Decay $\pi^0 \to \gamma\gamma$ to next to leading order in chiral perturbation theory

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The $\pi^0 \to \gamma\gamma$ decay width is analyzed within the combined framework of chiral perturbation theory and the $1/N_c$ expansion up to $\mathcal{O}(p^6)$ and $\mathcal{O}(p^4 \times 1/N_c)$ in the decay amplitude. The η' is explicitly included in the analysis. It is found that the decay width is enhanced by about 4.5% due to the isospin-breaking induced mixing of the pure $U(3)$ states. This effect, which is of leading order in low energy expansion, is shown to persist nearly unchanged at next to leading order. The chief prediction with its estimated uncertainty is $\Gamma_{\pi^0 \to \gamma\gamma} = 8.10 \pm 0.08$ eV. This prediction at the 1% level makes the upcoming precision measurement of the decay width even more urgent. Observations on the η and η' can also be made, especially about their mixing, which is shown to be significantly affected by next to leading order corrections.

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I. INTRODUCTION

In the chiral SU(2) limit ($m_{u,d}=0$) the $\pi^0 \to \gamma\gamma$ decay amplitude is precisely known to order α [1]. The amplitude is $\mathcal{O}(p^4)$ in the low energy chiral counting. It is determined entirely by the anomaly induced on the divergence of the axial vector current $A^3_\mu = \overline{q}\gamma_\mu\gamma_5\tau_3 q$ by the electromagnetic interaction, and is expressed in terms of the only two available quantities—the fine structure constant α and the pion decay constant F_π—the decay width in this limit is thus given by $\Gamma_{\pi^0 \to \gamma\gamma} = (\alpha/F_\pi)^2 (M_{\pi^0}/4\pi)^3$. The explicit breaking of chiral $SU_L(2) \times SU_R(2)$ symmetry induced by nonzero *u*- and *d*-quark masses generates corrections to the chiral limit result, and it is the purpose of the present work to evaluate these corrections as well as to understand their origin. In order to achieve this goal it is crucial to perform the analysis in the extended framework of three flavors supplemented by the $1/N_c$ expansion in order to include explicitly the η' degree of freedom. In this three flavor framework the corrections turn out to be of two types:

(i) those due to isospin breaking (mixing corrections) that are proportional to $(m_u - m_d)/m_s$ or to $N_c(m_u - m_d)/\Lambda_\chi$, both giving contributions to the decay amplitude that, according to the counting defined in the next section, are $\mathcal{O}(p^4)$, i.e., of the same order as the leading term, and

(ii) those proportional to $m_{u,d}/\Lambda_\chi$ that are of subleading order—$\mathcal{O}(p^6)$—and which stem from different sources, as shown below.

The inclusion of such corrections is crucial for a prediction of the $\pi^0 \to \gamma\gamma$ width at the 1% level, which is the level of theoretical precision required by the forthcoming dramatic improvement expected in the experimental measurement of the π^0 width via the Primakoff effect. The PRIMEX experiment at Jefferson Lab [2] is aiming at a measurement with an error about 1.5%, which is several times smaller than the 7.1% uncertainty in the current world-average value 7.74 ± 0.55 eV [3]. However, this quoted experimental uncertainty is open to question as can be seen by the large dispersion of the experimental results which suggests that the errors of the individual experiments have been underestimated. Indeed, a direct measurement gives $\Gamma_{\pi^0} = 7.25 \pm 0.23$ eV [4], a production experiment in $e^+ - e^-$ collisions yields $\Gamma_{\pi^0} = 7.74 \pm 0.66$ eV [5], while Primakoff effect experiments, all dating back to the early 1970s, give disparate values: a large number, 11.61 ± 0.55 eV [6], and two that are consistent with the current world average, 7.22 ± 0.55 eV [7] and 7.93 ± 0.39 eV [8]. This unsatisfactory experimental situation together with the rather precise theoretical prediction derived in the present work clearly lend great significance to the upcoming PRIMEX measurement.

Within the two-flavor framework, wherein the strange quark is integrated out, any corrections to the amplitude are $\mathcal{O}(p^6)$ and reside entirely in F_{π^0} [9,10] or in the $\mathcal{O}(p^6)$ odd-intrinsic parity chiral Lagrangian [11], also known as the $\mathcal{O}(p^6)$ Wess-Zumino (WZ) Lagrangian. At leading order, the ensuing theoretical prediction (taking $F_{\pi^0} = F_{\pi^+}$) is $\Gamma_{\pi^0 \to \gamma\gamma} = 7.725$ eV, a result that agrees well with the experimental world average within its generous error. The analysis within $SU(2)$, however, does not provide insight on the origin of the $\mathcal{O}(p^6)$ WZ contribution just mentioned. Such an insight *can* be gained by instead performing the analysis in the *three*-flavor framework, as has been shown by Moussallam [12]. In particular, he pointed out that the primary corrections to the π^0 width result from the leading order isospin breaking effects mentioned above in (i), which stem from the $m_u \neq m_d$-induced mixing between the pure isospin state π^0 and pure $SU(3)$ states η and η' (to be denoted below by π_3, π_8 and π_0 respectively). The present work confirms that such isospin breaking corrections persist as the dominant effect when next to leading order (NLO) corrections are included.

In this work then the $\pi^0\to\gamma\gamma$ decay rate is evaluated to NLO within $U_L(3)\times U_R(3)$ chiral perturbation theory wherein the η' meson is included consistently by means of the $1/N_c$ expansion since in the large N_c limit the η' becomes a Goldstone boson. Such a framework was recently developed by Herrera-Siklódy, Latorre, Pascual and Taron [13] and by Kaiser and Leutwyler [14,15], who showed that a simultaneous chiral and $1/N_c$ expansion leads to an effective theory for the pseudoscalar nonet that is not only internally consistent but is also very useful in practice, as the present work shows.

The chief result of this paper is that the $\pi^0\to\gamma\gamma$ width is enhanced by about 4.5% from the lowest order chiral anomaly prediction, a result expected to hold within an uncertainty of $\pm 1\%$ after NLO contributions are included. The magnitude of this enhancement agrees with that obtained in the analysis of Moussallam [12], where the NLO corrections were not implemented in a consistent fashion as in the present work.

