Black hole interaction energy

Sergio Dain*
Albert-Einstein-Institut, am Mühlenberg 1, D-14476, Golm, Germany
(Received 26 July 2002; published 29 October 2002)

The interaction energy between two black holes at a large separation distance is calculated. The first term in the expansion corresponds to the Newtonian interaction between the masses. The second term corresponds to the spin-spin interaction. The calculation is based on the interaction energy defined on the two black holes' initial data. No test particle approximation is used. The relation between this formula and cosmic censorship is discussed.

DOI: 10.1103/PhysRevD.66.084019 PACS number(s): 04.70.Bw, 04.20.Dw, 04.20.Ex, 04.25.Nx

I. INTRODUCTION

The purpose of this article is to prove, under appropriate assumptions, the following statement: the interaction energy, at a large separation distance l, between two black holes of masses M_1, M_2 and spins J_1, J_2 is given by

$$E=\frac{-M_1M_2}{l}+\frac{-J_1\cdot J_2+3(J_1\cdot\hat{n})(J_2\cdot\hat{n})}{l^3} + \text{higher order terms}, \tag{1}$$

where $\hat{n}$ is a unit vector which points inward along the line connecting the black holes. Before giving a precise definition of the parameters involved in Eq. (1), I want to discuss its physical meaning.

The first term in Eq. (1) has the Newtonian form. For two point particles of masses M_1 and M_2 separated by a Euclidean distance l, the Newtonian interaction energy between them is given by $-M_1M_2/l$. The fact that this term appears also for a two-black-hole system in general relativity is expected from the weak field limit of Einstein's equations. The second term in Eq. (1), which involves the spins, has an analogous form to the dipole-dipole electromagnetic interaction; there exists an analogy between the magnetic dipole in electromagnetism and spin in general relativity (see [1]). In the electromagnetic case, l is the Euclidean distance between two charge distributions and J_1, J_2 their corresponding dipole moments. However, the gravitational black hole spin-spin interaction has the opposite sign to the electromagnetic one. The first evidence of this fact was given by Hawking [2]. I want to reproduce Hawking's argument here because it points out the connection between Eq. (1) and the cosmic censorship conjecture (see also the discussion in [1]). In the argument, we assume the following two consequences of weak cosmic censorship and the theory of black holes (cf. [3,4]; see also [5]): (i) Every apparent horizon must be entirely contained within the black hole event horizon; (ii) if matter satisfies the null energy condition (i.e., if $T_{ab}k^ak^b \geqslant 0$ for all null k^a), then the area of the event horizon of a black hole cannot decrease in time. We also assume (iii) that all black holes eventually settle down to a final Kerr black hole.

Consider a system of two black holes such that, at a given time, the separation distance between them is large. Then there must exist a Cauchy surface in the asymptotically flat region of the space-time such that the intersection of the hypersurface with the event horizon has two disconnected component of areas A_1 and A_2. Since the black holes are far apart, these areas can be approximated by the Kerr formula

$$A_1=8\pi(M_1^2+\sqrt{M_1^4-J_1^2}),$$
$$A_2=8\pi(M_2^2+\sqrt{M_2^4-J_2^2}). \tag{2}$$

At late times, after the collision, the system will settle down to a Kerr black hole. Hence, there must exist another Cauchy hypersurface such that its intersection with the event horizon will have area

$$A_f=8\pi(M_f^2+\sqrt{M_f^4-J_f^2}), \tag{3}$$

where M_f is the mass of the final black hole and J_f is its final angular momentum. By (ii) we have

$$A_f \geqslant A_1+A_2. \tag{4}$$

Since gravitational waves have positive mass, we also have

$$M_f \leqslant M_1+M_2. \tag{5}$$

In general, gravitational waves will carry angular momentum. But in *axially symmetric* space-times the total angular momentum is a conserved quantity, since it can be defined by a Komar integral (cf. [6] and also [4]). Then in this case we have

$$J_f=J_1+J_2. \tag{6}$$

Using Eqs. (2), (3), (4), and (6) it is possible to obtain an upper bound, which depends on J_1 and J_2, to the total amount of radiation emitted by the system $M_1+M_2-M_f$. It can be seen that if J_1 and J_2 have the same sign this upper bound is smaller than if they have opposite sign. This suggests that there may be a spin-spin force between the black holes that is attractive if the angular momenta have opposite directions and repulsive if they have the same direction. Presumably, in the second case the system expends energy in doing work against the spin repulsive force, and for this reason this energy is not available to be radiated via gravitational radiation.

