Scaling properties of the perturbative Wilson loop in two-dimensional noncommutative Yang-Mills theory

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(Received 16 July 2002; published 18 October 2002)

Commutative Yang-Mills theories in $1+1$ dimensions exhibit an interesting interplay between geometrical properties and $U(N)$ gauge structures: in the exact expression of a Wilson loop with n windings a nontrivial scaling intertwines n and N. In the noncommutative case the interplay becomes tighter owing to the merging of space-time and "internal" symmetries in a larger gauge group $U(\infty)$. We perform an explicit perturbative calculation of such a loop up to $\mathcal{O}(g^6)$; rather surprisingly, we find that in the contribution from the crossed graphs (the genuine noncommutative terms) the scaling we mentioned occurs for large n and N in the limit of maximal noncommutativity $\theta=\infty$. We present arguments in favor of the persistence of such a scaling at any perturbative order and succeed in summing the related perturbative series.

DOI: 10.1103/PhysRevD.66.085012 PACS number(s): 11.15.Bt, 02.40.Gh

I. INTRODUCTION

One of the most interesting and intriguing features of noncommutative field theories is the merging of space-time and "internal" symmetries in a larger gauge group $U(\infty)$ [1,2]. Peculiar topological properties can find their place there and be conveniently described under the general frame provided by K theory [3].

On the other hand, some interplay occurs also when theories are defined on commutative spaces; in [4] it has been shown that in two space-time dimensions a nontrivial holonomy concerning the base manifold and the fiber $U(N)$ appears when considering a Wilson loop winding n times around a closed contour, leading to a peculiar scaling law intertwining the two integers n and N:

$$\mathcal{W}_n(\mathcal{A};N)=\mathcal{W}_N\left(\frac{n}{N}\mathcal{A};n\right), \tag{1}$$

$\mathcal{W}$ being the exact expression of the Wilson loop and $\mathcal{A}$ the enclosed area. When going around the loop the non-Abelian character of the gauge group is felt.

One may wonder whether similar relations are present in the noncommutative case and, in the affirmative, what they can teach us concerning the tighter merging occurring in such a situation.

Noncommutative field theories have been widely explored in recent years. Although their basic motivation relies, in our opinion, on their relation with string theories [5–7], they often exhibit curious new features and are therefore fascinating on their own [8,9].

The simplest way of turning ordinary theories into noncommutative ones is to replace the usual multiplication of fields in the Lagrangian with the Moyal $\star$ product. This product is constructed by means of a real antisymmetric matrix $\theta^{\mu\nu}$ which parametrizes the noncommutativity of Minkowski space-time:

$$[x^\mu,x^\nu]=i\,\theta^{\mu\nu}, \qquad \mu,\nu=0,\ldots,D-1. \tag{2}$$

The $\star$ product of two fields $\phi_1(x)$ and $\phi_2(x)$ can be defined by means of Weyl symbols

$$\phi_1\star\phi_2(x)=\int\frac{d^Dp\,d^Dq}{(2\pi)^{2D}}\exp\left[-\frac{i}{2}p_\mu\theta^{\mu\nu}q_\nu\right]\times\exp(ipx)\,\tilde{\phi}_1(p-q)\,\tilde{\phi}_2(q). \tag{3}$$

The resulting action obviously makes the theory nonlocal.

A particularly interesting situation occurs in $U(N)$ gauge theories defined in one-space, one-time dimensions (YM_{1+1}).

The classical Minkowski action reads

$$S=-\frac{1}{4}\int d^2x\,\mathrm{Tr}(F_{\mu\nu}\star F^{\mu\nu}) \tag{4}$$

where the field strength $F_{\mu\nu}$ is given by

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$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig(A_\mu\star A_\nu-A_\nu\star A_\mu) \tag{5}$$

and A_μ is a $N\times N$ Hermitian matrix.

The action in Eq. (4) is invariant under $U(N)$ noncommutative gauge transformations

$$\delta_\lambda A_\mu=\partial_\mu\lambda-ig(A_\mu\star\lambda-\lambda\star A_\mu). \tag{6}$$

We quantize the theory in the light-cone gauge $n^\mu A_\mu\equiv A_-=0$, the vector n_μ being light-like, $n^\mu\equiv(1/\sqrt{2})(1,-1)$. This gauge is particularly convenient since Faddeev-Popov ghosts decouple even in a noncommutative context [10], while the field tensor is linear in the field with only one nonvanishing component $F_{-+}=\partial_- A_+$.

In this gauge two *different* prescriptions are obtained for the vector propagator in momentum space: namely,

$$D_{++}=i[k_-^{-2}]_{PV} \tag{7}$$

and

$$D_{++}=i[k_-+i\epsilon k_+]^{-2}, \tag{8}$$

PV denoting the Cauchy principal value. The two expressions above are usually referred to in the literature as the 't Hooft [11] and Wu-Mandelstam-Leibbrandt (WML) [12,13] propagators. They correspond to two different ways of quantizing the theory, namely by means of a light-front or of an equal-time algebra [14–16], respectively and, obviously, coincide with the ones in the commutative case.

The WML propagator can be Wick-rotated, thereby allowing for an Euclidean treatment. A smooth continuation of the propagator to the Euclidean region is instead impossible when using the PV prescription.

In the commutative case, a perturbative calculation for a closed Wilson loop, computed with the 't Hooft propagator, coincides with the exact expression obtained on the basis of a purely geometrical procedure [16,17]

$$\mathcal{W}=\exp\left(-\frac{1}{2}g^2N\mathcal{A}\right). \tag{9}$$

The use instead of the WML propagator leads to a different, genuinely perturbative expression in which topological effects are disregarded [18,19]

$$\mathcal{W}_{WML}=\frac{1}{N}\exp\left(-\frac{1}{2}g^2\mathcal{A}\right)L_{N-1}^{(1)}(g^2\mathcal{A}), \tag{10}$$

$L_{N-1}^{(1)}$ being a Laguerre polynomial.