As this manuscript was being completed an analysis by Ananthanarayan and Moussallam [16] was posted where the electromagnetic corrections are studied in detail. Related work is also being completed by Kaiser and Leutwyler [17].

II. TWO-PHOTON DECAY AMPLITUDES

The decay amplitudes of π^0, η and η' into two photons can be obtained from the Ward identities satisfied by the three axial vector currents $A^a_\mu=\frac{1}{2}\bar{q}\gamma_\mu\gamma_5\lambda^a q$ $(a=3,8,0)$, where λ^a are $U(3)$ generators [λ^0 being the $U(1)$ generator] normalized via $\langle\lambda^a\lambda^b\rangle\equiv\mathrm{Tr}(\lambda^a\lambda^b)=2\delta^{ab}$. In the presence of the strong and electromagnetic interactions, the divergence of the axial vector current is given by

$$\partial^\mu A^a_\mu=\frac{\alpha N_c}{4\pi}\langle\lambda^a Q^2\rangle F\tilde{F}+\frac{\alpha_s}{4\pi}\langle\lambda^a\rangle G\tilde{G}+\frac{i}{2}\bar{q}\gamma_5\{\lambda^a,\mathcal{M}_q\}q+\cdots, \tag{1}$$

where $\mathcal{M}_q$ is the quark mass matrix and eQ is the electric charge operator. Here $F\tilde{F}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, $F_{\mu\nu}$ being the electromagnetic field tensor, and similarly $G\tilde{G}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{a\mu\nu}G^{a\rho\sigma}$, $G_{\mu\nu}$ being the gluon field,[1] and the ellipsis denotes terms irrelevant to this work.

The two-photon amplitudes can be obtained by considering the matrix elements

$$\langle\gamma\gamma|\partial^\mu A^a_\mu|0\rangle=C_a\frac{\alpha N_c}{12\pi}\langle\gamma\gamma|F\tilde{F}|0\rangle+\delta_{a0}\frac{\sqrt{6}\alpha_s}{4\pi}\langle\gamma\gamma|G\tilde{G}|0\rangle+\frac{i}{2}\langle\gamma\gamma|\bar{q}\gamma_5\{\lambda^a,\mathcal{M}_q\}q|0\rangle, \tag{2}$$

where $C_3=1$, $C_8=1/\sqrt{3}$, and $C_0=\sqrt{8/3}$.

[1] Throughout, the conventions in Bjorken and Drell are used.

If p denotes the total momentum of the final two-photon state, then in the limit of small p^2 Eq. (2) admits a low energy expansion and can be expressed as

$$\begin{aligned}\sum_{\bar{a}}\langle\gamma\gamma|\pi_{\bar{a}},p\rangle\langle\pi_{\bar{a}},p|\partial^\mu A^a_\mu|0\rangle\frac{i}{p^2-M^2_{\bar{a}}+i\epsilon}&=C_a\frac{\alpha N_c}{12\pi}\langle\gamma\gamma|F\tilde{F}|0\rangle+\delta_{a0}\frac{\sqrt{6}\alpha_s}{4\pi}\sum_{\bar{a}}\langle\gamma\gamma|\pi_{\bar{a}},p\rangle\\&\quad\times\langle\pi_{\bar{a}},p|G\tilde{G}|0\rangle\\&\quad\times\frac{i}{p^2-M^2_{\bar{a}}+i\epsilon}+\frac{i}{2}\sum_{\bar{a}}\langle\gamma\gamma|\pi_{\bar{a}},p\rangle\\&\quad\times\langle\pi_{\bar{a}},p|\bar{q}\gamma_5\{\lambda^a,\mathcal{M}_q\}q|0\rangle\\&\quad\times\frac{i}{p^2-M^2_{\bar{a}}+i\epsilon}+\cdots,\end{aligned} \tag{3}$$

where $\langle\gamma\gamma|\pi_{\bar{a}},p\rangle$ are the two-photon amplitudes, the ellipsis denotes contributions from excited mesons as well as from the continuum, all being of NLO or higher, and the mass eigenstates $\pi_{\bar{a}}$ that correspond to the physical π^0, η and η' are given by

$$\pi_{\bar{a}}=\sum_{a=3,8,0}\Theta_{\bar{a}a}\pi_a, \tag{4}$$

where the mixing matrix that diagonalizes the mass matrix is parametrized in terms of Euler angles θ_3, θ_8 and $-\theta_0$:

$$\Theta=\begin{pmatrix}c_3c_8-c_0s_3s_8 & c_3s_8+c_8c_0s_3 & -s_3s_0\\ -c_8s_3-c_3c_0s_8 & -s_3s_8+c_3c_8c_0 & -c_3s_0\\ -s_8s_0 & c_8s_0 & c_0\end{pmatrix},$$

where $c_i=\cos\theta_i$ and $s_i=\sin\theta_i$. Here, for small mixing, the projection of the physical π^0 onto π_8 is given by the angle $\epsilon\simeq\theta_3+\theta_8$, the projection of the physical η onto π_0 is approximately given by $-\theta_0$ (θ_0 can therefore be identified with the well known $\eta-\eta'$ mixing angle as it is customarily defined), and the projection of the physical π^0 onto π_0 is given by $\tilde{\epsilon}\simeq-\theta_3\theta_0$.

The NLO—i.e., $\mathcal{O}(p^6)$—corrections in Eq. (3) reside in the terms displayed explicitly through their dependence on the masses and decay constants, as well as in pieces that stem from continuum and excited states. In works that preceded that of Moussallam [12], such as Refs. [9] and [10], such mixing corrections as well as the NLO effects of the latter kind were disregarded. Ignoring such effects implies that only NLO corrections which are absorbed into the π^0 decay constant remain, as it was shown in [9]; in that case and taking $F_{\pi^0}=F_{\pi^+}$, the predicted π^0 width is the previously mentioned 7.725 eV. As shown here, however, disregarding mixing in particular constitutes a very poor approximation. In the presence of mixing the pseudoscalar decay constants form a 3×3 matrix defined by the matrix elements of the axial vector currents, which connect the pseudoscalar mesons to the vacuum:

$$\langle \pi_{\bar a},p|A^a_\mu|0\rangle = -ip_\mu F_{\bar a a}. \tag{5}$$

Indeed this decay-constant matrix contains all that is needed to calculate the three two-photon amplitudes, except the contributions stemming from the $\mathcal{O}(p^4)$ WZ Lagrangian, and its evaluation is the centerpiece of the present work.