*Email address: dain@aei-potsdam.mpg.de

Hawking's argument only suggests that the spin interaction energy between black holes has this sign dependence with respect to the spins. It is not a proof, first because there is no proof for the weak cosmic censorship conjectures (i), (ii) and for the assumption (iii); second, because even if we assume (i)–(iii) the argument only shows that an upper bound of the total amount of radiated energy has this sign dependence in terms of J_1 and J_2, but the real amount of gravitational radiation can, in principle, have a different dependence. In fact, the total amount of gravitational radiation produced by such systems, as numerical studies show, is much smaller than this bound. This upper bound is 50% of the total mass when the spins are antiparallel, the black holes are extreme ($J^2=M$), and have equal masses; when the spins are zero or when the black holes are extreme with parallel spins, the upper bound is 29% of the total mass. On the other hand, in the numerical calculations the maximum amount of radiation emitted by this type of system is about 3% of the total mass (see [7,8] for a recent calculation and also [9] for an up to date review on the subject). However, the numerical studies show that the system indeed radiates less when the spins are parallel than when they are antiparallel. Moreover, Wald [1] proves that the interaction energy between a test particle with spin J_2 and a stationary background of spin J_1 has precisely this sign dependence. Wald shows that the spin-spin interaction energy has the form

$$\frac{-J_1\cdot J_2+3(J_1\cdot\hat{n})(J_1\cdot\hat{n})}{l^3}, \tag{7}$$

where l and $\hat{n}$ are defined as follows. The stationary field is expanded at large distance with respect to Cartesian asymptotic coordinates x^i; here l is the Euclidean radius with respect to x^i and $\hat{n}^i=x^i/l$. Equation (7) has also been proved by D'Eath using post-Newtonian expansions [10]. It is important to note that Eq. (7) gives indirect evidence in support of (i)–(iii).

In this article I want to prove Eq. (7) without using either the particle or post-newtonian approximation. The proof is based on an interaction energy defined on the two-black-hole initial data. This interaction energy is genuinely nonlinear; it does not involve any approximation.

The plan of the paper is as follows. In Sec. II the main results are given. In Sec. III Theorem 2 is proved; in Sec. IV we prove Corollary 1. Finally, in Sec. V an alternative definition of the interaction energy is discussed.

II. MAIN RESULT

The strategy I will follow was given by Brill and Lindquist [11]. It is based on the analysis of an *initial data set* with many *asymptotic ends*. An initial data set for the Einstein vacuum equations is given by a triple $(\tilde{S},\tilde{h}_{ab},\tilde{K}_{ab})$ where $\tilde{S}$ is a connected three-dimensional manifold, $\tilde{h}_{ab}$ a (positive definite) Riemannian metric, and $\tilde{K}_{ab}$ a symmetric tensor field on $\tilde{S}$. They satisfy the vacuum constraint equations

$$\tilde{D}^b\tilde{K}_{ab}-\tilde{D}_a\tilde{K}=0, \tag{8}$$

$$\tilde{R}+\tilde{K}^2-\tilde{K}_{ab}\tilde{K}^{ab}=0 \tag{9}$$

on $\tilde{S}$, where $\tilde{D}_a$ is the covariant derivative with respect to $\tilde{h}_{ab}$, $\tilde{R}$ is the trace of the corresponding Ricci tensor, $\tilde{K}=\tilde{h}^{ab}\tilde{K}_{ab}$, and $a,b,c,\ldots$ denote abstract indices. Tensor indices of quantities with a tilde will be moved with the metric $\tilde{h}_{ab}$ and its inverse $\tilde{h}^{ab}$. The data will be called *asymptotically flat* with $N+1$ asymptotic ends, if for some compact set Ω we have that $\tilde{S}\backslash\Omega=\Sigma_{k=0}^{N}\tilde{S}_k$, where $\tilde{S}_k$ are open sets such that each $\tilde{S}_k$ can be mapped by a coordinate system x^j diffeomorphically onto the complement of a closed ball in $\mathbb{R}^3$ such that we have in these coordinates

$$\tilde{h}_{ij}=\left(1+\frac{2M_k}{r}\right)\delta_{ij}+O(r^{-2}), \tag{10}$$

$$\tilde{K}_{ij}=O(r^{-2}), \tag{11}$$

as $r=[\Sigma_{j=1}^{3}(x^j)^2]^{1/2}\to\infty$ in each set $\tilde{S}_k$; where $i,j\ldots$, which take values 1,2,3, denote coordinate indices with respect to the given coordinate system x^j, and δ_{ij} denotes the flat metric. We will call the coordinate system x^i an *asymptotic coordinate system* at the end k. Each asymptotic region $\tilde{S}_k$ has a different asymptotic coordinate system. The constant M_k denotes the Arnowitt-Deser-Misner (ADM) mass [12] of the data at the end k. These conditions guarantee that the mass, the linear momentum, and the angular momentum of the initial data set are well defined at every end.

For $N\geqslant 1$, this class of data contains, in general, apparent horizons. The existence of apparent horizons leads us to interpret these data as representing initial data for black holes. Their evolution will presumably contain an event horizon, according to the standard theory of black holes [3]. The validity of this picture depends, of course, on the cosmic censorship conjecture. The only statement about the evolution of the data that we can make is the geodesic incompleteness of the space-time. In general, in order to prove the geodesic incompleteness of a space-time, one needs to know that the data contain a trapped surface in order to apply the singularities theorems [3]. However, in this particular case, since the topology of the data is not trivial, the geodesic incompleteness of the space-time follows directly from a theorem proved by Gannon [13].