One can inquire to what extent these considerations can be generalized to a non-commutative $U(N)$ gauge theory, always remaining in $1+1$ dimensions. This was explored in Ref. [20] by performing a fourth order perturbative calculation of a closed Wilson loop.

In the noncommutative case the Wilson loop can be defined by means of the Moyal product as [21,22,1]

$$\mathcal{W}[C]=\frac{1}{N}\int\mathcal{D}Ae^{iS[A]}\int d^2x\,\mathrm{Tr}\,P_\star \times\exp\left(ig\int_C A_+(x+\xi(s))d\xi^+(s)\right), \tag{11}$$

where C is a closed contour in noncommutative space-time parametrized by $\xi(s)$, with $0\leqslant s\leqslant 1$, $\xi(0)=\xi(1)$ and $P_\star$ denotes noncommutative path ordering along $x(s)$ from left to right with respect to increasing s of $\star$ products of functions. Gauge invariance requires integration over coordinates, which is trivially realized when considering vacuum averages [23].

The perturbative expansion of $\mathcal{W}[C]$, expressed by Eq. (11), reads

$$\mathcal{W}[C]=\frac{1}{N}\sum_{n=0}^{\infty}(ig)^n\int_0^1 ds_1\dots\times\int_{s_{n-1}}^1 ds_n\dot{x}_-(s_1)\dots\dot{x}_-(s_n)\times\langle 0|\,\mathrm{Tr}\,\mathcal{T}[A_+(x(s_1))\star\dots\star A_+(x(s_n))]|0\rangle, \tag{12}$$

and it is shown to be an even power series in g, so that we can write

$$\mathcal{W}[C]=1+g^2\mathcal{W}_2+g^4\mathcal{W}_4+g^6\mathcal{W}_6+\cdots. \tag{13}$$

If we consider n windings around the loop, the result can be easily obtained by extending the interval $0\leqslant s\leqslant n$, $\xi(s)$ becoming a periodic function of s.

The main conclusion of [20] was that a perturbative Euclidean calculation with the WML prescription is feasible and leads to a regular result. We found indeed pure area dependence (we recall that invariance under area preserving diffeomorphisms holds also in a noncommutative context) and continuity in the limit of a vanishing noncommutative parameter. The limiting case of a large noncommutative parameter (maximal noncommutativity) is far from trivial: as a matter of fact the contribution from the nonplanar graph does not vanish in the large-θ limit at odds with the result in higher dimensions [8].

More dramatic is the situation when considering the 't Hooft's form of the free propagator. In the noncommutative case the presence of the Moyal phase produces singularities which cannot be cured [20]. As a consequence 't Hooft's context will not be further considered.

Another remarkable difference between 't Hooft's and WML formulations in commutative Yang-Mills theories was noticed in [4]. When considering n windings around the closed loop, a nontrivial holonomy concerning the base manifold and the fiber $[U(N)]$ [Eq. (1)] took place in the exact solution. The behavior of the WML solution was instead fairly trivial ($\mathcal{A}\to n^2\mathcal{A}$), as expected in a genuinely perturbative treatment. However, it is amazing to notice that expression (10) with n windings, when restricted to *planar* diagrams, becomes

$$\mathcal{W}_{WML}^{(pl)}=\sum_{m=0}^{\infty}\frac{(-g^2\mathcal{A}n^2N)^m}{m!(m+1)!}$$
$$=\frac{1}{\sqrt{g^2\mathcal{A}n^2N}}J_1(2\sqrt{g^2\mathcal{A}n^2N}). \tag{14}$$

Scaling (1) is recovered.

In the noncommutative case this issue acquires a much deeper interest thanks to the merging of space-time and ''internal'' symmetries in a large gauge group $U(\infty)$, or, better, in its largest completion $U_{cpt}(\mathcal{H})$ [2]. Also for the WML formulation we expect a nontrivial intertwining between n and N, which might help in clarifying some features of this merging. Actually this is the main motivation of the present research.

Lacking a complete solution, we limit ourselves to a perturbative context. A little thought is enough to be convinced that the function $\mathcal{W}_2$ in Eq. (13) is reproduced by the single-exchange diagram, which is exactly the same as in the commutative $U(N)$ theory. Actually all planar graphs contributions coincide with the corresponding ones of the commutative case [18], being independent of θ [see Eq. (14)]. Although they dominate for large N and n, they are a kind of ''constant'' background, which is uninteresting in this context. Therefore in the following we will concentrate ourselves in calculating and discussing the properties of nonplanar graphs $\mathcal{W}^{(cr)}$ in the WML (Euclidean) formulation.

The contributions $\mathcal{W}_4^{(cr)}$ and $\mathcal{W}_6^{(cr)}$ with n windings will be presented in detail. At $\theta=0$ the commutative result is recovered, together with its trivial perturbative scaling, the result being continuous (but probably not analytic there).

Surprisingly, at $\theta=\infty$ and at $\mathcal{O}(g^4)$, we recover the nontrivial scaling law (1) of the exact solution in the commutative case; however, for the sake of clarity, we stress that such a scaling is here realized in a quite different mathematical expression. At $\mathcal{O}(g^6)$ the scaling receives corrections, decreasing at large n; as a consequence we can say that it holds only at large θ and large n. We also realize that diagrams with a single crossing of propagators dominate, making possible the extension to higher perturbative orders. This evidence is partly based on a numerical evaluation of an integral occurring in the calculation of diagrams with a double crossing (see Appendix B).

We present arguments in favor of the persistence of such a scaling in the limits $(n,N,\theta)\to\infty$ at any perturbative order and eventually succeed in summing the related perturbative series.

As soon as we move away from the extreme values $\theta=0,\theta=\infty$, corrections appear which are likely to interpolate smoothly between small-θ and large-θ behaviors.