As already mentioned in the introduction, the present analysis includes the η' as an explicit degree of freedom, which, in order to be consistently implemented in an effective theory, requires the validity of the $1/N_c$ expansion, wherein, taking the chiral $SU_L(3)\times SU_R(3)$ limit, $M^2_{\eta'}=\mathcal{O}(1/N_c)$. Thus, in the framework of the $1/N_c$ expansion $M^2_{\eta'}$ should be considered as "small," and its explicit inclusion becomes consistent with having a simultaneous low energy chiral expansion. This explicit inclusion of the η' in the low energy expansion implies that $M^2_{\eta'}$ must count as a quantity of $\mathcal{O}(p^2)$, which in turn implies that $1/N_c$ should be counted as a quantity of the same order. Indeed, a consistent effective theory can be formulated with such a counting scheme [13–15], and it is interesting to note that $1/N_c$ and the magnitude of $SU(3)$ breaking are comparable in size in the real world.

Taking the chiral limit—$\mathcal{M}_q\to0$—in Eq. (3) and neglecting the electromagnetic piece, i.e., the $F\tilde{F}$ term, equating the residues at $p^2=M^2_{\eta'}$ leads to the well known relation

$$M_0^2=\sqrt{6}\,\frac{\alpha_s}{4\pi F_0}\langle \eta'|G\tilde{G}|0\rangle, \tag{6}$$

with M_0 being the η' mass in the chiral limit. Here the lowest order result

$$\langle \pi_{\bar a},p|\partial^\mu A^a_\mu|0\rangle = \delta_{\bar a a}p^2F_0$$

was used, where F_0 is the pion decay constant in the chiral limit. In the large N_c limit, F_0 scales as $\sqrt{N_c}$, while in the chiral limit the ratio $F_0/F_{\eta'}$ is equal to unity up to corrections of order $1/N_c$.

On the other hand, for nonvanishing quark masses equating the residues in Eq. (3) yields

$$\langle \pi_{\bar a},p|\bar{q}\gamma_5\{\lambda^a,\mathcal{M}_q\}q|0\rangle = -2i\left(\langle \pi_{\bar a},p|\partial^\mu A^a_\mu|0\rangle - \delta_{a0}\sqrt{6}\,\frac{\alpha_s}{4\pi}\langle \pi_{\bar a},p|G\tilde{G}|0\rangle\right) \tag{7}$$

where $p^2=M^2_{\bar a}$. As is well known, the p^2 dependence of the LHS appears first at $\mathcal{O}(p^6)$ [18], which would affect the two-photon amplitudes at $\mathcal{O}(p^8)$, i.e., beyond the accuracy needed in this work. Thus, it is consistent to use Eq. (7) at $p^2=M^2_{\bar a}$ to represent the LHS in the entire low p^2 domain.

At LO—$\mathcal{O}(p^4)$—then, Eqs. (3), (6), and (7) yield immediately the result for the two-photon amplitudes:

$$\langle \gamma\gamma|\pi_{\bar a}\rangle = \sum_{a=3,8,0} -i\frac{\alpha N_c}{12\pi}C_aF^{-1}_{\bar a a}\langle \gamma\gamma|F\tilde{F}|0\rangle. \tag{8}$$

At this order the decay constant matrix is simply given by

$$F_{\bar a a}=\Theta_{\bar a a}F_0 \tag{9}$$

where $\Theta_{\bar a a}$ is the mixing matrix obtained from the $\mathcal{O}(p^2)$ mass formulas. Of course, the result of Eq. (8) coincides with the result obtained by means of the $\mathcal{O}(p^4)$ WZ term including explicitly the singlet pseudoscalar. The purpose of carrying out the above Ward identity analysis is, however, to make more transparent the origin and structure of the higher order corrections.

III. LEADING ORDER RESULTS

The leading order mass formulas are obtained from the $\mathcal{O}(p^2)$ chiral Lagrangian. The $U(3)$ field is parametrized by the unitary matrix:

$$U=\exp\left(i\sum_{a=0}^{8}\frac{\pi_a\lambda^a}{F_0}\right) \tag{10}$$

where $F_0=92.42$ MeV at LO. The $\mathcal{O}(p^2)$ Lagrangian with the standard definitions of covariant derivatives and sources χ [18] is given by

$$\mathcal{L}^{(2)}=\frac{1}{4}F_0^2\langle D_\mu U D^\mu U^\dagger\rangle+\frac{1}{4}F_0^2\langle \chi U^\dagger+\chi^\dagger U\rangle-\frac{1}{2}M_0^2\pi_0^2 \tag{11}$$

and the mass matrix in the π_3,π_8,π_0 sector of interest resulting from $\mathcal{L}^{(2)}$ is

$$M^2_{LO}=B_0\begin{pmatrix} 2\hat{m} & \frac{1}{\sqrt{3}}(m_u-m_d) & \sqrt{\frac{2}{3}}(m_u-m_d)\\ \frac{1}{\sqrt{3}}(m_u-m_d) & \frac{2}{3}(\hat{m}+2m_s) & -\frac{\sqrt{8}}{3}(m_s-\hat{m})\\ \sqrt{\frac{2}{3}}(m_u-m_d) & -\frac{\sqrt{8}}{3}(m_s-\hat{m}) & \frac{M_0^2}{B_0}+\frac{2}{3}(2\hat{m}+m_s)\end{pmatrix}. \tag{12}$$

Using the leading order mass formulas—e.g., $M^2_{\pi^+}=2B_0\hat{m}$, with $2\hat{m}=m_u+m_d$—and extracting isospin breaking from the K^+-K^0 mass difference via Dashen's theorem to eliminate the EM contributions,[2] a best fit to the masses yields a singlet mass M_0 of approximately 850 MeV, and the Euler angles: $\theta_3=1.57°$, $\theta_8=-0.56°$, and $\theta_0=-18.6°$. This fit yields then $\epsilon=\theta_3+\theta_0\sim 1°$, which is substantially larger than the value [18]

$$\epsilon=\sqrt{\frac{3}{4}}\frac{m_d-m_u}{m_s}\simeq 0.56° \tag{13}$$

that arises in the limit $M_0\to\infty$, when only octet degrees of freedom are included. The LO mass matrix, however, gives a poor result for the masses. In particular the η mass is too low by almost 50 MeV, a problem that is generic at LO in the low energy and $1/N_c$ expansions [19].