For simplicity we will fix $N=2$ (see Fig. 1). In this case the data can be interpreted as initial data with two black holes. This interpretation is suggested by the following fact: when an appropriate distance parameter is large compared with the masses M_k, then it can be seen numerically that only two disconnected apparent horizons appear. For time symmetric data, these numeric calculations have been done in [11]; the non-time-symmetric case has been studied by Cook (see [14] and references therein). It is not clear that the number of apparent horizons is the number of black holes contained in the data, since, even when there are two discon-

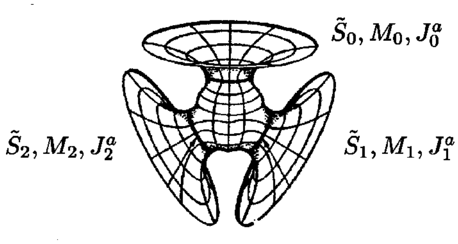

FIG. 1. Initial data with three asymptotic ends ($N=2$). For each asymptotic region $\tilde{S}_k$, we have the corresponding mass M_k and total angular momentum J_k^a.

nected apparent horizons, the intersection of the event horizon with the initial data can be connected. However, at large separation distance, this seems to be a reasonable assumption, which is confirmed by the numerical evolutions [9].

Brill and Lindquist define the following interaction energy at the end k:

$$E_k = M_k - \sum_{\substack{k'=0 \\ k' \neq k}}^{N} M_{k'} . \tag{12}$$

The energy E_k is a geometric quantity; its definition does not involve any approximation. The question now is how to calculate E_k in terms of physically relevant parameters. The first problem is how to define an appropriate separation distance between the black holes. When there are two apparent horizons, there is a well defined separation distance $l_{\tilde{h}}$ defined as the minimum geodesic distance between any two points in the two different horizons (see Fig. 2).

However, the distance $l_{\tilde{h}}$ is hard to compute. The location of the apparent horizons can be calculated only numerically. Since we are only interested in the energy at large separations, instead of $l_{\tilde{h}}$ we will use another parameter l, and we will argue that $l_{\tilde{h}} \approx l$ in this limit. The definition of the parameter l is related to the way in which one can construct solutions of the constraint equations with many asymptotic ends. The conformal method (see [15,16] and the references therein) is a general method for constructing solutions of the constraint equations. We assume that h_{ab} is a positive definite metric with covariant derivative D_a, and K^{ab} is a tracefree (with respect to h_{ab}) symmetric tensor, satisfying

$$D_a K^{ab} = 0 \quad \text{on} \quad \tilde{S}. \tag{13}$$

Let φ be a solution of

$$L_h \varphi = -\frac{1}{8} K_{ab} K^{ab} \varphi^{-7} \quad \text{on} \quad \tilde{S}, \tag{14}$$

where $L_h = D^a D_a - R/8$ and R is the scalar curvature of the metric h_{ab}. Then the physical fields $(\tilde{h}, \tilde{K})$ defined by $\tilde{h}_{ab} = \varphi^4 h_{ab}$ and $\tilde{K}^{ab} = \varphi^{-10} K^{ab}$ will satisfy the vacuum constraint equations on $\tilde{S}$. We have assumed that K^{ab} is trace free; hence $\tilde{K}^{ab}$ will also be trace free with respect to $\tilde{h}_{ab}$. That is, the initial data set will be *maximal*.

To ensure asymptotic flatness of the data at the each end k, we will require the following boundary conditions. Let i_1 and i_2 be two arbitrary points in $\mathbb{R}^3$, with coordinates x_1^j and x_2^j in some Cartesian coordinate system x^i. Define the manifold $\tilde{S}$ by $\tilde{S} = \mathbb{R}^3 \setminus \{i_1, i_2\}$. Assume that h_{ab} is regular on $\mathbb{R}^3$. At infinity we will impose the following falloff behavior:

$$h_{ij} = \delta_{ij} + O(r^{-2}), \tag{15}$$

$$K^{ab} = O(r^{-2}), \tag{16}$$

$$\varphi = 1 + O(r^{-1}). \tag{17}$$

At the points i_1 and i_2 we require

$$K^{ab} = O(r_1^{-4}), \quad K^{ab} = O(r_2^{-4}), \tag{18}$$

where

$$r_1 = \left(\sum_{i=1}^{3} (x^i - x_1^i)^2 \right)^{1/2},$$

$$r_2 = \left(\sum_{i=1}^{3} (x^i - x_2^i)^2 \right)^{1/2}, \tag{19}$$

and

$$\lim_{r_1 \to 0} r_1 \varphi = \frac{m_1}{2}, \quad \lim_{r_2 \to 0} r_2 \varphi = \frac{m_2}{2}, \tag{20}$$

where m_1 and m_2 are positive constants. Note that both φ and K^{ab} are singular at i_1, i_2.