In Sec. II we present the $\mathcal{O}(g^4)$ calculation; the $\mathcal{O}(g^6)$ results are reported in Sec. III together with our conjecture concerning the leading terms at large n,N and θ at any perturbative order. The details of the calculations are deferred to the Appendices. Final considerations are discussed in the Conclusions.

II. THE FOURTH ORDER CALCULATION

We concentrate our attention on $\mathcal{W}_4^{(cr)}$ and resort to a Euclidean formulation, generalizing to n windings the results reported in [20].

By exploiting the invariance of $\mathcal{W}$ under area-preserving diffeomorphisms, which holds also in this noncommutative context, we consider the simple choice of a circular contour

$$x(s)\equiv(x_1(s),x_2(s))=r(\cos(2\pi s),\sin(2\pi s)). \tag{15}$$

Were it not for the presence of the Moyal phase, a tremendous simplification would occur between the factor in the measure $\dot{x}_-(s)\dot{x}_-(s')$ and the basic correlator $\langle A_+(s)A_+(s')\rangle$ [18]. The Moyal phase can be handled in an easier way if we perform a Fourier transform, namely if we work in the momentum space. The momenta are chosen to be Euclidean and the noncommutative parameter imaginary $\theta\to i\theta$. In this way all the phase factors do not change their character.

We use WML propagators in the Euclidean form $(k_1-ik_2)^{-2}$ and parametrize the vectors introducing polar variables in order to perform symmetric integrations [12,18]. Then we are led to the expression

$$\mathcal{W}_4^{(cr)}=r^4\int_0^n ds_1\int_{s_1}^n ds_2\int_{s_2}^n ds_3\int_{s_3}^n ds_4\int_0^\infty\frac{dp}{p}\frac{dq}{q}\int_0^{2\pi}d\psi d\chi\,\exp(-2i(\psi+\chi))\exp(2ip\sin\psi\sin\pi(s_1-s_3))$$
$$\times\exp(2iq\sin\chi\sin\pi(s_2-s_4))\exp\left(i\frac{\theta}{r^2}pq\sin[\psi-\chi+\pi(s_2+s_4-s_1-s_3)]\right)=\mathcal{A}^2F\left(\frac{\theta}{\mathcal{A}},n\right). \tag{16}$$

Integrating over ψ and p, we get, after a trivial rescaling

$$\mathcal{W}_4^{(cr)}=\pi r^4n^4\int[ds]_4\int_0^\infty\frac{dq}{q}\oint_{|z|=1}\frac{dz}{iz^3}e^{-q\sin[n\pi(s_4-s_2)](z-1/z)}\frac{1-\dfrac{\gamma}{z}e^{-in\pi\sigma}}{1-\gamma ze^{in\pi\sigma}}, \tag{17}$$

where $\sigma = s_1 + s_3 - s_2 - s_4$ and

$$\gamma = \frac{\theta q}{2r^2 \sin[n\pi(s_3 - s_1)]},$$

$$\int [ds]_4 = \int_0^1 ds_1 \int_{s_1}^1 ds_2 \int_{s_2}^1 ds_3 \int_{s_3}^1 ds_4 .$$

We can further integrate over z, obtaining a series of Bessel functions. Integration over q and resummation of the series [24] lead to

$$\mathcal{W}_4^{(cr)} = 2n^4 \mathcal{A}^2 \int [ds]_4 \left[\frac{1}{2} + \frac{2}{\beta^2} (\exp[i\beta \sin \alpha] - 1 - i\beta \sin \alpha) \right] = 2n^4 \mathcal{A}^2 \int [ds]_4 \left[\frac{1}{2} + \frac{2}{\beta^2} \sum_{m=2}^{\infty} \frac{(i\beta \sin \alpha)^m}{m!} \right], \tag{18}$$

where

$$\alpha = n\pi(s_1 + s_3 - s_2 - s_4),$$

$$\beta = \frac{4\mathcal{A}}{\pi\theta} \sin[n\pi(s_4 - s_2)] \sin[n\pi(s_3 - s_1)]. \tag{19}$$

It is an easy calculation to check that the function F is continuous (but probably not analytic) at $\theta = 0$ with $F(0) = n^4/24$, exactly corresponding to the value of the commutative case obtained with the WML propagator [18].

The first order correction in θ can also be singled out

$$\mathcal{W}_4^{(cr)} \simeq 2n^4 \mathcal{A}^2 \int [ds]_4 \left[\frac{1}{2} - \frac{2i}{\beta} \sin \alpha \right]. \tag{20}$$

The calculation is sketched in Appendix A and the result is

$$\mathcal{W}_4^{(cr)} \simeq \frac{n^4 \mathcal{A}^2}{24} + i\theta \frac{n^3 \mathcal{A}}{4}. \tag{21}$$

One might recover the trivial scaling $\mathcal{A} \to \mathcal{A} n^2$ provided $\theta \to \theta n$; however, this is ruled out by the large-θ behavior we are going to explore.

The large-θ behavior can be obtained starting from Eq. (18); the first terms in the expansion turn out to be

$$\mathcal{W}_4^{(cr)} = -\frac{n^2 \mathcal{A}^2}{8\pi^2} + i\frac{n^3 \mathcal{A}^3}{8\pi^2 \theta} + \frac{8n^4 \mathcal{A}^4}{3\pi^2 \theta^2} \left(\frac{1}{256} + \frac{175}{3072} \frac{1}{n^2 \pi^2} \right) + \mathcal{O}(\theta^{-3}). \tag{22}$$

We notice that the large-θ limit [first term in Eq. (22)] obeys the scaling (1), which, in the commutative case, was present in the exact solution for the gauge group $U(N)$. This scaling is different from the trivial one at $\theta = 0$.