Using the relations

$$\langle\gamma\gamma|\pi_{\bar{a}}\rangle=\kappa_{\bar{a}}\langle\gamma\gamma|F\tilde{F}|0\rangle,\quad \Gamma_{\bar{a}}=|\kappa_{\bar{a}}|^2\frac{M^3_{\bar{a}}}{4\pi} \tag{14}$$

which connect the decay amplitudes and associated widths, the fitted parameters at LO lead to a $\pi^0\to\gamma\gamma$ decay width of 8.08 eV, which is 4.5% larger than the leading order result $\Gamma_{\pi^0\to\gamma\gamma}=7.725$ eV obtained in the two-flavor framework wherein mixing effects are moved to NLO. It should be noted, however, that the two-photon widths of the η and η' predicted in this leading order fit are too large, the first being 22% and the second 20% larger than the corresponding experimental values. One of the chief reasons for this disagreement is that $SU(3)$ breaking in the pseudoscalar decay constants is not included at LO. The $\eta-\eta'$ mixing angle θ_0 turns out to be $\sim -18.6°$ at LO, and will be reduced by almost a factor of two when NLO corrections are included. The magnitude of the observed LO enhancement of the π^0 width is in line with the ratio of isospin breaking versus m_s $[(m_d-m_u)/m_s\simeq 2.3\%]$ and versus M_0 $[B_0(m_d-m_u)/M_0^2\simeq 1.1\%]$, and is therefore not surprising. It is important to note that the corrections due to mixing with η and with η' are of the same sign and of similar magnitude. This point is the primary reason why the η' *must* be explicitly included for a full understanding of the mixing effects. Although $\eta'-\pi^0$ mixing is smaller than the $\eta-\pi^0$ mixing, the bare singlet state has an intrinsic two-photon amplitude larger by a factor $\sqrt{8}$ that compensates for the smaller mixing. Overall, however, the LO fit is poor, and dramatic improvement results when the NLO corrections are included, as shown in the following section.

IV. NEXT TO LEADING ORDER ANALYSIS AND RESULTS

At NLO the amplitudes receive corrections of two types—those that affect the decay constants and mixing angles, and those that stem from the presence of excited states and which are included in the $\mathcal{O}(p^6)$ WZ Lagrangian.

The first type of correction requires the determination of the masses and decay constants to NLO and can be obtained in the standard fashion by calculating the two-point functions of axial vector currents, where the relevant diagrams are shown in Fig. 1. Up to $\mathcal{O}(p^2)$ and $\mathcal{O}(p^0/N_c)$ such two-point functions require only the effective Lagrangians $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$. Chiral loop corrections are $\mathcal{O}(p^2/N_c)=\mathcal{O}(p^4)$, and therefore beyond the precision of the present calculation, but such loop corrections will be calculated merely as a means to estimate the size of possible contributions from terms of that order and also as a test on the practical validity of the $1/N_c$ expansion. The lowest order Lagrangian has already been given in Eq. (11), while the next to leading order Lagrangian $\mathcal{L}^{(4)}$ has the form given by [13–15,18]

$$\begin{aligned}\mathcal{L}^{(4)}=&\cdots+L_4\langle D_\mu U^\dagger D^\mu U\rangle\langle\chi^\dagger U+U^\dagger\chi\rangle\\&+L_5\langle D_\mu U^\dagger D^\mu U(\chi^\dagger U+U^\dagger\chi)\rangle+L_6\langle\chi^\dagger U+U^\dagger\chi\rangle^2\\&+L_7\langle\chi^\dagger U-U^\dagger\chi\rangle^2+L_8\langle\chi U^\dagger\chi U^\dagger+\text{H.c.}\rangle\\&+\frac{\Lambda_1}{2}D_\mu\pi_0 D_\mu\pi_0-\frac{iF_0\Lambda_2}{2\sqrt{6}}\pi_0\langle\chi U^\dagger-\chi^\dagger U\rangle\\&+iL_{18}\sqrt{6}D^\mu\pi_0\langle D^\mu U^\dagger\chi-D^\mu U\chi^\dagger\rangle\\&+iL_{25}\sqrt{6}\pi_0\langle U^\dagger\chi U^\dagger\chi-U\chi^\dagger U\chi^\dagger\rangle+\cdots,\end{aligned} \tag{15}$$

where only the terms relevant to this work are included. In the presence of the $SU(3)$ singlet axial vector source field a^0_μ, $D_\mu\pi_0=\partial_\mu\pi_0-F_0a^0_\mu$. At this point it is important to note that the singlet axial vector current has nonvanishing anomalous dimension [20], which implies that some low energy constants (LECs) as well as the singlet field π_0 must be renormalized (the corresponding renormalized quantities will thus depend on the QCD renormalization scale μ_{QCD}) [14,15]. Since the renormalization of the axial vector current is subleading in $1/N_c$, such dependence appears first at the level of $\mathcal{L}^{(4)}$ LECs. It has been found in particular that the LECs Λ_1, Λ_2, L_{18} and L_{25}, all of which are subleading in $1/N_c$, must be renormalized [14], implying that the values of these LECs will depend on the value of the scale μ_{QCD}. Other quantities such as $F_{\eta'}$ and the singlet π_0 field must be renormalized as well and depend on μ_{QCD} through the renormalization factor Z_A associated with the singlet axial current. It is very convenient to make use of the asymptotic freedom of QCD to set μ_{QCD} arbitrarily large and give the values of the LECs in that limit. Indeed, the renormalization factor of the axial vector current Z_A evolves to a fixed point that can

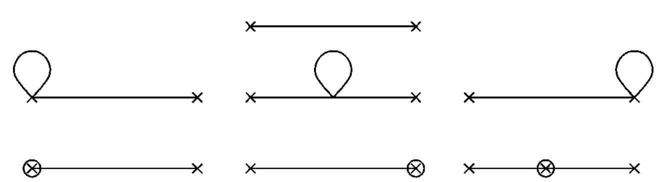

FIG. 1. Two-point function of axial vector currents to NLO. The last three diagrams involve the counterterm insertions from $\mathcal{L}^{(4)}$.

