} \times \sum_{n_\pm=0}^{\infty}\tilde{\psi}_\pm(x^+,k^+,n_\pm)W_{n_\pm}(x^\perp), \tag{42}$$

where we define

$$\tilde{\psi}_+(x^+,k^+,n_\pm) \equiv \int_{-L}^{\infty} dv\, e^{i(k^+ + i/L)v}\Big\{\langle\langle W_{n_\pm}|e^{-i\mathcal{H}\tau_+}|\psi_+(0,v)\rangle\rangle - \frac{i}{2}\int_0^{x^+} du\, e^{-ieA_-(u)(v+L)}\langle\langle W_{n_\pm}|(m + i\gamma^\perp\cdot\mathcal{D}_\perp)e^{-i\mathcal{H}\tau_-}\gamma^-|\psi_-(u,-L)\rangle\rangle\Big\}, \tag{43}$$

$$\tilde{\psi}_-(x^+,k^+,n_\pm) \equiv \int_{-L}^{\infty} dv\, e^{i(k^+ + i/L)v}\Big\{\Big\langle\Big\langle W_{n_\pm}\Big|\frac{i}{2\mathcal{H}}(m + i\gamma^\perp\cdot\mathcal{D}_\perp)\gamma^+ \times e^{-i\mathcal{H}\tau_+}\Big|\psi_+(0,v)\Big\rangle\Big\rangle + \int_0^{x^+} du\, e^{-ieA_-(u)(v+L)}\langle\langle W_{n_\pm}|e^{-i\mathcal{H}\tau_-}|\psi_-(u,-L)\rangle\rangle\Big\}. \tag{44}$$

Note that the inverse of $\mathcal{H}$ is straightforward to evaluate in the harmonic oscillator basis.

Taking the x^+ derivative of Eq. (43) gives

$$-i\partial_+\widetilde{\psi}_+(x^+,k^+,n_\pm) = -\frac{\frac{1}{2}m^2+(2n_-+1+\Sigma^3)\beta}{k^+-eA_-(x^+)+i/L}\widetilde{\psi}_+(x^+,k^+,n_\pm) - \frac{e^{-i(k^++i/L)L}}{k^+-eA_-(x^+)+i/L}\frac{i}{2}\langle\langle W_{n_\pm}|(m+i\gamma^\perp\cdot\mathcal{D}_\perp)\times\gamma^-|\psi_-(x^+,-L)\rangle\rangle. \quad (45)$$

The last term only contributes at $k^+=eA_-(x^+)$ in the large L limit since

$$\lim_{L\to\infty}\frac{e^{-i(k^+-eA_-(x^+)+i/L)L}}{k^+-eA_-(x^+)+i/L}=-2\pi i\,\delta(k^+-eA_-(x^+)). \quad (46)$$

We see that the large L limit of $\widetilde{\psi}_+(x^+,k^+,n_\pm)$ is an eigenoperator of $-i\partial_+$. Since the eigenvalues of Σ^3 are ±1 the sign of the eigenvalue is controlled by the denominator $k^+=eA_-(x^+)$. For $k^+>eA_-(x^+)$ the large L limit of $\widetilde{\psi}_+(x^+,k^+,n_\pm)$ annihilates electrons with spin $s=\frac{1}{2}\Sigma^3$; for $k^+<eA_-(x^+)$ it creates positrons with spin $s=-\frac{1}{2}\Sigma^3$.

III. LIGHT-CONE QUANTIZATION

The Lagrangian density for Dirac fermions in our background is

$$\mathcal{L}=\sqrt{2}\psi_+^\dagger\left(i\partial_+\psi_+-\frac{1}{2}(m+i\gamma^\perp\cdot\mathcal{D}_\perp)\gamma^-\psi_-\right)+\sqrt{2}\psi_-^\dagger\left((i\partial_--eA_-)\psi_- -\frac{1}{2}(m+i\gamma^\perp\cdot\mathcal{D}_\perp)\gamma^+\psi_+\right). \quad (47)$$

Using Eq. (47) we may read off the algebra that our operator solutions satisfy on the initial value surfaces. The conjugate momenta of these initial value fields, $\psi_+(0,v,x^\perp)$ and $\psi_-(u,-L,x^\perp)$, are the normal derivatives of Eq. (47) evaluated on the surfaces $x^+=0$ and $x^-=-L$, respectively. Therefore, the momentum conjugate to $\psi_+(0,v,x^\perp)$ is $i\sqrt{2}\psi_+^\dagger(0,v,x^\perp)$, and the corresponding conjugate momentum to $\psi_-^\dagger(u,-L,x^\perp)$ is $i\sqrt{2}\psi_-(u,-L,x^\perp)$. The two initial value surfaces are spacelike separated, and therefore the two nonzero anticommutators are

$$\{\psi_+(0,v,x^\perp),\psi_+^\dagger(0,w,y^\perp)\}=\frac{1}{\sqrt{2}}P_+\delta(v-w)\,\delta^2(x^\perp-y^\perp), \quad (48)$$

$$\{\psi_-(u,-L,x^\perp),\psi_-^\dagger(y,-L,y^\perp)\}=\frac{1}{\sqrt{2}}P_-\delta(u-y)\,\delta^2(x^\perp-y^\perp). \quad (49)$$

The anticommutation relations for arbitrary equal x^+ and equal x^- are not independent but follow from our solutions (28),(29),

$$\{\psi_+(x^+,x^-,x^\perp),\psi_+^\dagger(x^+,y^-,y^\perp)\}=\frac{1}{\sqrt{2}}P_+\delta(x^--y^-)\,\delta^2(x^\perp-y^\perp), \quad (50)$$

$$\{\psi_-(x^+,x^-,x^\perp),\psi_-^\dagger(y^+,x^-,y^\perp)\}=\frac{1}{\sqrt{2}}P_-\delta(x^+-y^+)\,\delta^2(x^\perp-y^\perp). \quad (51)$$

It remains to specify the Heisenberg state. For our purposes the natural "vacuum" $|\Omega\rangle$ is empty at $x^+=0$ and $x^-=-L$. This makes calculating expectation values of fermion bilinears straightforward. One first uses our solution to express the bilinear in terms of the initial value operators, and then computes the expectation value of these in the absence of the background fields using the standard free vacuum,

$$\langle\Omega|\psi_\alpha(x^+,x^-,x^\perp)\psi_\beta^\dagger(y^+,y^-,y^\perp)|\Omega\rangle_{A_\mu=0}=\int\frac{d^3p}{(2\pi)^3}\frac{(\not{p}\gamma^0+m\gamma^0)_{\alpha\beta}}{2\omega}\times e^{-ip^-(x^+-y^+)-ip^+(x^--y^-)+ip^\perp\cdot(x^\perp-y^\perp)}, \quad (52)$$

$$\langle\Omega|\psi_\beta^\dagger(y^+,y^-,y^\perp)\psi_\alpha(x^+,x^-,x^\perp)|\Omega\rangle_{A_\mu=0}=\int\frac{d^3p}{(2\pi)^3}\frac{(\not{p}\gamma^0-m\gamma^0)_{\alpha\beta}}{2\omega}\times e^{ip^-(x^+-y^+)+ip^+(x^--y^-)-ip^\perp\cdot(x^\perp-y^\perp)}. \quad (53)$$

The variable of integration above is p^i and we define $\omega\equiv\sqrt{m^2+\vec{p}\cdot\vec{p}}$.

In using Eqs. (52), (53) one first specializes to the desired initial value position and spinor component. Next change variables from p^3 to either p^+ or p^-,

$$\int_{-\infty}^{\infty}dp^3=\int_0^\infty dp^+\frac{\omega}{p^+}=\int_0^\infty dp^-\frac{\omega}{p^-}. \quad (54)$$

The complementary light-cone momentum is given by the mass shell condition, $2p^+p^-=m^2+p^\perp\cdot p^\perp$.

When the spinor indices are not explicitly written we shall understand expectation values of the form $\psi^\dagger M\psi$ to involve an implied spinor trace. Specializing to the initial value surfaces and taking $\pm$ components gives the various combinations of this form,

$$\langle\Omega|\psi_+^\dagger(0,w,y^\perp)\psi_+(0,v,x^\perp)|\Omega\rangle = \sqrt{2}\,\delta^2(x^\perp-y^\perp)\int_0^\infty\frac{dp^+}{2\pi}e^{ip^+(v-w)}, \tag{55}$$

$$\langle\Omega|\psi_+^\dagger(0,w,y^\perp)\gamma^-\psi_-(u,-L,x^\perp)|\Omega\rangle = -\sqrt{2}\int\frac{d^2p^\perp}{(2\pi)^2}e^{-ip^\perp\cdot(x^\perp-y^\perp)}\int_0^\infty\frac{dp^+}{2\pi}\frac{m}{p^+}\times e^{ip^-u-ip^+(w+L)}, \tag{56}$$

$$\langle\Omega|\psi_-^\dagger(y,-L,y^\perp)\gamma^+\psi_+(0,v,x^\perp)|\Omega\rangle = -\sqrt{2}\int\frac{d^2p^\perp}{(2\pi)^2}e^{-ip^\perp\cdot(x^\perp-y^\perp)}\times\int_0^\infty\frac{dp^+}{2\pi}\frac{m}{p^+}e^{-ip^-y+ip^+(v+L)}, \tag{57}$$

$$\langle\Omega|\psi_-^\dagger(y,-L,y^\perp)\psi_-(u,-L,x^\perp)|\Omega\rangle = \sqrt{2}\,\delta^2(x^\perp-y^\perp)\int_0^\infty\frac{dp^-}{2\pi}e^{ip^-(u-y)}, \tag{58}$$

$$= \sqrt{2}\,\delta^2(x^\perp-y^\perp)\frac{1}{2}\left\{\delta(u-y)+\frac{i}{\pi}\mathcal{P}\left(\frac{1}{u-y}\right)\right\}. \tag{59}$$

Notice that in the $L\to\infty$ limit Eqs. (56),(57) vanish. This means that the transverse coordinate dependence only contributes delta functions.

