Inflation: Flow, fixed points, and observables to arbitrary order in slow roll

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I generalize the inflationary flow equations of Hoffman and Turner to arbitrary order in slow roll. This makes it possible to study the predictions of slow roll inflation in the full observable parameter space of the tensor/scalar ratio r, spectral index n, and running $dn/d\ln k$. It also becomes possible to identify exact fixed points in the parameter flow. I numerically evaluate the flow equations to fifth order in slow roll for a set of randomly chosen initial conditions and find that the models cluster strongly in the observable parameter space, indicating a ''generic'' set of predictions for slow roll inflation. I comment briefly on the interesting proposed correspondence between flow in inflationary parameter space and renormalization group flow in a boundary conformal field theory.

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I. INTRODUCTION

Inflationary cosmology [1–3] has become the dominant paradigm for describing the very early universe. Over the past 20 years, inflationary model building has been a prolific enterprise [4]. Concurrently, cosmological observations have improved to the point that it is beginning to be possible to rule out models of inflation [5,6]. Future observations, particularly the Microwave Anisotropy Probe (MAP) [7] and Planck [8] cosmic microwave background (CMB) satellites, promise to dramatically improve the situation in the near future [9,10]. The key observational parameters for distinguishing among inflation models are the tensor/scalar ratio r, the scalar spectral index n, and the ''running'' of the spectral index, $dn/d\ln k$, since different inflation models predict different values for these parameters.

It is desirable, however, to gain some insight into what the *generic* predictions of inflation are without having to work within the context of some particular model. The standard lore of a small tensor/scalar ratio and nearly scale-invariant power spectrum is insufficient now that precision measurements of the CMB and large-scale structure are becoming a reality. Hoffman and Turner have proposed the method of inflationary ''flow'' to gain generic insight into the behavior of inflation models [11]. The flow equations relate the time derivatives of the slow roll parameters to other, higher order slow roll parameters. With a suitable choice of truncation, this makes it possible to study the dynamics of inflation models without having to specify a particular potential for the field driving inflation. In this paper we generalize the method from the lowest-order analysis of Hoffman and Turner and derive a simple set of flow equations which can be evaluated to arbitrarily high order, and which are in fact *exact* in the limit of infinite order in slow roll. We perform a numerical integration of 10^5 inflation models to fifth order in slow roll, and plot their predictions in the observable parameter space $(r, n, dn/d\ln k)$. The predictions of the models cluster strongly in the observable parameter space, in fact, even more strongly than was suggested by Hoffman and Turner. (However, the qualitative character of their analysis is preserved at higher order in slow roll.) We emphasize that in this paper we limit ourselves to inflation driven by a single scalar field ϕ. The case of multiple-field inflation is in general much more complex.

This idea of flow in the inflationary parameter space has taken on different significance with recent ideas arising from the ''holographic'' correspondence between de Sitter space and boundary conformal field theories proposed by Strominger [12]. Particularly interesting are efforts to interpret flow in the space of slow roll parameters as renormalization group flow in a boundary conformal field theory [13]. This raises the possibility that understanding the evolution of inflationary parameters is important not just for phenomenology, but for fundamental reasons as well.

The paper is organized as follows: Sec. II briefly reviews the very powerful Hamilton-Jacobi formalism for inflation. Section III discusses the generation of fluctuations in inflation and the relationship between the slow roll parameters and the observables in various exact and approximate solutions of the inflationary equations of motion. The hierarchy of flow equations is derived in Sec. IV. Section V discusses the fixed points in the slow roll parameter space. Section VI discusses the details of the numerical solution. Section VII presents conclusions.

II. INFLATION AND THE HAMILTON-JACOBI FORMALISM

The dominant component of an inflationary cosmology is a spatially homogeneous scalar field ϕ (the *inflaton*) with potential $V(\phi)$ and equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{1}$$

where $H \equiv (\dot{a}/a)$ is the Hubble parameter, and the Einstein field equations for the evolution of a flat background metric

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 \tag{2}$$

can be written as

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$$H^2=\left(\frac{\dot{a}}{a}\right)^2=\frac{8\pi}{3m_{\rm Pl}^2}\left[V(\phi)+\frac{1}{2}\dot{\phi}^2\right], \tag{3}$$

and

$$\left(\frac{\ddot{a}}{a}\right)=\frac{8\pi}{3m_{\rm Pl}^2}[V(\phi)-\dot{\phi}^2]. \tag{4}$$

Here $m_{\rm Pl}=G^{-1/2}\simeq 10^{19}$ GeV is the Planck mass. These background equations, along with the equation of motion (1), form a coupled set of differential equations describing the evolution of the universe. In the limit that $\dot{\phi}=0$, the expansion of the universe is of the de Sitter form, with the scale factor increasing exponentially in time:

$$H=\sqrt{\left(\frac{8\pi}{3m_{\rm Pl}^2}\right)V(\phi)}=\text{const},$$
$$a\propto e^{Ht}. \tag{5}$$

In general, the Hubble parameter H will not be exactly constant, but will vary as the field ϕ evolves along the potential $V(\phi)$. A convenient approach to the more general case is to express the Hubble parameter directly as a function of the field ϕ instead of as a function of time, $H=H(\phi)$. This is consistent as long as ϕ is monotonic in time. The equations of motion for the field and background are given by [14–17]

$$\dot{\phi}=-\frac{m_{\rm Pl}^2}{4\pi}H'(\phi),$$
$$[H'(\phi)]^2-\frac{12\pi}{m_{\rm Pl}^2}H^2(\phi)=-\frac{32\pi^2}{m_{\rm Pl}^4}V(\phi). \tag{6}$$

These equations are completely equivalent to the second-order equation of motion (1). The second of these is referred to as the *Hamilton-Jacobi* equation, and can be written in the useful form

$$H^2(\phi)\left[1-\frac{1}{3}\epsilon(\phi)\right]=\left(\frac{8\pi}{3m_{\rm Pl}^2}\right)V(\phi), \tag{7}$$

where the parameter ϵ is defined as

$$\epsilon\equiv\frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2. \tag{8}$$

