Slowly rotating charged fluid balls and their matching to an exterior domain

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The slow-rotation approximation of Hartle is developed to a setting where a charged rotating fluid is present. The linearized Einstein-Maxwell equations are solved on the background of the Reissner-Nordström space-time in the exterior electrovacuum region. The theory is put to action for the charged generalization of the Wahlquist solution found by García. The García solution is transformed to coordinates suitable for the matching and expanded in powers of the angular velocity. The two domains are then matched along the zero pressure surface using the Darmois-Israel procedure. We prove a theorem to the effect that the exterior region is asymptotically flat if and only if the parameter C_2, characterizing the magnitude of an external magnetic field, vanishes. We obtain the form of the constant C_2 for the García solution. We conjecture that the García metric cannot be matched to an asymptotically flat exterior electrovacuum region even to first order in the angular velocity. This conjecture is supported by a high precision numerical analysis.

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I. INTRODUCTION

There are a surprisingly few rotating perfect fluid solutions of Einstein's relativistic equations known to date. One of the most comprehensive of these is the rigidly rotating charged perfect fluid solution of García [1]. This metric is type D in the Petrov classification of the curvature. The Einstein-Maxwell equations are satisfied with a stress tensor which is the sum of that of the perfect fluid and the Maxwell field. The fluid medium carries electric charge, and it has a divergence-free four-current. Because of the tantalizing lack of explicit, relativistically rotating fluid stellar models in the literature, it is only natural to ask if it is possible to join a suitably chosen domain of the García solution to an external source-free Einstein-Maxwell region.

The García solution reduces to the Wahlquist space-time in the absence of electric charge. Unfortunately, as we have proven in an earlier publication [2], the Wahlquist solution is unsuitable as a model of isolated an relativistic object. *A priori* it is unclear if the presence of electric charge can supply the necessary ingredients for a smooth matching.

The purpose of the present paper is to search for the conditions of matching the García metric to an asymptotically well-behaved external electrovacuum domain. For the matching we use the Darmois-Israel procedure [3–6], stating that the induced metrics and induced extrinsic curvatures should agree along the matching surface. From these conditions it follows that the matching surface agrees with the zero pressure surface of the interior as desired. The condition on the extrinsic curvatures also ensures that there are no surface layers of matter. We further assume that there are no surface charges or currents implying continuity of the electromagnetic field tensor across the matching surface.

The García solution may be expanded in powers of the rotation parameter of the fluid and it becomes spherically symmetric in the non-rotating limit. It is hence appropriate to match it to rotating perturbations of the Reissner-Nordström solution. A suitable vehicle for performing this is the slow-rotation approximation scheme developed by Hartle [7]. If the fluid interior can be smoothly matched to the external region for arbitrary values of the rotation parameter then the matching conditions would be satisfied to any finite order in the expansion parameter. Conversely, when one can show to a given finite order that no exterior domain can be joined then there is no exact global solution either. We could successfully use this approach in our previous computations for the Wahlquist space-time [2]. In its original form, the Hartle method matches an internal fluid domain with an external vacuum. An essential tenet of this approach is the independence of the first-order perturbation function ω of the polar angle ϑ. For a generic setting in the presence of the Maxwell field, one would have to go through the proof of this independence. In the present case, however, we can omit this general theory since ω is a function of the radius alone in the slowly rotating García space-time. As a consequence of the matching conditions, then, this property is inherited by the electrovacuum exterior.

In this paper we apply the slow-rotation approximation to the García space-time. Our result is that the electric charge works against the conditions of matching. We find that matching to an asymptotically flat electrovacuum exterior is impossible already to the first order in the angular velocity. This comes as a surprise to us since all uncharged fluids can be matched to an asymptotically flat vacuum exterior to first order in the rotation parameter. This is because such vacuum space-times do not differ from the first-order form of the the Kerr metric.

The paper is organized as follows. In Sec. II we compute the García metric in various forms that will be necessary to launch the matching process. In Sec. II B we take the static limit which, of course, is spherically symmetric. Next, in Sec. II C we assemble the slowly rotating form of the metric. In Sec. III we prepare the field quantities in the external electrovacuum domain. Thus we get the general solution for

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the first-order perturbation function ω, and we investigate its asymptotic properties. The actual matching of the two domains takes place in Sec. IV. As before, we require that the Darmois-Israel conditions [3] are satisfied. The details are worked out in Sec. IV A, generalizing the theory in the presence of a Maxwell field. The solution of the matching conditions for the static state and to first order in the angular velocity is carried out in Secs. IV B and IV C, respectively. The results are further discussed in Sec. V. The Appendix is devoted to the issue of the choice of independent parameters.

II. THE INTERIOR METRIC

The charged generalization of the Wahlquist solution given by García [1] has the metric

$$ds^2=\frac{P}{\Delta}(d\tau+\delta N d\sigma)^2-\frac{Q}{\Delta}(d\tau+\delta M d\sigma)^2+\Delta\left(\frac{dx^2}{P}+\frac{dy^2}{Q}\right) \tag{1}$$

with the real functions

$$\Delta=M-N$$

$$M=\frac{1}{k^2}\sinh^2(kx)-\xi_0^2$$

$$N=-\frac{1}{k^2}\sin^2(ky)-\xi_0^2 \tag{2}$$

$$P=a+\frac{1}{2k}[2n+x(\alpha+\beta^2)]\sinh(2kx)+[b-(g+\beta x)^2]\cosh(2kx)$$

$$Q=a-\frac{1}{2k}[2m+y(\alpha+\beta^2)]\sin(2ky)+[b+(e+\beta y)^2]\cos(2ky).$$

Here a, b, e, g, k, m, n, α, β, δ and ξ_0 are eleven constants. The number of independent parameters modulo diffeomorphisms is eight (cf. the Appendix). This metric is a solution of Einstein's equations $G_{\alpha\beta}=8\pi(T^{(f)}_{\alpha\beta}+T^{(e)}_{\alpha\beta})$ with

$$T^{(f)}_{\alpha\beta}=(\mu+p)u_\alpha u_\beta+pg_{\alpha\beta}$$

$$T^{(e)}_{\alpha\beta}=\frac{1}{4\pi}\left(F_{\alpha\gamma}F^{\gamma}_{\beta}-\frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}\right)$$

where p is the pressure, μ the density, u^α the velocity of the fluid, and $F_{\alpha\beta}$ is the electromagnetic field tensor. In comoving coordinates, the velocity vector is $u=u^0\partial/\partial\tau$. The pressure and density are given by

$$8\pi p=-\frac{k^2}{\Delta}(Q-P)+\alpha k^2+\Sigma$$

$$8\pi\mu=3\frac{k^2}{\Delta}(Q-P)-\alpha k^2-\Sigma$$

$$\Sigma=-\frac{2\beta k}{\Delta}[(e+\beta y)\sin(2ky)+(g+\beta x)\sinh(2kx)].$$

The electromagnetic four-potential A_α determines the field tensor[1] as $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$, and its components are

$$A_x=A_y=0$$

$$A_\sigma=-\frac{\delta}{2\Delta k}[(e+\beta y)M\sin(2ky)+(g+\beta x)N\sinh(2kx)]$$

$$A_\tau=-\frac{1}{2\Delta k}[(e+\beta y)\sin(2ky)+(g+\beta x)\sinh(2kx)]=\frac{\Sigma}{4k^2\beta}.$$

The electric current vector $j^\alpha=\rho u^\alpha$ satisfies the field equation $4\pi j^\alpha=F^{\alpha\beta}_{;\beta}$. The current $4\pi j^\alpha=2k^2\beta\delta^\alpha_\tau$ has no divergence (as it should, since the Maxwell equations are satisfied) and the conserved charge density is

$$\rho=\frac{k^2\beta}{2\pi}\sqrt{\frac{Q-P}{\Delta}}.$$

Note that this solution has been named the *Wahlquist-Newman metric* in [8].