In addition to Eqs. (55)–(59), more complicated spinor traces will appear. It is convenient to list two of the operator reductions here to expedite derivations in later sections,

$$\langle\Omega|\psi_+^\dagger(0,w,y^{\perp\prime})e^{i\beta\Sigma^3\tau_{\pm\pm}}\psi_+(0,v,x^{\perp\prime})|\Omega\rangle = \sqrt{2}\,\delta^2(x^{\perp\prime}-y^{\perp\prime})\cos(\beta\tau_{\pm\pm})\int_0^\infty\frac{dp^+}{2\pi}e^{ip^+(v-w)}, \langle\Omega|\psi_-^\dagger(y,-L,y^{\perp\prime})$$

$$\times\gamma^+e^{i\beta\Sigma^3\tau_\pm^*}(m+i\gamma^\perp\cdot\mathcal{D}_\perp^*(x^\perp))\mathcal{G}^*(x^\perp,y^{\perp\prime};\tau_\pm^*)\times(m+i\gamma^\perp\cdot\mathcal{D}_\perp(x^\perp))$$

$$\times\mathcal{G}^*(x^\perp,x^{\perp\prime};\tau_\pm)e^{-i\beta\Sigma^3\tau_\pm}\gamma^-\psi_-(u,-L,x^{\perp\prime})|\Omega\rangle \tag{60}$$

$$= \sqrt{8}\,\delta^2(x^{\perp\prime}-y^{\perp\prime})\mathcal{G}^*(x^\perp,y^{\perp\prime};\tau_\pm^*)\{(m^2+\overleftarrow{\mathcal{D}}_\perp^*\cdot\vec{\mathcal{D}}_\perp)\cos(\beta\tau_{\pm\pm})$$

$$-\epsilon^{ij}\overleftarrow{\mathcal{D}}_{i\perp}^*\vec{\mathcal{D}}_{j\perp}\sin(\beta\tau_{\pm\pm})\}\mathcal{G}(x^\perp,x^{\perp\prime};\tau_\pm)\int_0^\infty\frac{dp^-}{2\pi}e^{ip^-(u-y)}, \tag{61}$$

where ϵ^{ij} is the antisymmetric Levi-Civita density in two dimensions with i and j running over the values 1 and 2.

IV. PAIR CREATION PROBABILITY

At the end of Sec. II we were able to identify an operator $\widetilde{\psi}_+(x^+,k^+,n_\pm)$ which gives exact eigenstates of the light-cone evolution operator $-i\partial_+$ in the large L limit. Its behavior changes abruptly at time $x^+=X(k^+)$, defined such that $k^+\equiv eA_-(X(k^+))$. For $x^+<X(k^+)$ the operator $\widetilde{\psi}_+(x^+,k^+,n_\pm)$ annihilates electrons of momentum k^+, Landau level n_- and spin $\frac{1}{2}\Sigma^3$. For $x^+>X(k^+)$ it creates positrons of momentum k^+, Landau level n_- and spin $-\frac{1}{2}\Sigma^3$.

The transition between these two regimes is a manifestation of particle creation, which is an instantaneous event on the light cone. Just as in the previous treatments [3,4], the newly created positron accelerates to the speed of light in the $+x^3$ direction, so its world line is asymptotically parallel to the x^+ axis. The electron goes the other way, so its world line is asymptotically parallel to the x^- axis. This has a curious effect when one regards x^+ as the evolution operator: electrons leave the light-cone manifold while the positrons accumulate.

In this section we compute the probability $\text{Prob}(k^+,n_-,s)$ for creating a positron of momentum k^+, Landau level n_- and spin s. From the previous section we see that the two nonzero spinor components of $\widetilde{\psi}_+(x^+,k^+,n_\pm)$ lack only a factor of $2^{1/4}$ to be canonically normalized. Therefore we can extract the creation probability [for $x^+>X(k^+)$] from the relation

$$\lim_{L\to\infty}\sqrt{2}\left\langle \Omega\left|\tilde{\psi}^{\dagger}_{+}(x^+,q^+,m_{\pm})\left(\frac{1}{2}-s\Sigma^3\right)\times\tilde{\psi}_{+}(x^+,k^+,n_{\pm})\right|\Omega\right\rangle=[1-\mathrm{Prob}(k^+,n_-,s)]2\pi\delta(k^+-q^+)\delta_{m_\pm,n_\pm}. \tag{62}$$

The procedure for evaluating Eq. (62) is to first express the operators in terms of the initial value operators using Eqs. (43), (44). For any bilinear this produces four kinds of operator products: the $++$ combination in which each term is from the $x^+=0$ surface; the $+-$ combination in which the first is from $x^+=0$ and the second from $x^-=-L$; and so on. We then compute the expectation values of each product from the free, sourceless theory as explained in the last section. Finally, the large L limit is taken. Since causality permits only the $++$ and $--$ products to survive this limit, we report only these terms.

The $++$ product is simple,

$$\sqrt{2}\left\langle \Omega\left|\tilde{\psi}^{\dagger}_{+}(x^+,q^+,m_{\pm})\left(\frac{1}{2}-s\Sigma^3\right)\tilde{\psi}_{+}(x^+,k^+,n_{\pm})\right|\Omega\right\rangle_{++}$$
$$=\int_{-L}^{+\infty}dv\,e^{i(k^++i/L)v}\int_{-L}^{+\infty}dw\,e^{-i(q^+-i/L)w}\left\langle\left\langle W_{n_\pm}\left|\mathrm{Tr}\left[e^{-i\mathcal{H}\tau_+}\left(\frac{1}{2}-s\Sigma^3\right)P_+\int_0^{\infty}\frac{dp^+}{2\pi}e^{ip^+(v-w)}e^{i\mathcal{H}\tau_+^*}\right]\right|W_{m_\pm}\right\rangle\right\rangle, \tag{63}$$

$$=\int_0^{\infty}\frac{dp^+}{2\pi}\,\frac{e^{-i(k^++p^++i/L)L}}{k^++p^++i/L}\,\frac{e^{i(q^++p^+-i/L)L}}{q^++p^+-i/L}\,\delta_{m_\pm,n_\pm}e^{i\epsilon(n_-,s)\tau_{++}}, \tag{64}$$

where $\epsilon(n_-,s)\equiv\frac{1}{2}m^2+(2n_-+1-2s)\beta$. We are interested in the limit $L\to\infty$, in which case,

$$\frac{e^{-i(k^++p^++i/L)L}}{k^++p^++i/L}\,\frac{e^{i(q^++p^+-i/L)L}}{q^++p^+-i/L}\to2\pi\delta(k^++p^+)2\pi\delta(q^++p^+). \tag{65}$$

When $q^+=k^+$ the large L limit of expression (22) for τ_{++} becomes

$$\lim_{L\to\infty}\int_0^{x^+}\left(\frac{du}{k^+-eA_-(u)-i/L}-\frac{du}{k^+-eA_-(u)+i/L}\right)=2\pi iX'(k^+)\,\theta(eA_--k^+)\,\theta(k^+). \tag{66}$$

This vanishes in Eq. (64) because the delta functions and the range of p^+ conspire to make k^+ negative whereas $eA_-(x^+)$ is assumed positive:

$$\lim_{L\to\infty}\sqrt{2}\left\langle \Omega\left|\tilde{\psi}^{\dagger}_{+}(x^+,q^+,m_{\pm})\left(\frac{1}{2}-s\Sigma^3\right)\tilde{\psi}_{+}(x^+,k^+,n_{\pm})\right|\Omega\right\rangle_{++}=2\pi\delta(k^+-q^+)\,\theta(-k^+)\,\delta_{m_\pm,n_\pm}. \tag{67}$$