The physical meaning of the parameter ϵ can be seen by expressing Eq. (4) as

$$\left(\frac{\ddot{a}}{a}\right)=H^2(\phi)[1-\epsilon(\phi)], \tag{9}$$

so that the condition for inflation $(\ddot{a}/a)>0$ is given by $\epsilon<1$. The evolution of the scale factor is given by the general expression

$$a\propto\exp\left[\int_{t_0}^{t}H\,dt\right], \tag{10}$$

where the number of e-folds N is defined to be

$$N\equiv\int_t^{t_e}H\,dt=\int_\phi^{\phi_e}\frac{H}{\dot{\phi}}d\phi=\frac{2\sqrt{\pi}}{m_{\rm Pl}}\int_{\phi_e}^{\phi}\frac{d\phi}{\sqrt{\epsilon(\phi)}}. \tag{11}$$

Here we take t_e and ϕ_e to be the time and field value at end of inflation. Therefore N increases as one goes *backward* in time, $dt>0\Rightarrow dN<0$. These expressions are exact, and do not depend on any assumption of slow roll. It is important to note the sign convention for $\sqrt{\epsilon}$, which we define to have the same sign as $H'(\phi)$:

$$\sqrt{\epsilon}\equiv+\frac{m_{\rm PL}}{2\sqrt{\pi}}\frac{H'}{H}. \tag{12}$$

In the next section we briefly discuss the generation of perturbations in inflation from the point of view of different exact and approximate solutions.

III. COSMOLOGICAL PERTURBATIONS: SCALAR AND TENSOR POWER SPECTRA

Cosmological density and gravitational wave perturbations in the inflationary scenario arise as quantum fluctuations which are "redshifted" to long wavelength by the rapid cosmological expansion [18–21]. The power spectrum of density perturbations is given by [22]

$$P_{\mathcal{R}}^{1/2}(k)=\sqrt{\frac{k^3}{2\pi^2}}\left|\frac{u_k}{z}\right|, \tag{13}$$

where k is a comoving wave number, and the mode function u_k satisfies the differential equation [23–25]

$$\frac{d^2u_k}{d\tau^2}+\left(k^2-\frac{1}{z}\frac{d^2z}{d\tau^2}\right)u_k=0, \tag{14}$$

where τ is the conformal time, $ds^2=a^2(\tau)(d\tau^2-d\mathbf{x}^2)$, and the quantity z is defined as

$$z\equiv\frac{2\sqrt{\pi}}{m_{\rm Pl}}\left(\frac{a\dot{\phi}}{H}\right)=-a\sqrt{\epsilon}. \tag{15}$$

We then have

$$\frac{1}{z}\frac{d^2z}{d\tau^2}=2a^2H^2\left(1+\epsilon-\frac{3}{2}\eta+\epsilon^2-2\epsilon\eta+\frac{1}{2}\eta^2+\frac{1}{2}\xi^2\right), \tag{16}$$

where the additional parameters η and ξ^2 are defined as [26,27]

$$\eta\equiv\frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H''(\phi)}{H(\phi)}\right) \tag{17}$$

and

$$\xi^2 \equiv \frac{m_{\rm Pl}^4}{16\pi^2}\left(\frac{H'(\phi)H'''(\phi)}{H^2(\phi)}\right). \tag{18}$$

These are often referred to as *slow roll* parameters, although they are defined here without any assumption of slow roll (discussed below). Note that despite the somewhat unfortunate standard notation used above, the parameter ξ^2 can be either positive or negative.

Equation (14) can be solved exactly for the case of power-law inflation, for which $\epsilon = \eta = \xi = \text{const}$, and the scale factor evolves as a power law in time,

$$a(t) \propto t^{1/\epsilon} = -\frac{1}{H(1-\epsilon)}\tau^{-1}. \tag{19}$$

(Note that during inflation, $\tau < 0$, with $\tau \to 0$ at late time.) The vacuum solution to the mode equation (14) is then

$$u_k \propto \sqrt{-k\tau}H_\nu(-k\tau), \tag{20}$$

where H_ν is a Hankel function of the first kind, and

$$\nu = \frac{3}{2} + \frac{\epsilon}{1-\epsilon}. \tag{21}$$

The power spectrum for modes with wavelength much larger than the horizon ($k \ll aH$) is an exact power law,

$$P_{\mathcal{R}}^{1/2} = \left.\frac{H}{2\pi\sqrt{\epsilon}}\right|_{aH=k} \propto k^{n-1}, \tag{22}$$

where the spectral index n is given by

$$n = 1 - \frac{2\epsilon}{1-\epsilon}. \tag{23}$$

Similarly, the tensor fluctuation amplitude is

$$P_{\rm T}^{1/2} = \left.\frac{H}{2\pi}\right|_{aH=k} \propto k^{n_{\rm T}}, \tag{24}$$

where

$$n_{\rm T} = -\frac{2\epsilon}{1-\epsilon}. \tag{25}$$

Other classes of exact solution are known [28].

There are also classes of approximate solution. The standard slow roll approximation is the assumption that the field evolution is dominated by drag from the expansion, $\ddot{\phi} \simeq 0$, so that $\dot{\phi}$ is approximately constant and $H(\phi)$ can be taken to vary as

$$H(\phi) = \sqrt{\left(\frac{8\pi}{3m_{\rm Pl}^2}\right)V[\phi(t)]}, \tag{26}$$

where $\phi(t)$ satisfies

$$\dot{\phi} = -\frac{V'(\phi)}{3H(\phi)}. \tag{27}$$

This approximation is consistent as long as the first two derivatives of the potential are small relative to its magnitude $V', V'' \ll V$. The parameters ϵ and η reduce in this limit to [29]

$$\begin{aligned} \epsilon &\simeq \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \\ \eta &\simeq \frac{m_{\rm Pl}^2}{8\pi}\left[\frac{V''(\phi)}{V(\phi)} - \frac{1}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2\right]. \end{aligned} \tag{28}$$