A. The Wahlquist form

Our goal in this section is to bring the metric (1) to a form which reduces to that of the Wahlquist solution in the no-charge limit. The static limit of the Wahlquist form will, in turn, yield the charged generalization of the Whittaker spacetime. To achieve this, we transform to the new coordinates ξ and ζ by setting

$$k\xi=\sinh(kx),\; k\zeta=\sin(ky). \tag{3}$$

Then we have

$$\Delta=\xi^2+\zeta^2. \tag{4}$$

In order to be able to introduce Wahlquist's functions h_1 and h_2, we rescale the functions P and Q using a new constant parameter r_0:

[1] García [1] uses the definition $F_{\alpha\beta}=2A_{[\alpha,\beta]}$, which gives rise to an overall sign difference in A_α.

$$h_1 = r_0^2 Q, \quad h_2 = r_0^2 P.$$

To choose the factors of the other terms appropriately, we also rescale the τ and σ coordinates as

$$\varphi = -\frac{1}{r_0^2}\sigma, \quad t = \frac{1}{r_0}\tau.$$

The metric takes the form

$$ds^2 = -\frac{h_1 - h_2}{\Delta}(dt - \mathbf{A} d\varphi)^2 + r_0^2 \delta^2 \Delta \frac{h_1 h_2}{h_1 - h_2} d\varphi^2 + \Delta r_0^2 \left[\frac{d\zeta^2}{(1-k^2\zeta^2)h_1} + \frac{d\xi^2}{(1+k^2\xi^2)h_2} \right] \tag{5}$$

where $\mathbf{A}$ is Wahlquist's function,

$$\mathbf{A} = \delta r_0 \left(\frac{h_1 \xi^2 + h_2 \zeta^2}{h_1 - h_2} - \xi_0^2 \right). \tag{6}$$

We introduce the rescaled constants

$$\overline{m} = m r_0^3, \quad \overline{b} = -n r_0^3$$
$$\overline{e} = e r_0, \quad \overline{g} = g r_0, \quad \overline{\beta} = \beta r_0 \tag{7}$$

and three new constants as follows. In place of α we shall use the constant κ, the constant a will be replaced by C and finally E will take the place of the constant b, using the following definitions[2]:

$$\frac{1}{\kappa^2} = r_0^2(\alpha + \beta^2)$$
$$C = r_0^2(a+b) \tag{8}$$
$$E = -2r_0^2 b k^2 - \frac{1}{\kappa^2}.$$

We get for the functions

$$h_1 = \frac{\zeta}{\kappa^2}\left[\zeta - \frac{1}{k}\sqrt{1-k^2\zeta^2}\arcsin(k\zeta)\right] - \frac{2\overline{m}}{r_0}\zeta\sqrt{1-k^2\zeta^2} + \left[\overline{e} + \frac{\overline{\beta}}{k}\arcsin(k\zeta)\right]^2 (1-2k^2\zeta^2) + C + E\zeta^2 \tag{9}$$

$$h_2 = -\frac{\xi}{\kappa^2}\left[\xi - \frac{1}{k}\sqrt{1+k^2\xi^2}\operatorname{arcsinh}(k\xi)\right] - \frac{2\overline{b}}{r_0}\xi\sqrt{1+k^2\xi^2} - \left[\overline{g} + \frac{\overline{\beta}}{k}\operatorname{arcsinh}(k\xi)\right]^2 (1+2k^2\xi^2) + C - E\xi^2. \tag{10}$$

The electromagnetic potential 1-form $A = A_\alpha dx^\alpha$ is

$$A = -\frac{1}{\Delta}\left[\overline{g} + \frac{\overline{\beta}}{k}\operatorname{arcsinh}(k\xi)\right]\xi\sqrt{1+k^2\xi^2}[dt + (\zeta^2 + \xi_0^2)\delta r_0 d\varphi] - \frac{1}{\Delta}\left[\overline{e} + \frac{\overline{\beta}}{k}\arcsin(k\zeta)\right]\zeta\sqrt{1-k^2\zeta^2} \times [dt - (\xi^2 - \xi_0^2)\delta r_0 d\varphi]. \tag{11}$$

The pressure and density become

$$8\pi p = -\frac{k^2}{\Delta r_0^2}(h_1 - h_2) + \frac{k^2}{r_0^2\kappa^2}(1 - \overline{\beta}^2\kappa^2) + \Sigma \tag{12}$$

$$8\pi\mu = 3\frac{k^2}{\Delta r_0^2}(h_1 - h_2) - \frac{k^2}{r_0^2\kappa^2}(1 - \overline{\beta}^2\kappa^2) - \Sigma \tag{13}$$

$$\Sigma = \frac{4\overline{\beta}k^2}{r_0^2}A_t. \tag{14}$$

The expressions (5)–(14) contain a large number of parameters: C, E, $\overline{m}$, $\overline{b}$, k, κ, $\overline{e}$, $\overline{g}$, $\overline{\beta}$, r_0, δ and ξ_0. How many of these 12 constants are necessary to uniquely describe the metric? This question will next be addressed.

By using linear coordinate transformations of t and φ we can obviously set $\delta = 1$ and $\xi_0 = 0$. The parameter r_0 has been introduced in order to enable ourselves to go to the slow-rotation limit. The scaling of this parameter can be chosen arbitrarily. Defining $r_0' = c r_0$, $C' = c^2 C$, $E' = c^2 E$, $\overline{m}' = c^3 \overline{m}$, $\overline{b}' = c^3 \overline{b}$, $\kappa' = \kappa / c$, $\overline{e}' = c\overline{e}$, $\overline{g}' = c\overline{g}$, $\overline{\beta}' = c\overline{\beta}$, where c is a constant, the $d\zeta^2$ and $d\xi^2$ terms in the metric remain unchanged with the primed quantities. However, we need to perform a transformation of the coordinates $t = ct'$, $\varphi = c^2\varphi'$ to keep the structure of the other components. This shows that the two sets of constants $\{C, E, \overline{m}, \overline{b}, k, \kappa, \overline{e}, \overline{g}, \overline{\beta}, r_0\}$ and $\{C', E', \overline{m}', \overline{b}', k, \kappa', \overline{e}', \overline{g}', \overline{\beta}', r_0'\}$ provide parametrizations of a single physical state equivalent under the diffeomorphism $\{\zeta', \xi', \varphi', t'\} = \{\zeta, \xi, c^{-2}\varphi, c^{-1}t\}$.

Another freedom of choice follows from the coordinate transformation $\zeta'' = f\zeta$, $\xi'' = f\xi$, where f is a constant. In the metric (5) we have

$$\Delta = \xi^2 + \zeta^2 = \frac{1}{f^2}(\xi''^2 + \zeta''^2) = \frac{1}{f^2}\Delta''. \tag{15}$$

[2] We have slight differences from the notation of García [1]: $\overline{m}$ and $\overline{b}$ have different factors, κ includes a term β^2, and E and C are not set to 1 yet.