III. THE SIXTH ORDER CALCULATION AND BEYOND

The motivation for exploring the sixth order is to see whether the scaling law we have found in the fourth order result at $\theta = \infty$ still persists in higher orders. In the affirmative case one would be strongly encouraged to resum the series in order to inquire about the persistence of such a scaling beyond a perturbation expansion. This, in turn, might have far-reaching consequences on the interpretation of the theory in the extreme noncommutative limit.

We organize the sixth order loop calculation according to the possible topologically different diagrams one can draw. If we order the six vertices on the circle from 1 to 6, we denote by $\mathcal{W}_{(ij)(kl)(mn)}$ the contribution of the graph corresponding to three propagators joining the vertices $(ij),(kl),(mn)$, respectively. Thus $\mathcal{W}_{(14)(25)(36)}$ corresponds to the maximally crossed diagram (i.e., the one in which all propagators cross); then we have three diagrams with double crossing, namely $\mathcal{W}_{(14)(26)(35)}$ $\mathcal{W}_{(13)(25)(46)}$, and $\mathcal{W}_{(15)(24)(36)}$. Finally we have six diagrams with a single crossing $\mathcal{W}_{(12)(35)(46)}$, $\mathcal{W}_{(16)(24)(35)}$, $\mathcal{W}_{(15)(23)(46)}$, $\mathcal{W}_{(15)(26)(34)}$, $\mathcal{W}_{(13)(26)(45)}$, and $\mathcal{W}_{(13)(24)(56)}$. Diagrams without any crossing are not interesting since they are not affected by the Moyal phase; they indeed coincide with the corresponding ones in the commutative case.

The diagrams with a single crossing can be fairly easily evaluated; surprisingly the most difficult diagrams are the ones with double crossing. All integrations can be performed analytically, except for a single one concerning the doubly crossed diagrams, which has been performed numerically.

The details of such a heavy calculation are described in Appendix B. Here we only report the starting point and the final results.

As an example of singly crossed diagram we consider $\mathcal{W}_{(16)(24)(35)}$

$$\mathcal{W}_{(16)(24)(35)} = -r^6 N n^6 \int [ds]_6 \int_0^{\infty} \frac{dp}{p} \frac{dq}{q} \frac{dk}{k} \int_0^{2\pi} d\phi d\chi d\psi \exp(-2i(\phi + \chi + \psi)) \exp(2ip \sin \phi \sin n\pi s_{16}^- + 2iq \sin \psi \sin n\pi s_{24}^- + 2ik \sin \chi \sin n\pi s_{35}^-) \exp\left(i\frac{\theta}{r^2} qk \sin[\psi - \chi + n\pi(s_{35}^+ - s_{24}^+)] \right), \tag{23}$$

where $s_{ij}^{\pm} = s_i \pm s_j$.

The doubly crossed diagram $\mathcal{W}_{(15)(24)(36)}$ leads to the expression

$$\mathcal{W}_{(15)(24)(36)} = -r^6 N n^6 \int [ds]_6 \int_0^\infty \frac{dp}{p}\frac{dq}{q}\frac{dk}{k}\int_0^{2\pi} d\phi d\chi d\psi \exp(-2i(\phi+\chi+\psi))\exp(2ip\sin\phi\sin n\pi s_{15}^-)$$
$$\times\exp(2iq\sin\psi\sin n\pi s_{24}^-)\times\exp(2ik\sin\chi\sin n\pi s_{36}^-)\exp\Bigg(i\frac{\theta}{r^2}(pk\sin[\phi-\chi+n\pi(s_{36}^+-s_{15}^+)]$$
$$+qk\sin[\psi-\chi+n\pi(s_{36}^+-s_{24}^+)])\Bigg). \tag{24}$$

Finally the maximally crossed diagram $\mathcal{W}_{(14)(25)(36)}$ reads

$$\mathcal{W}_{(14)(25)(36)} = -r^6 N n^6 \int [ds]_6 \int_0^\infty \frac{dp}{p}\frac{dq}{q}\frac{dk}{k}\int_0^{2\pi} d\phi d\chi d\psi \exp(-2i(\phi+\chi+\psi))\exp(2ip\sin\phi\sin n\pi s_{14}^-)$$
$$\times\exp(2iq\sin\psi\sin n\pi s_{25}^-)\times\exp(2ik\sin\chi\sin n\pi s_{36}^-)\exp\Bigg(i\frac{\theta}{r^2}(pq\sin[\phi-\psi+n\pi(s_{25}^+-s_{14}^+)]$$
$$+pk\sin[\phi-\chi+n\pi(s_{36}^+-s_{14}^+)]+qk\sin[\psi-\chi+n\pi(s_{36}^+-s_{25}^+)])\Bigg). \tag{25}$$

We notice that the $U(N)$ factor is the same in all three configurations.

The sum of the diagrams with a single crossing and n windings contributes at $\theta=\infty$ with the following expression:

$$\mathcal{W}^{(1)}(\theta=\infty) = \frac{\mathcal{A}^3 N n^4}{24\pi^2}\left(1-\frac{6}{n^2\pi^2}\right). \tag{26}$$

The maximally crossed diagram in turn leads to

$$\mathcal{W}^{(3)}(\theta=\infty) = -\frac{\mathcal{A}^3 N n^2}{64\pi^4}. \tag{27}$$

Finally the diagrams with double crossing give

$$\mathcal{W}^{(2)}(\theta=\infty) = \frac{\mathcal{A}^3 N n^2}{12\pi^4}(1+0.2088). \tag{28}$$

As we have anticipated, the last term has been evaluated numerically. Its n dependence has been checked up to $n=6$, within the incertitude due to the numerical integration (see Appendix B).