The $--$ term is a little more difficult:

$$\sqrt{2}\left\langle \Omega\left|\tilde{\psi}^{\dagger}_{+}(x^+,q^+,m_{\pm})\left(\frac{1}{2}-s\Sigma^3\right)\tilde{\psi}_{+}(x^+,k^+,n_{\pm})\right|\Omega\right\rangle_{--}$$
$$=\frac{1}{4}\int_{-L}^{+\infty}dv\,e^{i(k^++i/L)v}\int_0^{x^+}du\,e^{-ieA_-(u)(v+L)}\int_{-L}^{+\infty}dw\,e^{-i(q^+-i/L)w}\int_0^{x^+}dy\,e^{ieA_-(y)(w+L)}$$
$$\times\left\langle\left\langle W_{n_\pm}\left|\mathrm{Tr}\left[(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})e^{-i\mathcal{H}\tau_-}\left(\frac{1}{2}-s\Sigma^3\right)\gamma^-P_-\times\int_0^{\infty}\frac{dp^-}{2\pi}e^{-ip^-(u-y)}\gamma^+e^{i\mathcal{H}\tau_-^*}(m-i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\right]\right|W_{m_\pm}\right\rangle\right\rangle, \tag{68}$$

$$=\frac{1}{2}\int_0^{x^+}du\,\frac{e^{-i(k^++i/L)L}}{k^+-eA_-(u)+i/L}\int_0^{x^+}dy\,\frac{e^{i(q^+-i/L)L}}{q^+-eA_-(y)-i/L}\left\{\frac{1}{2}\delta(u-y)+\frac{i}{2\pi}P\left(\frac{1}{u-y}\right)\right\}\left\langle\left\langle W_{n_\pm}\left|\mathrm{Tr}\left[(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\right.\right.\right.\right.$$
$$\left.\left.\left.\left.\times P_+\left(\frac{1}{2}-s\Sigma^3\right)e^{i\mathcal{H}\tau_{--}}(m-i\gamma^{\perp}\cdot\mathcal{D}_{\perp})\right]\right|W_{m_\pm}\right\rangle\right\rangle. \tag{69}$$

We reduce the transverse structure using the identity

$$(m+i\gamma^{\perp}\cdot\mathcal{D}_{\perp})P_{+}\left(\frac{1}{2}-s\Sigma^{3}\right)e^{i\mathcal{H}\tau_{--}}(m-i\gamma^{\perp}\cdot\mathcal{D}_{\perp}) = P_{+}\left(\frac{1}{2}-s\Sigma^{3}\right)2\mathcal{H}e^{i\mathcal{H}\tau_{--}}. \tag{70}$$

This brings the $--$ term to the interesting form

$$\sqrt{2}\left\langle\Omega\left|\tilde{\psi}_{+}^{\dagger}(x^{+},q^{+},m_{\pm})\left(\frac{1}{2}-s\Sigma^{3}\right)\times\tilde{\psi}_{+}(x^{+},k^{+},n_{\pm})\right|\Omega\right\rangle_{--} = \int_{0}^{x^{+}}du\,\frac{e^{-i(k^{+}+i/L)L}}{k^{+}-eA_{-}(u)+i/L}\times\int_{0}^{x^{+}}dy\,\frac{e^{i(q^{+}-i/L)L}}{q^{+}-eA_{-}(y)-i/L}\times\left\{\frac{1}{2}\delta(u-y)+\frac{i}{2\pi}P\left(\frac{1}{u-y}\right)\right\}\times\delta_{m_{\pm},n_{\pm}}\epsilon(n_{-},s)e^{i\epsilon(n_{-},s)\tau_{--}}. \tag{71}$$

At this stage we observe that Eq. (71) is the same as the $1+1$ expression (4.8) of Ref. [4] with the trivial replacement,

$$\frac{1}{2}m^{2}\rightarrow\frac{1}{2}m^{2}+(2n_{-}+1-2s)\beta\equiv\epsilon(n_{-},s). \tag{72}$$

This means that the remaining analysis has already been done. We can read the final result from Ref. [4],

$$\lim_{L\to\infty}\sqrt{2}\left\langle\Omega\left|\tilde{\psi}_{+}^{\dagger}(x^{+},q^{+},m_{\pm})\left(\frac{1}{2}-s\Sigma^{3}\right)\times\tilde{\psi}_{+}(x^{+},k^{+},n_{\pm})\right|\Omega\right\rangle_{--} = 2\pi\delta(k^{+}-q^{+})\delta_{m_{\pm},n_{\pm}}\theta(k^{+})\theta(eA_{-}(x^{+})-k^{+})\times[1-e^{-2\pi\lambda(k^{+},n_{-},s)}], \tag{73}$$

where we define

$$\lambda(k^{+},n_{-},s)\equiv\frac{\epsilon(n_{-},s)}{|eE(X(k^{+}))|}. \tag{74}$$

Note that we could have used this same procedure to shorten the $++$ derivation as well. It is almost always the case that expressing the transverse coordinate dependence in the harmonic oscillator basis results in an expression which differs only by the replacement (72) from one already computed in Ref. [4] for the $(1+1)$-dimensional theory.

Combining Eqs. (67) and (73) gives

$$\lim_{L\to\infty}\sqrt{2}\left\langle\Omega\left|\tilde{\psi}_{+}^{\dagger}(x^{+},q^{+},m_{\pm})\left(\frac{1}{2}-s\Sigma^{3}\right)\times\tilde{\psi}_{+}(x^{+},k^{+},n_{\pm})\right|\Omega\right\rangle = 2\pi\delta(k^{+}-q^{+})\delta_{m_{\pm},n_{\pm}}\{\theta(-k^{+})+\theta(k^{+})\times\theta(eA_{-}(x^{+})-k^{+})[1-e^{-2\pi\lambda(k^{+},n_{-},s)}]\}. \tag{75}$$

The $\theta(-k^{+})$ term implies there is no particle creation for $k^{+}<0$. These modes start out as positron creation operators and they continue to have that meaning for $E(x^{+})>0$. Since the state was initially empty of these modes it remains so. The other term implies that positrons are created for $0<k^{+}<eA_{-}(x^{+})$ with probability,

$$\mathrm{Prob}(k^{+},n_{-},s)=e^{-2\pi\lambda(k^{+},n_{-},s)}. \tag{76}$$

Note that the spin dependence makes physical sense. It is more probable for a positron to be created with its spin aligned ($s=+\frac{1}{2}$) with the magnetic field than opposed ($s=-\frac{1}{2}$).

V. THE VECTOR CURRENTS

Our operator solutions (28), (29) enable us to calculate exactly the one-loop response to an external electromagnetic field. The light-cone currents are

$$J^{\pm}=\frac{e}{\sqrt{2}}(\psi_{\pm}^{\dagger}\psi_{\pm}-\mathrm{Tr}[\psi_{\pm}\psi_{\pm}^{\dagger}]). \tag{77}$$

As usual in quantum field theory, we must regulate these operators. We accomplish this by point splitting $J^{\pm}$ in $x^{\perp}$ and in $x^{\mp}$. To maintain gauge invariance we add a gauge string when needed,

$$J^{+}(x^{+};x^{-},y^{-};x^{\perp},y^{\perp})=\frac{e}{\sqrt{2}}e^{ieA_{-}(y^{-}-x^{-})+iey^{\perp}\cdot A_{\perp}(x^{\perp})}\times(\psi_{+}^{\dagger}(x^{+},x^{-},x^{\perp})\psi_{+}(x^{+},y^{-},y^{\perp})-\mathrm{Tr}[\psi_{+}(x^{+},y^{-},y^{\perp})\times\psi_{+}^{\dagger}(x^{+},x^{-},x^{\perp})]). \tag{78}$$

$$J^{-}(x^{+},y^{+};x^{-};x^{\perp},y^{\perp})=\frac{e}{\sqrt{2}}e^{iey^{\perp}\cdot A_{\perp}(x^{\perp})}(\psi_{-}^{\dagger}(x^{+},x^{-},x^{\perp})\times\psi_{-}(y^{+},x^{-},y^{\perp})-\mathrm{Tr}[\psi_{-}(y^{+},x^{-},y^{\perp})\times\psi_{-}^{\dagger}(x^{+},x^{-},x^{\perp})]). \tag{79}$$

Point splitting breaks Hermiticity. Therefore, our currents are the symmetric limits of Eqs. (78), (79),

$$J^{\pm}(x)=\lim_{y\to x}\frac{1}{2}(J^{\pm}(x;y)+J^{\pm}(y;x)). \tag{80}$$

Note that the expectation values of the transverse currents J^1,J^2 vanish. This can be seen by simple Dirac algebra. It corresponds physically to the zero average transverse current for a particle undergoing helical motion.

We begin with J^+ and compute the expectation values of the $++$ and $--$ terms as in the previous section. (As before, the $+-$ and $-+$ terms vanish in the large L limit.) The reductions are similar to those in the previous section, so we show the results (with $y^-=x^-+\Delta^-$) after performing the v and w integrations and taking the transverse expectation values,

$$\langle\Omega|J^+(x^+;x^-,y^-;x^\perp,y^\perp)|\Omega\rangle_{++}=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}\times\frac{e}{2}e^{ieA_-(x^+)\Delta^-}$$
$$\times\left\{\int_{-\infty}^{0}-\int_{0}^{\infty}\right\}\frac{dp^+}{2\pi}\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-p^++\dfrac{i}{L}}$$
$$\times\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{+i(q^+-i/L)(x^-+L)}}{q^+-p^+-\dfrac{i}{L}}e^{i\epsilon(n_-,s)\tau_{++}}, \tag{81}$$

$$\langle\Omega|J^+(x^+;x^-,y^-;x^\perp,y^\perp)|\Omega\rangle_{--}=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}\times\frac{e}{2}e^{ieA_-(x^+)\Delta^-}\int_0^{x^+}du\int_0^{x^+}$$
$$\times dy\left\{\int_{-\infty}^{0}-\int_{0}^{\infty}\right\}\frac{dp^-}{2\pi}e^{-ip^-(u-y)}\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}$$
$$\times\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-eA_-(u)+\dfrac{i}{L}}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{+i(q^+-i/L)(x^-+L)}}{q^+-eA_-(y)-\dfrac{i}{L}}\epsilon(n_-,s)e^{i\epsilon(n_-,s)\tau_{--}}. \tag{82}$$

Recall that τ_{++} and τ_{--} were defined in Eqs. (20)–(22) and that $\epsilon(n_-,s)\equiv\frac{1}{2}m^2+(2n_-+1-2s)\beta$.