[2]Throughout, the meson masses used are those with EM contributions subtracted.

be taken to be $Z_A=1$ as $\mu_{QCD}\to\infty$. All quantities given in the following that have a dependence on μ_{QCD} are then taken in this limit. It is well known that the low energy constants L_5 and L_8 are $\mathcal{O}(N_c)$, while L_4 and L_6 are subleading and $\mathcal{O}(N_c^0)$ [18] and are needed in order to renormalize the one-loop contributions from $\mathcal{L}^{(2)}$. With η' as an explicit degree of freedom, L_7 is also subleading in $1/N_c$. The renormalized pieces of the subleading LECs are therefore set to zero at the chosen chiral renormalization scale μ in our analysis. On the other hand, the LECs Λ_1 and Λ_2 are $\mathcal{O}(1/N_c)$ and the corresponding terms in the Lagrangian are $\mathcal{O}(p^2/N_c)=\mathcal{O}(p^4)$ and thus must be included in the calculation. The low energy constant Λ_1 provides an $\mathcal{O}(1/N_c)$ correction to the η' decay constant, while both Λ_1 and Λ_2 affect entries in the mass matrix involving the η' at order $\mathcal{O}(p^2/N_c)$. Finally, the terms involving the LECs L_{18} and L_{25} are of $\mathcal{O}(p^4/N_c)$ which is beyond the order of the present calculation and therefore their renormalized pieces are set to vanish as well. The renormalized LECs are defined in the usual $\overline{\text{MS}}$ renormalization scheme [18]:

$$L_i=L_i^r(\mu)+\Gamma_i\lambda(\mu),\qquad \Lambda_i=\Lambda_i^r(\mu)+\Delta_i\lambda(\mu)$$
$$\lambda(\mu)=\frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4}-\frac{1}{2}[\log 4\pi+1+\Gamma'(1)]\right). \tag{16}$$

The β functions Γ_i associated with the LECs L_i, and Δ_i associated with Λ_i that result from the chiral one-loop calculation are given by [13–15]: $\Gamma_4=1/8$, $\Gamma_5=3/8$, $\Gamma_6=1/16$, $\Gamma_7=0$, $\Gamma_8=3/16$, $\Gamma_{18}=-1/4$, $\Gamma_{25}=0$, $\Delta_1=-1/8$, and $\Delta_2=3/8$. The two-point functions of axial vector currents can be written in momentum space as

$$\int d^4x e^{ip\cdot x}\langle 0|T(A_\mu^a(x)A_\nu^b(0))|0\rangle$$
$$=p_\mu p_\nu\sum_{\bar{a}} F^T_{b\bar{a}}(p^2)\Delta_{\bar{a}}(p^2)F_{\bar{a}a}(p^2)+\cdots, \tag{17}$$

where the term explicitly shown contains the light pseudoscalar poles and $\Delta_{\bar{a}}(p^2)$ is the propagator of the corresponding mass eigenstate. From the location of the low energy poles and the residues of the two-point function the light pseudoscalar decay constants and masses are extracted, yielding

$$M^2_{ab}=M^2_{LO\,ab}-\left(\sigma_{CT}+\sigma_{loop}-\frac{1}{2F_0}\left\{M^2_{LO},\phi_{CT}+\frac{1}{2}\phi\right\}\right)_{ab},\qquad a,b=3,8,0 \tag{18}$$

$$M^2_a=M^2_{LO\,a}-\left((\sigma_{CT}+\sigma_{loop})_a-\frac{1}{F_0}M^2_{LO\,a}\left(\phi_{CT}+\frac{1}{2}\phi\right)_a\right),\qquad a\neq 3,8,0.$$

The corresponding decay constants are given by

$$F_{\bar{a}a}=\Theta_{\bar{a}b}F_{ba}$$
$$F_{ab}=F_{ba}=F_0\left(\delta_{ab}-\frac{1}{2}\left(\phi_{CT}+\frac{3}{2}\phi\right)_{ab}\right),\qquad a,b=3,8,0 \tag{19}$$
$$F_a=F_0-\frac{1}{2}\left(\phi_{CT}+\frac{3}{2}\phi\right)_a,\qquad a\neq 3,8,0$$
$$\phi_a\equiv\phi_{aa},\quad \text{etc., when } a\neq 3,8,0,$$

where the following definitions were made:

$$\phi_{CT\,ab}=-\left(\frac{4B_0L_5^r(\mu)}{F_0}\langle\{\lambda^a,\lambda^b\}\mathcal{M}_q\rangle+\Lambda_1\delta_{a0}\delta_{b0}\right)$$
$$\phi_{ab}=-\frac{1}{12F_0}\left(\sum_{c,d=3,8,0}\gamma^{abcd}\Theta^T_{c\bar{a}}\mu_{\bar{a}}\Theta_{\bar{a}d}+\sum_{c\neq 3,8,0}\gamma^{abcc}\mu_c\right)$$
$$\sigma_{CT\,ab}=-\frac{8B_0^2L_8^r(\mu)}{F_0^2}\langle\{\mathcal{M}_q,\lambda^a\}\{\mathcal{M}_q,\lambda^b\}\rangle-2\sqrt{\frac{2}{3}}\Lambda_2B_0(\delta_{a0}\langle\lambda^b\mathcal{M}_q\rangle+\delta_{b0}\langle\lambda^a\mathcal{M}_q\rangle)$$
$$\sigma_{ab}=\frac{1}{24F_0^2}\left(\sum_{c,d=3,8}\gamma^{abcd}\Theta^T_{d\bar{a}}\mu_{\bar{a}}M^2_{\bar{a}}\Theta_{\bar{a}d}+\sum_{c\neq 3,8}\gamma^{abcc}\mu_cM_c^2\right)+\frac{B_0}{24F_0^2}\left(\sum_{c,d=3,8,0}\mathcal{M}^{abcd}\Theta^T_{c\bar{a}}\mu_{\bar{a}}\Theta_{\bar{a}d}+\sum_{c\neq 3,8,0}\mathcal{M}^{abcc}\mu_c\right), \tag{20}$$

and

$$\mu_{\bar{a}}=\frac{1}{16\pi^2}M^2_{\bar{a}}\log\frac{M^2_{\bar{a}}}{\mu^2}$$
$$\gamma^{abcd}=\langle[\lambda^a,\lambda^c][\lambda^b,\lambda^d]\rangle \tag{21}$$
$$\mathcal{M}^{abcd}=\frac{1}{2}\sum_{perm\{\sigma\}}\langle\mathcal{M}_q\lambda^{\sigma_a}\lambda^{\sigma_b}\lambda^{\sigma_c}\lambda^{\sigma_d}\rangle.$$

Throughout, the terms whose renormalized LECs are set to vanish have not been displayed explicitly. It is interesting to note that ϕ_{ab} does not receive any loop contributions from the singlet pseudoscalar mode, implying that F_{ab} is also free of such contributions.