One can prove that the data so constructed will be asymptotically flat at the three ends. We have made an artificial distinction between the end 0, given by $r \to \infty$, and the ends 1 and 2. It is possible to discuss the same construction in a more geometrical way, such that all ends are treated equally; see [17–20]. However, since our final goal is to calculate the interaction energy at one end, it is convenient to make this distinction. The coordinate system x^i and the corresponding flat metric in the expansion Eq. (15) give the Euclidean distance l between i_1 and i_2:

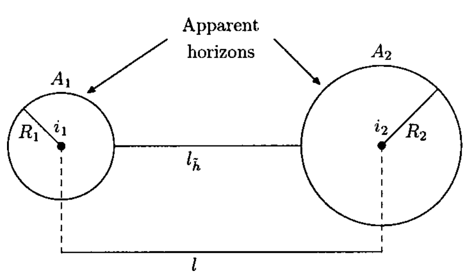

FIG. 2. An initial data set with three asymptotic end points and only two disconnected apparent horizons of area A_1, A_2, and radii R_1, R_2. The points i_1 and i_2 represent the two other infinities 1 and 2. The geodesic distance $l_{\tilde{h}}$ is computed with the physical metric $\tilde{h}_{ab}$. The parameter l is computed with the conformal metric. The geodesic distance between i_1 and i_2 with respect to the physical metric is infinite.

$$l=\left(\sum_{i=1}^{3}(x_2^i-x_1^i)^2\right)^{1/2}, \tag{21}$$

which will be our separation distance parameter (see Fig. 2).

In general, Eq. (14) is nonlinear. However, if we assume that the data are time symmetric, i.e., $K^{ab}=0$, then it becomes a linear equation for φ. If we assume that the conformal metric is flat, we obtain a Laplace equation for φ. The solution of this equation that satisfies the boundary conditions Eqs. (20) and (17) is given by

$$\varphi_0=1+\frac{m_1}{2r_1}+\frac{m_2}{2r_2}. \tag{22}$$

This solution was found by Brill and Lindquist in [11]. In this case it is possible to calculate explicitly the interaction energy (12) in terms of the masses and the separation distance. The result is the following.

Theorem 1 (Brill-Lindquist). Let h_{ab} be a flat metric and $\tilde{K}^{ab}=0$. Then the interaction energy defined by Eq. (12) is always negative. Moreover, when l is large compared with M_k the following expansion holds:

$$E_0=-\frac{M_1M_2}{l}+\text{higher order terms}. \tag{23}$$

Giulini [21] has computed the higher order terms for these data and other conformally flat time symmetric data with different topologies. In those examples the Newtonian term is invariant but the higher order terms depend on the particular initial data.

In order to discuss spin-spin interaction, we need initial data with nontrivial angular momentum, that is, we have to allow for nontrivial extrinsic curvature in the data. At each end we have the angular momentum J_k given by

$$J_k^a=\frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r} r\,K_{bc}n^b\epsilon^{cad}n_d dS_r, \tag{24}$$

where S_r is a two-sphere defined in the asymptotic region $\tilde{S}_k$ and n^a is its outward unit normal vector. In Eq. (24) we can use either K^{ab} or $\tilde{K}^{ab}$ because the conformal factor satisfies Eq. (17). In general the angular momentum at each end is *not* determined by the intrinsic angular momentum of each black hole. It includes also the angular momentum of the gravitational field surrounding the black holes. Then, in general, there is no relation between J_0, J_1, and J_2; these three quantities can be freely prescribed. But in the presence of symmetries these quantities cannot be given freely any more. Moreover, in the presence of *conformal symmetries* of the metric there exists a well defined quasilocal definition of angular momentum. Assume that ξ^a is a conformal Killing vector; that is, a solution of the equation $(\mathcal{L}_h\xi)^{ab}=0$, where

$$(\mathcal{L}_h\xi)^{ab}=D^a\xi^b+D^b\xi^a-\frac{2}{3}h^{ab}D_c\xi^c. \tag{25}$$

If the initial data are maximal, i.e., $K=0$, then the vector $K^{ab}\xi_b$ is divergence free. Hence, for each conformal symmetry ξ^a we have the associated integral

$$I_\xi=\int_S K^{ab}\xi_b n_a\,dS, \tag{26}$$

where S is a close two-surface and n^a its outward unit normal vector. This integral is a conformal invariant. It can be calculated also in terms of tilde quantities. The integral (26) will be nonzero only if the vector $K^{ab}\xi_b$ is singular at some points; in our case it will be singular at two points: the locations of the holes. Then the integral in Eq. (26) will have three different values depending on whether the surface S encloses one hole, two holes, or no hole. In the last case, $I_\xi=0$. If we chose ξ^a to be a rotation, I_ξ will give the corresponding component of the quasilocal angular momentum. If the data are conformally flat we have ten conformal Killing vectors. In particular, we have three rotations and hence a complete definition of the quasilocal angular momentum. These quantities will be defined only on this slice and will generally not be preserved in the evolution. They will only be preserved if the space-time admits a Killing vector. In this case they will coincide with the corresponding Komar integral. The space time will admit a Killing vector field if ξ^a is a Killing vector for the whole initial data set; that is, $\pounds_\xi\tilde{h}_{ab}=\pounds_\xi\tilde{K}^{ab}=0$, where $\pounds_\xi$ is the Lie derivative with respect to ξ^a. A conformally flat, maximal slice can be interpreted as an instant of time in which the gravitational field carries no angular momentum and no linear momentum itself, and hence these quantities are carried only by the "sources," which in this case are the black holes. Data containing matter with compact support can also be constructed.

There exist in the literature other definitions of quasilocal angular momentum [22–25], which are applicable for an arbitrary closed two-surface in the spacetime. It is not clear if any of these definitions will agree with Eq. (26) in the particular case of a two-surface lying on a conformally flat three-hypersurface.