Each of these results has the form of $\Sigma_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}$ times the corresponding $(1+1)$-dimensional result of [4] with the trivial replacement: $\frac{1}{2}m^2\to\epsilon(n_-,s)$. We can therefore read off the large L limits directly. That for the $++$ terms follows from equation (5.10) of [4]:

$$\lim_{L\to\infty}\langle\Omega|J^+(x^+;x^-,y^-;x^\perp,y^\perp)|\Omega\rangle_{++}=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}$$
$$\times\frac{e}{2}\left\{\frac{i}{\pi\Delta^-}-\int_0^{eA_-}\frac{dp^+}{2\pi}[1+e^{-2\pi\lambda(p^+,n_-,s)}]e^{-i(p^+-eA_-)\Delta^-}\right\}. \tag{83}$$

The large L limit of the $--$ terms derives from Eqs (5.11)–(5.13) of [4],

$$\lim_{L\to\infty}\langle\Omega|J^+(x^+;x^-,y^-;x^\perp,y^\perp)|\Omega\rangle_{--}=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}\frac{e}{2}\int_0^{eA_-}\frac{dp^+}{2\pi}[1-e^{-2\pi\lambda(p^+,n_-,s)}]e^{-i(p^+-eA_-)\Delta^-}. \tag{84}$$

Combining the $++$ and $--$ terms gives

$$\lim_{L\to\infty}\langle\Omega|J^+(x^+;x^-,y^-;x^\perp,y^\perp)|\Omega\rangle=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}$$
$$\times e\left\{\frac{i}{2\pi\Delta^-}-\int_0^{eA_-}\frac{dp^+}{2\pi}e^{-2\pi\lambda(p^+,n_-,s)}e^{-i(p^+-eA_-)\Delta^-}\right\}. \tag{85}$$

At this stage we can take $y^\perp \to x^\perp$. Hermitization discards the $1/\Delta^-$ term, at which point we can also take $\Delta^- \to 0$. The result is

$$\lim_{L\to\infty} \langle \Omega | J^+(x^+,x^-,x^\perp) | \Omega \rangle = -e \sum_{n_\pm,s} \| W_{n_\pm}(x^\perp) \|^2 \int_0^{eA_-} \frac{dp^+}{2\pi} e^{-2\pi\lambda(p^+,n_-,s)}. \tag{86}$$

This expression has a transparent physical interpretation based on the role of J^+ as the light-cone charge density. This charge density derives from the steady accumulation of positrons as the electron member of each newly created pair leaves the light-cone manifold. Hence the charge density is the sum over states of the $-e$ contributed by each positron, times the pair production probability (76) we derived in Sec. IV.

Since the expectation value of the current operators cannot depend upon the transverse coordinate we may as well set $x^\perp = 0$. The harmonic oscillator basis functions are especially simple at this point,

$$W_{n_\pm}(0) = (-)^{n_-} \delta_{n_-,n_+} \sqrt{\frac{\beta}{\pi}}. \tag{87}$$

Recalling that $\lambda(p^+,n_-,s) = \epsilon(n_-,s)/|eE(X(p^+))|$, we can perform the sums over $n_\pm$ and s,

$$\lim_{L\to\infty} \langle \Omega | J^+(x^+,x^-,x^\perp) | \Omega \rangle = -\frac{e\beta}{2\pi^2} \int_0^{eA_-} dp^+ e^{-\pi m^2/|eE|} [1 + e^{-4\pi\beta/|eE|}] \sum_{n=0}^{\infty} (e^{-4\pi\beta/|eE|})^n, \tag{88}$$

$$= \frac{e^2 B}{4\pi^2} \int_0^{eA_-} dp^+ e^{-\pi m^2/|eE|} \coth\left[\frac{\pi B}{E(X(p^+))} \right]. \tag{89}$$

Since $\beta \equiv -eB/2$ we see that the $B \to 0$ limit agrees with the result of [3]. The other new limit, that of large B, seems more interesting. In that case the hyperbolic cotangent goes to one, so J^+ grows linearly in the magnetic field. This might be phenomenologically relevant to astrophysics because very large, approximately homogeneous magnetic fields are known to occur. For example, the magnetic field strength in a neutron star can reach $B \sim 10^{13}$ G over a kilometer coherence length.

We turn now to J^-. After performing the v and w integrations and taking the transverse expectation values the $++$ and $--$ terms assume the form

$$\langle \Omega | J^-(x^+,y^+;x^-;x^\perp,y^\perp) | \Omega \rangle_{++} = \sum_{n_\pm,s} W^*_{n_\pm}(x^\perp) W_{n_\pm}(y^\perp) e^{iey^\perp \cdot A_\perp(x^\perp)} \frac{e}{2} \frac{\partial}{\partial x^+} \frac{\partial}{\partial y^+} \left\{ \int_{-\infty}^0 - \int_0^\infty \right\} \frac{dp^+}{2\pi}$$
$$\times \int_{-\infty}^\infty \frac{dk^+}{2\pi} \frac{e^{-i(k^+ + i/L)(x^- + L)}}{k^+ - p^+ + \frac{i}{L}} \int_{-\infty}^\infty \frac{dq^+}{2\pi} \frac{e^{+i(q^+ - i/L)(x^- + L)}}{q^+ - p^+ - \frac{i}{L}} \frac{e^{i\epsilon(n_-,s)\sigma_+}}{\epsilon(n_-,s)}, \tag{90}$$

$$\langle \Omega | J^-(x^+,y^+;x^-;x^\perp,y^\perp) | \Omega \rangle_{--} = \sum_{n_\pm,s} W^*_{n_\pm}(x^\perp) W_{n_\pm}(y^\perp) e^{iey^\perp \cdot A_\perp(x^\perp)} \frac{e}{2} \frac{\partial}{\partial x^+} \frac{\partial}{\partial y^+} \int_0^{y^+} du \int_0^{x^+} dy \left\{ \int_{-\infty}^0 - \int_0^\infty \right\}$$
$$\times \frac{dp^-}{2\pi} e^{-ip^-(u-y)} \int_{-\infty}^\infty \frac{dk^+}{2\pi} \frac{e^{-i(k^+ + i/L)(x^- + L)}}{k^+ - eA_-(u) + \frac{i}{L}}$$
$$\times \int_{-\infty}^\infty \frac{dq^+}{2\pi} \frac{e^{+i(q^+ - i/L)(x^- + L)}}{q^+ - eA_-(y) - \frac{i}{L}} e^{i\epsilon(n_-,s)\sigma_-}. \tag{91}$$

The quantities $\sigma_\pm$ are just τ_{++} and τ_{--} with the upper limits of the second integral in each changed from x^+ to y^+,

$$\sigma_+ \equiv \int_0^{x^+} \frac{du'}{q^+ - eA_-(u') - \frac{i}{L}} - \int_0^{y^+} \frac{du'}{k^+ - eA_-(u') + \frac{i}{L}}, \tag{92}$$

$$\sigma_- \equiv \int_y^{x^+} \frac{du'}{q^+ - eA_-(u') - \frac{i}{L}} - \int_u^{y^+} \frac{du'}{k^+ - eA_-(u') + \frac{i}{L}}. \tag{93}$$

The x^- derivative of J^- is ultraviolet finite so the oscillator sums again multiply the same $(1+1)$ dimensional currents whose large L limits were already computed in [4]. For example, the large L limit of the $++$ terms follows from equations (5.23), (5.24) of [4],

$$\lim_{L\to\infty}\partial_-\langle\Omega|J^-(x^+,x^+;x^-;x^\perp,y^\perp)|\Omega\rangle_{++}$$
$$=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}\frac{e^2E(x^+)}{4\pi}\times[1-e^{-2\pi\lambda(p^+,n_-,s)}]. \quad (94)$$

The large L limit of the $--$ terms derives from equations (5.25)–(5.28) of [4],

$$\lim_{L\to\infty}\partial_-\langle\Omega|J^-(x^+,x^+;x^-;x^\perp,y^\perp)|\Omega\rangle_{--}$$
$$=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{iey^\perp\cdot A_\perp(x^\perp)}\frac{e^2E(x^+)}{4\pi}\times[-1-e^{-2\pi\lambda(p^+,n_-,s)}]. \quad (95)$$

Once the two terms are combined we can take the transverse coordinates to coincidence and again exploit transverse translational invariance to perform the sums over $n_\pm$ and s at $x^\perp=0$,

$$\lim_{L\to\infty}\partial_-\langle\Omega|J^-(x^+,x^-,x^\perp)|\Omega\rangle=-\frac{e^2E(x^+)}{2\pi}\sum_{n_\pm,s}\|W_{n_\pm}\|^2e^{-2\pi\lambda(eA_-,n_-,s)}, \quad (96)$$

$$=\frac{e^3E(x^+)B}{4\pi^2}e^{-\pi m^2/|eE|}\coth\left[\frac{\pi B}{E(x^+)}\right]. \quad (97)$$

Comparison with Eq. (89) verifies current conservation.