The slow roll limit can then be equivalently expressed as $\epsilon, |\eta| \ll 1$. These expressions are frequently taken in the literature as definitions of the slow roll parameters, but here they are simply limits of the defining expressions (8) and (17). In the limit where slow roll is valid, the tensor and scalar spectra are again power laws, where the spectral index n is given by

$$n = 1 - 4\epsilon + 2\eta. \tag{29}$$

The tensor spectral index is just the $\epsilon \ll 1$ limit of the power-law case,

$$n_{\rm T} = -2\epsilon. \tag{30}$$

A second class of approximate models has $\epsilon \ll 1$ as in the slow roll case, but is characterized by a large parameter $\eta \simeq \text{const} \sim O(1)$. In this case, the slow roll expressions (28) do not apply, and it can be shown that ϵ can be expressed in terms of the potential by [30]

$$\epsilon(\phi) \simeq 3\left(1 - \frac{V(\phi)}{V(\phi_0)}\right), \tag{31}$$

where ϕ_0 is a stationary point of the field, $V'(\phi_0) = 0$. In this case, the scalar spectral index is

$$n \simeq 1 + 2\eta, \tag{32}$$

and the tensor spectral index is, as usual,

$$n_{\rm T} \simeq -2\epsilon. \tag{33}$$

Such models are strongly observationally disfavored, because they predict a rapidly varying power spectrum, $|n-1| \sim O(1)$, but they are nonetheless important as attractors in the inflationary parameter space.

Note that in all cases, the ratio of tensor to scalar perturbations is just the first slow roll parameter

$$r \equiv \frac{P_{\rm T}}{P_{\rm R}} = \epsilon. \tag{34}$$

This expression is exact in the power-law case, and valid to lowest order in the slow roll case.[1] Note that this is related to the tensor spectral index by the inflationary "consistency condition" [29] $n_{\rm T}=-2r$, so the shape of the tensor spectrum does not provide an additional independent observable. The scalar spectral index depends in general on η, and therefore is an independent observable. (In Sec. V we discuss the generalization of the expressions in this section to higher order in slow roll.)

IV. THE SLOW ROLL HIERARCHY AND FLOW IN THE INFLATIONARY PARAMETER SPACE

The slow roll parameters are not in general constant during inflation, but change in value as the scalar field driving the inflationary expansion evolves. From the definitions (8),(17),(18) of the parameters ϵ,η,ξ, we have

$$\frac{d\epsilon}{d\phi}=\left(\frac{2\sqrt{\pi}}{m_{\rm Pl}}\right)2\sqrt{\epsilon}(\eta-\epsilon),$$
$$\frac{d\eta}{d\phi}=\left(\frac{2\sqrt{\pi}}{m_{\rm Pl}}\right)\frac{1}{\sqrt{\epsilon}}(\xi^2-\epsilon\eta). \tag{35}$$

It will be convenient to use the number of e-folds before the end of inflation N as the evolution parameter instead of the field. From Eq. (11), it is straightforward to rewrite derivatives with respect to ϕ in terms of derivatives with respect to N,

$$\frac{d}{dN}=\frac{m_{\rm Pl}}{2\sqrt{\pi}}\sqrt{\epsilon}\,\frac{d}{d\phi}. \tag{36}$$

In terms of N, we then have

$$\frac{d\epsilon}{dN}=2\epsilon(\eta-\epsilon), \tag{37}$$

and

$$\frac{d\eta}{dN}=\xi^2-\epsilon\eta. \tag{38}$$

Note that the derivative of each slow roll parameter is itself higher order in slow roll. This suggests an infinite hierarchy of "Hubble slow roll" parameters[2] [27]

$${}^{\ell}\lambda_{\rm H}\equiv\left(\frac{m_{\rm Pl}^2}{4\pi}\right)^{\ell}\frac{(H')^{\ell-1}}{H^{\ell}}\frac{d^{(\ell+1)}H}{d\phi^{(\ell+1)}}. \tag{39}$$

For example, the $\ell=2$ parameter is just ξ^2:

$${}^{2}\lambda_{\rm H}=\frac{m_{\rm Pl}^4}{16\pi^2}\left(\frac{H'H'''}{H^2}\right)=\xi^2. \tag{40}$$

We can then define an infinite hierarchy of "flow" equations for the slow roll parameters by differentiating Eq. (39):

$$\frac{d({}^{\ell}\lambda_{\rm H})}{dN}=[(\ell-1)\eta-\ell\epsilon]({}^{\ell}\lambda_{\rm H})+{}^{\ell+1}\lambda_{\rm H}. \tag{41}$$

Together with Eqs. (37),(38), these form a system of differential equations that can be numerically integrated to arbitrarily high order in slow roll.

This flow equation approach to studying the inflationary parameter space was first suggested by Hoffman and Turner [11], who wrote the flow equations to lowest order in slow roll in terms of the tensor/scalar ratio (T/S) and the scalar spectral index n as

$$\frac{d(T/S)}{dN}=(n-1)\frac{T}{S}+\frac{1}{5}\left(\frac{T}{S}\right)^2,$$
$$\frac{d(n-1)}{dN}=-\frac{1}{5}(n-1)\frac{T}{S}-\frac{1}{25}\left(\frac{T}{S}\right)^2\pm\frac{m_{\rm Pl}^3}{16\pi^2}\sqrt{\frac{2\pi}{5}\frac{T}{S}}x'', \tag{42}$$

where

$$x(\phi)\equiv\frac{V'(\phi)}{V(\phi)}. \tag{43}$$

Here (T/S) is related to the parameter r defined in Eq. (34) by $(T/S)=10r$. Hoffman and Turner "closed" the flow equations by assuming that x'' is small and constant. It is straightforward to generalize these equations using the Hubble slow roll formalism above. We can define a new parameter

$$\sigma\equiv2\eta-4\epsilon, \tag{44}$$

which is equivalent to the spectral index parameter used by Hoffman and Turner: $\sigma\simeq n-1$ to lowest order in slow roll. The flow equations (37),(38) in terms of σ are

$$\frac{d\epsilon}{dN}=\epsilon(\sigma+2\epsilon),$$
$$\frac{d\sigma}{dN}=2\xi^2-5\epsilon\sigma-12\epsilon^2. \tag{45}$$

These expressions can be shown to be identical to Eqs. (42) by evaluating using the slow roll expressions (28) for ϵ and η. Using $\xi^2={}^{2}\lambda_{\rm H}$, the flow equations (45) along with Eq. (41) then represent a generalization of the flow equations of Hoffman and Turner to arbitrarily high order in slow roll.

