Fate of singularities and horizons in higher derivative gravity

Bob Holdom*
Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7
(Received 2 July 2002; published 16 October 2002)

We study static spherically symmetric solutions of high derivative gravity theories, with 4, 6, 8 and even 10 derivatives. Except for isolated points in the space of theories with more than 4 derivatives, only solutions that are nonsingular near the origin are found. But these solutions cannot smooth out the Schwarzschild singularity without the appearance of a second horizon. This conundrum, and the possibility of singularities at finite r, leads us to study numerical solutions of theories truncated at four derivatives. Rather than two horizons we are led to the suggestion that the original horizon is replaced by a rapid nonsingular transition from weak to strong gravity. We also consider this possibility for the de Sitter horizon.

DOI: 10.1103/PhysRevD.66.084010 PACS number(s): 04.20.Cv, 04.20.Dw

I. INTRODUCTION

The vacuum Schwarzschild (Schd) solution of general relativity suffers from a singularity at $r=0$, but perhaps a more meaningful observation is that the Schd solution cannot be trusted in a finite region enclosing the origin. At a nonzero radius inside the horizon a curvature invariant has grown as large as the Planck mass scale, at which point the Einstein action should no longer provide the correct description. For example in the standard Schd coordinate system $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}=48G^2M^2/r^6$, and thus the Schd solution breaks down at a radius of order $r_s=(48G^4M^2)^{1/6}$. For large M this is much smaller than the horizon size $2GM$, but it is much larger than the Planck length $\sqrt{G}$.

There have been discussions within general relativity on how the diverging curvatures can be tamed [1]. If a suitable matter distribution is postulated and allowed to violate the strong energy condition, then nonsingular black-hole-like solutions do arise. Although the curvatures in the core region are now finite they are still characterized by the Planck scale, and thus the Einstein action can only provide a crude description. But an interesting aspect of such solutions is the appearance of a second horizon, in the vicinity of r_s.

No matter what the theory is or what matter is present, it appears that nonsingular spherically symmetric static solutions can never have only one horizon. In the standard coordinate system the metric of interest is

$$ds^2=-B(r)dt^2+A(r)dr^2+r^2(d\theta^2+\sin^2\theta\phi^2). \quad (1)$$

If the curvature invariants R, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\ldots$ are to be nonsingular at the origin, what does this imply about the behavior of $A(r)$ and $B(r)$? For small r we find that[1]

$$A(r)=1+a_2r^2+\cdots$$
$$B(r)=b(1+b_2r^2+\cdots). \quad (2)$$

In this case $R=6(a_2-b_2)$ at $r=0$ and the higher invariants are polynomials in a_2 and b_2. (The constant b is affected by a rescaling of t and can be ignored.) The important point is that $A(0)$ is fixed to be unity. But inside the horizon of the Schd solution $A(r)$ is negative, a fact directly connected with the existence of the Schd horizon where the signs of $A(r)$ and $B(r)$ change. Then to match onto a nonsingular solution with $A(0)=1$ another sign change is required. To preserve a time-like direction $B(r)$ must also change sign again. Thus $A(r)$ and $B(r)$ must each have an even number of sign changes between $r=0$ and $r=\infty$, rather than the single sign change implied by a single horizon.

Given this result, a theory of gravity capable of describing Planck scale physics should address the following three questions. Are these metrics that describe large but nonsingular curvatures near the origin actually solutions to the theory? Do solutions that are singular at the origin continue to be present, or are they banished altogether? How does a nonsingular solution at the origin match onto a sensible solution at large r while experiencing an even number of sign changes, without encountering a singularity at a finite r?

To model Planck scale physics we consider theories where the Einstein action is extended to include terms with more factors of the curvature tensor and its covariant derivatives. We will explicitly study spherically symmetric static solutions of these higher derivative theories truncated at various orders in derivatives. In particular we will construct solutions as a series expansion about the origin for theories with 6, 8 and even 10 derivatives. And for the general theory truncated at four derivatives we will be able to construct numerical solutions that interpolate between the small curvature, weak gravity behavior at large r and the high curvature, strong gravity behavior at small r. Some aspects of our results may be true at any order in the derivative expansion.

In the $2+4$ derivative theory (the theory with the Einstein term and the general set of 4 derivative terms) the Schd solution is still present. Reference [2] applied the series expansion approach to this theory to find, in addition to the Schd solution, another class of singular solutions and a class

*Electronic address: bob.holdom@utoronto.ca

[1]In lieu of a proof we have scanned through a variety of power law behaviors of $A(r)$ and $B(r)$ near the origin and a variety of curvature invariants, similar to our scan for solutions in the next section.

of nonsingular solutions. But it was not known which, if any, of these alternative solutions near the origin could match onto the desired weak gravity solution at large r. Our numerical work shall clarify these issues.