We can obtain the undifferentiated current J^- by integrating with respect to x^-, just as in $1+1$ dimensions [4]. However, the $(3+1)$-dimensional integration constant must be treated with care. Although our choice of state makes the expectation value of $J^-(x^+,-L,x^\perp)$ vanish, moving even infinitesimally to the left of $x^-=-L$ results in an ultraviolet divergence. Of course this is the one loop photon field strength renormalization. To extract it we fix one of the fields at $x^-=-L$ and take the other just inside. Since there is no $++$ term, and the mixed terms always vanish for large L, we compute only the $--$ contribution,

$$\frac{e}{\sqrt{2}}e^{ieA_-(x^+)\Delta^-+ieA_\perp(x^\perp)\cdot\Delta^\perp}\langle\Omega|\{\psi^\dagger_-(x^+,-L,x^\perp)\psi_-(x^+,\Delta^--L,x^\perp+\Delta^\perp)-\mathrm{Tr}[\psi_-(x^+,\Delta^--L,x^\perp+\Delta^\perp)\times\psi^\dagger_-(x^+,-L,x^\perp)]\}|\Omega\rangle_{--}$$
$$=\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(x^\perp+\Delta^\perp)e^{ieA_\perp(x^\perp)\cdot\Delta^\perp}\frac{e}{4\pi}\int_{-\infty}^{\infty}dk^+\frac{e^{-i(k^+-eA_-(x^+)+i/L)\Delta^-}}{k^+-eA_-(x^+)+\frac{i}{L}}\times\int_0^{x^+}\frac{du}{k^+-eA_-(u)+\frac{i}{L}}\left\{\int_{-\infty}^0-\int_0^\infty\right\}\frac{dp^-}{2\pi}e^{ip^-(x^+-u)}\epsilon(n_-,s)e^{-i\epsilon(n_-,s)\tau_-}. \quad (98)$$

Because of its significance to this analysis we remind the reader of the function $\tau_-=\tau(u,x^+;k^+)$ from Eq. (20),

$$\tau(u,x^+;k^+)=\int_u^{x^+}\frac{du'}{k^+-eA_-(u')+\frac{i}{L}}. \quad (99)$$

It will be important to note that $\tau(u,x^+;k^+)$ has a negative imaginary part.

The next step is to perform the oscillator and spin sums using the relation

$$\sum_{n_\pm,s}W^*_{n_\pm}(x^\perp)W_{n_\pm}(y^\perp)e^{ieA_\perp(x^\perp)\cdot\Delta^\perp}\epsilon(n_-,s)e^{-i\epsilon(n_-,s)\tau_-}=i\frac{\partial}{\partial\tau_-}\left\{-\frac{i\beta}{\pi}e^{-(i/2)m^2\tau_-}\cot(\beta\tau_-)\times e^{(i/2)\beta\cot(\beta\tau_-)\|\Delta^\perp\|^2}\right\}. \quad (100)$$

Since $x^+\geqslant u$ the p^- integral gives

$$\left\{\int_{-\infty}^0-\int_0^\infty\right\}\frac{dp^-}{2\pi}e^{ip^-(x^+-u)}=-\frac{i}{2\pi}\frac{1}{x^+-u}. \quad (101)$$

Now change variables from u to $\tau(u,x^+;k^+)$ by recognizing the complex differential,

$$d\tau = \frac{\partial\tau}{\partial u}du = \frac{-du}{k^+ - eA_-(u) + \frac{i}{L}}. \tag{102}$$

Since $\tau(0,x^+;k^+) \equiv \tau_+$ and $\tau(x^+,x^+;k^+)=0$, expression (98) takes the form

$$-\frac{ie}{8\pi^3}\int_{-\infty}^{\infty}dk^+ \frac{e^{-i[k^+ - eA_-(x^+) + i/L]\Delta^-}}{k^+ - eA_-(x^+) + \frac{i}{L}}\int_0^{\tau_+}\frac{d\tau}{x^+ - u}$$
$$\times\frac{\partial}{\partial\tau}\{\beta\cot(\beta\tau)e^{-(i/2)m^2\tau + (i/2)\beta\cot(\beta\tau)\|\Delta^\perp\|^2}\}. \tag{103}$$

Note that the negative imaginary part of τ makes the integrand exponentially suppressed as $\tau\to 0$ as long as $\|\Delta^\perp\|^2 \neq 0$.

We must next express $1/(x^+ - u)$ in terms of τ. First expand $\tau(u,x^+;k^+)$ for small $\Delta u \equiv x^+ - u$,

$$\tau(u,x^+;k^+) = \frac{\Delta u}{k^+ - eA_-(x^+) + \frac{i}{L}} + \frac{\frac{1}{2}eA'_-(x^+)\Delta u^2}{\left[k^+ - eA_-(x^+) + \frac{i}{L}\right]^2} + \frac{\frac{1}{6}eA''_-(x^+)\Delta u^3}{\left[k^+ - eA_-(x^+) + \frac{i}{L}\right]^2} + \frac{\frac{1}{3}[eA'_-(x^+)]^2\Delta u^3}{\left[k^+ - eA_-(x^+) + \frac{i}{L}\right]^3} + O(\Delta u^4). \tag{104}$$

Since all the vector potentials are evaluated at x^+ we can suppress their arguments in subsequent expressions. We also define the complex parameter $K \equiv k^+ - eA_- + i/L$. Solving perturbatively for $1/\Delta u$ gives

$$\frac{1}{\Delta u} = \frac{1}{K\tau}\left\{1 + \frac{1}{2}eA'_-\tau + \frac{1}{6}eA''_-K\tau^2 + \frac{1}{12}(eA'_-\tau)^2 + O(\tau^3)\right\}. \tag{105}$$

Substituting this result and integrating by parts brings Eq. (98) to the form

$$-\frac{ie}{8\pi^3}\int_{-\infty}^{\infty}dk^+\frac{e^{-iK\Delta^-}}{K}\left\{\frac{\beta}{x^+}\cot(\beta\tau_+)\right.$$
$$\times e^{-(i/2)m^2\tau_+ + (i/2)\beta\cot(\beta\tau_+)\|\Delta^\perp\|^2} + \int_0^{\tau_+}d\tau\frac{\beta}{K\tau^2}\cot(\beta\tau)$$
$$\times e^{-(i/2)m^2\tau + (i/2)\beta\cot(\beta\tau)\|\Delta^\perp\|^2}\left[1 - \frac{1}{6}eA''_-K\tau^2\right.$$
$$\left.\left. - \frac{1}{12}(eA'_-\tau)^2 + O(\tau^3)\right]\right\}. \tag{106}$$

Note that the surface term is obviously finite in the unregulated limit.

The ultraviolet divergence derives from the integration over small τ. From the expansion $\cot(x) = 1/x - \frac{1}{3}x - \frac{1}{45}x^3 + \cdots$ we infer

$$\frac{\beta}{\tau^2}\cot(\beta\tau)e^{-(i/2)m^2\tau + (i/2)\beta\cot(\beta\tau)\|\Delta^\perp\|^2}$$
$$= e^{(i/2\tau)\|\Delta^\perp\|^2}\left\{\frac{1}{\tau^3} - \frac{i}{2}\left[m^2 + \frac{1}{3}\beta^2\|\Delta^\perp\|^2\right]\frac{1}{\tau^2} - \left[\frac{1}{8}m^4 + \frac{1}{12}m^2\beta^2\|\Delta^\perp\|^2 + \frac{1}{72}\beta^4\|\Delta^\perp\|^4 + \frac{1}{3}\beta^2\right]\frac{1}{\tau} + O(1)\right\}. \tag{107}$$

Only the following integrals can produce divergences:

$$\int_0^{\tau_+}\frac{d\tau}{\tau^3}e^{(i/2\tau)\Delta^2} = \left[-\frac{4}{\Delta^4} + \frac{2i}{\tau_+\Delta^2}\right]e^{(i/2\tau_+)\Delta^2} = -\frac{4}{\Delta^4} + O(1), \tag{108}$$

$$\int_0^{\tau_+}\frac{d\tau}{\tau^2}e^{(i/2\tau)\Delta^2} = \frac{2i}{\Delta^2}e^{(i/2\tau_+)\Delta^2} = \frac{2i}{\Delta^2} + O(1), \tag{109}$$

$$\int_0^{\tau_+}\frac{d\tau}{\tau}e^{(i/2\tau)\Delta^2} = -\mathrm{Ei}\left(\frac{i\Delta^2}{2\tau_+}\right) = -\ln(\Delta^2) + O(1). \tag{110}$$

We can therefore isolate the terms from Eq. (98) which diverge with the transverse point splitting $\Delta^2 \equiv \|\Delta^\perp\|^2$,

$$\frac{-ie}{8\pi^3}\int_{-\infty}^{\infty}dk^+\frac{e^{-iK\Delta^-}}{K^2}\left\{-\frac{4}{\Delta^4} + \frac{m^2}{\Delta^2} + \left[\frac{m^2}{8} + \frac{\beta^2}{3} + \frac{eA''_-K}{6} + \frac{(eA'_-)^2}{12}\right]\ln(\Delta^2) + O(1)\right\}. \tag{111}$$