TABLE I. Equivalent sets of parameters.

1	C	E	$\overline{m}$	$\overline{b}$	k	κ	$\overline{e}$	$\overline{g}$	$\overline{\beta}$	r_0
$cf=1$	c^2C	E	$\overline{m}$	$\overline{b}$	$\frac{1}{c}k$	κ	$c\overline{e}$	$c\overline{g}$	$\overline{\beta}$	$\frac{1}{c}r_0$
$cf^2=1$	C	f^2E	$f^3\overline{m}$	$f^3\overline{b}$	fk	$\frac{1}{f}\kappa$	$\overline{e}$	$\overline{g}$	$f\overline{\beta}$	f^2r_0

Introducing the constants $k''=k/f$, $C''=f^4C$, $E''=f^2E$, $\overline{m}''=f^3\overline{m}$, $\overline{b}''=f^3\overline{b}$, $\kappa''=\kappa/f$, $\overline{e}''=f^2\overline{e}$, $\overline{g}''=f^2\overline{g}$, $\overline{\beta}''=f\overline{\beta}$, and defining h_1'' and h_2'' similarly to h_1 and h_2, we get

$$h_1=\frac{1}{f^4}h_1'', \quad h_2=\frac{1}{f^4}h_2''. \tag{16}$$

Then the $d\zeta^2$ and $d\xi^2$ terms transform appropriately. However, to get the correct transformation for the other terms in the metric, we need to change the coordinates as $t=ft''$ and $\varphi=f^3\varphi''$. It follows that the two sets of constants, $\{C,E,\overline{m},\overline{b},k,\kappa,\overline{e},\overline{g},\overline{\beta},r_0\}$ and $\{C'',E'',\overline{m}'',\overline{b}'',k'',\kappa'',\overline{e}'',\overline{g}'',\overline{\beta}'',r_0\}$, are equivalent descriptors.

At this stage, the coordinates are completely fixed. Hence there are *eight* independent physical parameters. The corresponding analysis of the parameters in the original coordinates is not needed in the present paper, but for convenience, is included as an Appendix. Using the coordinate transformations $x^{i\prime}=x^{i\prime}(x^k)$ and $x^{i''}=x^{i''}(x^k)$, for example, the constants C and E can always be made ± 1 or 0. García [1] discusses the case $C=E=1$, which for zero $\overline{e}$, $\overline{g}$ and $\overline{\beta}$ yields the original uncharged Wahlquist solution. When C or E can only be rescaled to -1 or 0, some different solutions might arise, as was first noted by Mars and Senovilla [9] in the uncharged case.

There exist two combinations of these transformations which preserve many of the constants. Table I. shows the sets of constants giving metrics equivalent to the original set $\{C,E,\overline{m},\overline{b},k,\kappa,\overline{e},\overline{g},\overline{\beta},r_0\}$, when choosing $cf=1$ and $cf^2=1$.

B. The static charged fluid

In this section we consider the static limiting case of the García space-time. Following the procedure of Wahlquist [10], we introduce a new radial coordinate

$$r=\zeta r_0 \tag{17}$$

and a new constant γ in place of k by substituting

$$k=\gamma r_0 \tag{18}$$

everywhere. The constant γ is related to Wahlquist's ρ_s by $\gamma^2=\kappa^2\rho_s$. The static limit can be obtained then by going to zero with r_0. In the limit $r_0\to 0$,

$$\lim_{r_0\to 0}\left(\frac{r_0^2}{r^2}h_1\right)=\tilde{h}_1 \tag{19}$$

where

$$\tilde{h}_1=E-\frac{2\overline{m}}{r}\sqrt{1-\gamma^2r^2}+\frac{1}{\kappa^2}\left[1-\frac{1}{\gamma r}\sqrt{1-\gamma^2r^2}\arcsin(\gamma r)\right]+\left[\frac{\tilde{e}}{r}+\frac{\overline{\beta}}{\gamma r}\arcsin(\gamma r)\right]^2(1-2\gamma^2r^2) \tag{20}$$

and $\tilde{e}=r_0\overline{e}$. The function h_2 has the limiting form

$$\tilde{h}_2=\lim_{r_0\to 0}h_2=C-E\xi^2-2\tilde{b}\xi-(\overline{g}+\overline{\beta}\xi)^2 \tag{21}$$

where $\tilde{b}=\overline{b}/r_0$. Using the limits $r_0^2\Delta\to r^2$, and $\mathbf{A}\to 0$, the metric becomes

$$ds^2=-\tilde{h}_1dt^2+\frac{dr^2}{(1-\gamma^2r^2)\tilde{h}_1}+r^2\left(\frac{d\xi^2}{\tilde{h}_2}+\delta^2\tilde{h}_2d\varphi^2\right). \tag{22}$$

A simpler form arises if we introduce a new radial coordinate z by setting

$$\gamma r=\sin z. \tag{23}$$

Then

$$\tilde{h}_1=E-2\overline{m}\gamma\cot z+\frac{1}{\kappa^2}(1-z\cot z)+(\tilde{e}\gamma+\overline{\beta}z)^2(\cot^2 z-1) \tag{24}$$

and

$$ds^2=-\tilde{h}_1dt^2+\frac{dz^2}{\gamma^2\tilde{h}_1}+\frac{\sin^2 z}{\gamma^2}\left(\frac{d\xi^2}{\tilde{h}_2}+\delta^2\tilde{h}_2d\varphi^2\right). \tag{25}$$

The only nonvanishing component of the electromagnetic potential is

$$A_t=-(\tilde{e}\gamma+\overline{\beta}z)\cot z.$$

The pressure (12) and the density (13) are

$$8\pi p=\gamma^2\left(-\tilde{h}_1+\frac{1}{\kappa^2}-\overline{\beta}^2\right)+\Sigma$$

$$8\pi\mu=\gamma^2\left(3\tilde{h}_1-\frac{1}{\kappa^2}+\overline{\beta}^2\right)-\Sigma \tag{26}$$

$$\Sigma=4\overline{\beta}\gamma^2A_t.$$

The bracketed term in Eq. (25) is the metric of a two-surface. We next show that there is a parameter range for which this two-surface is the two-sphere S^2. In order to get the metric of the two-sphere, we introduce a new coordinate θ in place of ξ by $\delta^2\tilde{h}_2=c_1\sin^2\theta$, where c_1 is some constant. Then

$$\delta^2\frac{d\tilde{h}_2}{d\xi}=2c_1\sin\theta\cos\theta\frac{d\theta}{d\xi} \tag{27}$$

and

$$\begin{aligned}\frac{d\xi^2}{\tilde{h}_2}&=\frac{4c_1^2\sin^2\theta\cos^2\theta}{\delta^4\tilde{h}_2}\left(\frac{d\tilde{h}_2}{d\xi}\right)^{-2}d\theta^2\\&=\frac{4(c_1-\delta^2\tilde{h}_2)}{\delta^2}\left(\frac{d\tilde{h}_2}{d\xi}\right)^{-2}d\theta^2.\end{aligned} \tag{28}$$

The condition

$$\frac{d\xi^2}{\tilde{h}_2}=c_2 d\theta^2 \tag{29}$$

where c_2 is another constant, is satisfied if and only if

$$\begin{aligned}c_2&=\frac{1}{E+\overline{\beta}^2}\\c_1&=\delta^2\left[C-\overline{g}^2+\frac{(\tilde{b}+\overline{g\beta})^2}{E+\overline{\beta}^2}\right].\end{aligned} \tag{30}$$