From the discussion above, we conclude that in the case of conformally flat, maximal data we have

$$J_0^a+J_1^a+J_2^a=0. \tag{27}$$

For an observer placed at the asymptotic end 0 the system will look like two black holes with spins $-J_1^a$ and $-J_2^a$, and the total angular momentum will be $J_0^a=-J_1^a-J_2^a$. For a more general discussion of conformal symmetries in initial data, see [26] and [27]; in particular in those articles a generalization of Eq. (27) that includes linear momentum is proved.

Bowen and York obtain a simple model for a conformally flat data set which represents two black holes with spins [28]. Brandt and Brügmann [29] study these data with boundary conditions for the $N+1$ asymptotic ends given by Eqs. (18), (16), (20), and (17). For these data, the conformal second fundamental form is given by

$$K^{ab}=K_1^{ab}+K_2^{ab}, \tag{28}$$

where

$$K_1^{ab}=\frac{6}{r_1^3}n_1^{(a}\epsilon^{b)cd}J_{1\,c}n_{1\,d},$$

$$K_2^{ab}=\frac{6}{r_2^3}n_2^{(a}\epsilon^{b)cd}J_{2\,c}n_{2\,d},\tag{29}$$

and

$$n_1^i=\frac{x^i-x_1^i}{r_1},\quad n_2^i=\frac{x^i-x_2^i}{r_2},\tag{30}$$

where J_1^c, J_2^c are constants, and ϵ^{bcd} is the flat volume element. One can check that the constants J_1^c and J_2^c give the angular momentum at ends 1 and 2, respectively. The tensors (29) are divergence free and trace free with respect to the flat metric in $\mathbb{R}^3\setminus\{i_1,i_2\}$.

The first result of this article is the following theorem.

Theorem 2. Let h_{ab} be the flat metric and K^{ab} be given by Eq. (28). Then the interaction energy (12) is given by

$$E_0=\frac{-M_1M_2}{l}+\frac{-J_1\cdot J_2+3(J_1\cdot\hat{n})(J_1\cdot\hat{n})}{l^3}$$
$$+\text{higher order terms.}\tag{31}$$

The expansion of the dimensionless quantity E_0/l is made in terms of the dimensionless parameters M_1/l, M_2/l, J_1/l^2, and J_2/l^2. By "higher order terms" we mean terms of cubic order in those parameters. We prove Theorem 2 in Sec. III.

Using similar arguments, Gibbons [30,31] has obtained the qualitative sign dependence of the spin-spin interaction for a certain class of axially symmetric, conformally flat data. Note that in Theorem 2 the data are nonaxially symmetric in general, since the spins can point in arbitrary directions. Bonnor [32] studied the spin-spin interaction using an exact, axially symmetric solution of the Einstein-Maxwell equations; his result also qualitatively agrees with the spin-spin term in Eq. (31).

The Bowen-York data are, of course, very special. The natural question is how to generalize Theorem 2 for more general data. For general asymptotically flat data with three asymptotic ends we cannot even expect to recover the Newtonian interaction term. Take, for example, time symmetric initial data with only two ends. Choosing the conformal metric appropriately, one can easily construct data such that the difference M_1-M_0 is arbitrary. That means that we are putting more radiation in one end than in the other. If we add a third end with small mass, the new interaction energy will be dominated by the difference M_1-M_0 and hence will be not related to any Newtonian force. Hence, Theorem 1 is not true for general asymptotically flat metrics with many asymptotic ends.

The interaction energy defined by Eq. (12) can have the meaning of a two-black-hole interaction energy only if it is possible to distinguish in the data two objects that are similar to the Kerr black hole when the separation distance is large. I will call this class of data *two Kerr-like black hole initial data*. The existence of these data has been proved in [33,20]. The following two properties of the Kerr initial data, in the standard Boyer-Lindquist coordinates, are important: (i) The data are conformally flat up to order $O(J^2)$; (ii) the leading order term of the second fundamental form is given by the Bowen-York one (29). Then one can expect that Eq. (31) is unchanged in the principal terms for this class of data.

More precisely, two Kerr-like black hole initial data can be constructed as follows (see [33] and [20] for details). Take a slice of the Kerr metric, with parameters M_{K_1} and J_1, in the standard Boyer-Lindquist coordinates. Choosing the appropriate conformal factor, the conformal metric can be written in the following form:

$$h_{ab}^{K_1}=\delta_{ab}+h_{ab}^{R_1},\tag{32}$$

where $h_{ab}^{R_1}=O(J_1^2)$. In the same way the conformal second fundamental form can be written as

$$K_{K_1}^{ab}=K_1^{ab}+K_{R_1}^{ab},\tag{33}$$

where $K_{R_1}^{ab}=O(J_1^2)$ and K_1^{ab} is given by Eq. (29). Take another Kerr metric, with parameters M_{K_2} and J_2, and define the following conformal metric:

$$h_{ab}^{KK}=\delta_{ab}+h_{ab}^{R_1}+h_{ab}^{R_2},\tag{34}$$

and the following conformal second fundamental form:

$$K_{KK}^{ab}=\overline{K}_{K_1}^{ab}+\overline{K}_{K_2}^{ab}+(\mathcal{L}_{h_{KK}}w)^{ab},\tag{35}$$

where the bar means the trace-free part of the tensor with respect to the metric (34) and w^a is chosen such that K_{KK}^{ab} is divergence free and trace free with respect to the metric (34). In [33,20] it was proved that such a vector w^a exists and is unique. Using Eqs. (34) and (35), solve Eq. (14) with the boundary conditions (20) and (17) where

$$m_1=\sqrt{M_{K_1}^2-J_1^2/M_{K_1}^2},$$

$$m_2=\sqrt{M_{K_2}^2-J_2^2/M_{K_2}^2}.\tag{36}$$

The existence of a unique solution has been proved in [33,20].

