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Construction of nonsingular pre-big-bang and ekpyrotic cosmologies and the resulting density perturbations

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We consider the construction of nonsingular pre-big-bang and ekpyrotic type cosmological models realized by the addition to the action of specific higher-order terms stemming from quantum corrections. We study models involving general relativity coupled to a single scalar field with a potential motivated by the ekpyrotic scenario. We find that the inclusion of the string loop and quantum correction terms in the string frame makes it possible to obtain solutions of the variational equations which are nonsingular and bouncing in the Einstein frame, even when a negative exponential potential is present, as is the case in the ekpyrotic scenario. This allows us to discuss the evolution of cosmological perturbations without the need to invoke matching conditions between two Einstein universes, one representing the contracting branch, the second the expanding branch. We analyze the spectra of perturbations produced during the bouncing phase and find that the spectrum of curvature fluctuations in the model proposed originally to implement the ekpyrotic scenario has a large blue tilt ($n_{\mathcal{R}}=3$). Except for instabilities introduced on small scales, the result agrees with what is obtained by imposing continuity of the induced metric and of the extrinsic curvature across a constant scalar field (up to k^2 corrections equal to the constant energy density) matching surface between the contracting and the expanding Einstein universes. We also discuss nonsingular cosmological solutions obtained when a Gauss-Bonnet term with a coefficient suitably dependent on the scalar matter field is added to the action in the Einstein frame with a potential for the scalar field present. In this scenario, nonsingular solutions are found which start in an asymptotically flat state, undergo a period of superexponential inflation, and end with a graceful exit. The spectrum of fluctuations is also calculated in this case.

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I. INTRODUCTION

There has recently been a lot of interest in cosmological scenarios in which it is assumed that, instead of emerging from an initial big bang singularity, our universe has resulted from an Einstein frame bounce that connects a previous contracting phase with the present phase of cosmological expansion. A lot of this interest has been fueled by string cosmology, the attempt to merge string theory and cosmology. Pre-big-bang (PBB) cosmology [1,2] (see [3,4] for a comprehensive review) and the ekpyrotic scenario [5] are two well-known models in which our present phase of cosmological expansion is postulated to have emerged from a previous phase of cosmological contraction.[1] In both examples, however, the cosmological description in terms of an effective action breaks down at the bounce. In the case of PBB cosmology this bounce corresponds to a region of high curvature where higher-derivative and string corrections to the effective action will be important; in the case of the ekpyrotic scenario the bounce occurs when two four space-time dimensional branes collide in a five dimensional bulk.

Models with a cosmological bounce potentially provide an alternative to cosmological inflation in addressing the homogeneity problem of standard cosmology and in yielding a causal mechanism of structure formation, the latter since at times long before the bounce fixed comoving scales of cosmological interest today will have been inside the Hubble radius.[2] However, since in both PBB and ekpyrotic scenarios the Hubble parameter increases during the collapsing phase, symmetry arguments such as those used originally [12] to predict the scale invariance of cosmological fluctuations in inflationary cosmology would lead one to expect a blue spectrum of curvature perturbations in these models, at least in effective field theory models in which there is only one "matter" field. As outlined in Appendix A, in PBB cosmology one expects a spectrum with spectral index $n=4$,

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[1] In PBB cosmology this statement is true from the point of view of the Einstein frame metric; in the ekpyrotic scenario it is true from the point of view of the four space-time dimensional effective action which is used to describe the cosmology.

[2] See Refs. [6–11] for critical arguments on the ekpyrotic scenario.

whereas in the ekpyrotic scenario one expects $n=3$. In the case of PBB cosmology, this heuristic prediction was confirmed [13] by a general relativistic analysis (which is, however, subject to the caveats indicated below). In the case of the ekpyrotic scenario, there is a large disagreement in the results. Whereas the work of [14–17] yields results in agreement with the heuristic prediction (namely, $n=3$), others [18,19] obtain a scale-invariant spectrum of adiabatic fluctuations (thus also casting doubt on past results in the literature on the spectrum of fluctuations in the PBB scenario).

The singularity of the effective action at the time of the bounce makes it impossible to follow the evolution of the background cosmology and of the resulting cosmological perturbations rigorously [20]. In much of the previous work, in the contexts both of PBB cosmology [21] and of the ekpyrotic scenario [15–17,19], the fluctuations were computed by matching two Einstein universes (the first representing the contracting phase, the second the expanding phase) along a spacelike surface (representing the bounce region) and applying continuity of the induced metric and of the extrinsic curvature across the surface [22,21]. As emphasized in [19], the result will depend on how the matching surface is chosen.[3]

In the context of PBB cosmology, it was realized [26–28] (see also [29–31]) that higher-derivative corrections (defined in the string frame) to the action induced by inverse string tension and coupling constant corrections can yield a nonsingular background cosmology.[4] This then allows the study of the evolution of cosmological perturbations without having to use *ad hoc* matching prescriptions. The effect of the higher-derivative terms in the action on the evolution of fluctuations in the PBB cosmology was investigated in [36]. It was found that, for low-frequency modes, the spectrum of fluctuations is unaffected by the higher-derivative terms, and the result obtained is the same as what follows from the analysis using matching conditions between two Einstein universes [13,21] joined along a constant scalar field hypersurface.

Since the ekpyrotic scenario makes use of a negative exponential potential for the scalar matter field, which leads to an extra instability of the system, it is not clear that the higher-derivative terms used in [26–28] can in this case achieve a nonsingular cosmology. The first main result of this paper is that, with suitably chosen coefficients, the above mentioned terms are indeed sufficient to produce a nonsingular cosmology.

In this paper, we add the same higher-derivative terms used in [27] to the action (which includes a positive or negative potential for the scalar matter field) and construct nonsingular bouncing cosmologies. At the level of the effective action, our Lagrangian can be viewed as giving both nonsingular solutions of modified PBB type (the modification consisting of the addition of an exponential potential for the dilaton), and also nonsingular ekpyrotic solutions. The justification for adding these higher-derivative terms is different in the cases of modified PBB cosmology and in the ekpyrotic scenario. In the case of PBB cosmology, both the string coupling constant and the curvature become large as the dilaton increases, thus justifying the inclusion of both higher-derivative terms of the gravitational action and of quantum corrections. In the case of ekpyrotic cosmology (we have the initial scenario of [5] in mind in which a bulk brane impacts our physical space-time orbifold fixed plane at the time of the bounce and in which the dilaton and hence the string coupling constant are fixed), the density and hence curvature at the bounce are large, thus justifying including higher-derivative terms. In addition, the brane collision is a quantum mechanical process, thus justifying including loop corrections in the action. Note that our method yields a way of constructing a nonsingular bouncing universe which works even in a spatially flat universe and is thus different from the constructions of [37] in which a positive spatial curvature is used to generate a bouncing cosmology. Since tracing back the spatial curvature into the very early universe given the present date leads—under the assumption that there was no period of inflation after the bounce—to a highly suppressed curvature at early times, our approach in obtaining a bouncing cosmology appears more realistic.

We follow the fluctuations through the bounce, and study the spectrum of the resulting cosmological perturbations at late times. In this analysis, no matching conditions at the bounce are necessary. Note, however, that in principle the final spectrum could depend on the frame in which the higher-order correction terms are introduced, and on the specific form of the correction terms. In our nonsingular scenario discussed in Sec. IV, the correction terms are defined in the string frame and we find that the final spectrum of cosmological fluctuations on long wavelength scales has a shape which agrees with what is obtained when applying the matching conditions of [22,21] on a constant scalar field surface[5] (the most physical choice of the matching surface in both PBB and ekpyrotic models). In particular, for the ekpyrotic model of [5] rendered nonsingular by our construction, we obtain a blue spectrum of the curvature perturbation with index $n_{\mathcal{R}}=3$.

II. SINGLE SCALAR FIELD WITH AN EXPONENTIAL POTENTIAL

The Lagrangian considered in this paper can be used to describe both a modified PBB model in which the dilaton has

[3]As emphasized already in [15] and [23], there is a consistency check for proposed matching surfaces: when applied to the reheating surface in inflationary cosmology, the correct result should emerge. This does not happen with the prescription advocated in [19], nor does it with the matching prescription of [18] which is not based on a geometric analysis (see also [24,25] for a criticism of the latter matching prescription).

[4]Construction of nonsingular cosmologies in pure Einstein gravity by means of specific higher-derivative terms is also possible (see, e.g., [32,33] and its application to PBB cosmology in [34,35]).

[5]Note that up to terms of order k^2 the constant scalar field surface and the constant energy density surface are identical, as discussed in [15].

an exponential potential and the ekpyrotic scenario. Our Lagrangian describes gravity plus a single scalar matter field ϕ. In the case of PBB cosmology, the physical frame is the string frame, and ϕ is the dilaton field. In the case of the original version of the ekpyrotic scenario [5], the physical frame is the Einstein frame since the dilaton is fixed, and the field ϕ is related to the separation of a bulk brane from our four-dimensional space-time orbifold fixed plane. In the case of the second version of the ekpyrotic scenario [38] and in the cyclic variant thereof [39,40], ϕ is the modulus field denoting the size of the orbifold (the separation of the two orbifold fixed planes).

We begin with the Lagrangian of the four-dimensional effective theory in the string frame, which is

$$\mathcal{S}_S = \int d^4x \sqrt{-g} e^{-\phi}\left[\frac{1}{2}R + \frac{1}{2}(\nabla\phi)^2 - V_S(\phi)\right], \quad (2.1)$$

where R is the Ricci scalar and $V_S(\phi)$ is the scalar field potential in the string frame. In this form, the action looks reminiscent of the action for PBB cosmology. Note that $V_S(\phi)=0$ in the simplest version of the PBB scenario. We set the units such that $8\pi G \equiv 1$ with G being a four-dimensional gravitational constant. Making a conformal transformation

$$\hat{g}_{\mu\nu} = e^{-\phi} g_{\mu\nu}, \quad (2.2)$$

the action in the Einstein frame can be written as

$$\mathcal{S}_E = \int d^4x \sqrt{-\hat{g}}\left[\frac{1}{2}\hat{R} - \frac{1}{4}(\hat{\nabla}\phi)^2 - V_E(\phi)\right], \quad (2.3)$$

where

$$V_E(\phi) \equiv e^{\phi} V_S(\phi). \quad (2.4)$$

Introducing a rescaled field $\varphi = \pm\phi/\sqrt{2}$, the action (2.3) reads

$$\mathcal{S}_E = \int d^4x \sqrt{-\hat{g}}\left[\frac{1}{2}\hat{R} - \frac{1}{2}(\hat{\nabla}\varphi)^2 - V_E(\phi(\varphi))\right]. \quad (2.5)$$

In this form, the action is seen to describe both the PBB model in the Einstein frame, and the ekpyrotic scenario [19].

The ekpyrotic scenario is characterized by an exponential potential [5]

$$V_E = -V_0 \exp(-\sqrt{2/p}\,\varphi) \quad (2.6)$$

with $0 < p \ll 1$. The field φ denotes the separation of two parallel branes. According to the ekpyrotic scenario, the branes are initially widely separated but are approaching each other, which means that φ begins near $+\infty$ and is decreasing toward $\varphi=0$. In the PBB scenario, in contrast, the dilaton starts out from a weakly coupled regime with ϕ increasing from $-\infty$. Thus, if we want the potential (2.6) to describe a modified PBB scenario with a dilaton potential which is important when $\phi \to 0$ but negligible for $\phi \to -\infty$, we have to use the relation $\varphi = -\phi/\sqrt{2}$ between the field φ in the ekpyrotic case and the dilaton ϕ in the PBB case.

Adopting the Friedmann-Robertson-Lemaitre-Walker (FRLW) metric $ds^2 = -dt_E^2 + a_E^2 dx_E^2$ in the Einstein frame, the background equations are given by

$$3H_E^2 = \frac{1}{2}\dot{\varphi}^2 + V_E(\varphi), \quad \ddot{\varphi} + 3H_E\dot{\varphi} + V_E'(\varphi) = 0, \quad (2.7)$$

where a prime denotes a derivative with respect to a cosmic time t_E. For the exponential potential (2.4) we have the following exact solution:

$$a_E \propto (-t_E)^p, \quad H_E = \frac{p}{t_E},$$

$$V_E = -\frac{p(1-3p)}{t_E^2}, \quad \dot{\varphi} = \frac{\sqrt{2p}}{t_E}. \quad (2.8)$$

The solution for $t_E < 0$ describes the contracting universe in the Einstein frame prior to the collision of branes.

In the string frame the action is given by Eq. (2.1) with potential

$$V_S = -V_0 \exp\left[\left(\frac{1}{\sqrt{p}} - 1\right)\phi\right]. \quad (2.9)$$

The FRLW metric in the string frame is described by $ds^2 = -dt_S^2 + a_S^2 dx_S^2$, which is connected to the quantities in the Einstein frame by

$$dt_S = e^{-\varphi/\sqrt{2}} dt_E, \quad a_S = e^{-\varphi/\sqrt{2}} a_E, \quad (2.10)$$

where we used the relation $\phi = -\sqrt{2}\varphi$. Integrating the first relation gives

$$-(1-\sqrt{p})t_S = (-t_E)^{1-\sqrt{p}}. \quad (2.11)$$

Therefore the evolution of a_S and ϕ in the string frame is

$$a_S \propto (-t_S)^{-\sqrt{p}}, \quad \phi = -\frac{2\sqrt{p}}{1-\sqrt{p}}\ln[-(1-\sqrt{p})t_S]. \quad (2.12)$$

This illustrates the superinflationary solution with growing dilaton from $\phi = -\infty$. Note that singularities are inevitable in both frames as $t \to 0$. We wish to analyze whether this singularity can be avoided by including higher-order corrections.