We can use the map in the third row of Table I to set $c_2=1$ by making $E+\overline{\beta}^2=1$. Of course, this can be done only when $E+\overline{\beta}^2$ is positive. The Lorentzian signature of the metric (25) holds whenever it is possible to set

$$C-\overline{g}^2+\frac{(\tilde{b}+\overline{g\beta})^2}{E+\overline{\beta}^2}=1 \tag{31}$$

using the second row of Table I. Finally we set $\delta=1$ by rescaling the φ coordinate. With these choices $\tilde{h}_2=1-(\xi+\tilde{b}+\overline{g\beta})^2$ and we end up with the simple coordinate transformation

$$\xi+\tilde{b}+\overline{g\beta}=\cos\theta. \tag{32}$$

Then

$$\begin{aligned}\tilde{h}_1=&1-\overline{\beta}^2-2\overline{m}\gamma\,\text{cot}z+\frac{1}{\kappa^2}(1-z\cot z)+(\tilde{e}\gamma+\overline{\beta}z)^2\\&\times(\cot^2 z-1),\end{aligned} \tag{33}$$

and the metric becomes

$$ds^2=-\tilde{h}_1dt^2+\frac{1}{\gamma^2}\left[\frac{dz^2}{\tilde{h}_1}+\sin^2 z(d\theta^2+\sin^2\theta d\varphi^2)\right]. \tag{34}$$

The center, determined by $z=0$, can be regular only if $\overline{m}=0$ and $\tilde{e}=0$.

C. The combined transformation

In the previous sections we have been led to transforming the García metric to the Wahlquist coordinates and hence to a form which is amenable for accessing the static limit. We now combine these procedures into a direct transformation, sidestepping the Wahlquist form, and compute the quantities for slow rotation.

The coordinate transformations (3), (17) and (23), with $k=\gamma r_0$, can be combined to the single transformation

$$z=ky. \tag{35}$$

García's parameters are expressed in terms of ours as follows:

$$\begin{aligned}&k=\gamma r_0,\quad m=\frac{\overline{m}}{r_0^3}\\&n=-\frac{\tilde{b}}{r_0^2},\quad \alpha=\frac{1}{r_0^2}\left(\frac{1}{\kappa^2}-\overline{\beta}^2\right)\\&e=\frac{\tilde{e}}{r_0^2},\quad g=\frac{\overline{g}}{r_0}\\&\beta=\frac{\overline{\beta}}{r_0},\quad a=\frac{C}{r_0^2}+\frac{1}{2r_0^4\gamma^2}\left(E+\frac{1}{\kappa^2}\right)\\&b=-\frac{1}{2r_0^4\gamma^2}\left(E+\frac{1}{\kappa^2}\right).\end{aligned} \tag{36}$$

We now substitute the new parameters given by Eq. (36) into the metric form (1) and introduce new coordinates by

$$y=\frac{z}{\gamma r_0},\quad \tau=r_0t,\quad \sigma=-r_0^2\varphi. \tag{37}$$

Using Table I, we find that the coordinate system is fixed by setting

$$\begin{aligned}&\delta=1,\quad \xi_0=0,\quad E=1-\overline{\beta}^2\\&C=1+\overline{g}^2-(\tilde{b}+\overline{g\beta})^2.\end{aligned} \tag{38}$$

The regularity at the center is ensured by

$$\overline{m}=0,\quad \tilde{e}=0. \tag{39}$$

The angular coordinate ϑ is introduced by writing

$$x=\cos\vartheta-\tilde{b}-\overline{g\beta}. \tag{40}$$

The coordinate ϑ only equals the θ defined in Sec. II B when the fluid is static. Then for small r_0 to linear order we obtain the metric

$$ds^2=-\tilde{h}_1dt^2+\frac{dz^2}{\gamma^2\tilde{h}_1}+\frac{\sin^2 z}{\gamma^2}[d\vartheta^2+\sin^2\vartheta(d\varphi-\omega dt)^2] \tag{41}$$

with

$$\tilde{h}_1=1-\overline{\beta}^2+\frac{1}{\kappa^2}(1-z\cot z)+\overline{\beta}^2z^2(\cot^2 z-1) \tag{42}$$

and

$$\omega=r_0\frac{\gamma^2}{\sin^2 z}(\tilde{h}_1-1). \tag{43}$$

The first-order calculation shows that ω does not depend on the angular coordinate. To this order, the only component of the four-potential A to pick up a small new contribution is A_φ:

$$A=-\overline{\beta}z\cot z\,dt+r_0[\overline{\beta}\sin^2\vartheta(1-z\cot z)-\overline{\beta}-\overline{g}\cos\vartheta]d\varphi. \tag{44}$$

Note that to first order in r_0, the metric is independent of the magnetic monopole charge parameter $\overline{g}$. In fact, the monopole contribution affects only the Maxwell equations but does not affect the gravitational equations.

III. THE ELECTROVACUUM EXTERIOR

The metric of the ambient electrovacuum domain, to first order in the angular velocity, has the Reissner-Nordström form modified by the non-diagonal rotation term,

$$ds^2=-\left(1-2\frac{m}{r}+\frac{\mathrm{e}^2}{r^2}\right)dt^2+\left(1-2\frac{m}{r}+\frac{\mathrm{e}^2}{r^2}\right)^{-1}dr^2+r^2[d\vartheta^2+\sin^2\vartheta(d\varphi-\omega dt)^2]. \tag{45}$$

The four-potential $A=A_\alpha dx^\alpha$, where $A_\alpha=A_\alpha(r,\vartheta)$, has a time component

$$A_t=-\frac{\mathrm{e}}{r}. \tag{46}$$

From the (r,r) and (r,ϑ) components of Einstein's equations we get no contribution to A_t, and the (t,ϑ) component gives $A_{r,\vartheta}=A_{\vartheta,r}$. Hence there is a gauge in which

$$A_r=A_\vartheta=0. \tag{47}$$

The other perturbation components are obtained in the remainder of this section. Before doing that, we remark that for the special case of the slowly rotating Kerr-Newman solution the forms (46) and (47) of the four-potential components remain valid and we have

$$A_\varphi^{(KN)}=\frac{a\mathrm{e}\sin^2\vartheta}{r}$$

$$\omega^{(KN)}=\frac{2am}{r^3}-\frac{a\mathrm{e}^2}{r^4}. \tag{48}$$

We now proceed to computing the first-order contributions for a general electrovacuum. Fromt the (φ,t) component of Einstein's equations the second-order differential equation follows:

$$r^4\frac{d^2\omega}{dr^2}+4r^3\frac{d\omega}{dr}-\frac{4\mathrm{e}}{\sin^2\vartheta}\frac{\partial A_\varphi}{\partial r}=0. \tag{49}$$

For the uncharged case, $\mathrm{e}=0$, this has the solution $\omega=\omega_0+2am/r^3$ where a is the Kerr parameter. The value of the constant ω_0 can be set to zero by the transformation $\varphi\to\varphi+c_1t$.