Although the $2+4$ derivative theory has a variety of solutions near the origin, we find that only one class of solutions remains for the typical action with more derivatives. This turns out to be the nonsingular class of solutions, having the expected property $A(0)=1$. Such solutions are found for any action considered. It is perhaps not surprising that nonsingular solutions exist, since spherically symmetric static solutions that are everywhere weak, produced for example by a low density matter distribution, must be recovered. More surprising is that *singular* solutions, if they exist at all, only exist for very particular actions, of measure zero in the space of theories. (Among these very particular theories are those that have received some attention. For example the Schd solution continues to exist in theories where only powers of the curvature scalar R appear in the action [3].)

This is not to say that solutions with singularities don't exist; the typical unphysical solution would involve a curvature singularity at a finite r. But to find nonsingular black-hole-like solutions our previous discussion indicates that we must search for solutions with several sign flips. This appears difficult to accomplish in higher derivative theories, either analytically or numerically, and it is not the focus of this work. Rather we shall be more concerned with how a core region of strong gravity matches onto a weak gravity large r region without any change of sign of metric components, and thus with no horizons.

Before launching into the study of these solutions, it is natural to question the derivative expansion itself. Of course a derivative expansion truncated at some finite order has problems with physical interpretation; in particular it suffers from negative energy ghost modes when linearized. But this problem is entangled with the effects of truncation and linearization; it is not fundamental in the sense that the underlying theory, from which the derivative expansion is derived, should be quite sensible. Evidence of the role of nonlinearities in resolving the ghost problem was found for a different class of metrics of Friedmann-Robertson-Walker (FRW) form in [5], where ghost modes were found to decouple from positive energy matter when the nonlinear solutions were considered. In any case results that hold at arbitrary order in the nonlinear derivative expansion, such as the existence of nonsingular solutions near the origin as described above, would appear to carry some significance.

The solutions near the origin and their implications are discussed in more detail in the next section. In Secs. III and IV we turn to the general $2+4$ derivative theory and numerically analyze the non-Schd solutions that are respectively singular and nonsingular near the origin. We show how these solutions match onto the desired weak gravity solutions at large r, and in particular that they do so without encountering horizons. In Sec. V we turn to the de Sitter space, which in static coordinates also has a horizon at a finite radius. Here we again explore the idea that the horizon is replaced by a boundary where a transition from weak to strong gravity occurs; now the picture is inside-out with weak (strong) gravity on the inside (outside). We conclude in Sec. VI.

II. SOLUTIONS NEAR THE ORIGIN

We consider a general gravitational action with any number of derivatives. The field equations $H_{\mu\nu}=T_{\mu\nu}$ determining $A(r)$ and $B(r)$ can be found by substituting the metric (1) into the action and varying with respect to $A(r)$ and $B(r)$; this gives H_{rr} and H_{tt} respectively. $H_{\theta\theta}$ can then be found from the Bianchi identity [2]:

$$H_{\theta\theta}=\frac{r^3}{2}\left(\frac{H_{rr}}{A}\right)'+\frac{r^2H_{rr}}{A}+\frac{r^3B'H_{rr}}{4AB}+\frac{r^3B'H_{tt}}{4B^2}. \qquad (3)$$

We first look for nonsingular solutions in the presence of a smooth matter distribution with finite energy and pressure.

For our purposes we define nonsingular metrics as those without curvature singularities, and we have already said that such metrics must have $A(0)=1$. It is interesting to see how this emerges as a property of solutions to the field equations. When $A(0)$ and $B(0)$ are nonzero and finite the leading terms in the field equations for small r behave like $1/r^n$ where n is the maximum number of derivatives in the action. This behavior arises from the terms with no derivatives of $A(r)$ and $B(r)$, which nevertheless arise from n derivative terms in the action. The coefficient of the $1/r^n$ term in each equation is proportional to a polynomial in $A(0)$, and these polynomials vanish iff $A(0)=1$. This result is thus intrinsically connected with the nonlinearity of the theory.

All the terms behaving like inverse powers of r in the equations must vanish, and this leads to solutions of the form

$$A(r)=1+a_2r^2+\sum_{n\geqslant 4} a_nr^n,$$

$$B(r)=1+b_2r^2+\sum_{n\geqslant 4} b_nr^n. \qquad (4)$$

We find that solutions of this form exist for general actions, and we have tested actions up to 10 derivatives. a_2 and b_2 are free parameters of these solutions except for isolated points in theory space. One such exception is general relativity, where a_2 and b_2 are determined by the energy density and pressure at the origin.