It is useful at this point to give the explicit expressions of the masses and decay constants at NLO disregarding the chiral logarithms. For the decay constants the above expressions lead to

$$F_{\pi^+}=F_0+\frac{4L_5B_0}{F_0}(m_u+m_d) \tag{22}$$

$$F_{K^+}=F_0+\frac{4L_5B_0}{F_0}(m_u+m_s)$$

$$F_{33}=F_{\pi^+}$$

$$F_{88}=F_0+\frac{4L_5B_0}{3F_0}(m_u+m_d+4m_s)$$

$$F_{00}=F_0\left(1+\frac{\Lambda_1}{2}\right)+\frac{8L_5B_0}{3F_0}(m_u+m_d+m_s)$$

$$F_{38}=\frac{4L_5B_0}{\sqrt{3}F_0}(m_u-m_d)$$

$$F_{30}=\sqrt{2}F_{38}$$

$$F_{80}=\frac{\sqrt{32}L_5B_0}{3F_0}(m_u+m_d-2m_s)$$

while, for the masses,

$$M^2_{\pi^+}=2B_0\hat{m}+\frac{32(2L_8-L_5)}{F_0^2}B_0^2\hat{m}^2$$

$$M^2_{K^+}=B_0(m_u+m_s)+\frac{8(2L_8-L_5)}{F_0^2}B_0^2(m_u+m_s)^2$$

$$M^2_{K^0}=B_0(m_d+m_s)+\frac{8(2L_8-L_5)}{F_0^2}B_0^2(m_d+m_s)^2$$

$$M^2_{33}=2B_0\hat{m}+\frac{16(2L_8-L_5)}{F_0^2}B_0^2(m_u^2+m_d^2)$$

$$M^2_{88}=\frac{2}{3}B_0(\hat{m}+2m_s)+\frac{16(2L_8-L_5)}{3F_0^2}\times B_0^2(m_u^2+m_d^2+4m_s^2) \tag{23}$$

$$M^2_{00}=M_0^2(1-\Lambda_1)+\frac{32(2L_8-L_5)}{3F_0^2}\times B_0^2(m_u^2+m_d^2+m_s^2)+\frac{2}{3}(1+\rho)\times B_0(m_u+m_d+m_s)$$

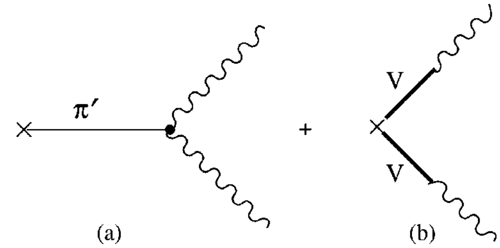

FIG. 2. Hadronic contributions to t_1: π' denotes excited pseudoscalar mesons and V denotes vector mesons.

$$M^2_{38}=\frac{1}{\sqrt{3}}B_0(m_u-m_d)+\frac{16(2L_8-L_5)}{\sqrt{3}F_0^2}B_0^2(m_u^2-m_d^2)$$

$$M^2_{30}=-\sqrt{\frac{2}{3}}\left(1+\frac{\rho}{2}\right)B_0(m_d-m_u)+\frac{16(2L_8-L_5)}{F_0^2}\sqrt{\frac{2}{3}}B_0^2(m_u^2-m_d^2)$$

$$M^2_{80}=\frac{\sqrt{2}}{3}\left(1+\frac{\rho}{2}\right)B_0(m_u+m_d-2m_s)+\frac{16(2L_8-L_5)}{F_0^2}\frac{\sqrt{2}}{3}B_0^2(m_u^2+m_d^2-2m_s^2),$$

where $\rho\equiv-\Lambda_1+2\Lambda_2-8L_5(M_0^2/F_0^2)$.

The second class of NLO corrections can be grouped into a single term contained in the $\mathcal{O}(p^6)$ odd-intrinsic parity WZ Lagrangian [12]:

$$\mathcal{L}^{(6)\gamma\gamma}_{WZ}=-i\pi\alpha t_1\langle\chi_-Q^2\rangle F\tilde{F}$$

$$\text{where}\quad \chi_-=u^\dagger\chi u^\dagger-u\chi^\dagger u\quad\text{with}\quad u=\sqrt{U}. \tag{24}$$

(There exists a second term [12] that, upon the explicit inclusion of the singlet pseudoscalar meson, becomes subleading in $1/N_c$ and is therefore neglected.) The low energy constant t_1 has vanishing β function and its value can be estimated by means of a QCD sum rule for the general three-point function involving the pseudoscalar density and two vector currents and saturating the spectral function in the hadronic sector with the states indicated in Fig. 2, yielding [12,21]

$$t_1=-\frac{1}{m_V^4}\left(F_0^2+\frac{\tau}{M^2_{\pi'}}\right). \tag{25}$$

Here the F_0^2 contribution is determined by the masses and decay constants of the vector mesons (the vector meson mass is naturally taken to be $m_V\simeq m_\rho$) and is represented by Fig. 2(b), while the contribution proportional to τ is determined by excited pseudoscalars, such as the $\pi'(1300)$, and is represented by Fig. 2(a). This latter contribution can be estimated within a model [12,22] and its expected size is at most one third of the magnitude of the vector meson contribution. Since this is similar to the level of uncertainty expected in the sum rule evaluation, the τ piece will be disregarded henceforth. As shown by the numerical analysis below, the effects on the π^0 width due to the $\mathcal{L}_{WZ}^{(6)\gamma\gamma}$ with t_1 as estimated above are of similar magnitude to the rest of the NLO corrections and in the range of 0.5%. At this point the NLO two-photon amplitudes can be explicitly given:

$$\langle\gamma\gamma|\pi_a\rangle = -i\alpha\left(\frac{N_c}{12\pi}C_aF_{aa}^{-1} + \pi\frac{B_0}{F_0}t_1\Theta_{\overline{a}a}\langle\{\lambda^a,\mathcal{M}_q\}Q^2\rangle\right)\times\langle\gamma\gamma|F\tilde{F}|0\rangle \tag{26}$$

and the term proportional to t_1 can be obtained in two equivalent ways, either by determining the contribution from $\mathcal{L}_{WZ}^{(6)}$ to the matrix elements

$$\langle\gamma\gamma|\overline{q}\gamma_5\{\lambda^a,\mathcal{M}_q\}q|0\rangle$$

in Eq. (3), or by directly calculating the contribution to $\langle\gamma\gamma|\pi_a\rangle$ due to that effective Lagrangian. Note that the contribution from $\mathcal{L}_{WZ}^{(6)}$ has the same scaling in N_c as the leading one. In Eq. (26) the factor $B_0\langle\{\lambda^a,\mathcal{M}_q\}Q^2\rangle$ can be expressed using the LO mass formulas, namely

$$B_0\langle\{\lambda^a,\mathcal{M}_q\}Q^2\rangle = \begin{cases}\frac{1}{9}(3M_\pi^2+5(M_{K^+}^2-M_{K^0}^2)), & a=3,\\ \frac{1}{9\sqrt{3}}(7M_\pi^2+M_{K^+}^2-5M_{K^0}^2), & a=8,\\ \frac{1}{9}\sqrt{\frac{8}{3}}(2M_\pi^2+2M_{K^+}^2-M_{K^0}^2), & a=0.\end{cases} \tag{27}$$

At NLO the extraction of the ratio $R=m_s/(m_d-m_u)$ that characterizes the size of isospin breaking should be improved by including NLO corrections to Dashen's theorem. Over time several works have shown that the corrections are sizeable. From the early works of Donoghue, Holstein and Wyler [23] and Bijnens [24], and more recent works [25], it is well established that the mass difference $M_{K^0}-M_{K^+}$ left after subtracting the EM contribution is larger than the one predicted by Dashen's theorem. While Dashen's theorem predicts $M_{K^0}-M_{K^+}=5.25$ MeV, after the corrections have been implemented it is estimated that $M_{K^0}-M_{K^+}=6.97$ MeV. The mass difference is slightly smaller if the chiral logarithms are neglected following the approach of this work. In such a case $M_{K^0}-M_{K^+}=6.47$ MeV. These corrections to Dashen's theorem translate naturally into a smaller value for R, as shown in the analysis that follows. There is additional evidence that R is overestimated by applying Dashen's theorem, and that comes from the decay $\eta\rightarrow\pi^+\pi^-\pi^0$. At lowest order the decay rate, which is proportional to $(m_d-m_u)^2$, is found to be a mere 66 eV [26], which is about a factor of four smaller than the experimental value of 281 ± 28 eV. At NLO in the chiral expansion the rate is increased to 167 ± 50 eV [18], while dispersive analyses [27,28] give 209 ± 56 eV, which is still substantially below the experimental value. Clearly one way to make up for the difference is to increase m_d-m_u. Because of the large uncertainties in both the experimental and theoretical side it is difficult to be precise, but it seems that the increment implied by the violations to Dashen's theorem mentioned above is in line with the enhancement required to explain the observed $\eta\rightarrow\pi^+\pi^-\pi^0$ width. In principle a precise measurement of the π^0 width could provide an independent determination of R. However, as shown by the results of the present analysis, the π^0 width is affected by the corrections to Dashen's theorem only at the level of 0.5%, which is unfortunately well below the experimental error of 1.4% aimed at by PRIMEX and about the same as the 0.6% uncertainty due to the 0.3% error in the experimental value of F_{π^+} [3].

There are nine low energy constants to be determined—F_0, B_0m_i $(i=u,d,s)$, M_0, L_5, L_8, Λ_1, and Λ_2 and these can be found by solving for the observables: $F_{\pi^+}=92.42\pm0.25$ MeV, $F_{K^+}=113.0\pm1.6$ MeV, $M_{\pi^0}=134.976$ MeV, $M_\eta=547.30$ MeV, $M_{\eta'}=957.78$ MeV, $M_{K^0}=497.78$ MeV, $M_{K^0}-M_{K^+}$ [which, as mentioned before, is 5.25 MeV at LO, while at NLO and disregarding (including) the chiral logarithms in the corrections to Dashen's theorem is 6.47 MeV (6.97 MeV), $\Gamma_{\eta\rightarrow\gamma\gamma}=464\pm45$ eV, and $\Gamma_{\eta'\rightarrow\gamma\gamma}=4.28\pm0.34$ keV]. Note that the tenth LEC t_1 cannot at this stage be extracted phenomenologically and thus its value is taken according to the estimate made above.

In Table I the second column displays the LO results, and the next three columns display three different NLO fits: namely,

(i) NLO No. 1 includes terms of $\mathcal{O}(p^6)$ and $\mathcal{O}(p^4/N_c)$ in the decay amplitude—i.e. chiral logarithms are omitted.

(ii) NLO No. 2 includes chiral logarithms, which are $\mathcal{O}(p^6/N_c)$, and the renormalization scale μ is set equal to M_η.

(iii) NLO No. 3 is identical to NLO No. 1 but sets $t_1=0$—i.e. excludes the $\mathcal{O}(p^6)$ WZ contributions.