The solution of Eq. (49) for a generic charge $\mathrm{e}\neq 0$ is

$$A_\varphi=\frac{1}{4\mathrm{e}}r^4\sin^2\vartheta\frac{d\omega}{dr}+f(\vartheta). \tag{50}$$

The Maxwell equation for the component A_φ is

$$(2mr-r^2-\mathrm{e}^2)\frac{\partial^2A_\varphi}{\partial r^2}+r^2\sin^2\vartheta\,\mathrm{e}\frac{d\omega}{dr}+2\left(\frac{\mathrm{e}^2}{r}-m\right)\frac{\partial A_\varphi}{\partial r}-\frac{\partial^2A_\varphi}{\partial\vartheta^2}+\cot\vartheta\frac{\partial A_\varphi}{\partial\vartheta}=0. \tag{51}$$

Substituting A_φ from Eq. (50)

$$\frac{1}{4}(2mr-r^2-\mathrm{e}^2)r^4\frac{d^3\omega}{dr^3}+\left(\frac{7}{2}rm-2r^2-\frac{3}{2}\mathrm{e}^2\right)r^3\frac{d^2\omega}{dr^2}+\left(4m-\frac{5}{2}r\right)r^3\frac{d\omega}{dr}=\frac{\mathrm{e}}{\sin^2\vartheta}\left(\frac{d^2f(\vartheta)}{d\vartheta^2}-\cot\vartheta\frac{df(\vartheta)}{d\vartheta}\right). \tag{52}$$

This is a separable equation. Introducing the separation constant K, we get

$$\frac{\mathrm{e}}{\sin^2\vartheta}\left(\frac{d^2f(\vartheta)}{d\vartheta^2}-\cot\vartheta\frac{df(\vartheta)}{d\vartheta}\right)=K \tag{53}$$

with the solution

$$f(\vartheta)=\frac{1}{2}\frac{K\cos^2\vartheta}{\mathrm{e}}+\frac{C_4}{\mathrm{e}}+\frac{C_5}{\mathrm{e}}\cos\vartheta. \qquad (54)$$

We next consider the radial part of Eq. (52).

(i) For the case $\mathrm{e}^2\neq m^2$ the general solution of the radial equation is

$$\omega=C_1+\frac{\mathrm{e}^2-2mr}{3m^3r^4}C_0+\frac{2}{r}C_2+\frac{\mathrm{e}^2(r^2-2mr+\mathrm{e}^2)[r+m+(\mathrm{e}^2-r^2)L]}{m^2(\mathrm{e}^2-m^2)^2r^4}C_3 \qquad (55)$$

where

$$C_0=Km^2+3\mathrm{e}^2m^2C_2+\frac{2\mathrm{e}^2}{\mathrm{e}^2-m^2}C_3 \qquad (56)$$

and

$$L(r)=\begin{cases}\dfrac{1}{2\sqrt{m^2-\mathrm{e}^2}}\ln\dfrac{r-m+\sqrt{m^2-\mathrm{e}^2}}{r-m-\sqrt{m^2-\mathrm{e}^2}} & \text{if} \quad m>\mathrm{e}\\ \dfrac{1}{\sqrt{\mathrm{e}^2-m^2}}\left(\dfrac{\pi}{2}-\arctan\dfrac{r-m}{\sqrt{\mathrm{e}^2-m^2}}\right) & \text{if} \quad m<\mathrm{e}\end{cases} \qquad (57)$$

with the derivative

$$\frac{dL(r)}{dr}=\frac{1}{2mr-r^2-\mathrm{e}^2}. \qquad (58)$$

(ii) For the equilibrium case $\mathrm{e}^2=m^2$, the radial solution is

$$\omega=C_1+\frac{m-2r}{3r^4}(K+3m^2C_2)+\frac{2}{r}C_2+\frac{2}{15}\frac{5r^2-4mr+m^2}{m(r-m)^2r^4}C_3. \qquad (59)$$

When an asymptotically nonrotating frame is chosen, we have that $C_1=0$, and we shall assume this to hold in the sequel.

Comparing with the Kerr-Newman metric with rotation parameter a, we obtain

$$K=-3am.$$

Substituting the solution (55), (54) and (58) in the potential (50) we get for the case $\mathrm{e}^2\neq m^2$:

$$A_\varphi=\frac{a\mathrm{e}}{r}\sin^2\vartheta+\frac{3\mathrm{e}^2mr-mr^3-2\mathrm{e}^4}{2\mathrm{e}mr}C_2\sin^2\vartheta+\frac{3m^2r(m+r)-2\mathrm{e}^2m^2-4\mathrm{e}^4}{6m^3r(\mathrm{e}^2-m^2)^2}C_3\mathrm{e}\sin^2\vartheta+\frac{3\mathrm{e}^2mr-mr^3-2\mathrm{e}^4}{2m^2r(\mathrm{e}^2-m^2)^2}C_3\mathrm{e}L(r)\sin^2\vartheta-\frac{3am}{2\mathrm{e}}+\frac{C_4}{\mathrm{e}}+\frac{C_5}{\mathrm{e}}\cos\vartheta. \qquad (60)$$

For $\mathrm{e}=\pm m$, we get identical limits from both values of L in Eq. (57). The limiting form of the potential A_φ can be obtained by substituting Eq. (59) in Eq. (50):

$$\pm A_\varphi=\frac{am}{r}\sin^2\vartheta+\frac{(r+2m)(r-m)^2}{2mr}C_2\sin^2\vartheta-\frac{(2r-m)(2m^2-5mr+5r^2)}{15m^2r(r-m)^3}C_3\sin^2\vartheta-\frac{3a}{2}+\frac{C_4}{m}+\frac{C_5}{m}\cos\vartheta. \qquad (61)$$

The constant C_4 is inessential since it does not appear in the Maxwell tensor.

In order to clarify the role of the constants C_2 and C_5, it will be helpful to introduce a tetrad with the components

$$\begin{aligned}e_0&=\left(1-2\frac{m}{r}+\frac{\mathrm{e}^2}{r^2}\right)^{-1/2}\frac{\partial}{\partial t}\\ e_1&=\left(1-2\frac{m}{r}+\frac{\mathrm{e}^2}{r^2}\right)^{1/2}\frac{\partial}{\partial r}\\ e_2&=\frac{1}{r}\frac{\partial}{\partial\vartheta}\\ e_3&=\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}-\omega r\sin\vartheta\left(1-2\frac{m}{r}+\frac{\mathrm{e}^2}{r^2}\right)^{-1}\frac{\partial}{\partial t}.\end{aligned} \qquad (62)$$

In this tetrad the leading terms for large r in the components of the Riemann tensor are given by

$$\begin{aligned}R_{0113}&=R_{0223}=\frac{C_2\sin\vartheta}{r^2}\\ \frac{1}{2}R_{0123}&=R_{0213}=-R_{0312}=\frac{C_2\cos\vartheta}{r^2}.\end{aligned} \qquad (63)$$

As seen the terms do not fall off sufficiently fast for the space-time to be asymptotically flat. Thus this object does not seem to be isolated. To see this we look at the asymptotic behavior of the electromagnetic field in this tetrad. From Eq. (60) we obtain

$$\lim_{r\to\infty} F_{13} = -\frac{C_2}{\mathrm{e}}\sin\vartheta \equiv -B_2$$

$$\lim_{r\to\infty} F_{23} = -\frac{C_2}{\mathrm{e}}\cos\vartheta \equiv B_1 . \tag{64}$$

This can be interpreted that for nonzero values of C_2, the fluid ball is immersed in a constant (in this approximation) external magnetic field, parallel to the axis of rotation, and extending to infinity. Note that the Riemann tensor falls off even though the electromagnetic field tends to a constant value. One would expect that the contribution of the electromagnetic stresses to the curvature will appear in a higher approximation. Hence we have the following:

Theorem. When C_2 has a nonzero value, the external solution is not asymptotically flat.