We now consider the possibility of singular solutions. To find other solutions near the origin we look for other cases where the leading terms in the $1/r$ expansion of the field equations vanish for special choices of the $A(r)$ and $B(r)$ expansion parameters:

$$A(r)=a_\alpha r^\alpha+a_{\alpha+1}r^{\alpha+1}+\cdots,$$

$$B(r)=b_\beta r^\beta+b_{\beta+1}r^{\beta+1}+\cdots. \quad (5)$$

Note that both the leading and next-to-leading parameters can appear in the leading terms in the field equations. If such a case is found then the expansion can be tested as a solution at higher orders. Since various powers of r^α and r^β appear in the nonlinear field equations it would seem miraculous to find solutions with α and β noninteger; in any case we have restricted the search to integer α and β. Our search is also blind to solutions that are not amenable to a series expansion at the origin.

We have performed a scan through a range of values of α and β for a range of actions. In the case of the general $2+4$ derivative theory the scan yields the three sets of solutions found in [2]: the nonsingular solutions in Eq. (4) with $(\alpha,\beta)=(0,0)$, the Schd solution with $(\alpha,\beta)=(1,-1)$, and another set of singular solutions to be described below with $(\alpha,\beta)=(2,2)$. For actions with more derivatives only the nonsingular solutions remain, except for specialized actions where various terms are set arbitrarily to zero.

The result then is that the static spherically symmetric solutions of a typical high derivative theory are not singular at the origin. If singularities occur, they occur at a nonzero radius.

We stress again that the Schd solution is a generic solution only in theories truncated at two derivatives (general relativity) or four derivatives. In typical theories with six or more derivatives the Schd solution will remain only as an approximate weak gravity solution in the large r region where the Einstein term dominates. In fact by examining a series expansion in $1/r$ of the field equations around the Schd solution one finds (see also [6]) that the corrections have the form

$$A(r)=1/(1-2GM/r)+\mathcal{O}(G^4M^2/r^6),$$

$$B(r)=1-2GM/r+\mathcal{O}(G^4M^2/r^6). \quad (6)$$

Once again we see that the Schd solution is a good approximation to an exact solution down to a radius well within the horizon, implying that $A(r)$ and $B(r)$ would have to change sign a second time if they were to match onto Eq. (4). When the exact solution is extended in towards the origin one must encounter a second horizon and/or a singularity at a finite radius.[2]

But there is a loophole in these arguments, related to the observation that an exact solution of the form in Eq. (6) is not the unique solution in the weak gravity region [2]. As discussed in the next section, there are other solutions involving the massive modes in the theory, giving rise to solutions involving terms of the Yukawa potential type. Thus the real question is what happens when a more general exterior solution, which is Newtonian plus Yukawa in form at large r, is continued in towards $r=0$.

To shed light on these issues it would be best to make a direct analysis of a theory with six or more derivatives, where the only solutions are nonsingular at the origin. But the complexity of those equations will constrain our present work to a numerical study of the $2+4$ derivative theory. We will investigate how the various solutions existing near the origin of the $2+4$ derivative theory join onto the weak-gravity large-r solutions. We are once again dealing with solutions of truncated theory in a region where the truncated theory is not justified, but it will be of interest to compare and contrast these solutions to the Schd solution.

III. DOMAINS OF STRONG GRAVITY

The action of the general $2+4$ derivative theory can be cast in the form

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}(R+aR^2+bR_{\mu\nu}R^{\mu\nu}). \quad (7)$$

We first consider the *linearized* version of this theory, where the equations for $A(r)$ and $B(r)$ have five independent solutions [2]. For a weak constraint on a and b, assumed here to be satisfied, four of the solutions have exponential dependence. The two that grow exponentially with r are discarded to satisfy boundary conditions at infinity. This leaves the Newtonian solution and two solutions that resemble Yukawa potentials. For any physical matter distribution the Yukawa potential terms will coexist with the Newtonian term in the exterior solution (an example is given in [2]). One of the Yukawa potential terms has a repulsive sign, reflecting the ghost-like nature of the massive mode. The Yukawa potentials are typically swamped by the Newtonian term and offer little chance of detection in weak gravity solutions [2,4].