[1] Conventions for the normalization of this parameter vary widely in the literature. In particular, the ratio T/S of the tensor and scalar contributions to the CMB depends on the current values of $\Omega_{\rm M}$ and Ω_{Λ} [31]. For the currently favored values $\Omega_{\rm M}\sim0.3$, $\Omega_{\lambda}\sim0.7$, the relationship is $T/S\simeq10r$, which is the normalization used in Refs. [11,6]. To compare with the normalization for r as defined in Refs. [5,9,10], take $r\rightarrow13.6r$.

[2] The slow roll parameters ${}^{\ell}\lambda_{\rm H}$ used here are related to the parameters ${}^{\ell}\beta_{\rm H}$ defined by Liddle *et al.* by ${}^{\ell}\lambda_{\rm H}=({}^{\ell}\beta_{\rm H})^{\ell}$.

This system of equations, taken to infinite order, is exact. In practice, these equations must be truncated at some finite order, by assuming ${}^{\ell}\lambda_{\rm H}=0$ for ℓ greater than some finite order M. The higher order the truncation, the weaker the implicit assumptions about the form of the potential. The next section discusses the fixed point structure of the inflationary parameter space.

V. FIXED POINTS IN THE INFLATIONARY PARAMETER SPACE

Summarizing the results of the preceding section, the hierarchy of inflationary flow equations is

$$\frac{d\epsilon}{dN}=\epsilon(\sigma+2\epsilon),$$

$$\frac{d\sigma}{dN}=-5\epsilon\sigma-12\epsilon^2+2({}^2\lambda_{\rm H}), \tag{46}$$

$$\frac{d({}^{\ell}\lambda_{\rm H})}{dN}=\left[\frac{1}{2}(\ell-1)\sigma+(\ell-2)\epsilon\right]({}^{\ell}\lambda_{\rm H})+{}^{\ell+1}\lambda_{\rm H}.$$

To lowest order in slow roll, these can be related to observables by $r=\epsilon$ and $n-1=\sigma$. To second order in slow roll, the observables are given by [23,27]

$$r=\epsilon[1-C(\sigma+2\epsilon)], \tag{47}$$

for the tensor/scalar ratio, and

$$n-1=\sigma-(5-3C)\epsilon^2-\frac{1}{4}(3-5C)\sigma\epsilon+\frac{1}{2}(3-C)({}^2\lambda_{\rm H}) \tag{48}$$

for the spectral index. Here $C\equiv 4(\ln 2+\gamma)$, where $\gamma\simeq 0.577$ is Euler's constant. Derivatives with respect to wave number k can be expressed in terms of derivatives with respect to N as [32]

$$\frac{d}{dN}=-(1-\epsilon)\frac{d}{d\ln k}. \tag{49}$$

The scale dependence of n is then given by the simple expression

$$\frac{dn}{d\ln k}=-\left(\frac{1}{1-\epsilon}\right)\frac{dn}{dN}, \tag{50}$$

which can be evaluated to third order in slow roll by using Eq. (48) and the flow equations. We wish to study flow in the parameter space of observables, r, n, and $dn/d\ln k$.

It is useful to identify fixed points of the system of Eqs. (46), for which all the derivatives vanish. Two classes of fixed points are easily obtained by inspection. First is the case of the vanishing tensor/scalar ratio, with

$$\epsilon={}^{\ell}\lambda_{\rm H}=0,\qquad \sigma=\text{const}. \tag{51}$$

The second class of fixed points is just the case of power-law inflation, $\epsilon=\eta=\xi^2=\text{const}$, or

$$\epsilon=\text{const},\qquad \sigma=-2\epsilon,\qquad {}^2\lambda_{\rm H}=\epsilon^2,\qquad {}^{\ell+1}\lambda_{\rm H}=\epsilon({}^{\ell}\lambda_{\rm H}),\quad l\geqslant 2. \tag{52}$$

Note that these are fixed points of the exact system of equations. It is straightforward to evaluate the stability of the fixed point (51), since

$$\left.\frac{d^2\epsilon}{dNd\epsilon}\right|_{\epsilon=0}=\sigma, \tag{53}$$

and

$$\left.\frac{d^2\epsilon}{dNd\sigma}\right|_{\epsilon=0}=0. \tag{54}$$

Therefore the fixed point at $\epsilon=0$ is stable with respect to perturbations in ϵ for $\sigma>0$, or spectral index $n>1$, and unstable for $\sigma<0$, or spectral index $n<1$. (This unusual sign convention for stability comes from the definition $dN<0$ for $dt>0$.) In general, inflationary evolution flows away from $r=0$ for $n<1$, and toward $r=0$ for $n>1$. This behavior can be easily understood in terms of simple inflaton potentials in slow roll. Using the slow roll expressions (28), taking $\epsilon=0$ implies that the field is at an equilibrium point $\dot\phi\propto V'(\phi)=0$, and the spectral index is

$$n-1=2\eta\simeq\frac{m_{\rm Pl}^2}{4\pi}\frac{V''(\phi)}{V(\phi)}. \tag{55}$$

The case $\epsilon=0$, $n<1$ is just that of the field sitting atop an unstable equilibrium, for example the point $\phi=0$ on a potential of the form $V(\phi)=\Lambda^4-m^2\phi^2$. The case $\epsilon=0$, $n>1$ is that of a field sitting at a stable equilibrium point $V''>0$, for example the point $\phi=0$ on a potential of the form $V(\phi)=\Lambda^4+m^2\phi^2$. In such models, inflation nominally continues forever. In practice, however, it is possible to end inflation by coupling to additional fields, as in "hybrid" inflation models [33–35]. The observables in this a case are given by their values near the late-time asymptote. The case of the fixed point (52) is more complex. It is, however, known that it is not in general a late-time attractor [36], a conclusion that is supported by numerical integration of the flow equations.