If we chose J_1^a and J_2^a to point in the same direction, the data will be axially symmetric. In this particular case, we can use the integral (26) to calculate the quasilocal angular momentum of each of the black holes. The result will be J_1^a and J_2^a. However, in general, this class of data will admit no conformal Killing vector. In this general situation, it is very hard to compute the quasilocal spins of each of the black holes. However, when the separation distance is large, J_1^a and J_2^a will give approximately the angular momentum of each of the black holes because this class of data has a far limit to the

Kerr initial data. In other words, J_1^a and J_2^a give the spins of one black hole when the parameters of the other are set to zero.

For this class of data we have the following result.

Collollary 1. For the two Kerr-like data defined above, the formula (31) for the interaction energy holds.

We prove this corollary in Sec. IV.

III. INTERACTION ENERGY FOR SPINNING BOWEN-YORK INITIAL DATA

In this section we will prove Theorem 2. For Bowen-York data, with the boundary conditions (20) and (17), Eq. (14) for the conformal factor can be written in the following form [29]:

$$\Delta u = -\frac{K^{ab}K_{ab}}{8\varphi^7}, \quad \varphi = \varphi_0 + u, \tag{37}$$

with the boundary condition

$$\lim_{r\to\infty} u = 0, \tag{38}$$

where φ_0 is defined in Eq. (22), with m_1 and m_2 arbitrary positive constants, and $K^{ab}=K_1^{ab}+K_2^{ab}$ is given by Eq. (29).

The coordinates x^i are asymptotic coordinates for the end 0. The total mass at 0 is given by

$$M_0 = m_1 + m_2 + 2u_\infty, \tag{39}$$

where u_∞ is the term that varies as $1/r$ in the solution u of Eq. (37) and is given by

$$u_\infty = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{K^{ab}K_{ab}}{8\varphi^7} dx^3. \tag{40}$$

We want to calculate the masses M_1 and M_2 for the other ends. The asymptotic coordinates for the other ends are

$$\hat{x}^i_{i_1} = \frac{m_1^2}{4}\frac{(x^i - x_1^i)}{r_1^2}, \quad \hat{x}^i_{i_2} = \frac{m_2^2}{4}\frac{(x^i - x_2^i)}{r_2^2}. \tag{41}$$

Take, for example, the end point i_1, in $\hat{x}^i_{i_1}$ coordinates we have

$$\tilde{h}_{\hat{i}\hat{j}} = \left(1 + \frac{m_1}{2\hat{r}_1} + \frac{m_1 m_2}{4l\hat{r}_1} + \frac{u(i_1)}{2\hat{r}_1}\right)\delta_{\hat{i}\hat{j}} + O(1/\hat{r}_1^2), \tag{42}$$

where we have used

$$r_2 = l + O(1/\hat{r}_1), \tag{43}$$

and $\hat{r}_1$ is the Euclidean radius with respect to $\hat{x}^i_{i_1}$. Then the mass at this end is given by

$$M_1 = m_1\left(1 + \frac{m_2}{2l} + u(i_1)\right), \tag{44}$$

where $u(i_1)$ denote the value of the function u at the point i_1. In an analogous way we obtain the mass at the end 2

$$M_2 = m_2\left(1 + \frac{m_1}{2l} + u(i_2)\right). \tag{45}$$

Then the interaction energy at the end i_0 is given by

$$\begin{aligned} E_0 &= M_0 - M_1 - M_2 \\ &= -\frac{m_1 m_2}{l} + 2u_\infty - m_1 u(i_1) - m_2 u(i_2). \end{aligned} \tag{46}$$

Using Eq. (37) and the Green's function for the Laplacian, we obtain the following integral representation of the terms involving u in Eq. (46):

$$\begin{aligned} &2u_\infty - m_1 u(i_1) - m_2 u(i_2) \\ &= \frac{1}{16\pi}\int_{\mathbb{R}^3} \frac{K^{ab}K_{ab}}{\varphi^7}\left(1 - \frac{m_1}{2r_1} - \frac{m_2}{2r_2}\right) dx^3. \end{aligned} \tag{47}$$

The formula (47) involves the unknown function u. Using the fact that u is $O(J^2)$, we make an expansion of this integral in terms of the parameters J_1/l^2, J_2/l^2, m_1/l, and m_2/l. We obtain that the first nontrivial term is given by

$$2u_\infty - m_1 u(i_1) - m_2 u(i_2) \approx E^s, \tag{48}$$

where

$$E^s = \frac{1}{8\pi}\int_{\mathbb{R}^3} K_1^{ab} K_{2\,ab}\, dx. \tag{49}$$

The interaction energy, up to this order, is given by

$$E = -\frac{M_1 M_2}{l} + E^s, \tag{50}$$

where we have used Eqs. (44) and (45) to replace m_k by M_k, since up to this order they are equal.