It remains to perform the k^+ integration. Most of the transverse divergences are proportional to $1/K^2$ so they vanish with Δ^-,

$$\int_{-\infty}^{\infty} dk^+ \frac{e^{-iK\Delta^-}}{K^2} = -2\pi\Delta^-. \tag{112}$$

They are also purely imaginary and would vanish upon Hermitization. The single exception is the term proportional to eA''_-. The k^+ integral for it is

$$\int_{-\infty}^{\infty} dk^+ \frac{e^{-iK\Delta^-}}{K} = -2\pi i. \tag{113}$$

This gives a real term which survives when $\Delta^- \to 0$,

$$\lim_{\Delta^-\to 0} \frac{e}{\sqrt{2}} e^{ieA_-\Delta^- + ieA_\perp\cdot\Delta^\perp} \langle \Omega | \{ \psi_-^\dagger(x^+,-L,x^\perp)\psi_-(x^+,\Delta^- - L,x^\perp+\Delta^\perp) - \mathrm{Tr}[\psi_-(x^+,\Delta^- - L,x^\perp+\Delta^\perp) \times \psi_-^\dagger(x^+,-L,x^\perp)]\} | \Omega \rangle$$

$$= \frac{e^2 A''_-(x^+)}{24\pi^2} \ln(\|\Delta^\perp\|^{-2}) + \text{finite}, \tag{114}$$

$$= -\delta Z_3 \partial_\nu F^{\nu -} + \text{finite}. \tag{115}$$

So we have recovered the standard one loop result for the photon field strength renormalization [5]. This is another impressive check on the correctness and consistency of the formalism. As one might expect, the divergence can be isolated without taking the large L limit.

VI. THE AXIAL VECTOR ANOMALY

The vector currents we have just obtained give the exact one-loop response to our electromagnetic background. Since they are not entire functions of the electric field they could never be obtained in a perturbative expansion. It seems obvious that we can also access some of the nonperturbative structure of the axial vector currents. This is interesting because it allows one to check for nonperturbative corrections to the axial vector anomaly, just as what has already been done in $1+1$ dimensions [4].

The axial vector anomaly is the violation of the naive divergence equation,

$$\partial_\mu J_5^\mu - 2imJ_5 = 0. \tag{116}$$

The anomaly in electrodynamics results from the one loop triangle diagram containing two vector and one pseudovector vertices. Adler and Bardeen showed that this diagram receives no *perturbative* corrections [7]. However, the possibility for nonperturbative corrections remains open.

Modulo operator ordering and regularization, the axial vector current operator and its pseudoscalar partner are

$$J_5^{\pm} = \sqrt{2}\,\psi_\pm^\dagger \gamma_5 \psi_\pm, \tag{117}$$

$$J_5 = \frac{1}{\sqrt{2}}(\psi_+^\dagger \gamma^- \gamma_5 \psi_- + \psi_-^\dagger \gamma^+ \gamma_5 \psi_+). \tag{118}$$

The conventions of Sec. II imply $\gamma_5 \equiv \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We regulate the axial vector currents the same as we did the vector currents,

$$J_5^+(x^+;x^-,y^-,x^\perp) \equiv \frac{1}{\sqrt{2}} e^{ieA_-(x^+)\Delta_-}\{\psi_+^\dagger(x^+,y^-,x^\perp)\gamma_5\psi_+(x^+,x^-,x^\perp) - \mathrm{Tr}[\gamma_5\psi_+(x^+,x^-,x^\perp)\psi_+^\dagger(x^+,y^-,x^\perp)]\}. \tag{119}$$

$$J_5^-(x^+,y+;x^-,x^\perp) \equiv \frac{1}{\sqrt{2}}\{\psi_+^\dagger(y^+,x^-,x^\perp)\gamma_5\psi_+(x^+,x^-,x^\perp) - \mathrm{Tr}[\gamma_5\psi_+(x^+,x^-,x^\perp) \times \psi_+^\dagger(y^+,x^-,x^\perp)]\}. \tag{120}$$

The pseudoscalar is regulated by point splitting in both null directions,

$$J_5(x^+,y^+;x^-,y^-,x^\perp) \equiv \frac{1}{\sqrt{8}} \exp\left[ie(x^- - y^-)\int_0^1 d\eta A_-(y^+ + \eta(x^+ - y^+))\right]\{\psi_+^\dagger(y^+,y^-,x^\perp)\gamma^-\gamma_5\psi_-(x^+,x^-,x^\perp) + \psi_-^\dagger(y^+,y^-,x^\perp)\gamma^+\gamma_5\psi_+(x^+,x^-,x^\perp) - \mathrm{Tr}[\gamma^-\gamma_5\psi_-(x^+,x^-,x^\perp)\psi_+^\dagger(y^+,y^-,x^\perp)] - \mathrm{Tr}[\gamma^+\gamma_5\psi_+(x^+,x^-,x^\perp)\psi_-^\dagger(y^+,y^-,x^\perp)]\}. \tag{121}$$

We Hermitize these operators as we did for the vector current,

$$J_5^+(x^+,x^-,x^\perp) \equiv \lim_{y^-\to x^-} \frac{1}{2}\{J_5^+(x^+;x^-,y^-,x^\perp) + J_5^+(x^+;y^-,x^-,x^\perp)\}, \tag{122}$$

$$J_5^-(x^+,x^-x^\perp)\equiv\lim_{y^+\to x^+}\frac{1}{2}\{J_5^-(x^+,y^+;x^-,x^\perp)+J_5^-(y^+,x^+;x^-,x^\perp)\}. \tag{123}$$

As with the vector currents the subscripts $\pm\pm$ denote which of the four initial value products is being considered. Also as before, only the $++$ and $--$ products contribute to the large L limit. We begin with $\langle\Omega|J_5^+|\Omega\rangle$. The $++$ and $--$ expectation values are

$$\langle\Omega|J_5^+(x^+;x^-,y^-,x^\perp)|\Omega\rangle_{++}=\left(\int_{-\infty}^0-\int_0^\infty\right)\frac{dp^+}{2\pi}e^{-i[p^+-eA_-(x^+)]\Delta_-}\times\int_{-\infty}^\infty\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-p^++i/L}\int_{-\infty}^\infty\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-p^+-i/L}$$
$$\times\int d^2x^{\perp\prime}\mathcal{G}(x^\perp,x^{\perp\prime};\tau_+)\mathcal{G}^*(x^\perp,x^{\perp\prime};\tau_+^*)i\sin(\beta\tau_{++}), \tag{124}$$

$$\langle\Omega|J_5^+(x^+;x^-,y^-,x^\perp)|\Omega\rangle_{--}=\frac{i}{2}e^{ieA_-\Delta_-}\int_0^{x^+}du\int_0^{x^+}dy\,\frac{i}{\pi}\mathcal{P}\left(\frac{1}{u-y}\right)$$
$$\times\int_{-\infty}^\infty\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-eA_-(u)+i/L}\int_{-\infty}^\infty\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-eA_-(y)-i/L}$$
$$\times\int d^2x^{\perp\prime}\mathcal{G}^*(x^\perp,x^{\perp\prime};\tau_-^*)\{(m^2-\overleftarrow{\mathcal{D}}_\perp^*\cdot\vec{\mathcal{D}}_\perp)\sin(\beta\tau_{--})$$
$$-\epsilon^{ij}\overleftarrow{\mathcal{D}}_{i\perp}^*\vec{\mathcal{D}}_{j\perp}\cos(\beta\tau_{--})\}\mathcal{G}(x^\perp,x^{\perp\prime};\tau_-). \tag{125}$$

The presence of γ_5 has interchanged the sines and cosines from where they would have resided had we computed the analogous vector current in transverse coordinate space. This small change allows us to obtain the result to all orders without going to the harmonic oscillator basis. For example, the $++$ term is

$$\langle\Omega|J_5^+(x^+;x^-,y^-,x^\perp)|\Omega\rangle_{++}=\frac{eB}{4\pi}\left(\int_{-\infty}^0-\int_0^{+\infty}\right)\frac{dp^+}{2\pi}e^{-i[p^+-eA_-(x^+)]\Delta_-}$$
$$\times\int_{-\infty}^\infty\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-p^++i/L}\int_{-\infty}^\infty\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-p^+-i/L}e^{(i/2)m^2\tau_{++}},$$
$$\to\frac{eB}{4\pi}\left(\int_{-\infty}^0-\int_0^{+\infty}\right)\frac{dp^+}{2\pi}e^{-i[p^+-eA_-]\Delta_-}e^{-2\pi\lambda(p^+)\theta(p^+)\theta(eA_--p^+)} \tag{126}$$

$$=\frac{eB}{4\pi}\left\{\frac{i}{\pi\Delta_-}-\int_0^{eA_-}\frac{dp^+}{2\pi}[1+e^{-2\pi\lambda(p^+)}]e^{-i(p^+-eA_-)\Delta_-}\right\}, \tag{127}$$

where $\lambda(p^+)\equiv\lambda(p^+,0,\frac{1}{2})$, and $\lambda(p^+,n_-,s)$ was defined in Eq. (74).