III. GENERAL ACTIONS AND EVOLUTION EQUATIONS

In this section we present the background and perturbed equations in the case of a generalized action containing higher-derivative terms. We write this action in the form [36,41]

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}f(R,\phi) - \frac{1}{2}\omega(\phi)(\nabla\phi)^2 - V(\phi) + \mathcal{L}_c\right], \quad (3.1)$$

where $f(R,\phi)$ is a function of the Ricci scalar R and a scalar field ϕ. $\omega(\phi)$ and $V(\phi)$ are general functions of ϕ. The

Lagrangian $\mathcal{L}_c$ represents the higher-order corrections to the tree-level action. Both higher-derivative gravitational terms and terms involving φ appear. The action (3.1) applies not only to low-energy effective string theories, but also to effective action that approaches to Einstein quantum gravity and to scalar tensor theories, among others.

As mentioned in the Introduction, our motivation for considering the addition of higher-derivative terms in the effective action is to construct a nonsingular bouncing model and thus to overcome the singularity problem ("graceful exit problem") which plagues both the PBB and the ekpyrotic scenarios. The higher-order contribution $\mathcal{L}_c$ can be written as the sum of the α' classical correction $\mathcal{L}_{\alpha'}$ and the quantum loop correction $\mathcal{L}_q$ [27,42]. Both involve the same gravitational and scalar field terms, but are multiplied by different powers of $e^{-\phi}$.

The leading α' (string) correction to the gravitational action we adopt is given by [26,27]

$$\mathcal{L}_{\alpha'}=-\frac{1}{2}\alpha'\lambda\xi(\phi)[cR_{\rm GB}^2+d(\nabla\phi)^4], \qquad (3.2)$$

where $\xi(\phi)$ is a general function of ϕ and $R_{\rm GB}^2=R^2-4R^{\mu\nu}R_{\mu\nu}+R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ is the Gauss-Bonnet term. The inverse string tension α' is set to unity. The Gauss-Bonnet term has the property of keeping the order of the gravitational equations of motion unchanged. It has been known since the early days of string theory that this term arises as the lowest string correction to the gravitational field equations in a string theory background. At tree level (lowest order in $\hbar$), we have $\xi(\phi)=-e^{-\phi}$. When applied to PBB cosmology, it turns out that as $t\to-\infty$ the invariants $R_{\rm GB}^2$ and $(\nabla\phi)^4$ decay faster than the coefficient function $\xi(\phi)$ blows up. Hence, the correction terms in the Lagrangian are unimportant for large negative values of ϕ, but become important as the system approaches the strongly coupled region ($\phi\sim 0$).

Following [27], we take the higher n-loop correction terms $\mathcal{L}_q$ in addition to the tree-level term $\mathcal{L}_{\alpha'}$. For the moment, however, we will keep $\xi(\phi)$ general. We will give specific forms for $\xi(\phi)$ and $\mathcal{L}_q$ later.

A. Background equations

Variation of the action (3.1) with respect to the scale factor, the lapse function (then set to 1 after the variation), and the scalar matter field leads to the following background equations [36]:

$$H^2=\frac{1}{6F}(\omega\dot\phi^2+RF-f+2V-6H\dot F+\rho_c), \qquad (3.3)$$

$$\dot H=\frac{1}{2F}\left(-\omega\dot\phi^2+H\dot F-\ddot F-\frac{1}{2}\rho_c-\frac{1}{2}p_c\right), \qquad (3.4)$$

$$\ddot\phi+3H\dot\phi+\frac{1}{2\omega}(\omega_\phi\dot\phi^2-f_\phi+2V_\phi-\Delta_\phi)=0, \qquad (3.5)$$

where $F\equiv\partial f/\partial R$ and $H\equiv\dot a/a$. ρ_c, p_c, and Δ_ϕ correspond to the higher-order curvature and derivative corrections with stress-energy tensor $T^\mu_\nu=(-\rho_c,p_c,p_c,p_c)$. Δ_ϕ comes from the variation of $\mathcal{L}_c$ with respect to ϕ. For the tree-level α' correction (3.2), one has

$$\rho_c=2\alpha'\lambda\left(12c\dot\xi H^3-\frac{3}{2}d\xi\dot\phi^4\right), \qquad (3.6)$$

$$p_c\equiv-2\alpha'\lambda\left\{4c[\ddot\xi H^2+2\xi H(\dot H+H^2)]+\frac{1}{2}d\xi\dot\phi^4\right\}, \qquad (3.7)$$

$$\Delta_\phi\equiv-\alpha'\lambda[24c\xi_\phi H^2(\dot H+H^2) -d\dot\phi^2(3\dot\xi\dot\phi+12\xi\ddot\phi+12\xi\dot\phi H)]. \qquad (3.8)$$

Note that taking into account quantum loop corrections provides additional source terms for ρ_c, p_c, and Δ_ϕ. We will discuss this issue in Sec. IV.

B. Perturbation equations

A perturbed space-time metric has the following form for scalar perturbations in an arbitrary gauge (see, e.g., [43], where the function A is denoted by ϕ):

$$ds^2=-(1+2A)dt^2+2a(t)B_{,i}dx^idt +a^2(t)[(1-2\psi)\delta_{ij}+2E_{,i,j}]dx^idx^j, \qquad (3.9)$$

where a comma denotes the usual flat space coordinate derivative. We introduce the curvature perturbation $\mathcal{R}$ in the comoving gauge [44]

$$\mathcal{R}\equiv\psi+\frac{H}{\dot\phi}\delta\phi. \qquad (3.10)$$

The perturbed Einstein equations for the action (3.1) are written in the form [41,36]

$$\frac{1}{a^3Q}(a^3Q\dot{\mathcal{R}})^{\cdot}-s\frac{\Delta}{a^2}\mathcal{R}=0, \qquad (3.11)$$

where

$$Q\equiv\frac{\omega\dot\phi^2+3I(\dot F-4\lambda c\dot\xi H^2)-6\lambda d\xi\dot\phi^4}{(H+I)^2}, \qquad (3.12)$$

$$s\equiv1+\frac{4\lambda c\xi\dot\phi^4-16\lambda c\dot\xi\dot H I+8\lambda c(\ddot\xi-\dot\xi H)I^2}{\omega\dot\phi^2+3I(\dot F-4\lambda c\dot\xi H^2)-6\lambda d\xi\dot\phi^4} \qquad (3.13)$$

with $I\equiv(\dot F-4\lambda c\dot\xi H^2)/(2F-8\lambda c\dot\xi H)$.

Introducing a new quantity,[6] $\Psi\equiv z\mathcal{R}$ with $z\equiv a\sqrt{Q}$, each Fourier component of Ψ satisfies the second order differential equation

[6]Note that Ψ is the variable in terms of which—for unmodified Einstein gravity—the action for fluctuations has the canonical form of a free field action with time dependent mass (see, e.g., [43] where the variable is denoted as v).

$$\Psi_\kappa''+\left(sk^2-\frac{z''}{z}\right)\Psi_k=0, \tag{3.14}$$

where a prime denotes the derivative with respect to conformal time, $\eta\equiv\int a^{-1}dt$. In the large scale limit, $|sk^2| \ll |z''/z|$, Eq. (3.14) is integrated to give

$$\mathcal{R}_k=C_k+D_k\int\frac{d\eta}{z^2}, \tag{3.15}$$

where C_k and D_k are integration constants. The curvature perturbation is conserved on super-Hubble scales as long as the second term in Eq. (3.15) is not strongly dominating, as in the case of the single field, slow-roll inflationary scenarios.

If the evolution of z before the bounce is given in the form[7]

$$z\propto(-\eta)^\gamma, \tag{3.16}$$

the second term in Eq. (3.15) yields $\int d\eta/z^2\propto(-\eta)^{1-2\gamma}$. Therefore curvature perturbations can be amplified for $\gamma \geqslant 1/2$ on super-Hubble scales, while they are not for $\gamma < 1/2$ [17] [note that $\mathcal{R}_k\propto\ln(-\eta)$ for $\gamma=1/2$]. Whether this enhancement occurs or not depends on the time evolution of z, and therefore on the string cosmological model.

We need to go to the next order solution of Eq. (3.14) in order to obtain the spectrum of curvature perturbations. If s is a positive constant (as it will be in the asymptotic limits), the solution for Ψ_k is expressed by the combination of the Hankel functions:

$$\Psi_k=\frac{\sqrt{\pi|\eta|}}{2}[c_1H_\nu^{(1)}(x)+c_2H_\nu^{(2)}(x)], \tag{3.17}$$

where

$$x\equiv\sqrt{s}k|\eta|, \quad \nu\equiv\frac{1}{2}(1-2\gamma). \tag{3.18}$$

The solution (3.17) corresponds to the Minkowski vacuum state in the small scale limit ($k\to\infty$).

We can expand the Hankel functions in the following form [45]:

$$H_\nu^{(1,2)}(x)=\sum_{n=0}^\infty\frac{1}{n!}\left(-\frac{x^2}{4}\right)^n\frac{1}{\sin\pi\nu}\left[\left(\frac{x}{2}\right)^\nu\frac{\pm ie^{\mp i\pi\nu}}{\Gamma(\nu+n+1)}+\left(\frac{x}{2}\right)^{-\nu}\frac{\mp i}{\Gamma(-\nu+n+1)}\right]. \tag{3.19}$$

The curvature perturbation $\mathcal{R}_k=\Psi_k/z$ has two solutions, which are proportional to $k^\nu|\eta|^{\nu-\gamma+1/2}$ and $k^{-\nu}|\eta|^{-\nu-\gamma+1/2}$, following from the first and second terms in Eq. (3.19), respectively. In the large scale limit ($k\to0$), the contribution of the second term dominates over the first one, thereby yielding the spectrum of the curvature perturbation as

$$P_\mathcal{R}\equiv\frac{k^3}{2\pi^2}|\mathcal{R}_k|^2\propto k^{3-2\nu}\propto k^{n_\mathcal{R}-1}, \tag{3.20}$$

in which case the spectral tilt is

$$n_\mathcal{R}-1=3-|1-2\gamma|. \tag{3.21}$$

Note that we have $k^{-\nu}|\eta|^{-\nu-\gamma+1/2}=k^{-\nu}|\eta|^0$ for $\gamma<1/2$, in which case the constant mode C_k in Eq. (3.15) corresponds to the solution that comes from the second term in Eq. (3.19). The ekpyrotic scenario with a negative potential ($0<p<1/3$) belongs to this case ($\gamma<1/2$) as we will show later. When $\gamma>1/2$ one has $k^{-\nu}|\eta|^{-\nu-\gamma+1/2}=k^{-\nu}|\eta|^{1-2\gamma}$, which means that $\mathcal{R}_k$ grows before the graceful exit ($\eta<0$). We will discuss a string-inspired model that belongs to this case in Sec. V. The PBB scenario corresponds to the marginal case with $\gamma=1/2$.

Note that s is exactly unity in Eq. (3.14) when the corrections $\mathcal{L}_c$ are not taken into account. In the presence of higher-order corrections ($\mathcal{L}_c\neq0$), s is generally a time-varying function, in which case the formula (3.21) can not be directly applied. Nevertheless it is still valid if s is a slowly varying positive function.

In subsequent sections we shall apply the above general formulas to concrete string-inspired models. In Sec. IV we apply the string loop and quantum corrections to the low-energy effective action in the string frame and find nonsingular bouncing cosmological solutions. In Sec. V, we consider a situation with fixed dilaton and add a Gauss-Bonnet term to the Einstein frame action. We find nonsingular cosmological solutions which begin in an asymptotically flat state, and undergo a period of superexponential inflation which terminates with a graceful exit.

IV. INCLUSION OF HIGHER-ORDER CORRECTIONS IN THE STRING FRAME: DILATON-DRIVEN CASE

In the context of PBB cosmology, the natural frame to use in order to define the correction term $\mathcal{L}_c$ in the Lagrangian is the string frame. In this case, we should use $f=e^{-\phi}R$ and $\omega=-e^{-\phi}$, i.e.,

$$\mathcal{S}_S=\int d^4x\sqrt{-g}\left\{e^{-\phi}\left[\frac{1}{2}R+\frac{1}{2}(\nabla\phi)^2-V_S(\phi)\right]+\mathcal{L}_c\right\}, \tag{4.1}$$

where ϕ corresponds to the dilaton. This Lagrangian was suggested in [26,27], and used to construct nonsingular background cosmological solutions of PBB cosmology in the absence of a potential for the dilaton. Fluctuations in this model were studied in [36] in the absence of a dilaton potential.