VI. EVALUATING THE FLOW EQUATIONS

With the flow equations in hand, it is possible to ask the question: what are the *generic* predictions of inflation? In principle, any model of inflation driven by a single, monotonic scalar field can be completely specified by selecting a point in the (infinite dimensional) slow roll parameter space,

$\epsilon,\sigma,{}^{\ell}\lambda_{\mathrm{H}}$.[3] For a model specified in this way, there is a straightforward procedure for determining its observable predictions, that is, the values of r, $n-1$, and $dn/d\ln k$ a fixed number N e-folds before the end of inflation. The algorithm for a single model is as follows:

Select a point in the parameter space $\epsilon,\eta,{}^{l}\lambda_{\mathrm{H}}$.

Evolve forward in time ($dN<0$) until either (a) inflation ends, or (b) the evolution reaches a late-time fixed point.

If the evolution reaches a late-time fixed point, calculate the observables r, $n-1$, and $dn/d\ln k$ at this point.

If inflation ends, evaluate the flow equations backward N e-folds from the end of inflation. Calculate the observable parameters at this point.

The end of inflation is given by the condition $\epsilon=1$ (*not* by the end of slow roll, although in practice these conditions are essentially equivalent). In the case where inflation ends in the late-time limit, there is another possibility: that one will find that inflation also ends when evolving back to early times. That is, the model is incapable of supporting N e-folds of inflation.

In principle, it is possible to carry out this program exactly, with no assumptions made about the convergence of the hierarchy of slow roll parameters. In practice, the series of flow equations (46) must be truncated at some finite order and evaluated numerically. In addition, for any given path in the parameter space, we do not know *a priori* the correct number of e-folds N at which to evaluate the observables, since this depends on details such as the energy density during inflation and the reheat temperature [17]. We adopt a Monte Carlo approach: we evaluate a large number of inflation models at order M in slow roll, where each model consists of a randomly selected set of parameters in the following ranges:

$$\begin{aligned} N&=[40,70]\\ \epsilon&=[0,0.8]\\ \sigma&=[-0.5,0.5]\\ {}^{2}\lambda_{\mathrm{H}}&=[-0.05,0.05]\\ {}^{3}\lambda_{\mathrm{H}}&=[-0.005,0.005],\\ &\cdots\\ {}^{M+1}\lambda_{\mathrm{H}}&=0 \end{aligned} \tag{56}$$

and so forth, reducing the width of the range by factor of ten for each higher order in slow roll. The series is closed to order M by taking ${}^{M+1}\lambda_{\mathrm{H}}=0$. The exact choice of ranges for the initial parameters does not have a large influence on the result of the Monte Carlo calculation, as long as they are chosen such that the slow roll hierarchy is convergent. For each model, we calculate observables according to the algorithm above, with two differences because of the finite nature of the calculation. When we evolve forward in time, there are now three possible late-time behaviors for a particular model: (1) the model reaches the late-time attractor $\epsilon=0$, $\sigma>0$, (2) inflation ends, or (3) none of the above, indicating that the integration failed to reach any identifiable asymptotic behavior within the limits of the integration, which we take to be 1000 e-folds. For models in which inflation ends at late time, we then evolve the model backward in time N *e*-folds from the end of inflation. If the choice of parameters supports N e-folds without inflation ending or slow roll failing, we calculate observable parameters r, n, and $dn/d\ln k$ at that point. We will call these points *nontrivial* points. In summary, there are four categories of outcome for a particular choice of initial condition:

Late-time attractor, $\epsilon=0$, $\sigma>0$.

Insufficient inflation.

Nontrivial point: Inflation ends at late time, supports N e-folds of inflation.

No identifiable asymptotic behavior at late time.

The numerical integration is implemented in C using a fifth-order adaptive step-size Runge-Kutta method to solve the system of equations. The Monte Carlo calculation is run by selecting initial conditions at random as described above for 100 000 points. We are interested in the models which converge to a late-time attractor or possess a nontrivial point. In addition, we require $n<1.5$ in order to be consistent with observations of the cosmic microwave background [5,6] and constraints from primordial black hole formation [37–40]. The results of a Monte Carlo run to order $M=5$ in slow roll are as follows:

Total iterations: 100 000.

Late-time attractor, $r=0$, $n>1.5$: 90 340.

Nontrivial points: 6999.

Late-time attractor, $r=0$, $n<1.5$: 2542.

Insufficient inflation: 116.

No identifiable asymptotic behavior: 3.

One surprising result is that more than 90% of the models evaluated result in an unacceptably blue spectral index, $n>1.5$: the most "generic" prediction of inflation from this point of view is already ruled out. Figure 1 shows the remaining models plotted on the (n,r) plane. (Note that the normalization for r used here differs from elsewhere in the literature. To compare with Refs. [5,9,10], take $r\rightarrow13.6r$. To compare with Refs. [11,6], take $r\rightarrow T/S=10r$.) The models cluster strongly near (but *not* on) the power-law fixed point, and on the $r=0$ fixed point. This is qualitatively consistent with the results of Hoffman and Turner, except that the models appear to be much more strongly clustered in the parameter space than they concluded from a lowest-order analysis. Also, models sparsely populate the regions that Hoffman and Turner label "excluded" and "poor power law," suggesting that these categorizations do not generalize to higher order in slow roll. (We note that a poor power law was a rare result in the integrations, with of order 0.1% of the models predicting $|dn/d\ln k|>0.05$.) However, consistent with Hoffman and

[3]Strictly speaking, this statement is true only if the slow roll expansion is nonsingular to all orders.

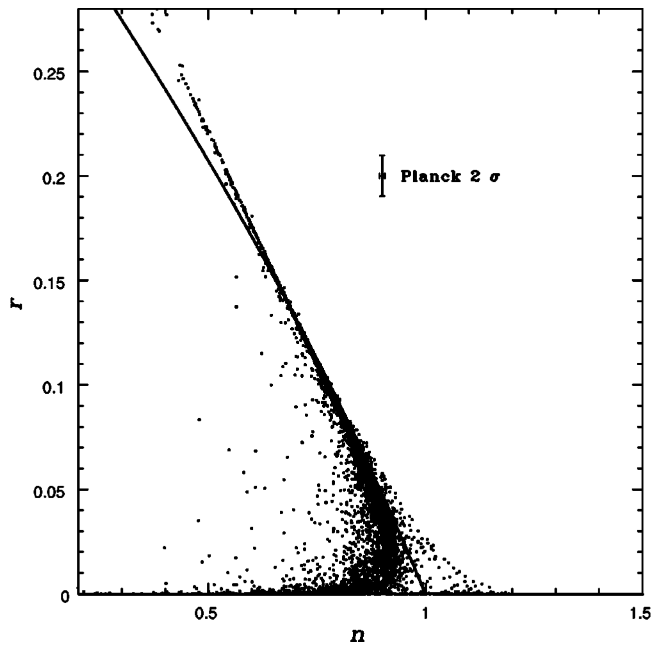

FIG. 1. Models plotted in the (n,r) plane for an $M=5$ Monte Carlo calculation. The solid line is the power-law fixed point $n=1-2r/(1-r)$. The error bar shows the size of the expected 2σ error from Planck. (See the note in text regarding the somewhat unconventional normalization of r used here.)