The $--$ term can be greatly simplified by the identity

$$(\mathcal{D}_\perp^*\mathcal{G}^*(x^\perp,x^{\perp\prime};\tau_-^*))(\mathcal{D}_\perp\mathcal{G}(x^\perp,x^{\perp\prime};\tau_-))\sin(\beta\tau_{--})+\epsilon^{ij}(\mathcal{D}_i^*\mathcal{G}^*(x^\perp,x^{\perp\prime};\tau_-^*))(\mathcal{D}_j\mathcal{G}(x^\perp,x^{\perp\prime};\tau_-))\cos(\beta\tau_{--})=0. \tag{128}$$

Using this identity and taking the large L limit gives

$$\langle\Omega|J_5^+(x^+;x^-,y^-,x^\perp)|\Omega\rangle_{--}=\frac{im^2}{2}e^{ieA_-\Delta_-}\int_0^{x^+}du\int_0^{x^+}dy\,\mathcal{P}\left(\frac{1}{u-y}\right)$$
$$\times\int_{-\infty}^\infty\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(y^-+L)}}{k^+-eA_-(u)+i/L}\int_{-\infty}^\infty\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-eA_-(y)-i/L}$$
$$\times\int d^2x^{\perp\prime}\sin(\beta\tau_{--})\mathcal{G}(x^\perp,x^{\perp\prime};\tau_-)\mathcal{G}^*(x^\perp,x^{\perp\prime},\tau_-^*), \tag{129}$$

$$\to\frac{eB}{4\pi}\int_0^{eA_-}dp^+\lambda(p^+)e^{-i(p^+-eA_-)\Delta_-}e^{-2\pi\lambda(p^+)}\times\int_{-\infty}^{\infty}\frac{da}{2\pi}\frac{e^{-i(a+i)}}{a+i}e^{-i\lambda(p^+)\ln(a+i)}\int_{-\infty}^{\infty}\frac{db}{2\pi}\frac{e^{i(b-i)}}{b-i}e^{i\lambda(p^+)\ln(b-i)},\tag{130}$$

$$=\frac{eB}{4\pi}\int_0^{eA_-}dp^+\lambda(p^+)e^{-i(p^+-eA_-(x^+))\Delta_-}\left[\frac{1-e^{-2\pi\lambda(p^+)}}{2\pi\lambda(p^+)}\right],\tag{131}$$

$$=\frac{eB}{8\pi^2}\int_0^{eA_-}dp^+[1-e^{-2\pi\lambda(p^+)}].\tag{132}$$

Combining Eqs. (127) and (132) and Hermitizing gives us the large L expression for J_5^+,

$$\lim_{L\to\infty}\langle\Omega|J_5^+(x^+,x^-,x^\perp)|\Omega\rangle=-\frac{eB}{4\pi^2}\int_0^{eA_-}dp^+e^{-2\pi\lambda(p^+)}.\tag{133}$$

J_5^- involves many of the same procedures. Beginning with the $++$ term, it has the following reduction:

$$\langle\Omega|J_5^-(x^+,y^+;x^-,x^\perp)|\Omega\rangle_{++}=-\frac{im^2}{2}\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(x^-+L)}}{k^+-eA_-(x^+)+i/L}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-eA_-(y^+)-i/L}\times\left(\int_{-\infty}^{0}-\int_0^{\infty}\right)\frac{dp^+}{2\pi}\frac{1}{k^+-p^++i/L}\frac{1}{q^+-p^+-i/L}\times\int d^2x^{\perp\prime}\sin(\beta\tau_{++})\mathcal{G}(x^\perp,x^{\perp\prime},\tau_+)\mathcal{G}^*(x^\perp,x^{\perp\prime},\tau_+^*),\tag{134}$$

$$=-\frac{eBm^2}{8\pi}\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(x^-+L)}}{k^+-eA_-(x^+)+i/L}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}}{q^+-eA_-(y^+)-i/L}\times\frac{1}{k^+-q^++2i/L}\left[-i+\frac{1}{\pi}\ln\left(\frac{k^++i/L}{q^+-i/L}\right)\right]e^{(i/2)m^2\tau_{++}}.\tag{135}$$

We again take the x^- derivative to complete the calculation, this time requiring the axial vector currents to vanish at $x^-=-L$.[1] Acting ∂_- on the $++$ term, taking the large L limit, and enforcing coincidence gives

$$\lim_{L\to\infty}\partial_-\langle\Omega|J_5^-(x^+,x^-,x^\perp)|\Omega\rangle_{++}=\frac{eBm^2}{8\pi}\int_{-\infty}^{\infty}\frac{da}{2\pi}\frac{e^{-i(a+i)}}{a+i}e^{i\lambda(eA_-)\ln(a+i)}\int_{-\infty}^{\infty}\frac{db}{2\pi}\frac{e^{i(b-i)}}{b-i}e^{-i\lambda(eA_-)\ln(b-i)},\tag{136}$$

$$=-\frac{e^2E(x^+)B}{8\pi^2}[1-e^{-2\pi\lambda(eA_-(x^+))}].\tag{137}$$

Integrating this last expression gives us the final result for the $++$ term

$$\langle\Omega|J_5^-(x^+,x^-,x^\perp)|\Omega\rangle_{++}\to-\frac{e^2E(x^+)B}{8\pi^2}(x^-+L)[1-e^{-2\pi\lambda(eA_-)}].\tag{138}$$

Note that this is not properly the infinite L limit, but rather the two leading terms—one of which diverges linearly in L.

We pass now to the $--$ term. Reducing the transverse coordinates gives

[1]That this is so can be seen in Eq. (135) from the fact that the k^+ and q^+ integrals can be closed above and below to avoid each's respective poles.

$$\langle\Omega|J_5^-(x^+,y^+;x^-,x^\perp)|\Omega\rangle_{--}=\frac{eB}{4\pi}\frac{\partial}{\partial x^+}\frac{\partial}{\partial y^+}\int_0^{x^+}du\int_0^{y^+}dy\mathcal{P}\left(\frac{1}{u-y}\right)$$
$$\times\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)(x^-+L)}}{k^+-eA_-(u)+i/L}\int_{-\infty}^{+\infty}\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)(x^-+L)}e^{(i/2)m^2\tau_{--}}}{q^+-eA_-(y)-i/L}. \tag{139}$$

This can be recognized as $B/2\pi$ times expression (5.19) in Ref. [4]. So we can read off the result of the subsequent reductions from expressions (5.25) and (5.28) of that paper,

$$\lim_{L\to\infty}\partial_-\langle\Omega|J_5^-(x^+,x^-,x^\perp)|\Omega\rangle_{--}=\frac{e^2E(x^+)B}{8\pi^2}[1+e^{-2\pi\lambda(eA_-)}]. \tag{140}$$

Integrating from $x^-=-L$ gives

$$\langle\Omega|J_5^-(x^+,x^-,x^\perp)|\Omega\rangle_{--}\to\frac{e^2E(x^+)B}{8\pi^2}(x^-+L)[1+e^{-2\pi\lambda(eA_-)}]. \tag{141}$$

Adding the $++$ terms (138) gives the final result for J_5^-,

$$\langle\Omega|J_5^-(x^+,x^-,x^\perp)|\Omega\rangle\to\frac{e^2E(x^+)B}{4\pi^2}(x^-+L)e^{-2\pi\lambda(p^+)}. \tag{142}$$

As was the case for the vector current, the only divergence in the axial vector currents resides in J_5^-. Before computing the pseudoscalar it is worth noting that in the massless limit the anomaly equation in 3+1 is simply

$$\partial_\mu J_5^\mu=\frac{\alpha}{4\pi}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu}=\frac{e^2EB}{2\pi^2}. \tag{143}$$

Whereas our axial currents contain factors that are completely *nonperturbative*, the limiting case satisfies Eq. (143),

$$\lim_{m\to0}[\partial_+J_5^++\partial_-J_5^-]=\lim_{m\to0}\frac{e^2E(x^+)B}{2\pi^2}e^{-2\pi\lambda(p^+)}=\frac{e^2E(x^+)B}{2\pi^2}. \tag{144}$$

Notice how Eq. (144) does not follow if the $--$ terms are suppressed.