In the following, we extend these analyses to the case of a nonvanishing dilaton potential. We will first construct nonsingular solutions which in the Einstein frame correspond to nonsingular bouncing cosmologies. We then study how the

[7]Note that as long as the additional terms $\mathcal{L}_c$ in the action are negligible, then $\gamma=p/(1-p)$, and thus $0<\gamma\leqslant1$ for the collapsing phase of ekpyrotic cosmology, and $\gamma=1/2$ ($p=1/3$) for the collapsing phase of PBB cosmology.

fluctuations evolve across the bounce and compute the spectrum of fluctuations. The analysis in this section thus applies immediately to the modified PBB scenario in which the dilaton has a negative exponential potential. We can also apply the results to the initial version [5] of the ekpyrotic scenario. In this case, the brane collision occurs at $\varphi=0$, and since thus the gravitational coupling constant does not change significantly near the bounce, the difference in the role of the higher-derivative terms between the Einstein frame (the frame in which it appears most logical to define the correction terms in the Lagrangian) and the "string frame" (quotation marks used here because in the case of the ekpyrotic scenario φ is not the dilaton, the dilaton being fixed) is not expected to be significant. The application of the results of this section to the version of the ekpyrotic scenario with moving boundary branes [38] and to the cyclic scenario [39] is more problematic since in this case φ is the dilaton, $\varphi\to-\infty$ at the bounce, and thus the difference in the evolution of models with correction terms defined in the Einstein and string frames is expected to be important.

The reason why nonsingular solutions are possible in the presence of the correction term $\mathcal{L}_c$ is that such a term can lead to violations of the null energy condition (from the perspective of an observer using unmodified Einstein equations). Thus, it is expected to lead to a successful graceful exit, in the same way that introducing matter violating the null energy condition allowed the construction of nonsingular bouncing models in [46,47].

In this model the background equations are written as

$$6H^2-6H\dot{\phi}+\dot{\phi}^2-2V_S=e^{\phi}\rho_c, \tag{4.2}$$

$$4\dot{\phi}H-4\dot{H}-6H^2-\dot{\phi}^2+2\ddot{\phi}+2V_S=e^{\phi}p_c, \tag{4.3}$$

$$6\dot{H}+12H^2+\dot{\phi}^2-2\ddot{\phi}-6H\dot{\phi}-2(V_S-V_S')=e^{\phi}\Delta_{\phi}. \tag{4.4}$$

The dilatonic corrections $\mathcal{L}_c$ are the sum of the tree-level α' corrections and the quantum n-loop corrections ($n=1,2,3,...$), with the function $\xi(\phi)$ [see Eq. (3.2)] given by

$$\xi(\phi)=-\sum_{n=0} C_n e^{(n-1)\phi}, \tag{4.5}$$

where C_n ($n\geqslant 1$) are the coefficients of n-loop corrections with $C_0=1$. In this case the source terms due to $\mathcal{L}_c$ on the right hand side of Eqs. (4.2)–(4.4) are given by [42]

$$\rho_c=\sum_{n=0} C_n\{\rho_c\}_n, \quad p_c=\sum_{n=0} C_n\{p_c\}_n,$$

$$\Delta_{\phi}=\sum_{n=0} C_n\{\Delta_{\phi}\}_n, \tag{4.6}$$

where

$$\{\rho_c\}_n=\alpha'\lambda\dot{\phi}e^{(n-1)\phi}\{-24c(n-1)H^3+3d\dot{\phi}^3\}, \tag{4.7}$$

$$\{p_c\}_n=\alpha'\lambda e^{(n-1)\phi}\{8c(n-1)H[(n-1)\dot{\phi}^2H+\ddot{\phi}H+2\dot{\phi}(\dot{H}+H^2)]+d\dot{\phi}^4\}, \tag{4.8}$$

$$\{\Delta_{\phi}\}_n=\alpha'\lambda e^{(n-1)\phi}\{24c(n-1)H^2(\dot{H}+H^2)-3d\dot{\phi}^2[4\ddot{\phi}+4\dot{\phi}H+(n-1)\dot{\phi}^2]\}, \tag{4.9}$$

with $\lambda=-1/4$. Following Ref. [36] we choose the coefficients as $c=-d=-1$. Note that the above corrections include the α' corrections (3.6)–(3.8), corresponding to $n=0$.

It is also convenient to relate the Hubble parameter and its derivative in the Einstein frame with those in the string frame by using Eq. (2.10):

$$H_E=e^{\phi/2}\left(H_S-\frac{\dot{\phi}}{2}\right),$$

$$\dot{H}_E=e^{\phi}\left(\dot{H}_S-\frac{\ddot{\phi}}{2}+\frac{1}{2}\dot{\phi}H_S-\frac{\dot{\phi}^4}{4}\right). \tag{4.10}$$

Here the overdots on the right hand sides denote the time derivatives with respect to t_S. The energy density ρ_E and the pressure p_E in the Einstein frame are expressed as

$$\rho_E=3H_E^2, \quad \rho_E=-3H_E^2-2\dot{H}_E. \tag{4.11}$$

Once we know the evolution of the background in the string frame, it is easy to find the evolution of H_E, a_E, $\varphi=-\phi/\sqrt{2}$, and to check whether the null energy condition $\rho_E+p_E>0$ holds or not in the Einstein frame by using Eqs. (4.10) and (4.11). Note that in the absence of higher-order corrections ($\mathcal{L}_c=0$) one has $2\dot{H}_E=-(\rho_E+p_E)=-\dot{\phi}^2<0$. In this case once the contraction begins ($\dot{H}_E<0$) the Hubble parameter is *always* negative. Therefore it is not possible to have the bouncing solutions required for the nonsingular ekpyrotic scenario unless higher-order corrections $\mathcal{L}_c$ are taken into account.

A. Background evolution

In the absence of a negative exponential potential ($V_S=0$), it was found in Ref. [26] that curvature singularities can be avoided by taking into account higher-order corrections $\mathcal{L}_c$. In this case we have nonsingular bouncing solutions in the Einstein frame due to the violation of the null energy condition. We are interested in whether singularity avoidance is possible or not in the presence of the ekpyrotic potential (2.9). Note that, since near the bounce $H^2\sim t^{-2}$, higher-curvature corrections to the Einstein action will likewise be important in the presence of a potential.

When $V_S\neq 0$ and $\mathcal{L}_c=0$ the background solutions are described by Eqs. (2.8) and (2.12). In the string frame the scale factor evolution is superinflationary with growing Hubble rate ($\dot{H}_S>0$). We plot in Fig. 1 the evolution of background quantities in both the string and Einstein frames [see the case (i)]. The dilaton ϕ starts out from the weakly coupled regime $g_{\rm string}^2\equiv e^{\phi}\ll 1$, corresponding to widely separated branes in the ekpyrotic scenario, $\varphi=-\phi/\sqrt{2}\gg 1$. In the Einstein frame the universe is contracting with a negative Hubble rate. The solution inevitably meets a curvature singularity as ϕ grows

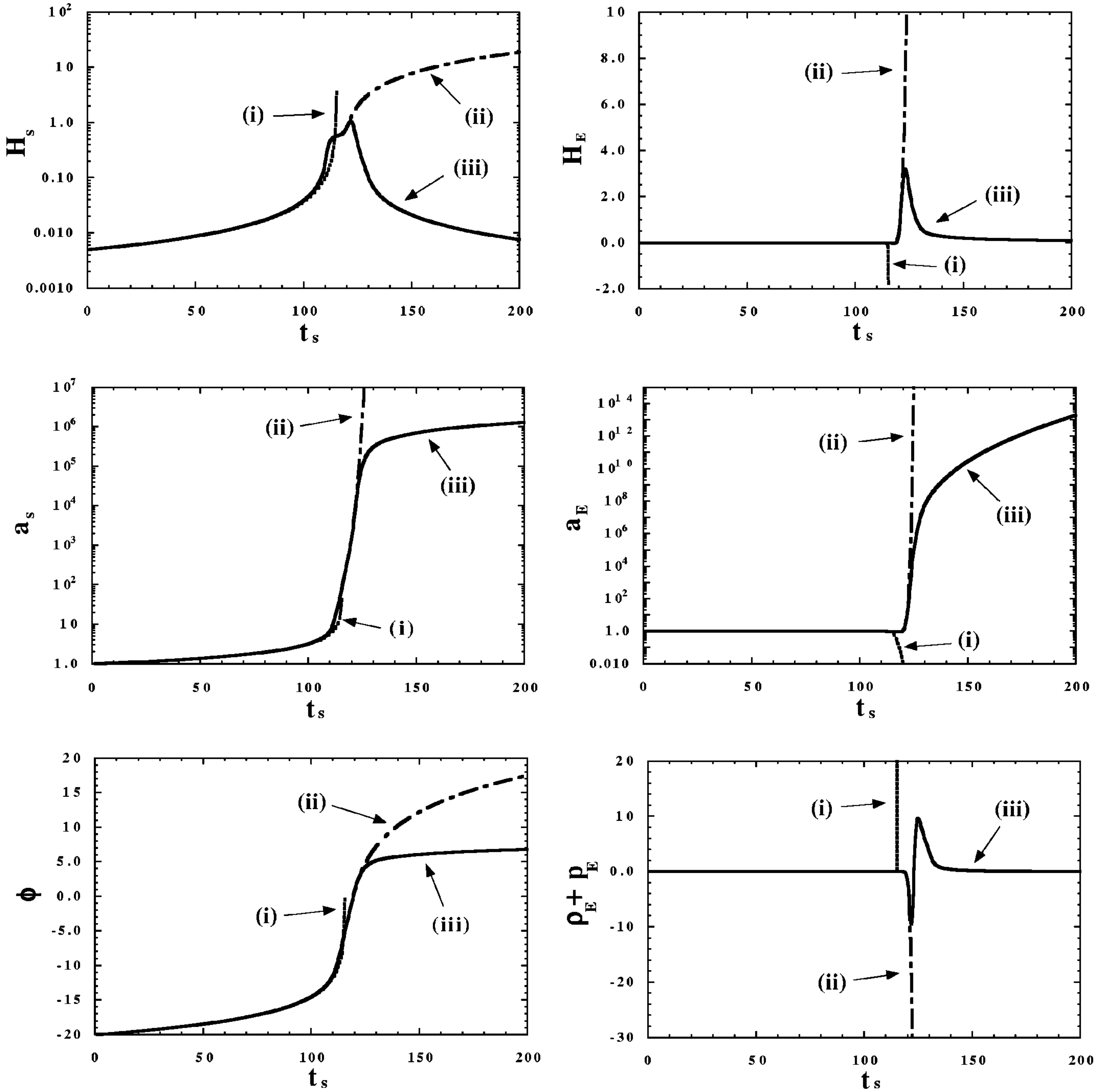

FIG. 1. The evolution of H_S, H_E, a_S, a_E, ϕ, and ρ_E+p_E with $c=-1$, $d=1$, $p=0.1$. We choose initial conditions $\phi=-20$, $H=5.0\times10^{-3}$. The cases correspond to (i) only tree-level correction terms but no higher-order corrections $(C_1=C_2=0)$, (ii) tree-level and one-loop corrections present $(C_1=1.0,C_2=0)$, and (iii) tree-level and one- and two-loop corrections present with $C_1=1.0$ and $C_2=-1.0\times10^{-3}$.

toward the strongly coupled regime $(g_{\rm string}^2\sim1)$.

Our first main finding is that with $V_S(\phi)\neq0$ there exist nonsingular trajectories in the presence of higher-order corrections $(\mathcal{L}_c\neq0)$. Thus, the presence of the potential for the dilaton does not prevent the higher-derivative terms from being able to smooth out the curvature singularity. The details depend on the value of the power-law index p. When $p\ll1$ the ekpyrotic potential (2.6) is exponentially suppressed for $\varphi\gtrsim1$, in which case the dynamics of the system is hardly affected by the negative potential except for the region $\varphi\sim0$. However, in this region the higher-derivative terms play a crucial role.

In our simulations, we have adopted the potential (2.6) for $\varphi>0$ and $V_E=0$ for $\varphi<0$. This is in the spirit of the first version of the ekpyrotic scenario [5] in which the potential vanishes at the brane collision, the bulk brane is absorbed by the orbifold fixed plane via a small instanton transition, and there is no potential left afterward. We show in Fig. 1 the dynamical evolution of the system for $p=0.1$. The case (i) is the one in which only tree-level terms are present and in which singularity avoidance is not possible. The case (ii) corresponds to the one where both tree-level and one-loop corrections are taken into account ($C_1=1.0$ and $C_2=0$). Inclusion of one-loop corrections makes it possible to have nonsingular cosmological solutions. In fact ρ_E+p_E becomes negative around $t_S\sim 115$ in Fig. 1, after which the Hubble parameter H_E begins to grow. The universe starts to expand once H_E crosses zero; namely, the violation of the null energy condition allows us to have nonsingular bouncing solutions in the Einstein frame. Nevertheless, we should notice that the evolution of the scale factors is superinflationary in both the string and Einstein frames due to unbounded increase of H_S and H_E, together with rapid growth of the field ϕ. Therefore we are faced with another problem, namely, how to connect to the stage of a decreasing Hubble parameter.