All that remains is to compute the integral E^s. This integral can, in principle, be calculated explictly from the expressions (29). However, such a calculation is very complicated. Instead of this we will calculate Eq. (49) in the following way. The tensors (29) can be written as

$$(\mathcal{L}_\delta v_1)^{ab} = K_1^{ab}, \quad (\mathcal{L}_\delta v_2)^{ab} = K_2^{ab}, \tag{51}$$

where $\mathcal{L}_\delta$ is the conformal Killing operator defined in Eq. (25) for the flat metric, and

$$v_1^i = -\epsilon^{ijk} J_{1j} n_{1\,k} r_1^{-2}, \tag{52}$$

$$v_2^i = -\epsilon^{ijk} J_{2j} n_{2\,k} r_2^{-2}. \tag{53}$$

Let B_{ϵ_1} and B_{ϵ_2} be small balls centered at i_1 and i_2, respectively, of radii ϵ_1 and ϵ_2. We have that

$$E^s = \frac{1}{8\pi}\lim_{\epsilon_1,\epsilon_2\to 0}\int_{\mathbb{R}^3 - B_{\epsilon_1} - B_{\epsilon_2}} K_1^{ab} K_{2ab} dx. \tag{54}$$

Using the Gauss theorem in $\mathbb{R}^3 - B_{\epsilon_1} - B_{\epsilon_2}$ we obtain

$$\int_{\mathbb{R}^3 - B_{\epsilon_1} - B_{\epsilon_2}} K_1^{ab} K_{2ab}\, dx = -2 \int_{\partial B_{\epsilon_1}} K_2^{ab} v_{1b} n_{1a} dS_{\epsilon_1} - 2 \int_{\partial B_{\epsilon_2}} K_2^{ab} v_{1b} n_{2a} dS_{\epsilon_2}, \tag{55}$$

where n_1^a and n_2^a are the outward normals to the two surfaces B_{ϵ_1} and B_{ϵ_2}. In the limit $\epsilon_1 \to 0$ the first integral vanishes. We use here that K_2^{ab} is regular in B_{ϵ_1}. The second integral can easily be calculated in the limit $\epsilon_2 \to 0$. We obtain

$$E^s = \frac{-J_1 \cdot J_2 + 3(J_1 \cdot \hat{n})(J_1 \cdot \hat{n})}{l^3}. \tag{56}$$

IV. INTERACTION ENERGY FOR TWO KERR-LIKE BLACK HOLES INITIAL DATA

The two Kerr-like black hole initial data are solutions of the following equation:

$$L_{h^{KK}} \varphi_{KK} = -\frac{K_{KK\,ab} K_{KK}^{ab}}{8 \varphi_{KK}^7} \quad \text{on} \quad \tilde{S}, \tag{57}$$

with the boundary conditions (20) and (17), where h_{ab}^{KK} and K_{KK}^{ab} are given by Eqs. (34),(35) and m_1, m_2 are given by Eq. (36) in terms of the Kerr parameters.

The conformal factor for the Kerr initial data with parameters M_{K_1}, J_1 can be written in the following form:

$$\varphi_{K_1} = 1 + \frac{m_1}{2 r_1} + \varphi_{R_1}, \tag{58}$$

where $\varphi_{R_1} = O(J_1^2)$. We can decompose the two Kerr-like solution in the following form:

$$\varphi_{KK} = \varphi_0 + \varphi_{R_1} + \varphi_{R_2} + u. \tag{59}$$

Using that Eq. (58) is a solution for a one Kerr-like initial data set, we obtain that the first term in the expansion in J_1 and J_2 of the function u satisfies the following linear equation:

$$\Delta u \approx -\frac{K_1^{ab} K_{2\,ab}}{8 \varphi_0^7}. \tag{60}$$

Hence, the spin-spin interaction term has the same form as the Bowen-York one. In an analogous way to the previous section, we obtain that the interaction energy is given by

$$E_0 \approx -\frac{M_1 M_2}{l} + E^s, \tag{61}$$

where E^s is given by Eq. (49).

V. DISCUSSION

We have shown, using the interaction energy defined by Eq. (12), that the spin-spin interaction between black holes of arbitrary masses and spins has an expansion of the form (31). This formula was previously derived using a test particle approximation [1] and post-Newtonian expansions [10]. The main improvement of the present calculation, apart from its simplicity, is that no approximation is used in the definition of the interaction energy. Moreover, the very definition of the interaction energy involves black holes, in contrast with previous calculations where the black holes appear indirectly.