Summing together all the contributions of diagrams with crossed propagators, we get

$$\mathcal{W}_6^{(cr)}(\theta=\infty) = \frac{\mathcal{A}^3 N n^4}{24\pi^2}\left(1-\frac{1}{n^2\pi^2}\left(\frac{35}{8}-0.4176\right)\right). \tag{29}$$

We remark that the leading term at large n

$$\mathcal{W}_6^{(cr)}(\theta=\infty) \simeq \frac{\mathcal{A}^3 N n^4}{24\pi^2}$$

exhibits the scaling (1). It comes only from diagrams with a single crossing. Diagrams with such a topological configuration can also be computed in higher orders; for instance, at $\mathcal{O}(g^8)$ they lead to the result

$$\mathcal{W}_8^{(cr)}(\theta=\infty) \simeq -\frac{\mathcal{A}^4 N^2 n^6}{192\pi^2}+\mathcal{O}(n^4). \tag{30}$$

The integral over the loop variables provides a factor n^{-2}, turning the trivial n^8, due to the kinematical rescaling, into the factor n^6. Details are reported in Appendix C.

We are led to argue that the dominant term at the $(2m+4)$th perturbative order increases with n no faster than n^{2m+2}. In turn it exhibits the highest $U(N)$ contribution, behaving like N^m

$$g^{2m+4}\mathcal{W}_{2m+4}^{(cr)}(\theta=\infty) \simeq \mathcal{K}_m (nN)^m (g^2\mathcal{A}n)^{m+2}, \tag{31}$$

which obeys the scaling (1).

Further we conjecture that diagrams with a single crossing dominate; then the weights $\mathcal{K}_m$ can be evaluated (see Appendix D) and lead to

$$g^{2m+4}\mathcal{W}_{2m+4}^{(cr)}(\theta=\infty) \simeq -\frac{(g^2\mathcal{A}n)^2}{4\pi^2}\frac{1}{m!}\frac{(-g^2\mathcal{A}Nn^2)^m}{(m+2)!}; \tag{32}$$

the related perturbative series can be easily resummed

$$\mathcal{W}^{(cr)}(\theta=\infty) = -\frac{g^2\mathcal{A}}{4\pi^2 N}J_2(2\sqrt{g^2\mathcal{A}n^2N}). \tag{33}$$

If we compare Eq. (32) with the corresponding term due to planar diagrams, which are insensitive to θ [see Eq. (14)], we notice that, in the 't Hooft's limit $N\to\infty$ with fixed g^2N, the planar diagrams dominate by a factor n^2N^2, as expected.

Our conjecture is open to more thorough perturbative tests as well as to possible non-perturbative derivations which might throw further light on its ultimate meaning and related consequences. For recent papers on nonperturbative approaches, see [25–27].

IV. CONCLUSIONS

Summarizing our perturbative investigation, we can say that, when winding *n* times around the Wilson loop, the non-Abelian nature of the gauge group in the noncommutative case is felt, even in a perturbative calculation making use of the WML prescription for the vector propagator. This is due to the merging of space-time properties with "internal" symmetries in a large invariance group $U_{cpt}(\mathcal{H})$ [2,27].

One gets the clear impression that in a noncommutative formulation what is really relevant are not separately the space-time properties of the "base" manifold and of the "fiber" $U(N)$, but rather the overall algebraic structure of the resulting invariance group $U_{cpt}(\mathcal{H})$. To properly understand its topological features is certainly beyond any perturbative approach. Rather one should possibly resort to suitable $\mathcal{N}$ truncations of the Hilbert space in the form of matrix models leading to the invariance groups $U(\mathcal{N})$.

It is not clear how many perturbative features might eventually be singled out in those contexts, especially in view of the difficulty in performing the inductive limit $\mathcal{N}\to\infty$.

For this reason we think that our perturbative results are challenging. They indicate that the intertwining between *n*, controlling the space-time geometry, and *N*, related to the gauge group, is far from trivial. The presence of corrections to the scaling laws occurring at $\theta=0$ and at $\theta=\infty$, while frustrating at first sight in view of a generalization to all values of θ, might be taken instead as a serious indication that *n* and *N* separately are not perhaps the best parameters to be chosen unless large values for both (and for θ!) are considered. In such a situation, perhaps surprisingly, the relation (1) is recovered.

Equations (14), (32) are concrete realizations of the more general structure

$$\mathcal{W}_{2m+4}=(\mathcal{A}n^2N)^{m+2}f_m(n,N), \tag{34}$$

f_m being a symmetric function of its arguments. We stress that Eq. (14) concerns only *planar* diagrams; crossed graph contributions in the commutative case cannot be put in the form (34) and violate the relation (1).[1] In the noncommutative case, for large n,N and maximal noncommutativity ($\theta=\infty$), the structure (34) is instead restored for the leading contribution of *crossed* diagrams. The presence of the function f_m in the WML context might be thought of as a sign of the merging of space-time and internal symmetries.

All these difficult, but intriguing questions are worthy in our opinion of thorough investigations and promise further exciting, unexpected developments.

ACKNOWLEDGMENTS

We thank Matteo Viel for help in the numerical calculation. One of us (A.B.) acknowledges a partial support by the European Community network HPRN-CT-2000-00149.

APPENDIX A: SMALL θ LIMIT

The first significant term in the small θ expansion is the second one in Eq. (20), that is the $\mathcal{O}(\theta)$ term in $\mathcal{W}_4^{(cr)}$,

$$\mathcal{W}_\theta=-in^4\mathcal{A}\pi\theta\int_0^1[ds]\frac{\sin[n\pi(s_1+s_3-s_2-s_4)]}{\sin[n\pi(s_4-s_2)]\sin[n\pi(s_3-s_1)]}$$

$$\equiv -in^4\mathcal{A}\pi\theta I, \tag{A1}$$

with measure $[ds]=ds_1ds_2ds_3ds_4\,\theta(s_4-s_3)\,\theta(s_3-s_2)\,\theta(s_2-s_1)$. Integrating in ds_1 and ds_4 leads to