If two-loop terms are added (keeping the previous tree-level and one-loop terms) phenomenologically more appealing nonsingular solutions can be obtained. When C_2 is positive, the evolution of the system does not differ significantly compared to the case (ii). However, it is possible to obtain a decreasing Hubble rate if we take a negative value of C_2. The case (iii) of Fig. 1 corresponds to the coefficients $C_1=1.0$ and $C_2=-1.0\times 10^{-3}$. We find that the growth rates of the scale factor and of ϕ are slowed compared to the case (ii). We see that ρ_E+p_E becomes positive and begins to decrease toward $+0$ after the short period of violations of the null energy condition. Although this case does not correspond to the radiation-dominated universe after the graceful exit, it is possible to connect to it by taking into account the decay of the dilaton to radiation.[8] However, including radiation in the ekpyrotic cosmology has some subtle points, and we do not consider this problem in the present work.

We have checked that the addition of three-loop terms with coefficients chosen to be of the order 10^{-7} (roughly the same hierarchy of coefficients between the two- and three-loop terms as between the one- and two-loop terms) does not change the results of the two-loop analysis in a significant way. With a coefficient of the three-loop term of order 1, the background solution ceases to be nonsingular.

We emphasize that we have nonsingular bouncing solutions in the Einstein frame even in the presence of a negative exponential potential. When $p\ll 1$ the potential is vanishingly small for $\varphi\gg 1$, in which case the dynamics of the system is practically the same as that of the zero potential discussed in Ref. [27]. In this case the dilaton starts out from the low-curvature regime $|\phi|\gg 1$, which is followed by the string phase with linearly growing dilaton and nearly constant Hubble parameter. During the string phase one has [26]

[8]If we were to include production of radiation at a fixed time during the expanding phase, we could use the well-known results on the constancy of $\mathcal{R}$ in the expanding phase [44,48,49] to argue that the spectrum of fluctuations on large scales will be the same as that obtained in this paper. The crucial fact about our bounce is that it is not symmetric in time (see Fig. 1).

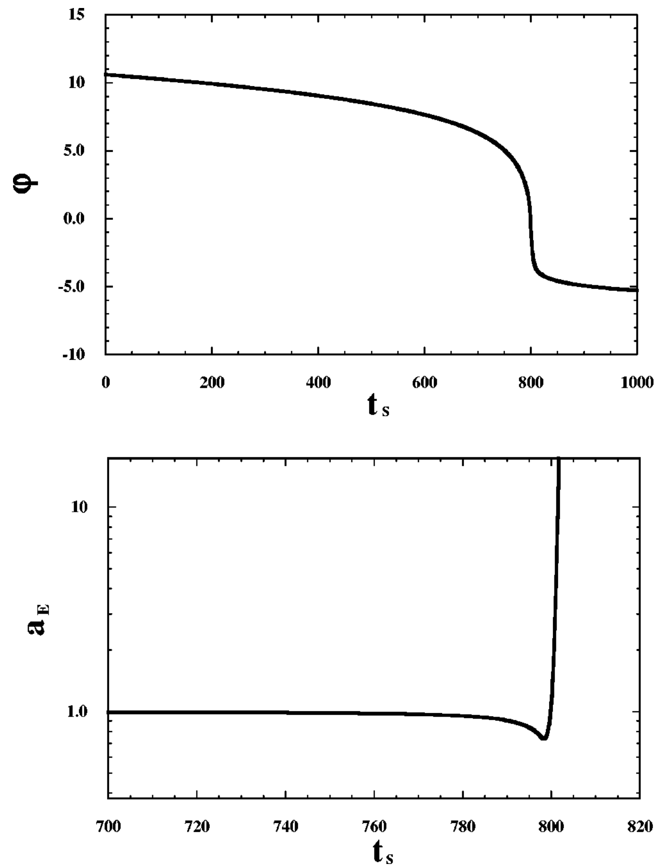

FIG. 2. The evolution of φ and a_E with $c=-1$, $d=1$, $p=0.1$, $C_1=1.0$, and $C_2=-1.0\times 10^{-3}$. In this case we include the correction term $\mathcal{L}_c$ only for $\varphi<1$. We choose initial conditions $\phi=-15$, $H=1.5\times 10^{-3}$. Prior to the collision of branes at $\varphi=0$, the universe is slowly contracting, which is followed by the bouncing solution through higher-order corrections.

$$a_S\propto(-\eta_S)^{-1}, \quad \phi=-\frac{\dot{\phi}_f}{H_f}\ln(-\eta_S)+\text{const}, \tag{4.12}$$

where $\dot{\phi}_f\simeq 1.40$ and $H_f\simeq 0.62$. In the Einstein frame this corresponds to a contracting universe with

$$a_E\propto(-\eta_E)^{\dot{\phi}_f/(2H_f)-1}. \tag{4.13}$$

On the other hand, we can consider the scenario where the negative ekpyrotic potential dominates initially but the higher-order correction becomes important when two branes approach sufficiently. Numerically we confirmed that it is possible to have nonsingular solutions (see Fig. 2). In the simulations we included the correction terms of $\mathcal{L}_c$ only for $\varphi\lesssim 1$. In this case the background solutions are described by Eq. (2.8) or Eq. (2.12) before the higher-order correction terms begin to work. Given this background solution, one can obtain the spectra of curvature perturbations analytically, as we will see in the next section. The spectra depend on whether the higher-order terms are always dominant or not relative to the negative potential before the bounce.

We have also studied ekpyrotic potentials with other values of p, and found that if the potential is negative, corresponding to $p<1/3$, then singularity avoidance is possible for suitable choices of C_1 and C_2 as in the case $p=0.1$ shown in Fig. 1. In the case of $p>1/3$ the field ϕ climbs up a positive exponential potential due to the Hubble contraction term. When $p\gtrsim 1/2$ with $V_0=p(1-3p)$ we found that the field ϕ returns back before it reaches the strongly coupled region, $\phi\gtrsim 0$. This is equivalent to the fact that two parallel branes do not approach each other sufficiently. In such cases the positive exponential potential makes the field bounce back before the higher-order correction becomes important (this may be related to the instability discussed recently in [50]). If we choose smaller values of V_0, it is possible to have nonsingular bouncing solutions which are similar to those in Fig. 1. This case corresponds to the one where the effect of the positive potential is negligible compared to higher-order corrections, in which case the background solutions are given by Eq. (4.12). When the positive potential is dominant from the beginning, it is difficult to obtain a solution where a successful graceful exit is realized by higher-order corrections.

B. Density perturbations

Let us proceed to the analysis of the evolution and the spectra of density perturbations. We shall consider two cases: (i) the effect of the potential $V(\phi)$ is always negligible relative to the correction term $\mathcal{L}_c$, and (ii) the effect of the correction term $\mathcal{L}_c$ becomes important only around the graceful exit $(\phi\sim 0)$. Note that the second case is the physically more interesting one for applications to ekpyrotic cosmology.

1. Case (i): $|V(\phi)|\ll|\mathcal{L}_c|$

When the correction terms (3.6)–(3.8) always dominate relative to the exponential potential (2.6), the spectra of density perturbations are similar to the ones discussed in Ref. [36]. During the string phase with linearly growing dilaton and nearly constant Hubble parameter with $\dot{\phi}_f\simeq 1.40$ and $H_f\simeq 0.62$, we have a sufficient amount of inflation with e-folds $N\equiv\ln(a/a_i)>60$ provided that the dilaton field satisfies $|\phi|\gg 1$ initially [36,42]. In this stage Q defined in Eq. (3.13) is proportional to $e^{-\phi}$ by making use of Eq. (4.12), thereby leading to

$$z\propto(-\eta_S)^{\gamma} \quad \text{with} \quad \gamma=-1+\frac{\dot{\phi}_f}{2H_f}\simeq 0.13. \qquad (4.14)$$

Making use of the relation (3.21), which is valid for positive s, the spectral tilt of the large scale curvature perturbation is

$$n_{\mathcal{R}}-1=3-\left|3-\frac{\dot{\phi}_f}{H_f}\right|\simeq 2.26. \qquad (4.15)$$

The evolution of the frequency shift s is nontrivial (see Fig. 3). In the low-curvature regime where the higher-order terms are not important, s is positive $(s\simeq 1)$, as in the usual PBB scenario. It then changes sign and becomes negative during a short transition from the low-curvature regime to

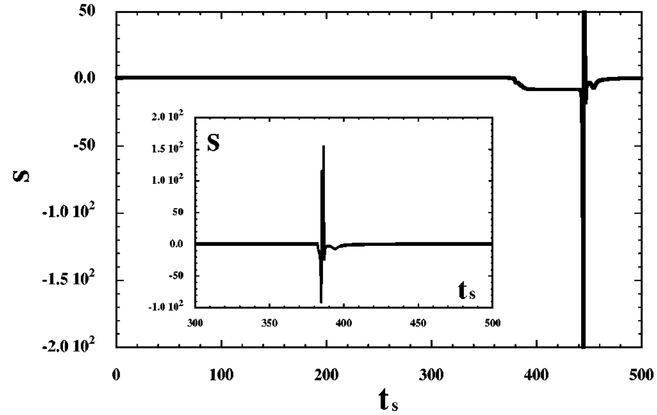

FIG. 3. The evolution of the frequency shift s for $c=-1$, $d=1$, $p=0.1$ with initial conditions $\phi=-100$, $H=1.5\times 10^{-3}$. We include the quantum correction $\mathcal{L}_c$ from the beginning. The shift s is approximately constant (and negative) during the string phase, which is followed by the stage of decreasing curvature with positive s. Inset: The evolution of s in the case where the quantum correction is taken into account only for $\varphi\lesssim 1$. Note that s rapidly changes sign around the graceful exit.

the string phase. During the string phase, $\dot{\phi}$ and H_S are constant ($\dot{\phi}\simeq 1.40$ and $H_S\simeq 0.62$), and $\xi\sim -e^{-\phi}$. The correction term on the right hand side of Eq. (3.13) for s dominates in this phase. It follows from Eq. (3.13) that the ϕ dependence of the leading term cancels out between $\xi(\phi)$ and $\omega(\phi)$, and that hence s is constant and negative until the graceful exit. In a stage with negative constant s, the solution of Eq. (3.14) can be written in the form

$$\Psi_k=\sqrt{|\eta|}[c_1 I_\nu(x)+c_2 K_\nu(x)], \qquad (4.16)$$

where x and ν are given in Eq. (3.17) (with s replaced by the absolute value of s), and I_ν and K_ν are modified Bessel functions, whose asymptotic solutions are $I_\nu\propto x^\nu$, $K_\nu\propto x^{-\nu}$ for $x\to 0$, and $I_\nu\sim e^x/\sqrt{2\pi x}$, $K_\nu\sim\sqrt{\pi/(2x)}e^{-x}$ for $x\to\infty$. Then one reproduces the spectral tilt (4.15) in the large scale limit $(|sk^2|\ll|z''/z|)$. For small scale modes curvature perturbations show exponential instability due to negative frequency shift. After the horizon crossing $(|sk^2|\lesssim|z''/z|)$, curvature perturbations are frozen, since γ is smaller than 1/2 in this case.

It was shown in Ref. [31] that the ratio $\dot{\phi}_f/H_f$ is required to lie in the range $2\lesssim\dot{\phi}_f/H_f\lesssim 3$ for a successful graceful exit in the presence of other forms of higher-order α' correction. Therefore the spectral tilt lies in the range

$$2\lesssim n_{\mathcal{R}}-1\lesssim 3, \qquad (4.17)$$

which is valid for large scale modes $(|sk^2|\ll|z''/z|)$. Therefore we have blue-tilted spectra as long as the correction $\mathcal{L}_c$ dominates compared to the exponential potential.

2. Case (ii): $|V(\phi)|\gg|\mathcal{L}_c|$ but $|V(\phi)|<|\mathcal{L}_c|$ for $\varphi\sim 0$

When the correction term $\mathcal{L}_c$ becomes important only around the graceful exit $(\phi\sim 0)$, the spectra of density perturbations generated before the bounce are mainly determined by the exponential potential. In this case the evolution of the background can be characterized by Eq. (2.12). Then the quantity Q in Eq. (3.12) evolves as

$$Q=\frac{2\dot{\phi}^2 e^{-\phi}}{(2H_S-\dot{\phi})^2}\propto(-\eta_S)^{2\sqrt{p}/(1-p)}. \tag{4.18}$$

Therefore we find

$$z\propto(-\eta_S)^{\gamma} \quad \text{with} \quad \gamma=\frac{p}{1-p}, \tag{4.19}$$

and the spectral tilt for the curvature perturbation is

$$n_{\mathcal{R}}-1=\begin{cases}\dfrac{2}{1-p} & (\text{for } 0<p<1/3),\\[2mm] \dfrac{4-6p}{1-p} & (\text{for } 1/3<p<1).\end{cases} \tag{4.20}$$

This coincides with the result in the Einstein frame obtained in Refs. [14–17]. For very slow contraction with a negative ekpyrotic potential ($p\ll 1$), one has blue-tilted spectra with $n_{\mathcal{R}}-1=2$. Since γ is less than 1/2 for $p<1/3$ (i.e., negative potential), curvature perturbations are not enhanced in the large scale limit even in the presence of the correction $\mathcal{L}_c$ around the graceful exit. The simplest PBB scenario with zero potential corresponds to $p=1/3$ and $\gamma=1/2$, in which case one has $n_{\mathcal{R}}-1=3$. In this case curvature perturbations evolve as $\mathcal{R}_k\propto\ln(-\eta)$ as found from Eq. (3.15) with Eq. (3.16).