Turner, there is a large region for $n>1$ and $r>0$ that is entirely unpopulated by models. Figure 2 shows the (n,r) plane zoomed in on the observationally favored region near $n=1$. Figure 3 shows the same models plotted vs log(r) to

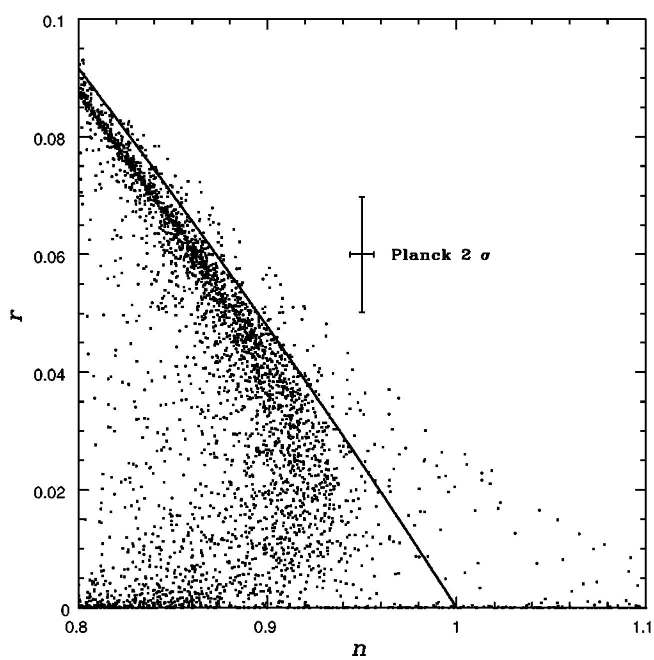

FIG. 2. Figure 1 zoomed in to the region preferred by observation. The models populate the entire area below the power-law line. There is, however, a large apparently excluded region above the power-law line. This overlaps with (but is not identical to) the region labeled "poor power law" by Hoffman and Turner.

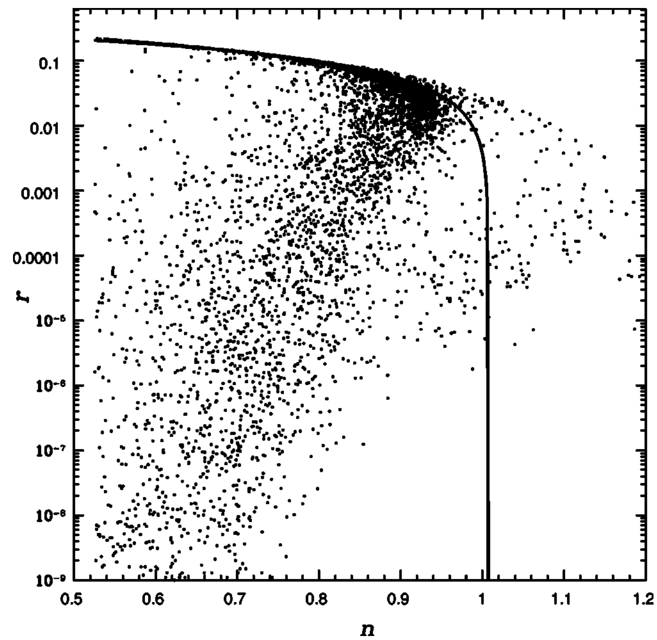

FIG. 3. Spectral index vs log(r), showing the behavior of the attractor region for small r.

show the small-r behavior of the attractor region. Figure 4 shows the same models plotted on the $n, dn/d\ln k$ plane, also showing noticeable clustering behavior in the parameter space. In particular, $dn/d\ln k<0$ is favored.

Figure 5 shows $dn/d\ln k$ as a function of r. Especially interesting is that the models with large r (the ones close to the power-law line in Fig. 1), also have significant variation in the spectral index. This suggests that the models are not flowing to the power-law fixed point, which has $dn/d\ln k=0$. This raises an interesting question: are the models converging slowly to the power-law line at early times, or are

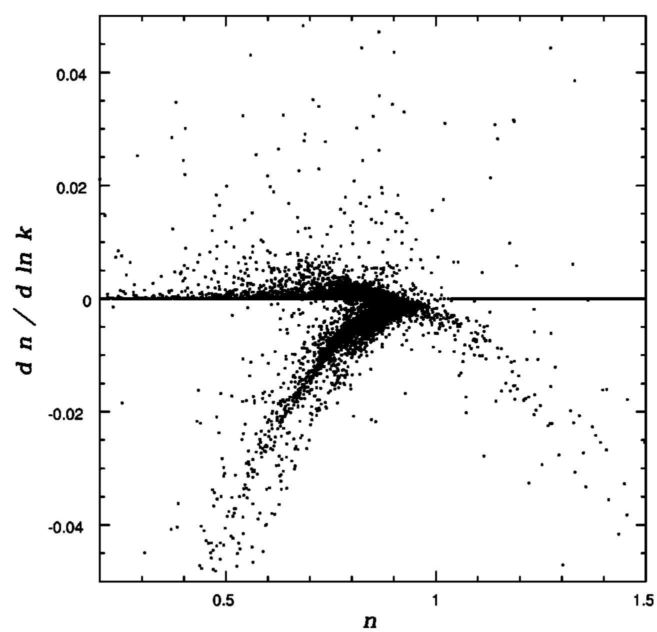

FIG. 4. Models plotted in the $(n,dn/d\ln k)$ plane for an $M=5$ Monte Carlo calculation.