$$I=\frac{1}{n^2\pi^2}\int_0^1 ds_2ds_3\,\theta(s_3-s_2)\Bigg\{-n\pi\cos[2n\pi(s_3-s_2)]\left[s_2\log\left|\frac{\sin n\pi s_2}{\sin n\pi(s_3-s_2)}\right|+(1-s_3)\log\left|\frac{\sin n\pi s_3}{\sin n\pi(s_3-s_2)}\right|\right]+\sin[2n\pi(s_3-s_2)]\left[-n^2\pi^2s_2(1-s_3)+\log|\sin n\pi(s_3-s_2)|\log\left|\frac{\sin n\pi(s_3-s_2)}{\sin n\pi s_2\sin n\pi s_3}\right|+\log|\sin n\pi s_2|\log|\sin n\pi s_3|\right]\Bigg\}\equiv I_1+I_2, \tag{A2}$$

where I_1 and I_2 refer to the first and second square brackets in Eq. (A2), respectively. The two integrals in I_1 coincide. They can be easily performed leading to

$$I_1=-\left(\frac{1}{6n\pi}+\frac{1}{4n^3\pi^3}\right). \tag{A3}$$

Concerning I_2, the first term is trivial, and provides us with a factor $1/(8n^3\pi^3)-1/(12n\pi)$, whereas in the remaining integrals it is more convenient to integrate first on one variable and then to add the integrands together before performing the final integration, i.e.

$$\begin{aligned}I_2=&\frac{1}{8n^3\pi^3}-\frac{1}{12n\pi}-\frac{1}{2n^3\pi^3}\int_0^1 ds\,\sin n\pi s\\&\times[2n\pi s\cos n\pi s\log|\sin n\pi s|\\&+\sin n\pi s(\log|\sin n\pi s|-1)]\\=&\frac{1}{4n^3\pi^3}-\frac{1}{12n\pi}.\end{aligned} \tag{A4}$$

Adding Eqs. (A3) and (A4) and taking Eq. (A1) into account, Eq. (21) follows.

APPENDIX B: SIXTH ORDER CALCULATION

1. The singly crossed diagram

We show in some detail the formulas for $\mathcal{W}_{(16)(24)(35)}$, the other five diagrams being simply obtainable by renaming the variables.

Integrating Eq. (23) over ϕ and p we recover an expression analogous to Eq. (16). Therefore we can use the result obtained at $\mathcal{O}(g^4)$ to get

$$\begin{aligned}&\mathcal{W}_{(16)(24)(35)}\\&=-2n^6\mathcal{A}^3N\int[ds]_6\left[\frac{1}{2}+\frac{2}{\beta'^2}(\exp[i\beta'\sin\alpha']\right.\\&\left.-1-i\beta'\sin\alpha')\right],\end{aligned} \tag{B1}$$

where now

$$\alpha'=n\pi(s_2+s_4-s_3-s_5),$$

$$\beta'=\frac{4\mathcal{A}}{\pi\theta}\sin[n\pi(s_4-s_2)]\sin[n\pi(s_5-s_3)]. \tag{B2}$$

The large-θ limit is easily derived from this formula; summing all the singly crossed diagrams we find

$$-\mathcal{A}^3Nn^6\left[\frac{1}{4\pi^4n^4}-\frac{1}{24\pi^2n^2}\right]. \tag{B3}$$

2. The doubly crossed diagram

Integrating Eq. (24) over ϕ and p, and then over ψ and q, we get

$$\begin{aligned}\mathcal{W}_{(15)(24)(36)}=&-r^6Nn^6\pi^2\int[ds]_6\int_0^\infty\frac{dk}{k}\oint_{|z|=1}\frac{dz}{iz^3}\\&\times e^{-k\sin[n\pi(s_6-s_3)](z-1/z)}\times\frac{1-\frac{\gamma'}{z}e^{-in\pi\sigma'}}{1-\gamma' ze^{in\pi\sigma'}}\\&\times\frac{1-\frac{\gamma''}{z}e^{-in\pi\sigma''}}{1-\gamma'' ze^{in\pi\sigma''}},\end{aligned} \tag{B4}$$

where

$$\sigma'=s_1+s_5-s_3-s_6,\qquad \gamma'=\frac{\theta k}{2r^2\sin[n\pi(s_5-s_1)]},$$

$$\sigma''=s_2+s_4-s_3-s_6,\qquad \gamma''=\frac{\theta k}{2r^2\sin[n\pi(s_4-s_2)]}.$$

We consider the identity $e^{-k\sin[n\pi(s_6-s_3)](z-1/z)}\equiv[(e^{-k\sin[n\pi(s_6-s_3)](z-1/z)}-1)+1]$ in Eq. (B4); in the first term it is possible to send θ to infinity in the integrand, obtaining the result

$$-\frac{r^6Nn^6\pi^3}{3}\int[ds]_6e^{2\pi i(2s_3+2s_6-s_1-s_5-s_2-s_4)}. \tag{B5}$$

The other contribution can be exactly integrated over z and k, leading to the sum of two expressions

$$\begin{aligned}&-\frac{r^6Nn^62\pi^3}{3}\int[ds]_6\exp(-i(\lambda+\omega))\\&\times[\cos(\lambda-\omega)]^3\end{aligned} \tag{B6}$$

and

$$\begin{aligned}&\frac{ir^6Nn^62\pi^3}{3}\int[ds]_6\exp(-i(\lambda+\omega))\\&\times\frac{\left(1+\left|\frac{d}{c}\right|\exp(i(\lambda-\omega))\right)}{\left(1-\left|\frac{d}{c}\right|\exp(i(\lambda-\omega))\right)}[\sin(\lambda-\omega)]^3,\end{aligned} \tag{B7}$$

where

$$c=-\sin[n\pi(s_1-s_5)]\exp(-in\pi\sigma')=|c|exp(i\omega),$$

$$d=-\sin[n\pi(s_2-s_4)]\exp(-in\pi\sigma'')=|d|\exp(i\lambda).$$

The integrals (B5) and (B6) can be easily computed; when summed with the corresponding ones from $\mathcal{W}_{(14)(26)(35)}$ and $\mathcal{W}_{(13)(25)(46)}$, they give