We have solved the evolution equation (3.11) for the cosmological fluctuations numerically. Experience from studying fluctuations in inflationary cosmology teaches us that following the evolution equation for Ψ instead of for the gravitational potential Φ is less likely to be affected by numerical noise. Since in a contracting universe one of the modes of Φ increases much more rapidly than the dominant mode of Ψ, we believe that it is advantageous to use Ψ in our case as well. In addition, from a more conceptual point of view, the variable $\mathcal{R}$ is preferable since it is more closely related to Ψ in terms of which the action for cosmological fluctuations takes on its canonical form. Note that Ψ is also the good variable to use when following cosmological fluctuation from inflation through reheating [70]. In Fig. 4 we plot the resulting evolution of the spectra of curvature perturbations for several different frequencies. The higher-order correction $\mathcal{L}_c$ is included only when two branes approach sufficiently, i.e., $\varphi\lesssim 1$. We find that large scale modes ($k\ll 1$) are not enhanced as predicted by Eq. (3.15). In contrast, small scale curvature perturbations exhibit rapid growth around the graceful exit ($\varphi\sim 0$).

There are two reasons for this instability. The first is the fact that the frequency shift s becomes negative for a short period where the higher-curvature effect is dominant (see the inset of Fig. 3). As is obvious from Eq. (3.14), an exponential instability for Ψ is induced by negative s, which is stronger for larger k. We expect this instability will become important for modes with $|s|_{\max}k^2\gtrsim 1$, where $|s|_{\max}$ is the

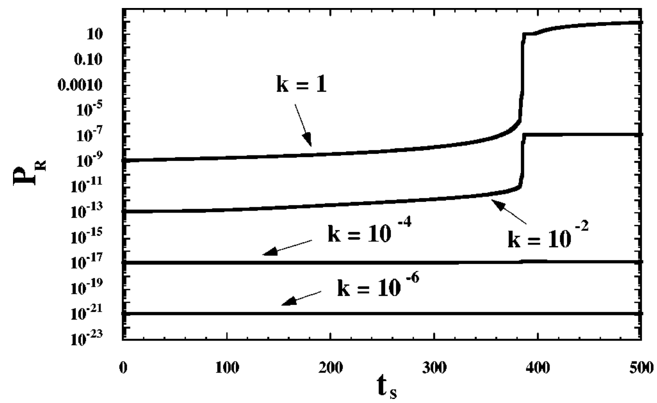

FIG. 4. The evolution of the spectra of curvature perturbations, $P_{\mathcal{R}}$, for $c=-1$, $d=1$, $p=0.1$, $C_1=1.0$, and $C_2=-1.0\times10^{-3}$. The initial conditions are chosen to be $\phi=-15$, $H=1.5\times10^{-3}$. We include the higher-order correction $\mathcal{L}_c$ only for $\varphi\lesssim 1$. The curvature perturbation does not exhibit significant variation during the contracting phase. However small scale modes are enhanced around the graceful exit.

maximal negative value of the function s. From the inset of Fig. 3 the maximal absolute value of s during the negative branch is about $|s|_{\max}\sim 10^2$. Hence, the s instability is expected to be important only for modes with $k\gtrsim 10^{-1}$. By comparing runs with s given by the general formula and runs with $s=1$, we were able to determine numerically that the actual cutoff value of k below which the instability due to the s term is negligible is $k\sim 10^{-2}$. Thus, we conclude that the main source of the short wavelength instability of the fluctuation modes around the bounce must be a second one, namely, the nontrivial nature of the bouncing background and its result on the quantity z''/z.

After the transition to the expanding universe, the curvature perturbation is nearly conserved as found in Fig. 4.

We show in Fig. 5 the spectra of curvature perturbations for $p=10^{-3}$ in the case where the correction $\mathcal{L}_c$ is included only for $\varphi\lesssim 1$. We find that the numerical value of the spectral tilt is $n_{\mathcal{R}}-1\sim 2$ for $k\lesssim 10^{-4}$, which coincides with the analytic estimation (4.20). However, this estimate is no longer valid for small scale modes due to the negative frequency shift and the instability around the graceful exit. The spectra are highly blue tilted for $k\gtrsim 10^{-4}$ as found in Fig. 5. This growth of small scale fluctuations obviously works as the gravitational back reaction to the background evolution. Although we did not consider the effect of the back reaction here, it is certainly of interest to investigate how the background evolution is modified around the graceful exit. We have performed the simulations with various choices of time steps to make sure that the effects we find are not numerical artifacts. The spectra obtained are independent of the specific value of the time step.

In Fig. 5 we have also plotted the induced fluctuations of Φ_k^E in the Einstein frame, determined from the results for $\mathcal{R}_k^S$ in the string frame, using the relation

$$\Phi_k^E\propto\dot{\mathcal{R}}_k^S/k^2. \tag{4.21}$$

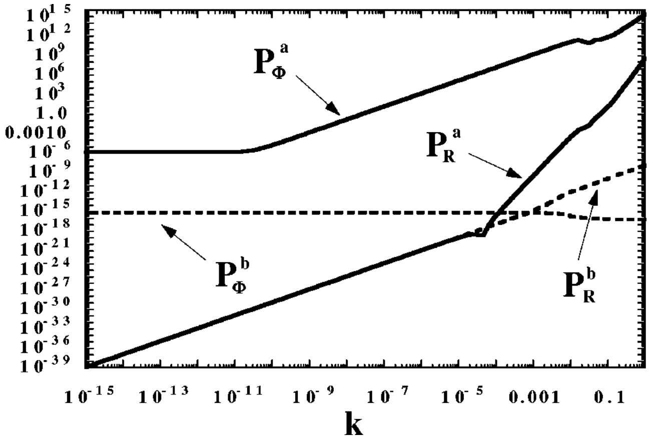

FIG. 5. The final spectra of the curvature perturbation, $P_{\mathcal{R}}$, and of the gravitational potential, P_Φ, in a simulation with $c=-1$, $d=1$, $p=10^{-3}$, $C_1=1.0$, and $C_2=-1.0\times10^{-3}$ with same initial conditions as in Fig. 4. The superscript "*b*" denotes the quantities *before* the bounce (at $t_S=300$), while "*a*" indicates the quantities *after* the bounce (at $t_S=1000$). We have included the quantum correction $\mathcal{L}_c$ for values $\varphi\lesssim1$. The spectral tilt is $n_{\mathcal{R}}-1-2$ for $k\lesssim10^{-4}$, which agrees with the analytic estimation of Eq. (4.20). For the modes $k>10^{-4}$, the spectra are highly blue tilted, due to an instability of small scale modes during the graceful exit. The fluctuations in Φ are nearly scale invariant on large scales before the bounce.

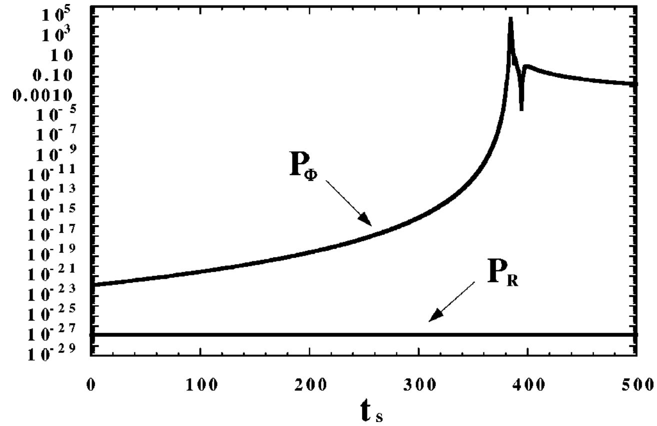

FIG. 6. The evolution of the curvature perturbation, $P_{\mathcal{R}}$, and of the gravitational potential, P_Φ, for the fluctuation mode corresponding to $k=10^{-9}$. The model parameters and initial conditions are the same as in Fig. 5. The amplitude of the gravitational potential near the bounce when the higher-derivative terms dominate cannot be trusted since Φ_k^E is computed from $\mathcal{R}_k^S$ via Eq. (4.21) which is only valid in the absence of such higher-derivative terms. As follows from this plot, the dominant growing mode of Φ_k^E during the period of contraction couples only to the post bounce decaying mode. At the time of the bounce, the curve for Φ is dominated by numerical noise. However, since Φ is computed at each time separately from the value of $\mathcal{R}$ at that time, this does not introduce numerical errors in the late time values of Φ.

This corresponds to Eq. (B3) in Appendix B, which follows from the relation (B1) in the Einstein frame. Note that this relation is valid in the absence of higher-curvature corrections to the Lagrangian, and will therefore be good at times long before and long after the bounce. For the negative ekpyrotic potential ($0<p<1/3$), one has $0<\gamma<1/2$ from Eq. (4.19), in which case the second term in Eq. (B4) completely vanishes. In this case we have the relation (B6), namely,

$$\Phi_k^E\propto H_{\nu-1}^{(1,2)}/k. \tag{4.22}$$

Note that $H_{\nu-1}^{(1,2)}$ can be written as the sum of two terms which are proportional to $(k|\eta_S|)^{\nu-1}$ and $(k|\eta_S|)^{-\nu+1}$.

Since $0<\nu<1/2$ for $0<\gamma<1/2$ (i.e., $0<p<1/3$), the term proportional to $(k|\eta_S|)^{\nu-1}$ is the growing mode during the contracting phase on large scales. Therefore the spectrum of Φ_k^E before the bounce can be estimated as

$$P_\Phi^b\propto k^{2\nu-1}\propto k^{-2\gamma}\propto k^{n_\Phi-1}, \tag{4.23}$$

from which we have

$$n_\Phi-1=-\frac{2p}{1-p}. \tag{4.24}$$

Then we have a scale-invariant spectrum before the bounce for $p\sim0$, as first pointed out in [18]. This agrees with our numerical result shown in Fig. 5.

The term proportional to $(k|\eta_S|)^{\nu-1}$, however, decays after the graceful exit as long as $\nu<1$. The dominant mode in Φ_k^E long after the bounce is described by the term $(k|\eta|)^{-\nu+1}$, in which case the spectrum of Φ is written as

$$P_\Phi^a\propto k^{3-2\nu}. \tag{4.25}$$

From Fig. 5 we find that the spectrum of Φ is blue tilted with $n_\Phi\sim3$ for $k\gtrsim10^{-10}$ (small scale modes for $k\gtrsim10^{-2}$ exhibit larger blue tilt with $n_\Phi>3$). This corresponds to the value $\nu\sim1/2$ in Eq. (4.25) after the bounce.

Our numerical calculations show that large scale modes with $k\lesssim10^{-10}$ do not exhibit such a blue spectrum. This can be understood to mean that the term proportional to $(k|\eta_S|)^{\nu-1}$ which is dominant in the contracting phase does not become smaller than the one proportional to $(k|\eta_S|)^{-\nu+1}$ in the expanding branch for very small k, unless we evolve the fluctuations until long after the bounce. However, it is rather difficult to follow such a large amount of time numerically. In addition, the second term in Eq. (B4) is not numerically negligible relative to the first term for these large scale modes due to the modification of the equation of state after the bounce [when $\gamma>1/2$ the second term in Eq. (B4) is nonvanishing as found by Eq. (B5)]. Nevertheless, we expect that the term proportional to $(k|\eta_S|)^{-\nu+1}$ in the first term in Eq. (B4) eventually dominates long after the bounce, in which case the spectrum is given by Eq. (4.25). Therefore the final spectrum of Φ is not generally scale invariant. The spectral index is dependent on the evolution of the background after the bounce (i.e., γ). In this sense including radiation is necessary in order to evaluate the spectrum of Φ in realistic cases where the solution connects to our Friedmann branch.

From Fig. 6 we find that the amplitude of Φ decreases after the bounce, thus showing that the dominant prebounce mode of Φ couples exclusively to the decaying mode of Φ

after the bounce, as derived in [15] using matching conditions on a constant scalar field hypersurface.

Equations (3.20) and (4.19) indicate that a scale-invariant spectrum may be obtained for $p=2/3$ for the modes that are enhanced during the bouncing phase [23]. In order to obtain such a spectrum, the exponential potential (positive in this case) is required to dominate the higher-order term except around the graceful exit. However we have found that, for some likely initial conditions, the field φ bounces back toward larger φ before the higher-order correction begins to work. In language appropriate to ekpyrotic cosmology, this means that the branes never collide. If the higher-order term always dominates compared to the positive exponential potential, we have the blue-tilted spectra (4.15).