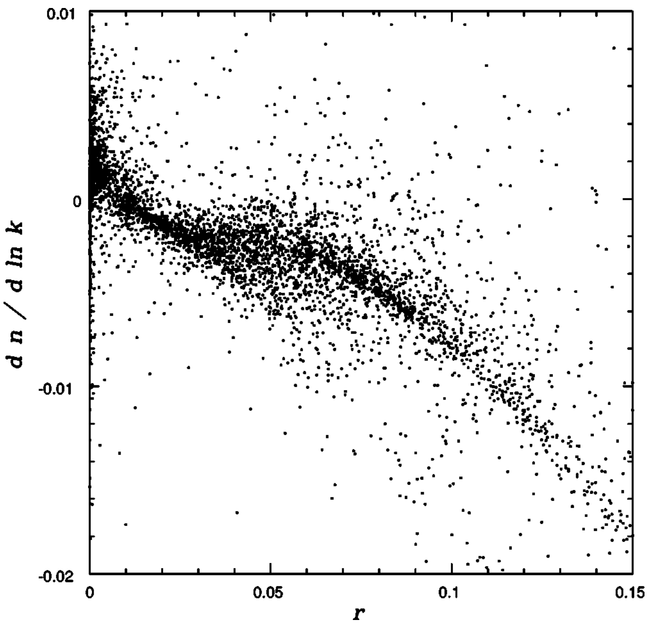

FIG. 5. Models plotted in the $(r, dn/d\ln k)$ plane for an $M=5$ Monte Carlo calculation.

they converging to some other fixed point? To answer this question, we evolve the models to very early times, $N \gg 70$. Figure 6 shows models plotted on the (n,r) plane for $N=125$, 250, 500, and 1000. Instead of flowing to the power-law fixed point at early time, the models instead flow down to the $r=0$ line. We therefore find that the power-law fixed point is not an attractor at early or late time. It is important to note that this conclusion is not in conflict with the analysis of Copeland *et al.* [41], which concluded that power-law inflation is a unique late-time attractor in a cosmology consisting of a scalar field and a second fluid component. Copeland *et al.* assumed a scalar field with an exponential potential—that is, a model lying exactly on the power-law fixed point—and showed that the scalar field would generically dominate the cosmological evolution at late times. Figure 7 shows examples of flow in the (σ,ϵ) plane.

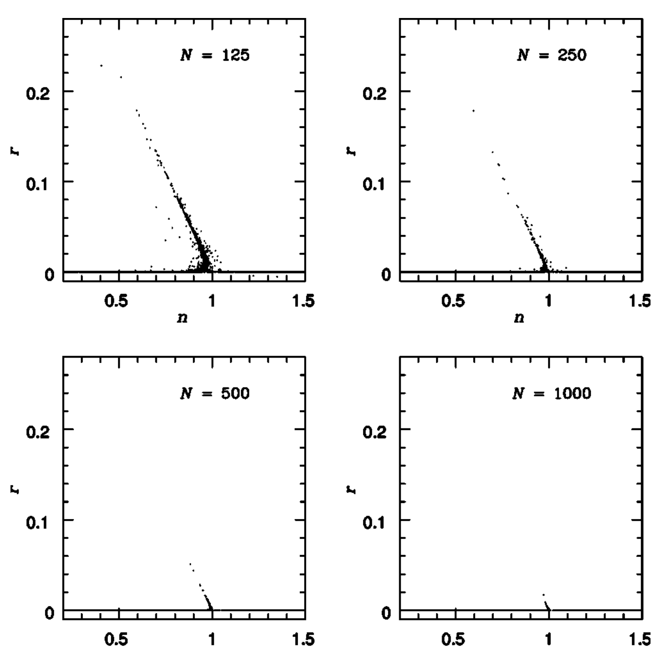

FIG. 6. Results of the Monte Carlo calculation with a large number of e-folds *N*, illustrating the behavior of the flow at early times (larger N). The models flow not to the power-law fixed point $n=1-2r/(1-r)$ but to the $r=0$ fixed point.

VII. CONCLUSIONS

We have derived a set of inflationary "flow" equations based on the Hubble slow roll expansion of Liddle *et al.* [27] that is in principle exact when taken to all orders. These equations completely specify the dynamics of the inflationary system, so that any particular inflationary potential can be specified as a point in this parameter space. The past and future dynamics of the model are then determined by evaluating the flow of the parameters away from this point. It is possible to identify two classes of fixed points of the exact flow equations: power-law inflation, with $n=1-2r/(1-r)$, and models with vanishing tensor/scalar ratio, $r=0$. This latter class is unstable for $n<1$ and stable for $n>1$.

In practice, the flow equations must be truncated to some order and evaluated numerically, which was done to lowest order by Hoffman and Turner [11]. Extending the system of flow equations to higher order makes it possible to consider the running of the spectral index $dn/d\ln k$ as well as r and n. We perform a Monte Carlo integration of the flow equations to fifth order in slow roll, and show that the distribution of models in the parameter space of observables r, n and $dn/d\ln k$ is strongly clustered around particular values. 90% of the models selected in the Monte Carlo calculation converge to the observationally unacceptable asymptote $r=0$, $n>1.5$. The remaining models cluster around two classes of early-time "attractor," the first class at the $r=0$ fixed point and the second with $r>0$ and $n<1$. Interestingly, the $r>0$ attractor *cannot* be identified with the power-law fixed point, since they generally have $dn/d\ln k<0$, and the variation in the spectral index vanishes at the fixed point. Evaluation of the models at very early times, $N\gg 70$, indicates that the power-law fixed point is not an attractor at early times, since the models generically flow to the $r=0$ line for large N. We therefore interpret the $r>0$ attractor as simply an artifact of the fact that observable perturbations are generated relatively late in the inflationary evolution, when slow roll has begun to measurably break down. In addition, we see that power-law inflation is not in general an attractor for either early *or* late times. At higher order, models cluster much more strongly than is suggested by the "favored" region of the parameter space derived by Hoffman and Turner. Also, models sparsely populate the regions labeled by Hoffman and Turner as "excluded" and "poor power law," suggesting that these categorizations do not generalize to higher order. However, consistent with previous analysis, there is a region for $r>0$, $n>1$ which is entirely unpopulated by models.