It is therefore important to investigate the values of C_2 that the matching provides. This investigation will be carried out in the next section.

The term containing C_5 is the potential of a magnetic monopole. In the given frame, it gives rise to a purely radial magnetic field which has the asymptotic form $\sim 1/r^2$.

IV. MATCHING

In this section the matching procedure is described. The next section outlines the method in general terms. We then join the static, spherically symmetric external and internal domains in Sec. IV B. The static internal state is parametrized by the constants $\overline{\beta}$, κ and γ. As a result of the matching, these three constants determine the radius r_1 of the matching surface and the parameters m and e of the vacuum exterior. It is a consequence of the matching conditions that the surface of matching coincides with the zero pressure surface of the interior. In Sec. IV C we carry out the matching to first order in the rotation parameter. This will yield the parameters of the slowly rotating electrovacuum region in terms of r_0, the parameter describing the angular velocity of the fluid, and in terms of the parameter $\overline{g}$.

A. The matching conditions

We want a global model from which (i) any surface charges and currents and, furthermore, (ii) the surface layers of matter are absent. From the first condition it follows [11] that the electromagnetic stress tensor can be continuously matched at the surface Σ if we assume that both the permeablity and dielectric coefficients are equal in the electrovacuum and in the interior. This means with (ii) that there is no discontinuity in the pressure p across Σ.

We write the metric $ds^2 \equiv g_{\alpha\beta}dx^\alpha dx^\beta$ for both the interior and the exterior regions in curvature coordinates, $\{x^\alpha\} = \{t,r,\vartheta,\varphi\}$, in the following form:

$$ds^2 = -\mathcal{A}^2 dt^2 + \mathcal{B}^2 dr^2 + r^2[d\vartheta^2 + \sin^2\vartheta(d\varphi - \omega dt)^2] \tag{65}$$

where $\mathcal{A}$, $\mathcal{B}$ and ω are functions of the radial coordinate r alone. Both to order zero and one in the angular velocity, the constant-pressure surfaces of the perfect fluid coincide with the constant r surfaces.

We match the hypersurface given by $r=r_1$ of the interior region with the corresponding matching surface at $r=r_1$ of the exterior region, such that the induced metrics $ds^2|_\Sigma$ and induced extrinsic curvatures $K|_\Sigma$ are equal. The continuity of the functions $\mathcal{A}$ and ω across the matching surface can be achieved by transforming the coordinates such that

$$t = C_6 t', \quad \varphi = \varphi' + \Omega t' \tag{66}$$

where C_6 and Ω are suitably chosen constants and $x^1 = r$ and $x^2 = \vartheta$ are unchanged. We shall, however, drop the primes from the new coordinates.

The normal of the hypersurface Σ has the form

$$n = \frac{1}{\mathcal{B}}\frac{\partial}{\partial r}. \tag{67}$$

The extrinsic curvature has the nonvanishing components [2]

$$K_{00} \equiv \frac{1}{2} g_{00,1} n^1 = -\frac{\mathcal{A}}{\mathcal{B}}\mathcal{A}_{,r} \tag{68}$$

$$K_{03} \equiv \frac{1}{2} g_{03,1} n^1 = -\frac{1}{2\mathcal{B}}\sin^2\vartheta (r^2\omega)_{,r} \tag{69}$$

$$K_{22} \equiv \frac{1}{2} g_{22,1} n^1 = \frac{r}{\mathcal{B}} \tag{70}$$

$$K_{33} \equiv \frac{1}{2} g_{33,1} n^1 = \frac{r}{\mathcal{B}}\sin^2\vartheta . \tag{71}$$

The junction of K_{22} with Eq. (70) implies that the function $\mathcal{B}$ must be $\mathcal{C}^0$ at $r=r_1$. (This implies the matching of K_{33}.) Next we conclude from Eqs. (68) and (69), respectively, that $\mathcal{A}$ and ω are $\mathcal{C}^1$ functions at r_1.

The four-potential $A = A_\alpha dx^\alpha$ in both regions has a small A_φ component, and $A_r = A_\vartheta = 0$. The timelike component A_t retains its spherically symmetric form to first order in the angular velocity. From condition (i), the following components of the electromagnetic stress tensor $F_{\alpha\beta}$ are continuous across the matching surface Σ: $F_{\alpha\beta}h^\alpha_\gamma h^\beta_\delta$ and $F_{\alpha\beta}h^\alpha_\gamma n^\beta$, where $h^\alpha_\gamma = \delta^\alpha_\gamma - n^\alpha n_\gamma$ is the projection tensor into the orthogonal complement to the normal n of the tangent space.[3] We must match the following nonvanishing components:

$$F_{rt} = A_{t,r} \tag{72}$$

$$F_{r\varphi} = A_{\varphi,r} \tag{73}$$

[3]If we allow for surface currents and charges (but still assume that the permeability and dielectricity are that of the vacuum) these conditions take the general form $[F_{\alpha\beta}]h^\alpha_\gamma h^\beta_\delta = 0$ and $[F_{\alpha\beta}]h^\alpha_\gamma n^\beta = 4\pi j^{(surface)}_\gamma$ where the jump is indicated by a bracket.

$$F_{\vartheta\varphi}=A_{\varphi,\vartheta}. \tag{74}$$

B. Spherical matching

We first carry out the matching of the electrovacuum region at the sphere $r=r_1$ with the internal region at $z=z_1$ to order zero in the rotation parameter r_0. The metric of the perfect fluid takes a simpler form when using the coordinate z. However, the matching process is more transparent when using the radial coordinate r. These dual pictures are connected by the transformation $\gamma r=\sin z$ [cf. Eq. (23)]. From the continuity of the metric component $g_{\vartheta\vartheta}$ at the junction surface we find

$$r_1=\frac{\sin z_1}{\gamma}. \tag{75}$$

Eliminating the mass m between the junction conditions of the metric components g_{tt} and $K_{\vartheta\vartheta}$, we get the simple result for the value of the parameter

$$C_6=\cos z_1. \tag{76}$$

Continuity of the radial component of the electromagnetic field yields the value of the total charge,

$$\mathrm{e}=\frac{\overline{\beta}}{\gamma}(z_1-\sin z_1\cos z_1). \tag{77}$$

Integration of the charge density over the proper volume of the fluid gives consistently the same charge e.