On the other hand, the mere existence of the Yukawa potential terms for some matter distribution for which linearized gravity is applicable proves that the exact Schd solution of the full $2+4$ derivative theory is not the true exterior solution [4]. There is also no reason to expect it to be the relevant solution when there is sufficient matter density to cause the Schd horizon to form. We must study other exact solutions of the theory that take a Newtonian plus Yukawa form in the exterior region.

Such solutions can be markedly different from the Schd solution in the interior region. In particular the region of high curvature and strong gravity can be larger than in the Schd solution. The idea that a region of Planck scale curvature can be of macroscopic size is not conceptually new since, as described in the introduction, this is already is implied by the Schd solution. If strong gravity extends out to nearly the radius of the would-be Schd horizon, then $A(r)$ and $B(r)$ could retain their weak gravity signs for all r, and a horizon need not exist.

This possibility turns out to be illustrated by the other class of solutions in the $2+4$ derivative theory that are singular near the origin. Close to the origin they have the form

[2]The singularity could be impassable. There is also the possibility of a violent singularity at the origin of a type not amenable to a series expansion. Note that some kind of singularity is expected in the case that M is negative, since a nonsingular negative energy solution would imply that Minkowski space is not stable [7].

$$A(r)=a_2r^2+a_3r^3+a_4r^4+\sum_{n\geqslant 5} a_nr^n,$$

$$B(r)=b_2r^2+\frac{a_3}{a_2}b_2r^3+\frac{6a_2a_4+a_3^2+2a_2^3}{8a_2^2}b_2r^4+\sum_{n\geqslant 5} b_nr^n. \tag{8}$$

We shall be concerned with $a_2,b_2>0$, where b_2 is sensitive to a rescaling of t. The existence of these solutions up to r^3 was shown in [2]. We have found that these are 5 parameter solutions where the parameters (a_2,a_3,a_4,a_5,b_5) determine (a_n,b_n) for $n\geqslant 6$, for a given action. The leading behavior of the curvature invariants $[R,R_{\mu\nu}R^{\mu\nu},R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}]$ is $[(22a_3a_2^3+10a_2a_3a_4-a_3^3-60b_5a_2^3+36a_5a_2^2)/(4a_2^4r), 12/(a_2^2r^8),24/(a_2^2r^8)]$, to be compared with $[0, 0, \sim 1/r^6]$ for the Schd solution. We note here that any nonsingular mass distribution only affects (a_n,b_n) for $n\geqslant 10$.

It is an interesting coincidence that there are five parameters here, just as in the linearized gravity solutions. This lends support to the idea that full solutions exist which join together the strong gravity small r solutions with the weak gravity large r solutions. A numerical verification is made somewhat difficult by the existence of exponentially growing solutions at large r, and by the extremely singular behavior of the equations at small r away from the exact solutions. In particular, a numerical analysis based on the initial value problem at $r=0$ is not feasible. Instead the following strategy was adopted.

At some finite r_0 in the *weak* gravity region initial conditions are chosen so as to deviate only very slightly from the Schd solution, and the equations are numerically integrated for r both larger and smaller than r_0. The Yukawa potential modes are induced and they grow for decreasing r. This can cause the solution to deviate significantly from the Schd solution around the would-be horizon region, and rather than changing sign, both $A(r)$ and $B(r)$ can stay finite and positive throughout this region and down to $r=0$. The initial conditions at r_0 are then finely tuned so that $A(r)$ and $B(r)$ near $r=0$ take the form of the known solution in Eq. (8) to $\mathcal{O}(r^4)$.

Additional fine tuning is needed to remove the unwanted modes that grow exponentially with r. But it is easier just to repeat the whole process for a larger r_0 where the required deviation from the Schd solution is smaller; then the amplitude of the unwanted modes is smaller and a sensible numerical solution extends out to larger r. It appears that the extent to which one can push this is a purely numerical limitation, and that these numerical results are sufficient to demonstrate the existence of exact solutions of this type.

We display one of these numerical vacuum solutions in Fig. 1, where we plot the functions $-\ln(A(r))$ and $\ln(B(r))$. These functions coincide (after a suitable rescaling of t) with each other and with the Schd solution in the weak gravity region, exterior to the would-be horizon. We have chosen $a=G/2$ and $b=-G$ in Eq. (7) to simplify the equations somewhat (corresponding, in the linearized theory, to equal masses for the two massive modes), but the basic properties of the solutions are independent of this choice. The mass $\mathcal{M}$ as deduced by the large r behavior in this example is $10G^{-1/2}$. Larger $\mathcal{M}$ has also been considered and there does not seem to be any limitation, other than numerical, to recover arbitrarily large mass solutions.