V. INCLUSION OF HIGHER-ORDER CORRECTIONS IN THE EINSTEIN FRAME: MODULUS-DRIVEN CASE

In this section we consider adding higher-derivative terms defined in the Einstein frame. We add a Gauss-Bonnet term proportional to $R_{\rm GB}^2$ multiplied by a function of the modulus field φ to the action. Such a term arises as the one-loop correction in the context of orbifold compactifications of the heterotic superstring [51]. Since the initial version of ekpyrotic cosmology [5] is based on an orbifold compactification of a theory dual to heterotic superstring theory, the correction terms used in this section are well motivated in the context of the scenario of [5]. Indeed, it was found in Ref. [52] (see also [53–60]) that the inclusion of the Gauss-Bonnet term coupled to a modulus field in the Einstein frame leads to the possibility of obtaining nonsingular solutions. In the work of [52], the potential for the modulus field was taken to vanish. In this section we will include an exponential potential.[9] More specifically, the correction Lagrangian we consider here corresponds [in the notation of Eqs. (3.1) and (3.2)] to $f=R$, $\omega=1$, $c=-1$, $d=0$, $\xi(\varphi)=\ln[2e^{\varphi}\eta^4(ie^{\varphi})]$ with $\eta(ie^{\varphi})$ being the Dedekind η function [52]. Here $\xi(\varphi)$ is approximately given by

$$\xi(\varphi)\simeq-\frac{\pi}{3}(e^{\varphi}+e^{-\varphi}). \tag{5.1}$$

The sign of λ is chosen to be positive, which is different from the one discussed in the previous section. Note also that, even though the coefficient $\xi(\varphi)$ becomes large at large brane separation (large negative values of the dilaton in the case of PBB cosmology), this increase is outweighed by the falloff of the curvature invariant, as in the case of the model considered in the previous section. Thus, in ekpyrotic cosmology the correction terms are expected to become important only in the high-curvature region.

Let us analyze the one-field system of a modulus φ, keeping the dilaton fixed. We will consider solutions starting in an asymptotically flat region and beginning in the expanding branch. We have not found solutions which begin in a contracting phase and undergo a successful bounce. However, note that in the original ekpyrotic scenario of [5], the scale factor on the orbifold fixed plane corresponding to our four-dimensional space-time corresponds to an initially asymptotically flat region, and is always expanding. Thus, the solutions found here might be applicable to a version of ekpyrotic cosmology formulated entirely in terms of physics on the orbifold fixed plane.

A. Background evolution

As was discussed in [52], when $V_E=0$ the PBB singularity can be avoided for positive values of λ when the α' corrections introduced above are taken into account. The sign of λ is crucial for the existence of nonsingular cosmological solutions. For negative values of λ, the α' corrections do not help to lead to a successful graceful exit, as was analyzed in Ref. [59].

In the absence of the ekpyrotic potential, the background evolution for $t_E<0$ is given by [56]

$$a_E\simeq a_0,\quad H_E=\frac{H_0}{t_E^2},\quad \dot{\varphi}=\frac{5}{t_E}, \tag{5.2}$$

where a_0 and H_0 (>0) are constants. The Gauss-Bonnet term leads to a violation of the null energy condition ($\rho_E+p_E<0$) at sufficiently large curvatures and thus enables a graceful exit. If we start in an expanding branch (contrary to the spirit of PBB and ekpyrotic cosmology), this leads to a superinflationary solution ($\dot{H}_E>0$) until a "graceful exit" (see Fig. 7). The universe is initially expanding very slowly with a nearly constant scale factor. After the Hubble parameter reaches its peak value $H_E=H_{\rm max}$, the system connects to a Friedmann-like universe with $H_E\simeq 1/(3t_E)$, $a_E\propto t_E^{1/3}$, and $\varphi\propto-\ln t_E$.

If the ekpyrotic potential is present, the situation is quite different. We have adopted the potential (2.6) for $\varphi>0$ and $V_E=0$ for $\varphi<0$. Once again, we start in an expanding phase. Initially, the potential term is not important and the universe evolves in a superinflationary trajectory until a graceful exit after which the Hubble expansion rate begins to decrease. When $p<1/3$, corresponding to the case of a negative exponential potential, then as $\varphi\to 0$ the potential becomes important and leads to a change in sign of H_E. We find that the system enters a stage of slow contraction [see the case (ii) of Fig. 7]. Note that in Fig. 7 $\dot{H}_E$ changes sign twice. After the negative Hubble peak, the Hubble rate begins to grow toward $H_E\to -0$ without changing sign. Then the system enters a very slowly contracting phase with a nearly constant scale factor. In this stage the field φ evolves rapidly toward large negative values. In the presence of negative exponential potential ($p<1/3$) we have found that the contracting stage eventually appears even when V_0 is small.

When $p>1/3$ there exists a positive potential barrier as the field φ approaches zero. The case (iii) in Fig. 7 corresponds to $p=1/2$ with $V_0=p(1-3p)$. The effect of the positive potential is important around $\varphi\sim 0$, which works to return the field back toward larger φ. After the graceful exit

[9]The authors of Ref. [60] analyzed nonsingular cosmological solutions in the presence of some positive potentials (not the exponential potential).

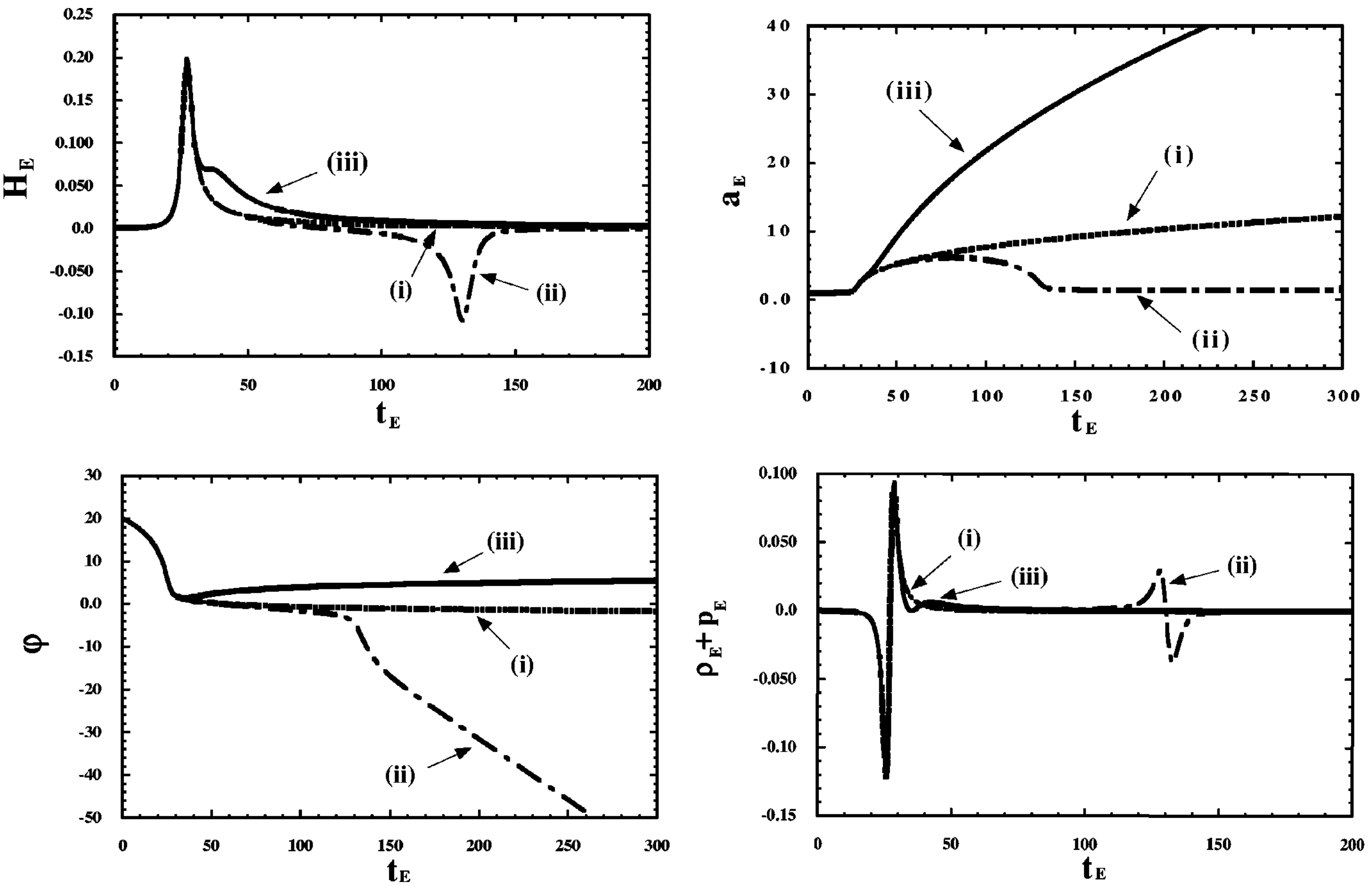

FIG. 7. The evolution of H_E, a_E, φ, and ρ_E+p_E in the modulus-driven case with $c=-1$, $d=0$. We choose initial conditions $\varphi=20$ and $H=5.087\times10^{-4}$. Each case corresponds to (i) $V_0=0$ with $p=0.1$, (ii) $V_0=0.01p(1-3p)$ with $p=0.1$, and (iii) $V_0=p(1-3p)$ with $p=0.5$.

the Hubble rate is always positive with slowly changing φ. The scale factor evolves as a power law ($a\propto t^p$) due to a positive exponential potential. The $p>1/3$ case provides us with reasonable nonsingular cosmological solutions. Nevertheless, we need to caution that these nonsingular solutions are different from the bouncing ones where the contraction of the universe occurs before the graceful exit.

One may argue that the bouncing trajectories may be found by including the correction $\mathcal{L}_c$ only around $\varphi\sim0$. However, we have numerically found that this is not the case. The superinflationary evolution characterized by Eq. (5.2) is typically required for the construction of nonsingular solutions in the present scenario.

B. Density perturbations

When the Gauss-Bonnet term is dominant relative to the ekpyrotic potential, the spectra of density perturbations can be analyzed as in the case of the zero potential ($p=0$ or $p=1/3$). In this case the evolution of the background during the phase of modulus-driven inflation is given by Eq. (5.2), thereby leading to $\dot{\xi}(\varphi)\simeq-(\pi/3)\dot{\varphi}e^{\varphi}\propto(-t_E)^4$. Making use of this relation together with Eq. (3.12), we find the evolution of Q and z as

$$Q\propto(-t_E)^2,\quad z\propto(-t_E)\propto(-\eta_E). \tag{5.3}$$

This means that $\gamma=1$ in Eq. (3.16), in which case curvature perturbations are enhanced on super-Hubble scales during superinflation [$\mathcal{R}_k\propto(-\eta_E)^{-1}$] due to the growth of the second term in Eq. (3.15) [61]. We show in Fig. 8 the evolution of curvature perturbations in the case of zero potential ($p=1/3$) for two different modes ($k=10^{-3}$ and $k=10^{-1}$). We find that curvature perturbations are amplified before the graceful exit.

In order to obtain the spectral tilt of density perturbations, we have to caution that the function s defined by Eq. (3.13) is a time-varying function and is proportional to $(-t_E)$. Therefore the formula (3.21) cannot be directly applied. Instead one is required to consider the evolution equation for curvature perturbations:

$$\ddot{\mathcal{R}}_k+\frac{2}{t_E}\dot{\mathcal{R}}_k-\alpha\frac{k^2}{a_0^2}t_E\mathcal{R}_k=0, \tag{5.4}$$

where α (>0) is a constant that depends on H_0 in Eq. (5.2). The solution of this equation is written in terms of the Bessel functions

$$\mathcal{R}_k=(-t_E)^{-1/2}[c_1J_{-1/3}(x)+c^2J_{1/3}(x)], \tag{5.5}$$

where $x\equiv2/3\sqrt{\alpha}(k/a_0)(-t_E)^{3/2}$. Notice that this solution asymptotically approaches the Minkowski vacuum for $x\to\infty$.

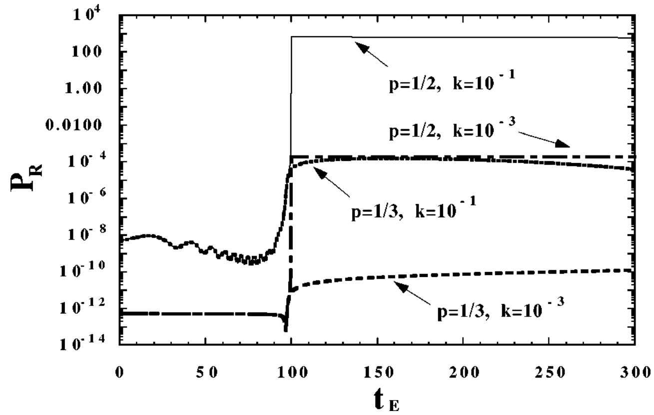

FIG. 8. The evolution of the spectra of curvature perturbations in the modulus-driven case ($c=-1$ and $d=0$) for $(p,k)=(1/3,10^{-1}),(1/3,10^{-3}),(1/2,10^{-1}),(1/2,10^{-3})$. We choose initial conditions $\varphi=23.888$ and $H=4.158\times10^{-4}$. Note that the $p=1/2$ case corresponds to the positive exponential potential while the zero potential corresponds to $p=1/3$. Around the graceful exit curvature perturbations exhibit rapid growth especially when the potential is positive ($p>1/3$).

Since $J_{\pm 1/3}(x)\propto k^{\pm 1/3}$ in the $x\to 0$ limit, the spectrum of large scale curvature perturbation is proportional to $P_{\mathcal{R}}\propto k^{7/3}$. Therefore the spectral index is

$$n_{\mathcal{R}}-1=\frac{7}{3}, \tag{5.6}$$

which is a blue-tilted spectrum.