We next eliminate m between the junction condition of g_{tt} and that of the extrinsic curvature component K_{tt}. Solving for κ^2, we get

$$\frac{1}{\kappa^2}=\frac{\tan z_1}{z_1}+2\overline{\beta}^2(2+z_1\cot 2z_1). \tag{78}$$

The mass is then obtained from the condition of continuity of the g_{tt} component as

$$m=\frac{r_1}{2}\left(1-\frac{\cos^2 z_1}{\kappa^2}\right)+\frac{\overline{\beta}^2}{2\gamma\sin z_1}\left(z_1^2+z_1\sin 2z_1\cos 2z_1+\frac{1}{2}\sin^2 2z_1\right). \tag{79}$$

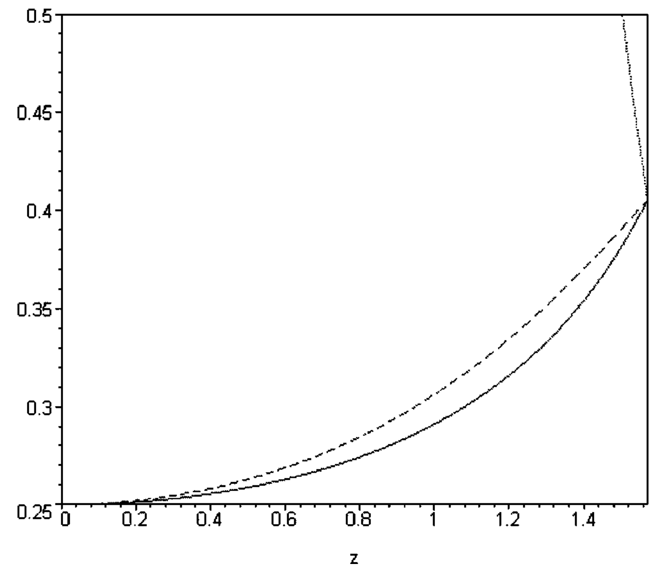

FIG. 1. The physically allowed region of the specific charge $\overline{\beta}^2$: The solid curve gives the maximally allowed value as a function of the radius.

The specific charge of the body is characterized by the function $m^2-\mathrm{e}^2$, given by Eqs. (77), (78) and (79). Taylor expanding about the origin, $z_1=0$, we get

$$m^2-\mathrm{e}^2=\frac{1-4\overline{\beta}^2}{9\gamma^2}z_1^6+\mathcal{O}(z_1^8). \tag{80}$$

Hence we find that for small stars, $\mathrm{e}^2>m^2$ for the values $|\overline{\beta}|>1/2$. In other cases, we need to treat the different types of specific charge separately.

The pressure at the center is required to be non-negative. In Fig. 1 we plot $\overline{\beta}^2$ as a function of the radius z_1 of the star for the limiting case when the pressure at the center vanishes (solid curve). The allowed region of $\overline{\beta}^2$ lies under this curve. The other two curves represent the two solutions for $\overline{\beta}^2$ for the extremely charged star, $m^2=\mathrm{e}^2$. However, the extremely charged state lies outside the physically allowed domain.

C. First-order matching

From the continuity of the metric component $g_{t\varphi}$ we get the angular velocity of the fluid:

$$\Omega=\frac{r_0\gamma^2}{\sin^2 z_1}[\overline{\beta}^2 z_1^2(\cot^2 z_1-1)-\overline{\beta}^2+\kappa^{-2}(1-z_1\cot z_1)]+\frac{\gamma^3(2m\sin z_1-\gamma\mathrm{e}^2)}{m^3\sin^4 z_1\cos z_1}\left(\mathrm{e}^2m^2C_2+\frac{2\mathrm{e}^2}{3(\mathrm{e}^2-m^2)}C_3-am^3\right)$$
$$-C_3\frac{\gamma^4}{m^2\sin^4 z_1\cos z_1}\frac{\mathrm{e}^2}{(\mathrm{e}^2-m^2)^2}\left(\mathrm{e}^2+\frac{\sin^2 z_1}{\gamma^2}-2m\frac{\sin z_1}{\gamma}\right)\left[\frac{\sin z_1}{\gamma}+m+\left(\mathrm{e}^2-\frac{\sin^2 z_1}{\gamma^2}\right)L\left(\frac{\sin z_1}{\gamma}\right)\right]-\frac{4\gamma}{\sin 2z_1}C_2. \tag{81}$$

The condition of continuity of the Maxwell field $F_{\varphi\vartheta}$ can be investigated by using the form of the four-potential in Eqs. (44) and (60). The matching of the magnetic monopole terms, proportional to cos ϑ, yields that

$$C_5=-r_0\mathrm{e}\overline{g}. \tag{82}$$

The rest of this matching equation, taken together with the continuity of $F_{\varphi r}$ and $K_{t\varphi}$ can be solved for the parameters a, C_2 and C_3 as follows:

$$a=\frac{r_0}{6\gamma m\kappa^2}\{\mathrm{e}\overline{\beta}^{-1}\gamma(4\overline{\beta}^2\kappa^2-3)-2\overline{\beta}^2\kappa^2 z_1\cos 2z_1+2z_1\sin^2 z_1+\overline{\beta}\kappa^2 4z_1(z_1\overline{\beta}-\mathrm{e}\gamma)\cot z_1-(1+2z_1^2)\overline{\beta}\sin 2z_1\}. \tag{83}$$

$$C_2=C_{L0}-16r_0\mathrm{e}\gamma\overline{\beta}^3 z_1^2\cos^2 z_1\sin^4 z_1\frac{FGH}{D^2}L\left(\frac{\sin z_1}{\gamma}\right) \tag{84}$$

$$C_3=\frac{r_0\overline{\beta}^4\cos z_1\cot z_1}{4096\gamma^4 z_1^3\mathrm{e}}FGH^2. \tag{85}$$

The detailed form of the quantities here is

$$\begin{aligned}C_{L0}=&\frac{\overline{\beta}\mathrm{e}\gamma^2 r_0}{3D^2}z_1\tan z_1\{-512\mathrm{e}^4\overline{\beta}^{-4}\gamma^4\sin^6 z_1-70\overline{\beta}^2 z_1\sin^4 z_1\sin 2z_1+4\overline{\beta}^2 z_1^2\sin^4 z_1(39+448z_1^2+384z_1^4)\\&+7\overline{\beta}^2 z_1\sin^4 z_1(5\sin 6z_1-\sin 10z_1)+2\overline{\beta}^2 z_1^2\sin^4 z_1(6\cos 2z_1+144z_1^2\cos 2z_1)-2\overline{\beta}^2 z_1^2\sin^4 z_1(104\cos 4z_1+9\cos 6z_1)\\&+2\overline{\beta}^2 z_1^2\sin^4 z_1(26\cos 8z_1+3\cos 10z_1)-64\overline{\beta}^2 z_1^3\sin^4 z_1(28z_1\cos 4z_1+21\sin^3 2z_1)-32\overline{\beta}^2 z_1^3\sin^4 z_1(12\cos 2z_1\sin^3 2z_1\\&+9z_1\cos 6z_1)-32\overline{\beta}^2 z_1^3\sin^4 z_1(128z_1^2\sin 2z_1+24z_1^2\sin 4z_1)-\overline{\beta}^4 z_1^2\sin^2 z_1(40-162z_1^2+1136z_1^4+1536z_1^6)\\&-8\overline{\beta}^4 z_1^2\sin^2 z_1(z_1^2+56z_1^4)\cos 2z_1-\overline{\beta}^4 z_1^2\sin^2 z_1\cos 4z_1(221z_1^2-1088z_1^4-60)-4\overline{\beta}^4 z_1^2\sin^2 z_1(6\cos 8z_1-\cos 12z_1)\\&+32\overline{\beta}^4 z_1^3\sin^2 z_1(7\cos 2z_1-2)\sin^5 2z_1+4\overline{\beta}^4 z_1^4\sin^2 z_1(3+112z_1^2)\cos 6z_1-3\overline{\beta}^4 z_1^4\sin^2 z_1\cos 12z_1\\&+256\overline{\beta}^4 z_1^7\sin^2 z_1(10\sin 2z_1+\sin 4z_1)+128\overline{\beta}^4 z_1^5\sin^2 z_1(5+6\cos 2z_1)\sin^3 2z_1+2\overline{\beta}^4 z_1^4\sin^2 z_1[(31+24z_1^2)\cos 8z_1\\&-2\cos 10z_1]+\overline{\beta}^6 z_1^3 B^2(32z_1^3-12z_1-9\sin 2z_1+3\sin 6z_1)+2\overline{\beta}^6 z_1^4 B^2(\cos 2z_1+6\cos 4z_1-\cos 6z_1)\\&+128\overline{\beta}^6 z_1^5 B^2\cos^3 z_1\sin z_1\}\end{aligned} \tag{86}$$