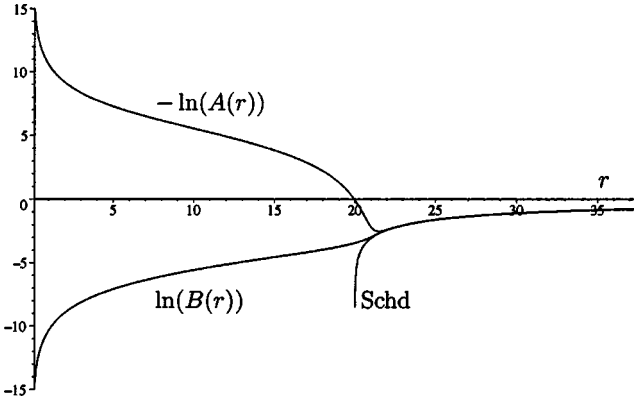

FIG. 1. A solution of the $2+4$ derivative theory with no horizon and a singularity at the origin, which matches onto the Schd solution with $\mathcal{M}=10$ close to its horizon at $r=20$ (in units with $G=1$). $A(r)$ and $B(r)$ exhibit an r^2 dependence near $r=0$.

The new solutions have no horizon and a singularity at the origin. Their interior form, and the point where deviation from the Schd solution occurs, is not uniquely determined by $\mathcal{M}$. This is as expected given the additional parameters governing the strength of the Yukawa potentials in the exterior solution, and the five parameters in Eq. (8). We also find that $a_2\sim G^{-3}\mathcal{M}^{-4}$, which is consistent with the curvature invariant $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ being of Planck size close to the would-be horizon, even for a large mass object. Of course this is how the usual arguments for the existence of a horizon are avoided, since the uniqueness of the Schd solution relies on small curvatures and the applicability of general relativity down to radii well within the horizon.

This solution is intriguing, but it is occurring in a theory arbitrarily truncated in derivatives, and the solution involves a region of high curvature where higher derivative terms would be important. On the other hand, the Schd solution suffers from exactly the same problems. As in that case one may want to presume that the higher derivative terms being ignored would serve to smooth out the singularity at the origin while retaining the main qualitative feature, the fact that the transition between weak and strong gravity has moved out to the would-be horizon radius.

IV. NONSINGULAR SOLUTIONS

We shall now turn to the nonsingular solutions of the $2+4$ derivative theory, where additional insights will emerge.[3] A spherically symmetric nonsingular matter distribution will act as a source for these solutions. We choose to

[3]We keep in mind though that these solutions need not be truly representative of the nonsingular solutions in theories with even more derivatives.

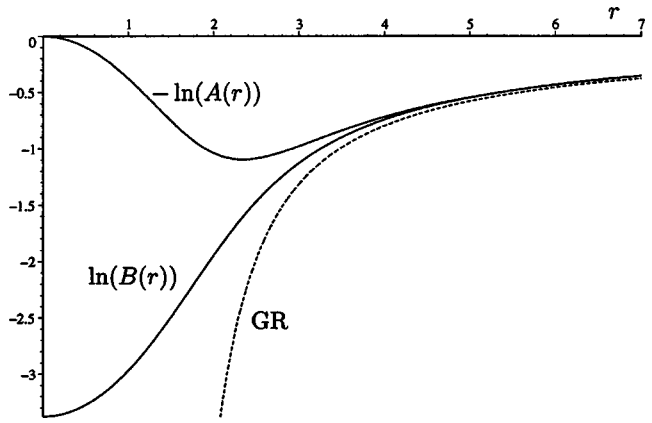

FIG. 2. A nonsingular solution of the $2+4$ derivative theory for the smooth mass distribution with $\mathcal{R}=4/3$ and $M=1.1$ in Planck units. For the same distribution the dotted line shows $\ln(B(r))$ from general relativity, which yields vanishing $B(r)$ and infinite pressure at some radius.