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[1]The structure (34) is shared also by the exact geometrical solution of the commutative case [4].

$$\frac{\mathcal{A}^3 N n^2}{12\pi^4}. \tag{B8}$$

Expression (B7) instead, together with the corresponding ones from $\mathcal{W}_{(14)(26)(35)}$ and $\mathcal{W}_{(13)(25)(46)}$, is difficult to deal with. We can prove their sum is real and have evaluated such a sum numerically, for $n=1,\ldots,6$. We present the result in the form

$$\frac{4\pi r^6 N n^4}{3}\times J_{NUM}, \tag{B9}$$

where

$$J_{NUM}(n=1)=1.32236(80\pm 37)\times 10^{-3},$$

$$J_{NUM}(n=2)=0.330(49\pm 16)\times 10^{-3}, \quad \frac{J_{NUM}(n=1)}{4}=0.330592(01\pm 93)\times 10^{-3},$$

$$J_{NUM}(n=3)=0.146(97\pm 35)\times 10^{-3}, \quad \frac{J_{NUM}(n=1)}{9}=0.146929(80\pm 41)\times 10^{-3},$$

$$J_{NUM}(n=4)=0.08(17\pm 29)\times 10^{-3}, \quad \frac{J_{NUM}(n=1)}{16}=0.082648(00\pm 23)\times 10^{-3},$$

$$J_{NUM}(n=5)=0.05(10\pm 40)\times 10^{-3}, \quad \frac{J_{NUM}(n=1)}{25}=0.052894(72\pm 15)\times 10^{-3},$$

$$J_{NUM}(n=6)=0.03(88\pm 79)\times 10^{-3}, \quad \frac{J_{NUM}(n=1)}{36}=0.036732(45\pm 10)\times 10^{-3}.$$

All the errors are three standard deviations. Within the numerical error, J_{NUM} scales as $1/n^2$.

3. The maximally crossed diagram

Integrating Eq. (25) over ϕ and p, and then over χ and k, we get, after a simple rescaling,

$$\mathcal{W}_{(14)(25)(36)}=-r^6 N n^6 2\pi^2\int [ds]_6\int_0^\infty \frac{dq}{q}\int_0^{2\pi} d\psi \exp(-2i\psi)\exp\left(4i\frac{qr^2}{\theta}\sin\psi\sin[n\pi(s_2-s_5)]\right)\left[\frac{1}{2}\frac{\bar{\alpha}}{\alpha}\frac{\bar{\beta}}{\beta}\right.$$
$$\left.+\frac{\theta^2}{8r^4\alpha^2\beta^2}\left(\exp\left(\frac{4ir^2}{\theta}\mathrm{Im}(e^{in\pi\sigma'''}\bar{\alpha}\beta)\right)-\frac{4ir^2}{\theta}\mathrm{Im}(e^{in\pi\sigma'''}\bar{\alpha}\beta)-1\right)\right], \tag{B10}$$

where $\sigma'''=s_1+s_4-s_3-s_6$, the bars denote complex conjugation and

$$\alpha=\sin[n\pi(s_1-s_4)]+q\exp i(\psi+n\pi(s_1+s_4-s_2-s_5)),$$

$$\beta=\sin[n\pi(s_3-s_6)]-q\exp i(\psi+n\pi(s_3+s_6-s_2-s_5)).$$

We can recognize in Eq. (B10) the same structure we have found in Eq. (18). We rewrite it as follows:

$$\mathcal{W}_{(14)(25)(36)}=-r^6 N n^6 2\pi^2\int [ds]_6\int_0^\infty\frac{dq}{q}\int_0^{2\pi}d\psi\exp(-2i\psi)\times\exp\left(4i\frac{qr^2}{\theta}\sin\psi\sin[n\pi(s_2-s_5)]\right)\exp(-2i(\gamma_\alpha+\gamma_\beta))$$
$$\times\left[\frac{1}{2}\cos(2n\pi\sigma'''-2\gamma_\alpha+2\gamma_\beta)+\frac{1}{\pi i}\int_{\mu-i\infty}^{\mu+i\infty}ds\Gamma(-s)e^{-i(\pi/2)s}\left[\frac{4|\alpha||\beta|r^2}{\theta}\right]^{s-2}[\sin(n\pi\sigma'''-\gamma_\alpha+\gamma_\beta)]^s\right],$$
$$2<\mu<3, \tag{B11}$$

where we have defined $\alpha=|\alpha|\exp(i\gamma_\alpha)$ and $\beta=|\beta|\exp(i\gamma_\beta)$.

One can prove that, in the large-θ limit, the last integral goes to zero. Then, in this limit, Eq. (B11) can be easily evaluated:

$$\mathcal{W}_{(14)(25)(36)}\to_{\theta\to\infty}-\frac{r^6Nn^6\pi^3}{3}\int [ds]_6(e^{2in\pi(2s_2+2s_5-s_1-s_4-s_3-s_6)}+e^{2in\pi(2s_1+2s_4-s_2-s_5-s_3-s_6)}+e^{2in\pi(2s_3+2s_6-s_1-s_4-s_2-s_5)})$$
$$=-\frac{\mathcal{A}^3Nn^2}{64\pi^4}. \tag{B12}$$

Here we notice that the integrand is completely symmetric in the three propagators (14)(25)(36), as it should.