where

$$D=16\mathrm{e}^2\overline{\beta}^{-2}\gamma^2\sin^4 z_1+\overline{\beta}^4 z_1^2 B^2+2\overline{\beta}^2 z_1\sin^2 z_1[(3+24z_1^2)\sin 2z_1-\sin 6z_1]+2\overline{\beta}^2 z_1^2\sin^2 z_1(4z_1\sin 4z_1-\cos 2z_1+8\cos 4z_1)+2\overline{\beta}^2 z_1^2\sin^2 z_1(\cos 6z_1-8-16z_1^2)$$

and

$$B=4z_1^2+\cos 4z_1+z_1\sin 4z_1-1$$

$$F=4\overline{\beta}^2 z_1^2+(1+2\overline{\beta}^2 z_1^2)\cos 2z_1-3\overline{\beta}^2 z_1\sin 2z_1-1$$

$$G=(8z_1^2-1)\cos z_1+\cos 3z_1-4z_1\sin z_1$$

$$H=2z_1(2+\overline{\beta}^2 B)-4z_1\cos 2z_1-2\sin 2z_1+\sin 4z_1.$$

We next investigate the value of the constant C_2 as a function of the radius z_1 and the charge density parameter $\overline{\beta}$. The values of C_2 are strictly negative in the physically allowed region $\{z_1\in(0,\pi/2),\overline{\beta}\in(0,0.5)\}$. In Fig. 2 we display the values of C_2 on a three-dimensional diagram. For clarity we have chosen the parameter region $\{z_1\in(0,0.2),\overline{\beta}\in(0,0.1)\}$. (The values have been obtained by the software MAPLE of the University of Waterloo, using a precision of 33 significant digits.)

Based on this numerical result, we are in position to state the following.

Conjecture. The García solution cannot be used as the model of an isolated rotating body. Our conjecture does not prohibit physical applications of the García solution. For a nonvanishing parameter C_2, the fluid domain is embedded in an external homogeneous magnetic field parallel to the axis of rotation. This is a typical setting in the interior of galactic disks.

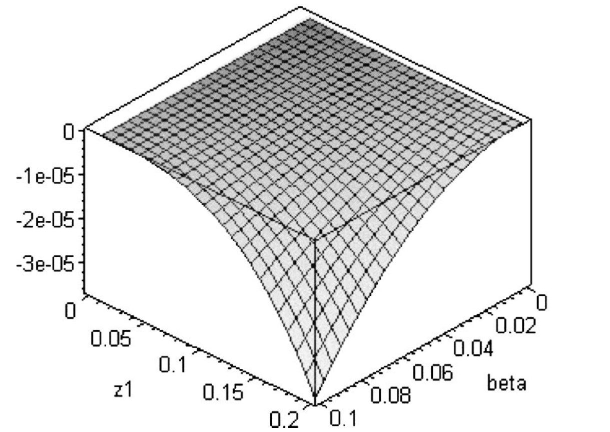

FIG. 2. The coefficient C_2 as a function of the specific charge $\bar{\beta}$ and the radius z_1.

V. DISCUSSION OF RESULTS

The García solution is the electrically charged generalization of the Wahlquist space time and it carries the extra parameter $\bar{\beta}$ determining the charge density. The new degrees of freedom that the presence of electric charge bring in the model would seem to raise the possibility of a successful matching to an empty exterior domain. However, the appearance of these new degrees of freedom is compensated for by a larger number of matching conditions. In addition to the surface gravity and first curvature, one must match also the electromagnetic field, to rule out surface charges and currents. As we demonstrated in the main part of this paper, the net effect is that the charged generalization of the Wahlquist metric is even less likely to serve as a model of an isolated star. An alternative approach would be to assume that the dielectricity or the permeability of the interior solution differs from that of the vacuum. We could then solve the junction conditions for the electric polarizability $\vec{P}$ and/or magnetization $\vec{M}$. However, this would probably not give rise to any physically realistic model of the substance occupying the interior region.

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APPENDIX

An important question is how many physical parameters there are in the metric (1). A coordinate transformation $\tau = \tau' + c\sigma$, where c is a constant, can be used to set ξ_0 to an arbitrary value. A transformation $\sigma = c\sigma'$ rescales the constant δ. The determinant of the (τ,σ) part of the metric is $-\delta^2 PQ$, showing that the symmetry axis is at those values of $x = x_0$ where $P(x_0) = 0$. The coefficient of the $d\tau d\sigma$ part of the metric is $(\delta/\Delta)(PN - QM)$. In order to obtain a coordinate system which can be made regular at the axis we must have $M(x_0) = 0$, i.e.

$$\xi_0 = \frac{1}{k}\sinh(kx_0).$$

The constant δ should be set by requiring the usual perimeter per radius ratio 2π for small circles near the axis, taking into account the range of the cyclical coordinate σ.

We should decide whether or not two different sets of constants a, b, e, g, k, m, n, α, β, δ and ξ_0 determine different spacetimes. Let us assume for the time being that we set $\xi_0 = 0$ and $\delta = 1$. If we perform the coordinate transformation $x' = cx$, $y' = cy$ and introduce the constant $k' = k/c$ then the functions M and N transform as

$$M = \frac{1}{k^2}\sinh^2(kx) = \frac{1}{c^2 k'^2}\sinh^2(k'x') = \frac{1}{c^2}M'$$

and $N = N'/c^2$. Introducing the constants $a' = c^4 a$, $b' = c^4 b$, $e' = c^2 e$, $g' = c^2 g$, $m' = c^3 m$, $n' = c^3 n$, $\alpha' = c^2\alpha$, $\beta' = c\beta$, and defining P' and Q' similarly to Eq. (2) for P and Q, we get

$$P = \frac{1}{c^4}P', \quad Q = \frac{1}{c^4}Q'.$$

Then the (x,y) block of the metric becomes

$$\Delta\left(\frac{dx^2}{P} + \frac{dy^2}{Q}\right) = \Delta'\left(\frac{dx'^2}{P'} + \frac{dy'^2}{Q'}\right).$$

To get the appropriate transformation for the other terms in the metric, we have to change the coordinates as $\tau = c\tau'$ and $\sigma = c^3\sigma'$. It follows that the two sets of constants, a, b, e, g, k, m, n, α, β and a', b', e', g', k', m', n', α', β' describe diffeomorphic metrics. Since no other similar freedom exists, there are eight physical parameters in the charged Wahlquist metric. One can use the above freedom to set one of the constants to some prescribed value; for example, one may set $k = 1$ or $\beta = 1$. The constants a and b can only be scaled by a positive constant. Hence they may be put equal to ± 1 or 0.

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