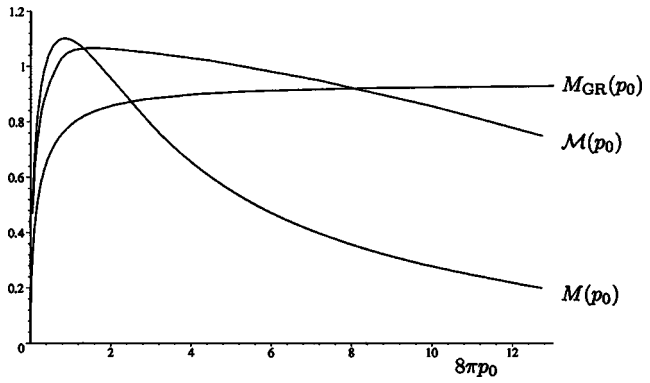

FIG. 3. The mass in a fixed volume versus the pressure p_0 at the origin. M is the volume integral of the matter energy density and $\mathcal{M}$ is the mass deduced from the exterior solution. Both definitions apply to the mass in general relativity, $M_{\rm GR}$.

study incompressible matter, where $\rho(r)$ will be given a fixed profile and the theory will determine the required pressure $p(r)$. In particular we take $\rho(r)=\rho_0\exp(-r^2/\mathcal{R}^2)$. Thus far we are only successful by taking $\mathcal{R}$ of order the Planck length $\sqrt{G}$, but this seems to be purely a limitation of the numerical strategy which we now describe. Noting that equations cannot be directly integrated from $r=0$, we use the series expansion in Eq. (4) to $\mathcal{O}(r^{12})$ and express all coefficients in terms of a_2, b_2, p_0 and ρ_0. This is used to provide the initial conditions for the differential equations at a small finite value of r, where the series solution can be trusted. The equations can then be numerically integrated out from this point to large r. For a given ρ_0, a tedious search of the (a_2,b_2,p_0) parameter space is then needed to match onto the weak gravity exterior solution, with the additional constraint that $p(r)\to 0$ for $r\gg\mathcal{R}$. We display one of these solutions in Fig. 2; our choice here and in the following is $\mathcal{R}=4\sqrt{G}/3$.

With the usual definition of mass $M\equiv\int\rho(r)4\pi r^2dr$ we have $M\propto\rho_0$ for fixed $\mathcal{R}$. In general relativity the pressure at the origin p_0 increases with increasing M, and p_0 becomes infinite at a finite M. The whole notion of gravitational collapse is based on this basic behavior, but we find a qualitatively different behavior in the $2+4$ derivative theory. We refer to our derived $M(p_0)$ in Fig. 3 where for small p_0 and M, p_0 increases with M, but more slowly than in general relativity (GR). The higher derivative terms are acting to reduce the tendency to collapse, and p_0 is still finite at the particular mass that first causes infinite p_0 in GR. p_0 continues to increase for increasing M until the strong gravity effects cause a peculiar phenomenon; a maximum value $M_{\rm max}$ occurs at a finite p_0. After this point p_0 continues to increase while M *decreases*.

Connected with this behavior is the fact that, unlike GR, the physical mass $\mathcal{M}$ of the object as deduced by the exterior solution is not the same as M, the volume integral of $\rho(r)$. From Fig. 3 we see that $\mathcal{M}$ is smaller than M for small p_0, while the reverse happens for sufficiently large p_0. In the latter case strong gravity contributions are responsible for an increasing fraction of the total $\mathcal{M}$, which decreases much less rapidly than M for increasing p_0. We note that the maximum value $M_{\rm max}$ is achieved when the scalar curvature R at the origin switches sign, being positive (negative) on the low (high) pressure side of $M_{\rm max}$. The example in Fig. 2 corresponds to $M\approx M_{\rm max}$.

Even though the $M(p_0)$ curve in Fig. 3 is derived for an incompressible material, it has implications for massive objects with more physical equations of state. For the more standard part of the curve where p_0 is growing with ρ_0 (remember $\rho_0\propto M$), p_0/ρ_0 is also growing. Thus when more matter is added to a given volume the object may or may not be able to support the required increase in pressure, depending on its equation of state. If not, gravitational collapse ensues. On the other part of the curve, p_0 *decreases* with increasing ρ_0. Thus adding more matter to a given volume will tend to decrease the required pressure, and the required p_0/ρ_0 decreases even more. Then the matter exerts too much pressure and the object will tend to expand in size rather than contract. Thus the usual tendency for gravitational collapse is absent on this part of the curve. The crossover between the two types of behavior occurs at the value of p_0/ρ_0 at $M_{\rm max}$; for our example with $\mathcal{R}=4\sqrt{G}/3$ this is 0.39.

It remains to be checked whether this basic picture survives for much larger $\mathcal{R}$. The question is whether a $M(p_0)$ curve corresponding to a larger volume, which would lie above the one in Fig. 3, has the same general shape. But our present numerical approach is not powerful enough to deal with this question.