The only thing left to compute is the pseudoscalar. We begin with the $++$ term,

$$\langle\Omega|J_5(x^+,y^+;x^-,y^-)|\Omega\rangle_{++}=-\frac{eBm}{8\pi}e^{ie(x^--y^-)\int_0^1 d\eta A_-(y^++\eta(x^+-y^+))}$$
$$\times\left(\int_{-\infty}^0-\int_0^\infty\right)\frac{dp^+}{2\pi}\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)x^-}}{k^+-p^++\frac{i}{L}}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)y^-}}{q^+-p^+-\frac{i}{L}}$$
$$\times\left(\frac{1}{k^+-eA_-(x^+)+\frac{i}{L}}-\frac{1}{q^+-eA_-(y^+)-\frac{i}{L}}\right)e^{(i/2)m^2[\tau^*(0,y^+;q^+)-\tau_+]}, \tag{145}$$

$$\to-\frac{ieB}{4\pi m}e^{ie(x^--y^-)A_-}\left(\int_{-\infty}^0-\int_0^\infty\right)\frac{dp^+}{2\pi}$$
$$\times\left(\frac{\partial}{\partial x^+}\right)\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\frac{e^{-i(k^++i/L)x^-}}{k^+-p^++\frac{i}{L}}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\frac{e^{i(q^+-i/L)y^-}}{q^+-p^+-\frac{i}{L}}e^{(i/2)m^2\tau_{++}}, \tag{146}$$

$$\to-\frac{ieB}{4\pi m}e^{-ieA_-\Delta_-}\partial_+\left(\int_{-\infty}^0-\int_0^\infty\right)\frac{dp^+}{2\pi}e^{ip^+\Delta_-}e^{-2\pi\lambda(p^+)\theta(p^+)\theta(eA_--p^+)}, \tag{147}$$

$$=\frac{ieB}{8\pi^2 m}e^{-ieA_-\Delta_-}\partial_+\left[\frac{i}{\Delta_-}+\frac{ie^{ieA_-\Delta_-}}{\Delta_-}+\int_0^{eA_-}dp^+e^{-2\pi\lambda(p^+)+ip^+\Delta_-}\right], \tag{148}$$

$$\to -\frac{ie^2A'_-(x^+)B}{8\pi^2 m}[1-e^{-2\pi\lambda(eA_-(x^+))}]. \tag{149}$$

In these reductions we sequentially took $y^+=x^+$, the large L limit, and then $y^-=x^-$. The final result is

$$\lim_{L\to\infty}\langle\Omega|J_5(x^+,x^-,x^\perp)|\Omega\rangle_{++}=\frac{ie^2E(x^+)B}{8\pi^2 m}[1-e^{-2\pi\lambda(eA_-(x^+))}]. \tag{150}$$

The $--$ term is perfectly regular at x^+ and x^- coincidence, so we can begin at coincidence,

$$\langle\Omega|J_5(x^+,x^+;x^-,y^-)|\Omega\rangle_{--}=-\frac{ieBm}{8\pi}\left(\frac{\partial}{\partial x^+}\right)\int_0^{x^+}du\int_0^{x^+}dy\,\frac{i}{\pi}\mathcal{P}\left(\frac{1}{u-y}\right)$$
$$\times\int_{-\infty}^{\infty}\frac{dk^+}{2\pi}\,\frac{e^{-i(k^++i/L)x^-}}{k^+-eA_-(u)+i/L}\int_{-\infty}^{\infty}\frac{dq^+}{2\pi}\,\frac{e^{i(q^+-i/L)y^-}e^{(i/2)m^2\tau_{--}}}{q^+-eA_-(y)-i/L}, \tag{151}$$

$$\to -\frac{ieB}{8\pi^2 m}\left(\frac{\partial}{\partial x^+}\right)\int_0^{eA_-(x^+)}dp^+[1-e^{-2\pi\lambda(p^+)}] \tag{152}$$

$$=-\frac{ie^2A'_-(x^+)B}{8\pi^2 m}[1-e^{-2\pi\lambda(eA_-(x^+))}]. \tag{153}$$

Combining Eqs. (150) and (153) gives J_5,

$$\lim_{L\to\infty}\langle\Omega|J_5(x^+,x^-,x^\perp)|\Omega\rangle$$
$$=\frac{ie^2E(x^+)B}{4\pi^2 m}[1-e^{-2\pi\lambda(eA_-(x^+))}]. \tag{154}$$

With our results for the axial vector current, our divergence equation becomes

$$\lim_{L\to\infty}\langle\Omega|\partial_+J_5^+ + \partial_-J_5^- - 2imJ_5|\Omega\rangle=\frac{e^2EB}{2\pi^2}. \tag{155}$$

So the axial vector anomaly equation is satisfied and there are no nonperturbative corrections.

VII. DISCUSSION

This paper had three basic purposes. The first of these was to compute the positron creation probability and the vector current expectation values using operator solutions (28),(29) which are exact for any L. This is important because one cannot properly take the large L limit—or any other limit—of an operator. The correct procedure is first to take the expectation value in the presence of some state and then take L to infinity in the resulting $\mathbb{C}$-number function.

As in previous treatments [3,4] pair creation in a homogeneous electric field is a discrete and instantaneous event. For momentum k^+ it occurs at the time $x^+=X(k^+)$ such that $k^+=eA_-(x^+)$. Electrons accelerate to the speed of light in the minus z direction and leave the light-cone manifold. In Sec. IV we obtained the following probability for the appearance of a positron of momentum k^+, Landau level n_- and spin s:

$$\mathrm{Prob}(k^+,n_-,s)=e^{-2\pi\lambda(k^+,n_-,s)}, \tag{156}$$

where we define

$$\lambda(k^+,n_-,s)\equiv\frac{\frac{1}{2}m^2+(2n_-+1-2s)\left|\frac{eB}{2}\right|}{|eE(X(k^+))|}. \tag{157}$$

It is reassuring that creation is more probable when the spin lines up with the magnetic field field ($s=+\frac{1}{2}$).

In Sec. V we obtained the following results for the nonzero currents:

$$\langle\Omega|J^+(x^+,x^-,x^\perp)|\Omega\rangle=\frac{e^2B}{4\pi^2}\int_0^{eA_-}dk^+e^{-\pi m^2/|eE}(X(k^+))$$
$$\times\coth\left[\frac{\pi B}{E(X(k^+))}\right], \tag{158}$$

$$\langle\Omega|J^-(x^+,x^-,x^\perp)|\Omega\rangle_{\rm ren} = \frac{e^3 BE(x^+)}{4\pi^2}(x^-+L)e^{-\pi m^2/|eE(x^+)|}\coth\left[\frac{\pi B}{E(x^+)}\right]. \tag{159}$$

We have removed the charge renormalization from J^-. Our results are conserved, and they correctly reduce to the currents of Ref. [3] when $B=0$. It may be that the extra magnetic field endows them with some phenomenological significance. Whereas it is very difficult to maintain large electric fields over long distances, there are many astrophysical sources which have large and quite extensive magnetic fields.

Our second objective was to check the axial vector anomaly in $(3+1)$-dimensional light-cone QED. Whereas an electric background suffices for checking the $(1+1)$-dimensional anomaly [4], increasing the dimensionality by 2 requires the addition of a colinear magnetic field. Although we chose this to be constant it seems feasible to consider more general backgrounds. For example, our solution (16) can be made valid for an x^+ dependent magnetic field $B(x^+)$ by the replacements

$$A_\perp(x^\perp)\rightarrow A_\perp(x^+,x^\perp) = \frac{B(x^+)}{2}(x^2\hat{x}_1 - x^1\hat{x}_2), \tag{160}$$

$$\mathcal{U}(x^\perp,\tau)\rightarrow \exp\left[-i\int_u^{x^+} du' \times \frac{\mathcal{H}[eA_\perp(u',x^\perp)]}{k^+ - eA_-(u') + i/L}\right]. \tag{161}$$

This background entails transverse electric and magnetic fields,

$$E^\perp = \frac{1}{\sqrt{8}}B'(x^+)(x^1\hat{x}^1 - x^2\hat{x}^2),$$

$$B^\perp = \frac{1}{\sqrt{8}}B'(x^+)(x^2\hat{x}^1 + x^1\hat{x}^2). \tag{162}$$

Although these make no contribution to the anomaly they do introduce an interesting breaking of translation invariance in the transverse directions.

Our final purpose was to catalog the various disasters which ensue when the operators at $x^-=-L$ are suppressed. One loses unitarity, current conservation and the axial vector anomaly. Not surprisingly, one also loses renormalizability. For example, when we point-split on both x^- and $x^\perp$ and then Hermitize, the $++$ part of the expectation value of J^+ is

$$-\frac{e}{2\pi}\sum_{n_\pm,s} W^*_{n_\pm}(x^\perp)W_{n_\pm}(x^\perp+\Delta^\perp) \times\int_0^{eA_-} dp^+[1+e^{-2\pi\lambda(p^+,n_-,s)}] \times\cos[(p^+-eA_-)\Delta^-]. \tag{163}$$

The first term in the square brackets diverges quadratically like $\|\Delta^\perp\|^{-2}$. Yet the only counterterm QED allows for the current vector J^μ is $\partial_\nu F^{\nu\mu}$, which is only nonzero for $\mu=-$ in our background.

What do these problems mean? There is a ‘‘folk theorem’’ to the effect that anything one can see by studying the free theory with a nontrivial background must occur as well, in some way or another, for the interacting theory in a trivial background. Of course the theory is fine if one includes the operators on the $x^-=-L$ surface, but then much of the simplicity of light-cone quantum field theory is sacrificed. The best thing would be if the effects of the extra operators could be subsumed into some simple extra interactions, at least for certain purposes. Quantifying the problem and deriving an appropriate fix are the subject of on-going research.

Two extensions of this work seem worth making. The first is to compute the one loop effective action with the addition of a static magnetic field. This can no doubt be accomplished using the same techniques which worked for the case of only an electric field [8]. It would be interesting to check whether the Schwinger form persists in this larger class of backgrounds.

The second extension is to re-compute the large L limits of the vector currents under the assumption that $A_-(x^+)$ obeys the Maxwell equation,

$$-A''_-(x^+) = \langle J^-\rangle. \tag{164}$$

Since the term on the right-hand side grows linearly with L, it is apparent that the back-reacted vector potential must do the same. Our work of Sec. V assumed that $A_-(x^+)$ is fixed as L goes to infinity.

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