APPENDIX C: HIGHER ORDERS

First we prove that singly crossed diagrams behave in the large-θ limit at least as $1/n^2$ or subleading in the limit of a large number of windings n. We start by realizing that we can always express the integral of a generic diagram with m propagators and a single crossing generalizing Eq. (18)

$$\mathcal{I}\equiv\int_0^1 dt\int_0^t dz\int_0^z dy\int_0^y dx\int [ds]_{2m-4} \times\cos[2\pi n(x+z-y-t)], \tag{C1}$$

$[ds]_{2m-4}$ being a measure depending on x,y,z,t only through the extremes of integration. As a matter of fact, it is always possible to single out the variables linked to the propagators which cross, suitably rearranging the other kinematical integrations. These integrations lead to polynomials

$$\mathcal{I}=\int_0^1 dt\int_0^t dz\int_0^z dy\int_0^y dx \times\sum_{k_1k_2k_3k_4} c_{k_1k_2k_3k_4}x^{k_1}y^{k_2}z^{k_3}t^{k_4} \times\cos[2\pi n(x+z-y-t)]. \tag{C2}$$

Now we perform the change of variables $\alpha=y+x,\ \beta=y-x,\ \gamma=t+z,\ \delta=t-z$

$$\mathcal{I}=\int_0^1 d\delta\int_\delta^{2-\delta}d\gamma\int_0^{(\gamma-\delta)/2}d\beta\int_\beta^{\gamma-\delta-\beta}d\alpha \times\sum_{q_1q_2q_3q_4} c'_{q_1q_2q_3q_4}\alpha^{q_1}\beta^{q_2}\gamma^{q_3}\delta^{q_4}\cos[2\pi n(\beta+\delta)] \tag{C3}$$

and then integrate over α. Changing again variables to $\psi=\beta+\delta,\ \xi=\delta-\beta$, we end up with

$$\mathcal{I}=\int_0^1 d\psi\int_{-\psi}^{\psi}d\xi\int_{(3\psi-\xi)/2}^{2-(\psi+\xi)/2}d\gamma \times\sum_{p_1p_2p_3} C_{p_1p_2p_3}\psi^{p_1}\xi^{p_2}\gamma^{p_3}\cos[2\pi n\psi]. \tag{C4}$$

The integrals over ξ and γ can be easily performed giving, of course, a polynomial in ψ

$$\mathcal{I}=\sum_r C'_r\int_0^1\psi^r\cos[2\pi n\psi]d\psi. \tag{C5}$$

Integrating by parts, we realize that only even inverse powers of n are produced, starting from n^{-2}.

Now we turn our attention to the $U(N)$ factors. A direct computation of the traces involved in the diagrams with a single, a double or the triple crossing ($\mathcal{O}(g^6)$), shows that they all share the common factor N^2 [our normalization being $t^0=\mathbf{1}/\sqrt{N},\mathrm{Tr}(t^at^b)=\delta^{ab};\ \ a,b=1,\dots,N^2-1$]. As the Wilson loop is normalized with N^{-1}, at $\mathcal{O}(g^6)$ the single factor N ensues.

It is now trivial to realize that any insertion of $m-3$ lines no matter where in such diagrams, provided that further crossings are avoided, produces the factor N^{m-3}.

APPENDIX D: COMPUTATION OF THE WEIGHTS

In the previous appendix we have shown that the n dependence of singly crossed diagrams in the large-θ limit takes the form $\sum_{p=1}^{P}c_pn^{-2p}$. To find the leading contribution at large n we have to evaluate c_1. This can be done as follows: at $\mathcal{O}(g^{2m+4})$ we start drawing a cross and then add the remaining m propagators in such a way they do not further cross. From Eqs. (C2)–(C5) one can realize that c_1 is different from zero only for a particular subset of these diagrams: if we label the four sectors in which the cross divides the circular loop as North (the sector containing the origin of the loop variables s_i), West, South and East, then only diagrams with r propagator in the southern sector and $m-r$ in the northern one contribute to c_1; moreover, these contributions are all equal. Therefore we can evaluate this integral once and then multiply it by the number of configurations in this subset.

We choose as representative the diagram with all the m nonintersecting propagators in the northern sector, starting from the origin and connecting s_1 with $s_2,\dots,s_{2m-1}$ with s_{2m}. In this way the crossed variables are $s_{2m+1},\dots,s_{2m+4}$. We obtain the integral

$$\mathcal{I}=(-\pi)^{m+2}(gr)^{2m+4}N^mn^{2m+4} \times\int_0^1 dt\int_0^t dz\int_0^z dy\int_0^y dx\frac{x^{2m}}{(2m)!} \times\cos[2\pi n(x+z-y-t)]. \tag{D1}$$

Following the procedure described in Appendix C we get

$$\mathcal{I}=\frac{(-\pi)^{m+2}(gr)^{2m+4}}{(2m)!}N^m n^{2m+4} \times\int_0^1 d\psi \frac{1}{(2m+1)(2m+2)}\psi(1-\psi)^{2m+2} \times\cos[2\pi n\psi] \tag{D2}$$

and finally

$$\mathcal{I}=-N^m\frac{(-g^2\mathcal{A}n^2)^{m+2}}{(2m+2)!}\left(\frac{1}{4\pi^2n^2}+\mathcal{O}\left(\frac{1}{n^4}\right)\right). \tag{D3}$$

Now we have to count. We denote by S_{2r} the ways in which the r propagators in the southern sector can be arranged without crossing. A little thought provides the recursive relation

$$S_0=1, \quad S_{2r}=\sum_{k=1}^{r} S_{2k-2}S_{2r-2k}, \tag{D4}$$

which can easily be solved

$$S_{2r}=\frac{2^{2r}\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(r+2)}. \tag{D5}$$

The $m-r$ propagators in the northern sector lead to the weight $S_{2(m-r)}$ times the number of possible insertions of the origin, namely $[2(m-r)+1]$. The number of relevant diagrams is therefore

$$\mathcal{N}_m=\sum_{r=0}^{m} S_{2r}S_{2(m-r)}[2(m-r)+1] =\frac{2^{2m+2}(m+1)\Gamma\left(m+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(m+3)}. \tag{D6}$$

Multiplying Eqs. (D3) and (D6) we are led to Eq. (32).

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