Finally, the concept of a maximum mass at a finite pressure leads to the following question. Suppose we start with the maximum mass configuration; what happens if the mass in the fixed volume is increased still further? As in general relativity, we may expect that a curvature singularity should develop at the origin. If so then the solution should transform into one of the solutions of the previous section; we have already mentioned that the presence of a nonsingular mass distribution would only affect those solutions at order $\mathcal{O}(r^{10})$. By continuity it seems appropriate that the new configuration would maintain a finite p_0. In this way it appears that the ''maximum mass at finite pressure'' property of the

nonsingular solution is related to the existence of the singular class of solutions. Unlike general relativity, the transition from a nonsingular solution to a singular one does not lead to the appearance of a horizon.

V. THE de SITTER HORIZON

Another horizon of physical interest occurs in the de Sitter metric, which in the static coordinate system has

$$B(r)=1-\frac{\Lambda}{3}r^2, \quad A(r)=1/B(r). \tag{9}$$

The physical region is interior to the horizon at $r=\sqrt{3/\Lambda}$ and for small cosmological constant it is a weak gravity, small curvature region with $R=4\Lambda$. But we may wonder whether the weak gravity region can border on a strong gravity region in such a way that there is no horizon, with the border being roughly the location of the would-be horizon. We will find that actions containing terms with six or more derivatives are necessary to deal with this question. Thus we must rely less on numerical analysis and more on series expansions, both in the weak and strong gravity regions. We find results that are consistent with the idea of horizon elimination.

First we note that the de Sitter metric when expanded about the origin is a member of the nonsingular class of solutions in Eq. (4). We are interested in other solutions of this class where the scalar curvature R is close to being constant, departing significantly from 4Λ only when the would-be horizon is approached. We can write such solutions to theories with a cosmological constant and no matter in the form

$$A(r)=1\Big/\left(1-\left(\frac{\Lambda}{3}-\varepsilon\right)r^2\right)+\sum_{i\geqslant 2} a_{2i}r^{2i},$$

$$B(r)=1-\left(\frac{\Lambda}{3}+\varepsilon\right)r^2+\sum_{i\geqslant 2} b_{2i}r^{2i}. \tag{10}$$

The parameter ε characterizes the deviation from the de Sitter metric. The coefficients a_i and b_i vanish with ε and they also depend on Λ and the action being considered.

For the $2+4$ derivative action we find that solutions with a nonzero ε do not in fact produce a departure of R from 4Λ. Actions with more derivatives are required for this to happen; in addition such actions cause the Λ in Eq. 10 to shift by an amount of order Λ^3 from the cosmological constant in the action. When this is taken into account we find that an ε expansion gives[3]

$$R=4\Lambda+\varepsilon^2 f_\Lambda(r^2)+\cdots. \tag{11}$$

$f_\Lambda(r^2)$ has a nonvanishing $\Lambda=0$ limit and $f_\Lambda(0)=0$. By taking ε appropriately small we can suppose that $|R-4\Lambda|$ will remain small until r approaches the would-be horizon. Numerically we find that ε has to be exponentially small; this is reminiscent of the exponentially small departure from the Schd solution in the exterior region discussed in the previous sections.

To explore the large r strong gravity region we expand the equations in powers of $1/r$. We find a new one-parameter family of solutions that can only arise when six or more derivative terms appear in the action. These new solutions are typically a corrected form of the anti-de Sitter–Schd metric, since pure AdS-Schd solutions only occur for specialized actions. The appearance of corrections are not unexpected given the corrected Schd metric in Eq. (6). The leading corrections are deduced by solving the equations to the appropriate order in $1/r$, and we find[4]

$$A(r)=1\Big/\left(1+\frac{\hat{\Lambda}}{3}r^2-2GM/r\right)+\hat{a}_8 G^5M^2/r^8+\cdots,$$

$$B(r)=1+\frac{\hat{\Lambda}}{3}r^2-2GM/r+\hat{b}_6 G^4M^2/r^6+\cdots. \tag{12}$$

These are strong gravity solutions since the positive $\hat{\Lambda}$ is fixed by the theory to be Planckian in size, $\hat{\Lambda}\approx 1/G$. The dimensionless $\hat{b}_6$ and $\hat{a}_8$ constants are also determined by the action being considered, whereas M is the free parameter.

If the corrections were absent then the scalar curvature would be $-4\hat{\Lambda}$ and there would be a horizon at $r=(6GM/\hat{\Lambda})^{1/3}\approx G^{2/3}M^{1/3}$ for large M. But at large r the corrections modify the scalar curvature as follows:

$$R=-4\hat{\Lambda}-\frac{2\hat{a}_8}{3}\frac{G^5\hat{\Lambda}^2M^2}{r^6}+\cdots. \tag{13}$$

Thus in the vicinity of the would-be horizon the departures from the pure AdS-Schd metric are becoming large, and so what actually happens there is not known. Note that this modification of the horizon region has emerged automatically from the general solution. The one parameter M can then be adjusted so that the radius of this would-be horizon is in the vicinity of the would-be horizon of the weak gravity interior solution. In both the interior and exterior regions $A(r)$ and $B(r)$ are positive, and so the matching of solutions can occur without a horizon.