In the absence of the ekpyrotic potential, the evolution of the background in the asymptotic future is given by $\varphi\sim\sqrt{3/2}\ln t_E$, $H_E\propto 1/(3t_E)$, and $a\propto t_E^{1/3}$. Therefore one has $z\propto t_E^{1/3}\propto\eta_E^{1/2}$ in Eq. (3.15), in which case curvature perturbations exhibit logarithmic growth,

$$\mathcal{R}\propto\ln\eta_E. \tag{5.7}$$

This indicates that the second term in Eq. (3.15), which we call the "D mode," dominates even after the graceful exit. In the case where the D mode decays after the graceful exit, the surviving spectra observed in an expanding universe should correspond to the first term in Eq. (3.15) ("C mode"). In the present model, however, the D mode survives in an expanding branch. Therefore the spectrum of the curvature perturbation during superinflation can be preserved even after the graceful exit. In fact, the numerical value of the final spectral tilts of $\mathcal{R}$ are found to be $n_{\mathcal{R}}-1\sim 2.3$ for the modes $k\ll 1$ (see Fig. 9). This agrees well with the analytic result (5.6).

When a positive ekpyrotic potential is present ($p>1/3$), the dynamics of the system is more unstable around the graceful exit. This leads to the violent growth of curvature perturbations when the field bounces back due to the potential barrier. This threatens the viability of the cosmological perturbation theory around the graceful exit. Nevertheless the perturbations are not singular as long as the background is smoothly joined to the expanding branch. We have numerically evaluated the power spectra of $\mathcal{R}$ for the modes

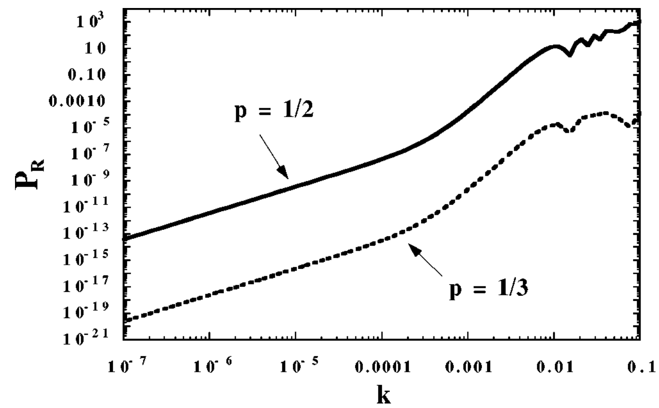

FIG. 9. The final spectra of curvature perturbations in the modulus-driven case ($c=-1$ and $d=0$) for $p=1/3$ and $p=1/2$. The initial conditions are the same as in Fig. 8. When the positive potential is present ($p>1/3$), the amplitude of the spectrum is larger than in the case of the zero potential. We numerically find that the spectral tilt is $n_{\mathcal{R}}-1\sim 2.3$ for $k\ll 1$, which agrees well with the analytic estimation, $n_{\mathcal{R}}-1=7/3$.

which left the horizon during superinflation. Although the amplitude is larger compared to the case of the zero potential ($p=1/3$), the final spectral tilts are similar for large scale modes ($k\ll 1$); see Fig. 9. Again the final spectra are found to be blue tilted. We should also mention that the frequency shift s becomes negative for the Hubble rate which is larger than unity around the graceful exit [61]. In this case the small scale modes show exponential instability as we pointed out in the dilaton-driven case. The negative ekpyrotic potential ($p<1/3$) is not worth studying, since this case does not connect to the expanding branch as analyzed in the previous subsection.

Finally, we should mention that we have neglected the effect of radiation in all our analysis. However, this is expected to appear at some moment of time. This can also alter the final spectra of curvature perturbations due to the dominance of the C mode in Eq. (3.15). We leave to future work investigation of these realistic situations.

VI. DISCUSSIONS AND OPEN ISSUES

We have studied the effects of higher-derivative terms in the joint gravitational and matter action for theories motivated by pre-big-bang and ekpyrotic cosmology with a single scalar matter field with an exponential potential. Applied to PBB cosmology, our model corresponds to a theory with an exponential potential for the dilaton. In the language of the initial version of ekpyrotic cosmology [5], our scalar field is the modulus field corresponding to the separation of the bulk brane from our orbifold fixed plane; in the second version [38] and in its cyclic version [39,40] the field is the radius of the extra spatial dimension. The higher-derivative terms introduced are the leading string and quantum corrections to the low-energy effective action of string theory.

When applying the correction terms in the string frame, and for suitable choices of the coefficients of the higher-order corrections, we find nonsingular cosmological solutions which in the Einstein frame correspond to bouncing universes. We thus find that higher-derivative terms can smooth out the singularities in PBB and ekpyrotic cosmology and lead to a graceful exit (in the language of PBB cosmology) or a nonsingular bounce (in the language of ekpyrotic cosmology). We have thus generalized the results of [26] to models with exponential scalar field potentials.

We have studied the evolution of fluctuations in our nonsingular bouncing cosmologies. This analysis is not plagued by the matching ambiguities inherent to analyses where the contracting and expanding cosmologies are matched across a singular space-time surface. For all potentials with $0<p \ll 1$ we find a blue spectrum of curvature fluctuations. The precise spectral index depends, as expected, on whether the higher-derivative correction terms are important at times when the scales on which we compute the fluctuation spectrum exit the Hubble radius during the phase of contraction.

If the higher-derivative terms are not dominant when the scales exit the Hubble radius, the index of the spectrum agrees with what is obtained by applying the general relativistic matching conditions on a uniform density hypersurface [15–17]. The only difference is an instability of small scale fluctuation modes during the bounce (see also [61]) which leads to a further steepening of the spectrum. Our result implies that the growing mode of Φ during the contracting phase, which is scale invariant for $0<p \ll 1$, is effectively uncoupled with the dominant constant mode of Φ in the expanding phase, a result obtained in the context of matching conditions in [13] (in the case of PBB cosmology) and in [15,16] for the ekpyrotic scenario. If the higher-derivative terms dominate when scales of interest exit the Hubble radius, then the spectrum is blue with a slope of $n_{\mathcal{R}}-1 \simeq 2.26$. Note that our result implies that it is the curvature fluctuation $\mathcal{R}$ (more precisely, the variable $\tilde{\zeta}$ originally introduced by Bardeen in [62] and used in [15], which equals $\mathcal{R}$ up to terms that are suppressed by k^2 for large scale fluctuations) which is effectively conserved for large scale perturbations across the bounce.

We have also studied nonsingular cosmological models obtained by adding a Gauss-Bonnet term (defined in the Einstein frame) multiplied by a suitably chosen function of the single scalar matter field in the problem (a modulus field). Once again, we have included an exponential potential for the modulus field. Although we do not find bouncing cosmologies, we find interesting nonsingular cosmological solutions which begin in an asymptotically flat region, undergo superexponential inflation followed by a graceful exit to a phase with decreasing Hubble radius. In the presence of a negative exponential potential $(0<p<1/3)$, the solutions reach a maximal radius and begin to contract as the field crosses $\varphi=0$. During this period of contraction, the Hubble parameter remains finite. Such solutions might be applicable to ekpyrotic universe models formulated in terms of physics on the four-dimensional orbifold fixed plane corresponding to our visible space-time. When the potential is positive $(p>1/3)$, the modulus φ bounces back around the brane collision toward larger φ due to the barrier of the positive potential. Although singularities can be avoided around $\varphi \sim 0$, this model does not correspond to bouncing solutions where the contraction of the universe occurs due to the ekpyrotic potential before the graceful exit.

We have also studied the spectrum of curvature fluctuations $\mathcal{R}$ in this modulus-driven cosmology. When higher-order corrections are important before the bounce (as they must be for the existence of nonsingular solutions), one has $n_{\mathcal{R}}-1 \sim 7/3$. This result is again in agreement with what can be obtained by neglecting the graceful exit and matching two Einstein universes at a constant density hypersurface.

Note that we have chosen to evolve the curvature perturbation $\mathcal{R}$ on comoving hypersurfaces, and found that the spectral index is given by Eq. (4.20) when the ekpyrotic potential is dominant. Note that this spectrum in the $0<p<1/3$ case comes from the C mode in Eq. (3.15), which is blue tilted for very slow contraction $(0<p \ll 1)$. Since the large-scale D modes are enhanced for $1/3<p<1$ during the contracting phase, the spectrum of $\mathcal{R}$ will be scale invariant for $p \sim 2/3$ *right after* the bounce [23].

If we follow instead the gravitational potential Φ in the longitudinal gauge, its spectral index generated during the collapsing phase is estimated as Eq. (4.24), which is different from that of $\mathcal{R}$ [see Eq. (4.20)]. When $p \sim 0$, corresponding to a very slow contraction, the growing mode (D mode) of Φ is approximately scale invariant. The authors of Refs. [18], [19] claimed that a scale-invariant spectrum of the dominant postbounce mode of Φ would inherit this scale-invariant spectrum.

However, we know that when computed at late times long after the bounce, in an expanding universe, the spectra of $\mathcal{R}$ and Φ must be identical. Thus, given our results concerning the spectrum of $\mathcal{R}$, we know that the spectrum of Φ long after the bounce cannot be scale invariant. Our numerical simulations show that the contribution from the prebounce D mode decays after the system enters the expanding branch, and thus show that the prebounce growing mode of Φ couples exclusively to the postbounce decaying mode. This results in a blue-tilted spectrum of Φ when evaluated long after the bounce (see Fig. 5). For very large scales with $k \lesssim 10^{-10}$, we need to solve the equation of fluctuations up to sufficient amount of time in order to find the complete decay of the D mode relative to the C mode. In addition the second term in Eq. (B4) is not numerically negligible for very small k when γ is greater than 1/2. Nevertheless, the term proportional to $(k|\eta_S|)^{-\nu+1}$ in the first term in Eq. (B4) eventually dominates long after the bounce $(\eta_S \to \infty)$, thereby yielding the spectrum (4.25). Therefore the spectrum of Φ long after the bounce is not generally expected to be scale invariant; its spectral index depends on the evolution of the background in an expanding branch.

Since near the bounce the magnitudes of the two modes of Φ and $\mathcal{R}$ differ by such a large ratio, we must worry about the possibility of numerical errors. In particular, if one were to follow the evolution equation for Φ, it would be difficult to ensure that numerical noise does not lead to an artificial coupling between the prebounce growing mode and the postbounce dominant (constant) mode. We have checked that our results do not seem to suffer from a similar problem by repeating the simulations with different values of the time step Δt. We did not find any dependence of the results within the range of time steps we have chosen $(10^{-5} \leqslant \Delta t \leqslant 10^{-3})$.

Let us compare our findings to results that have already appeared in the literature. As mentioned repeatedly, our results concerning the spectrum of fluctuations obtained in the classes of nonsingular bouncing universe models considered in this paper agree with the results of [15–17] obtained when removing the higher-derivative correction terms (thus going back to a singular background) and matching the fluctuations on a constant scalar field matching surface. The results imply that the growing mode of Φ in the contracting phase does not source the postbounce dominant mode of Φ. Our results thus indicate that the conjecture of [18,19], namely, that the growing mode of Φ in the contracting phase (which in ekpyrotic cosmology has a scale-invariant spectrum) should *generically* determine the amplitude and spectrum of the dominant mode of Φ in the postbounce phase, is not valid. As emphasized in [24] and [19], in the case of a singular background the spectrum of fluctuations in the expanding phase depends sensitively on the details of the matching conditions used. Since we have only used one class of ways to smooth out the singularity, the sensitive dependence on the matching surface might not have been completely eliminated, but might find itself reflected in a sensitive dependence of the final spectrum on the specific form of the correction terms in the action. We leave the study of this issue to future work.

Our work indicates that it is difficult to obtain a scale-invariant spectrum of curvature fluctuations for a single field PBB or ekpyrotic cosmology. However, in the case of ekpyrotic cosmology there is the intriguing fact that the growing mode of the gravitational potential Φ during the phase of contraction has a scale-invariant spectrum. To obtain a scale-invariant spectrum of Φ and thus also of the curvature fluctuation $\mathcal{R}$ at late times in the expanding phase, one suggestion [18,19] was to nontrivially connect the growing mode of Φ during the contracting phase with the constant mode in the expanding phase. We have shown that this does not occur in the single field case with our choice of correction terms to the action (needed to obtain a nonsingular bounce).