The interaction energy defined by Eq. (12) uses the fact that we are choosing a particular topology for the initial data, but other topologies are possible. Examples of different kinds of topologies where we cannot use Eq. (12) are the Misner topology of two isometric sheets [34], the Misner wormhole [35], or even initial data with trivial topologies which contain an apparent horizon [36]. If the initial data have k disconnected apparent horizons of area A_k, we can define the individual masses as follows:

$$M_{H_k} = \sqrt{\frac{A_k}{16\pi}}. \tag{62}$$

Then the interaction energy is given by the formula

$$E_H = M - \sum_{\substack{k'=0 \\ k' \neq k}}^{N} M_{H'_k}. \tag{63}$$

What is the relation between E_k defined by Eq. (12) and E_H defined by Eq. (63)? Note that E_H is much harder to compute than E_k. I want to argue that E_k will presumably give the same result as E_k for the leading order terms in the expansion given by Theorem 2. Assume that the data are such that, when the separation distance parameter is large, then the areas can by approximated by the Schwarzschild formula $A_k \approx 16\pi M_k^2$ plus terms of order J^2. Then the radii of the horizons will be $R_k \approx 2 M_k$. The distance $l_{\tilde{h}}$ will differ from l in terms of order $O(M, J^2)$. Then we can replace l by $l_{\tilde{h}}$ in theorem $l_{\tilde{h}}$ and M_{H_k} by M_k in Eq. (12), up to this order.

It is interesting to note that the interaction energy E_H has been used in a different context, namely, to determine the last stable circular orbit in a black hole collision [37].

ACKNOWLEDGMENTS

I would like to thank J. Isenberg and J. Valiente-Kroon for their reading of the first draft of the paper and L. Szabados for discussions concerning quasilocal angular momentum. I would also like to acknowledge the friendly hospitality of the American Institute of Mathematics (AIM), where part of this work was done.

[1] R. Wald, Phys. Rev. D **6**, 406 (1972).
[2] S.W. Hawking, Commun. Math. Phys. **25**, 152 (1972).
[3] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).
[4] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[5] R. Wald, in *Black Holes, Gravitational Radiation and the Universe*, edited by B.R. Iyer and B. Bhawal, Fundamental Theories of Physics Vol. 100 (Kluwer Academic, Dordrecht, 1999), pp. 69–85; gr-qc/9710068.
[6] A. Komar, Phys. Rev. **119**, 934 (1959).
[7] J. Baker, B. Brügmann, M. Campanelli, C.O. Lousto, and R. Takahashi, Phys. Rev. Lett. **87**, 121103 (2001).
[8] J. Baker, M. Campanelli, C.O. Lousto, and R. Takahashi, Phys. Rev. D **65**, 124012 (2002).
[9] L. Lehner, Class. Quantum Grav. **18**, R25 (2001).
[10] P.D. D'Eath, Phys. Rev. D **12**, 2183 (1975).
[11] D. Brill and R.W. Lindquist, Phys. Rev. **131**, 471 (1963).
[12] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), pp. 227–265.
[13] D. Gannon, J. Math. Phys. **16**, 2364 (1975).
[14] G.B. Cook, Living Rev. Relativ. **3**, 5 (2001), http://www.livingreviews.org/Articles/Volume3/2000-5cook/
[15] Y. Choquet-Bruhat, J. Isenberg, and J.W. York, Jr., Phys. Rev. D **61**, 084034 (2000).
[16] Y. Choquet-Bruhat and J.W. York, Jr., in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980), Vol. 1, pp. 99–172.
[17] R. Beig and N.O. Murchadha, Class. Quantum Grav. **11**, 419 (1994).
[18] H. Friedrich, Commun. Math. Phys. **119**, 51 (1988).
[19] H. Friedrich, J. Geom. Phys. **24**, 83 (1998).
[20] S. Dain, Phys. Rev. D **64**, 124002 (2001).
[21] D. Giulini, Class. Quantum Grav. **7**, 1271 (1990).
[22] R. Penrose, Proc. R. Soc. London **A381**, 53 (1982).
[23] M. Ludvigsen and J.A.G. Vickers, J. Phys. A **16**, 1155 (1983).
[24] A. Dougan and L. Mason, Phys. Rev. Lett. **67**, 2119 (1991).
[25] L.B. Szabados, Class. Quantum Grav. **13**, 1661 (1996).
[26] R. Beig and N.O. Murchadha, Commun. Math. Phys. **176**, 723 (1996).
[27] S. Dain and H. Friedrich, Commun. Math. Phys. **222**, 569 (2001).
[28] J.M. Bowen and J.W. York, Jr., Phys. Rev. D **21**, 2047 (1980).
[29] S. Brandt and B. Brügmann, Phys. Rev. Lett. **78**, 3606 (1997).
[30] G.W. Gibbons (private communication).
[31] S.W. Hawking, in *Black Holes*, edited by C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1973), pp. 1–56.
[32] W.B. Bonnor, Class. Quantum Grav. **19**, 149 (2002).
[33] S. Dain, Phys. Rev. Lett. **87**, 121102 (2001).
[34] C.W. Misner, Ann. Phys. (N.Y.) **24**, 102 (1963).
[35] C.W. Misner, Phys. Rev. **118**, 1110 (1960).
[36] R. Beig and N.O. Murchadha, Phys. Rev. Lett. **66**, 2421 (1991).
[37] G.B. Cook, Phys. Rev. D **50**, 5025 (1994).