These results are of course not sufficient to prove the existence of an exact solution that joins together the small and large r behaviors we have described, but the results are consistent with such a solution. A numerical analysis of this horizon-free possibility remains to be performed. The resulting picture is of a large mass object in a strong AdS space that has as its interior a weak dS space. The "mass of the universe" $M\approx G^{-2}\Lambda^{-3/2}$ is enormous, since it is of order a Planckian mass density times the volume of the interior space.

VI. CONCLUSION

We have studied static spherically symmetric solutions of higher derivative gravity, and have found that when there are

[4]Actions with 6 and 8 derivatives were considered.

six or more derivatives the solutions are typically nonsingular near the origin. But the nature of these solutions implies that the complete solutions with Newtonian large r behavior must have an even number of horizons. Our study of $2+4$ derivative gravity provides some insights into how a solution with a strong gravity core region can realize the zero horizon option. The region of strong gravity extends out to the radius of the would-be horizon, which then negates the usual arguments for the existence of a horizon. We exhibited horizon-free solutions of this type that were singular at the origin. It remains to show how this picture persists in the higher derivative theories where solutions are always nonsingular at the origin.

We also discussed some unexpected properties of nonsingular solutions in $2+4$ derivative gravity that affect the inevitability of gravitation collapse. A high pressure accumulation of matter can have the property that increasing energy density corresponds to decreasing pressure. Associated with this is a maximum for the amount of matter in the given volume. This sounds familiar, but in this case horizons and black holes play no role. Rather, the maximum mass configuration corresponds to a particular finite pressure at the origin, and this configuration can be approached from both the low pressure and high pressure sides. Numerical limitations confined our study of this phenomena to Planck mass objects, and so it remains to extend this picture to much larger masses.

It could be expected that strong gravity and higher derivatives would help to resolve singularities. But we have seen that a transition region from weak to strong gravity can also take the place of a horizon. Of most interest are solutions that are nonsingular and same-sign everywhere; such solutions have to be investigated in theories with six or more derivatives to see whether they exist for smooth mass distributions of arbitrary energy density. We have seen in the previous section that theories with six or more derivatives are necessary to describe the possible elimination of the de Sitter horizon. In both the Schd and de Sitter cases the deviations in the weak gravity regions are exponentially small until the would-be horizon is reached. It remains to be seen whether regions of strong gravity can be intruding on our weak gravity universe.

ACKNOWLEDGMENTS

I thank E. Poppitz for discussions. This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

[1] J. Bardeen, presented at GR5, Tiflis, U.S.S.R., 1968; E. Poisson and W. Israel, Class. Quantum Grav. **5**, L201 (1988); Phys. Lett. B **233**, 74 (1989); Phys. Rev. D **41**, 1796 (1990); V.P. Frolov, M.A. Markov, and V.F. Muchanov, *ibid.* **41**, 383 (1990); I. Dymnikova, Gen. Relativ. Gravit. **24**, 235 (1992); M. Mars, M.M. Martin-Prats, and J.M.M. Senovilla, Class. Quantum Grav. **13**, L51 (1996); G. Magli, Rep. Math. Phys. **44**, 407 (1999); E. Elizalde and S.R. Hildebrandt, Phys. Rev. D **65**, 124024 (2002); E. Ayon-Beato and A. A. Garcia, Phys. Lett. B **464**, 25 (1999).

[2] K.S. Stelle, Gen. Relativ. Gravit. **9**, 353 (1978).

[3] F.C. Michel, Ann. Phys. (N.Y.) **76**, 281 (1973); B. Whitt, Phys. Lett. **145B**, 176 (1984); S. Mignemi and D.L. Wiltshire, Phys. Rev. D **46**, 1475 (1992).

[4] E. Pechlaner and R. Sexl, Commun. Math. Phys. **2**, 165 (1966).

[5] B. Holdom, Phys. Rev. D, **66**, 045001 (2002).

[6] M. Lu and M. Wise, Phys. Rev. D **47**, 3095 (1993).

[7] G.T. Horowitz and R. Myers, Gen. Relativ. Gravit. **27**, 915 (1995).