Note that there are examples where a large growth of Φ during the phase of contraction persists after the bounce (see, e.g., [23,47,37]). A criterion for when this occurs has been proposed recently in [37]. The condition is that the relative variation of $\mathcal{R}$ over a Hubble time scale should be appreciable, i.e., the following relation [37]:

$$\frac{\dot{\mathcal{R}}}{H\mathcal{R}} \geqslant 1, \tag{6.1}$$

should hold close to or right at the bounce. We use Eq. (3.15) and restrict consideration to the case of an exponential potential, in which case one has

$$\frac{\dot{\mathcal{R}}}{H\mathcal{R}} = \frac{D_k(1-p)\eta}{pa^2[C_k+D_k(-\eta)^{1-2\gamma}]} = \begin{cases} \dfrac{D_k(1-p)}{pC_k}(-\eta)^{(1-3p)/(1-p)} & \text{for } 0<p<1/3, \\[2ex] \dfrac{1-p}{p} & \text{for } 1/3<p<1. \end{cases} \tag{6.2}$$

The marginal case $p=1/3$ (important for the PBB and also for the ekpyrotic scenario, in which the potential disappears close to the bounce) should be treated separately, and leads to

$$\frac{\dot{\mathcal{R}}}{H\mathcal{R}} = \frac{2D_k}{C_k+D_k\ln(-k\eta)}. \tag{6.3}$$

Surprisingly enough, both the results (6.2) and (6.3) indicate that Φ could *never* match to $\mathcal{R}$ nontrivially. For $p \leqslant 1/3, \dot{\mathcal{R}}/H\mathcal{R} \to 0$ as $\eta \to 0$. For $1/3<p<1, \dot{\mathcal{R}}/H\mathcal{R} \to (1-p)/p \sim \mathcal{O}(1)$ as $\eta \to 0$. This latter case is interpreted as a variation of $\mathcal{R}$ rather than a change induced by Φ. Interestingly enough, if one takes seriously the ratio (6.1), the singularity at the bounce (i.e., if $\eta=0$ is reached or not) does not matter in the impossibility of matching Φ to $\mathcal{R}$.

Recently several authors [47,37] considered models of a bouncing universe (realized in [47] by introducing matter violating the weak energy condition and in [37] by making use of spatial curvature in the background metric) in which $\mathcal{R}$ grows dramatically across the bounce and there is a coupling between the growing mode of Φ in the contracting phase and the dominant mode of Φ in the expanding phase. In this case, it may be possible to obtain a scale-invariant spectrum, as already realized in [23].

Although we have concentrated on the density perturbation in the single field scenario, the situation can be changed by taking into account a second scalar field [23,63,64]. A system of multicomponent scalar fields generally induces isocurvature perturbations, which can be the source of adiabatic perturbations. In such a case the relation (6.1) could be satisfied, since isocurvature perturbations act as source term for $\dot{\mathcal{R}}$ in addition to Φ. In fact the authors of Ref. [63] considered a specific two-field system with a brane modulus φ and a dilaton χ. When the dilaton has a negative exponential potential with a suppressed ekpyrotic potential for φ, the entropy ''field'' perturbation can be scale invariant if the model parameters are fine-tuned [63]. It was also pointed out in Ref. [23] that the quantum fluctuation of a light scalar field (with a noncanonical kinetic term as studied in [65]) such as axion may lead to the flat spectra of isocurvature perturbations. If the correlation between adiabatic and isocurvature perturbations is strong, adiabatic perturbations may be scale invariant.

We wish to stress that our work is not conclusive. In particular, in order to fully evaluate the final power spectra, one should solve the equations of motion for fluctuations in a nonsingular bouncing model including radiation. Important issues which should be investigated further include the following.

The final power spectra of the curvature perturbation are found to be blue tilted for the nonsingular ekpyrotic models we have considered, which rely on specific higher-derivative correction terms. Are there other correction terms to the action which are motivated by string theory, lead to nonsingular bouncing scenarios, and yield a flat spectrum even in the single field case? Perhaps toy bouncing models using exotic scalar fields or matter in Refs. [46], [47] can be a good starting point to construct viable nonsingular ekpyrotic models.

We do not include the effect of radiation (or particles) which can be efficiently produced near the bounce. Particle production around the transition region is expected to be quite efficient [66], and this could lead to an additional instability of small scale metric perturbations. This effect may also nontrivially alter the nonsingular background evolution by the back-reaction effect of created particles. It is also required to include the radiation after the bounce in order to evaluate the surviving spectra accurately, although the coupling between the scalar field and radiation should be chosen carefully in that case.

It is of interest to study the effect of isocurvature perturbations in the two-field system of nonsingular ekpyrotic scenarios. In particular, isocurvature perturbations can be affected by the instability of the background near the bounce. In order to obtain the final spectra of adiabatic perturbations, we need to solve the coupled equations of adiabatic and entropy perturbations through the nonsingular bounce including radiation. It is important to investigate whether nearly scale-invariant spectra are obtained by conversion from isocurvature to adiabatic perturbations.

Our analysis also applies to cyclic universe models proposed in Refs. [39], [40] in which the bounce has been regularized by including higher-order corrections. Thus, our conclusions about the difficulty in obtaining a scale-invariant spectrum of fluctuations carry over to single field realizations of the cyclic scenario. In fact, we have done some simulations in the case of a simple negative potential $V=m^2(\phi^2-\phi_c^2)$ for $|\phi|<\phi_c$, and found that the solutions can be nonsingular so long as the higher-order effect dominates around the graceful exit. Note, however, that the spectra of density perturbations will be the same as in the ekpyrotic scenario.

Recently, a paper has appeared [67] in which in the context of a brane world scenario a nonsingular bouncing cosmology is obtained by considering the motion of a D3-brane as a boundary of a five-dimensional charged anti–de Sitter black hole. In this model, computed in linear theory, the spectrum of gravitational wave fluctuations was shown not to be scale invariant. This result supports the conclusions we have reached.

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APPENDIX A: HEURISTIC DERIVATION OF THE SPECTRUM OF FLUCTUATIONS

In this appendix we give a heuristic derivation of the spectral index of cosmological perturbations in the PBB and ekpyrotic scenarios. This analysis is based on two key assumptions. The first is the assumption that the amplitude of the fluctuations when they exit the Hubble radius during the phase of contraction (in the Einstein frame) is given by the Hubble constant. This assumption is reasonable assuming that the fluctuations are quantum vacuum perturbations which freeze when their wavelength crosses the Hubble radius.

The second assumption is that the "physical magnitude" of the fluctuations remains unchanged while the wavelength of the fluctuation is larger than the Hubble radius. This assumption is much less obvious, although at first sight this assumption may seem obvious based on causality, namely, the fact that microphysics cannot influence physics on scales larger than the Hubble radius. However, in inflationary cosmology and in models with a contracting period such as the PBB and ekpyrotic scenarios, the forward light cone (causal horizon) is much larger than the Hubble radius, and the spatial coherence of background fields over scales of the forward light cone can lead to nontrivial effects on fluctuation modes on these scales, one of the most dramatic manifestations of this effect being the parametric amplification of super-Hubble (but subhorizon) cosmological fluctuations during reheating in certain two-field inflationary models [68–74]. Furthermore, the term "physical magnitude" of cosmological fluctuations is not well determined. On super-Hubble scales, the magnitude of the density fluctuations depends sensitively on the coordinate system chosen. It is possible to choose coordinate-invariant (gauge-invariant) variables to describe the fluctuations, but there are many choices, and even in single field inflationary models many of these gauge-invariant fluctuation variables increase on super-Hubble scales [however, the increase between initial Hubble radius crossing during inflation at $t_i(k)$ and final Hubble radius crossing during the late time (FRW) cosmology at $t_f(k)$ is by a factor that depends only on the ratio of the equations of state at the two Hubble radius crossings]. This increase is a self-gravitational effect.

In spite of the above caveats, let us proceed with the heuristic discussion of the amplitude of density fluctuations, applying it first to inflationary cosmology (exponential expansion to be specific). The quantity we wish to calculate is the mean square mass fluctuation on a scale k when the corresponding wavelength enters the Hubble radius at final Hubble radius crossing $t_f(k)$. This quantity, denoted $|(\delta M/M)[k,t_f(k)]|^2$, is given by the power spectrum of fluctuations [see Eq. (3.20)], and its k dependence on the spectral index n is given by

$$\left|\frac{\delta M}{M}[k,t_f(k)]\right|^2 \sim k^{n-1}. \tag{A1}$$

Based on the first assumption,

$$\left|\frac{\delta M}{M}[k,t_i(k)]\right|^2 \sim H^2[t_i(k)] \sim \text{const}, \tag{A2}$$

and using the second assumption we infer that

$$\left|\frac{\delta M}{M}[k,t_f(k)]\right|^2 \sim \left|\frac{\delta M}{M}[k,t_i(k)]\right|^2 \sim \text{const}, \tag{A3}$$

and that hence the power spectrum is scale invariant with an index $n=1$.

PBB cosmology is characterized (in the Einstein frame) by a scale factor which scales as

$$a(t) \sim t^{1/3}, \tag{A4}$$

and thus

$$H(t) = \frac{1}{3t}. \tag{A5}$$

The condition of the initial Hubble radius crossing during the period of contraction

$$ka^{-1}[t_i(k)] = H[t_i(k)], \tag{A6}$$

leads to

$$t_i(k) \sim k^{-3/2}, \quad H[t_i(k)] \sim k^{3/2}, \tag{A7}$$

and thus, applying our two basic assumptions as in the case of inflationary cosmology, to

$$\left|\frac{\delta M}{M}[k,t_f(k)]\right|^2 \sim k^3, \tag{A8}$$

which corresponds to a blue spectrum with index $n=4$.

The analysis for ekpyrotic cosmology is analogous. The only difference is that the value of p is different, $0<p\ll 1$, and hence

$$t_i(k) \sim k^{-1/(1-p)}, \quad H[t_i(k)] \sim k^{1/(1-p)}, \tag{A9}$$

and thus, taking $p=0$ at the end,

$$\left|\frac{\delta M}{M}[k,t_f(k)]\right|^2 \sim k^2, \tag{A10}$$

which corresponds to a blue spectrum with index $n=3$.

Obviously, given the caveats discussed at the beginning of this appendix, the results for PBB and ekpyrotic cosmology cannot be trusted without a fully relativistic analysis. The growth rates of cosmological fluctuations are very different in expanding and contracting cosmologies, and thus even given that the above heuristic analysis works in the case of inflationary cosmology, this does not mean it has to work for PBB and ekpyrotic cosmologies. However, the results of our paper are in agreement with those derived from the heuristic analysis.

APPENDIX B: ANALYTIC ESTIMATES FOR THE GRAVITATIONAL POTENTIAL

Let us analyze the gravitational Φ in more details. In the Einstein frame, the gravitational potential Φ_k^E in the longitudinal gauge is expressed in terms of $\mathcal{R}_k^E$ in the absence of higher-order corrections [70,75]:

$$\Phi_k^E = \frac{a_E^2 \dot{H}_E}{k^2 H_E} \dot{R}_k^E. \tag{B1}$$

The gravitational potential Φ_k^S in the string frame is related to the one in the Einstein frame as [76]

$$\Phi_k^S = \Phi_k^E + \frac{\delta F}{2F} = \Phi_k^E - \frac{1}{2}\delta\phi_k. \tag{B2}$$

Making use of Eqs. (2.10) and (4.10), we find that the curvature perturbation in the Einstein frame is exactly the same as that in the string frame (i.e., $\mathcal{R}_k^E = \mathcal{R}_k^S$). Therefore the gravitational potential in the Einstein frame is expressed in terms of $\dot{\mathcal{R}}_k^S$:

$$\Phi_k^E = \frac{a_S^2(\dot{H}_S - \ddot{\phi}/2 + \dot{\phi}H_S/2 - \dot{\phi}^4/4)}{k^2(H_S - \dot{\phi}/2)} \dot{\mathcal{R}}_k^S. \tag{B3}$$

Note that the overdots in Eq. (B3) denote the time derivatives with respect to t_S. This is the equation that we solve numerically.

Taking note of the relation $H_\nu'(x) = H_{\nu-1}(x) - (\nu/x)H_\nu(x)$, one finds

$$\dot{\mathcal{R}}_k^S = \frac{\sqrt{\pi}}{4a_S z}\Bigg[2\sqrt{s|\eta_S|}k[c_1 H_{\nu-1}^{(1)}(x) + c_2 H_{\nu-1}^{(2)}(x)] \pm \frac{1}{\sqrt{|\eta_S|}}\left(1 - 2\nu \mp 2\left|\eta_S\right|\frac{z'}{z}\right)[c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x)]\Bigg], \tag{B4}$$

where each sign corresponds to the case with $\eta_S>0$ and $\eta_S<0$, respectively. When the evolution of z is given by $z \propto |\eta_S|^\gamma$, we have

$$1 - 2\nu \mp 2|\eta_S|\frac{z'}{z} = 1 - 2\nu - 2\gamma = 1 - 2\gamma - |1 - 2\gamma| = \begin{cases} 0 & \text{for } \gamma<1/2, \\ 2(1-2\gamma) & \text{for } \gamma>1/2. \end{cases} \tag{B5}$$

This term completely vanishes during the contracting phase in the ekyprotic cosmology with $p<1/3$, since γ is less than 1/2. In this case the gravitational potential in the Einstein frame can be expressed as

$$\Phi_k^E = \frac{a_S\sqrt{\pi s|\eta_S|}(\dot{H}_S - \ddot{\phi}/2 + \dot{\phi}H_S/2 - \dot{\phi}^4/4)}{2z(H_S - \dot{\phi}/2)k} \times [c_1 H_{\nu-1}^{(1)}(x) + c_2 H_{\nu-1}^{(2)}(x)]. \tag{B6}$$

This relation is used to estimate the spectrum of Φ_k^E in Sec. IV.

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