It is important to consider questions of generality with respect to both the choice of the order in slow roll *M* and the choice of initial conditions for the Monte Carlo calculation

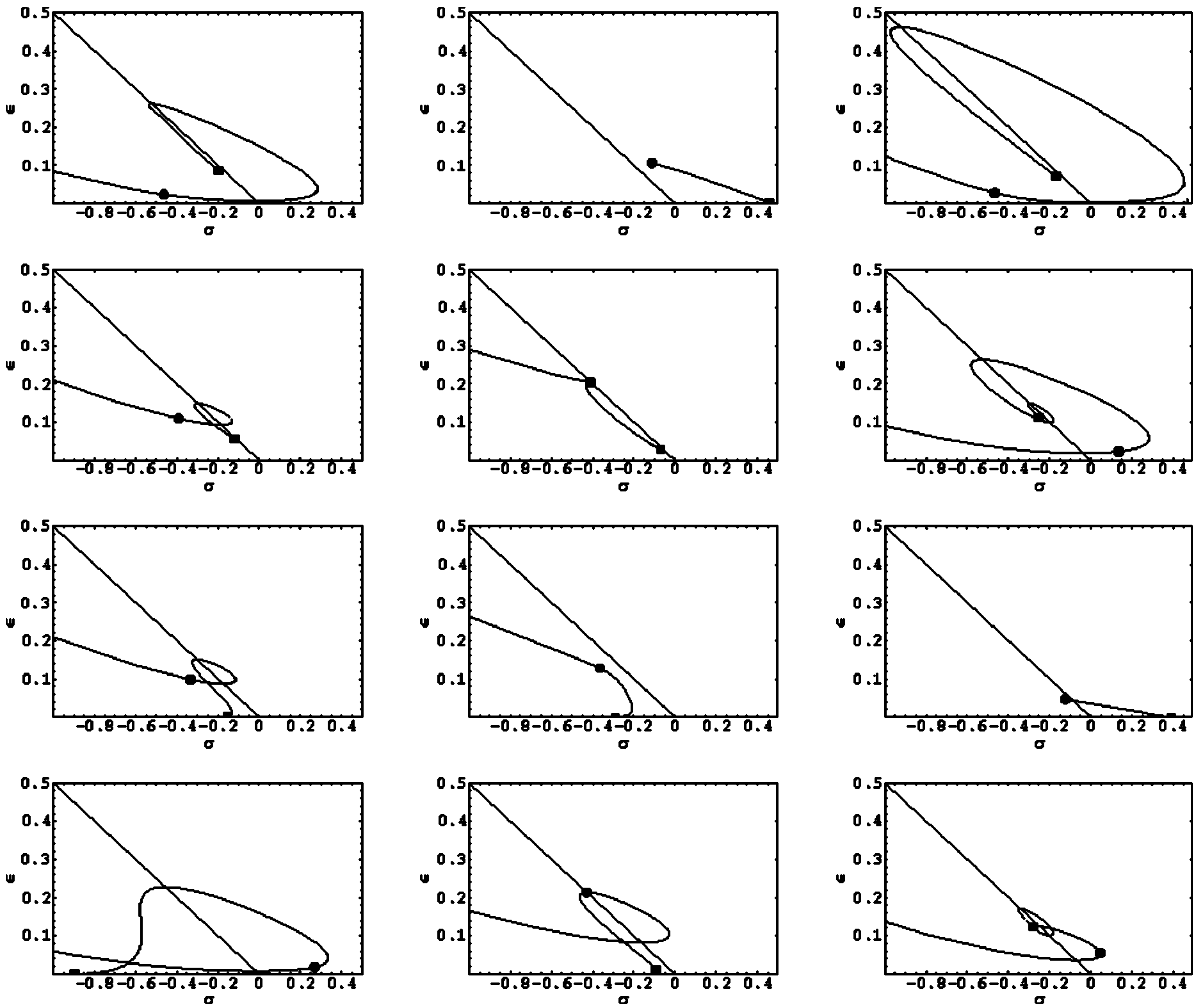

FIG. 7. Examples of flow plotted in the (σ,ϵ) plane. The circles indicate the randomly selected initial value, and the squares indicate the value N *e*-folds before the end of inflation. The straight line is the power-law fixed point, $\sigma=-2\epsilon$. Integrating to high order in slow roll allows for a variety of complex flows.

(56). By ''closing'' the hierarchy of flow equations at finite order, we are implicitly limiting ourselves to a restricted class of potentials, although for $M=5$, that class of potentials is large. However, models with potentials that contain features [42,43] or for which the slow roll expansion is not convergent [44] will not be captured by solutions at finite order in slow roll. In addition, inflation might not be driven by only a single scalar field. The effect of different choices of initial conditions can be studied empirically, simply by trying different constraints on the space of initial conditions. Choosing ''looser'' initial conditions does not alter the characteristics of the result. Instead, models that fail to support sufficient inflation become much more numerous. Perhaps most importantly, absent a metric on the space of initial conditions, one should use caution when attempting to interpret these ''scatter plots'' statistically. We do not know how the initial conditions for the universe were selected. However, if observations determine that the relevant cosmological parameters lie outside the ''favored'' region, it will be an indication of highly unusual dynamics during the inflationary epoch.

Finally, we note an interesting recent body of literature connecting flow in inflationary models to a proposed ''holographic'' correspondence between quasi–de Sitter spaces and boundary conformal field theories (CFTs) [12,45–50]. In particular, Larsen *et al.* [13] have proposed a correspondence between slow roll parameters and couplings in the boundary CFT, interpreting flow in the inflationary parameter space as renormalization group flow in the associated CFT [51]. The fixed points at $r=0$ are interpreted as ultraviolet $(n>1)$ and infrared $(n<1)$ fixed points in the renormalization group flow. In this picture, studying inflationary dynamics is equivalent to studying the structure of the underlying CFT. (It is not immediately clear, however, how one interprets the power-law fixed point in the context of the boundary CFT.)

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