It should be noted that the mass of the η' in the chiral limit and at NLO in $1/N_c$ is given by $\sqrt{1-\Lambda_1}M_0\sim940$ MeV, which is slightly high leaving not enough room for the piece linear in the quark masses. This linear contribution is suppressed by the rather large value of Λ_2 which leads to a very small value of $1+\rho$. This cancellation between the leading and subleading in $1/N_c$ contributions seems to indicate some difficulty with the $1/N_c$ expansion for the masses. The first manifestation of this problem is of course in the problem found with the η and η' masses at LO. Although this problem has a minor impact on the π^0 width, it certainly deserves further study. It should be noted that the mass difference $M_{\pi^+}-M_{\pi^0}$ has been given as input in fit No. 2 since its value emerges as too large if left unconstrained. The reason why it is too large can be traced back to the chiral logarithms generated by the η'. It seems, there-

TABLE I. Results for the LECs for several different fits, the LO fit and three NLO fits as explained in the text. The LECs that depend of the QCD renormalization scale μ_{QCD} correspond to the limit $\mu_{QCD}\to\infty$. Although all the LECs are the renormalized ones, only those in the NLO fit No. 2 depend on the chiral renormalization μ that is taken to be equal to M_η.

LEC	LO Fit	NLO No. 1	NLO No. 2	NLO No. 3
F_0 (MeV)	92.5 ± 0.6	90.73 ± 0.47	84.54 ± 0.85	90.69 ± 0.38
M_0 (MeV)	848 ± 40	1047 ± 5	1142 ± 33	1044 ± 4
$2B_0\hat{m}$ (GeV^2)	0.0366 ± 0.0001	0.03656 ± 0.00001	0.0362 ± 0.0002	0.03648 ± 0.00001
$B_0(m_d-m_u)$ (GeV^2)	0.0235 ± 0.0006	0.0237 ± 0.0002	0.0245 ± 0.0006	0.0244 ± 0.0002
B_0m_s (GeV^2)	0.236 ± 0.006	0.2349 ± 0.0006	0.197 ± 0.002	0.2291 ± 0.0005
$2L_5+L_8$	0	$(5.26\pm0.01)\times10^{-3}$	$(6.3\pm1.1)\times10^{-3}$	$(5.44\pm0.07)\times10^{-3}$
$2L_8-L_5$	0	$(0.8\pm0.9)\times10^{-5}$	$(-0.53\pm0.04)\times10^{-3}$	$(0.11\pm0.01)\times10^{-3}$
Λ_1	0	0.19 ± 0.01	0.29 ± 0.04	0.209 ± 0.006
Λ_2	0	0.74 ± 0.02	1.4 ± 0.4	0.81 ± 0.02

fore, that requiring the subleading renormalized LECs to vanish is not such a good approximation when such chiral logarithms are included.

Table II lists the associated predictions for various quantities of interest, in particular the π^0 width. It is evident from the NLO fits that $\Gamma_{\pi^0\to\gamma\gamma}$ is rather stable and always within 1% of the leading order result, which is within the expected range of the NLO corrections. As shown by comparison of the first and third NLO fits, the correction from the $\mathcal{O}(p^6)$ WZ Lagrangian $\mathcal{L}_{WZ}^{(6)\gamma\gamma}$ reduces the π^0 width by 0.5%, a magnitude in line with the fact that such a correction is controlled by the ratio $m_{u,d}/\Lambda_\chi$. The chiral-logarithm contributions to the amplitudes as shown by fit No. 2, provide an increase of order 0.5% to the π^0 width. Since these are subleading corrections of $\mathcal{O}(p^6/N_c)$ to the decay amplitude, they are somewhat larger than the 0.2–0.3 % expected from the ratio $m_{u,d}/(N_c\Lambda_\chi)$ that determines them. This problem is similar to the one with the pion mass difference just mentioned; in this case the η' loops affect the mixing angles producing a larger than expected correction to the rate. Indeed, turning off the chiral logarithms generated by the η' the π^0 width is essentially identical to the result in fit No. 1.

It is important to note that the mixing angles are substantially modified at NLO: in particular, the $\pi^0-\eta$ mixing angle is found to be $\epsilon\sim\theta_3+\theta_8=0.8°-0.9°$ in the three fits, which is $\sim$10–20 % smaller than the LO result of 1°, but still larger than that obtained at LO within $SU(3)$ and given in Eq. (13). The latter is chiefly a consequence of the corrections to Dashen's theorem. The $\pi^0-\eta'$ mixing angle $\tilde{\epsilon}$ goes from 0.5° at LO to approximately 0.3° at NLO; finally, the $\eta-\eta'$ mixing angle is dramatically reduced to about $-10°$ from its LO value of $-18.6°$. In view of these substantial corrections to the mixing angles, the stability of the π^0 width is nontrivial: besides the corrections due to the $\mathcal{O}(p^6)$ WZ Lagrangian, the decay amplitude is determined by the decay constant matrix $F_{\bar{a}a}$, which is affected by the mixing of states as well as by the NLO corrections contained in the decay constant matrix F_{ab} given in Eq. (19). It turns out that the entries in $F^{-1}_{a\bar{a}}$—namely $F^{-1}_{a\pi^0}$—affecting the π^0 amplitude remain stable well within the natural size of the NLO corrections.

In order to assess the theoretical uncertainty of the analysis of the π^0 width, an estimate of the magnitude of EM corrections beyond the ones taken into account by Dashen's theorem should also be given. Such corrections are of order $\alpha/2\pi$, which puts them in the 0.2–0.3 % range. Note also that the value of F_π being used is that of F_{π^+} which has an EM correction. This correction can be estimated with the results from Refs. [25,29] and is given by $\delta_{\rm EM}F_\pi\sim\kappa4\pi\alpha F_0$ where the low energy constants that determine the coefficient κ can be estimated in a resonance saturation

TABLE II. Results implied by the different fits displayed in Table I. The ⋆ indicates that the quantities are inputs.

	LO Fit	NLO No. 1	NLO No. 2	NLO No. 3
$\Gamma_{\pi^0\to\gamma\gamma}$ (eV)	8.08	8.10	8.16	8.14
$\Gamma_{\eta\to\gamma\gamma}$ (eV)	565	464 ⋆	464 ⋆	464 ⋆
$\Gamma_{\eta'\to\gamma\gamma}$ (keV)	5.1	4.28 ⋆	4.28 ⋆	4.28 ⋆
$M_{\pi^+}-M_{\pi^0}$ (MeV)	0.32	0.24	0.16 ⋆	0.21
$m_s/\hat{m}$	25.9	25.7	21.7	25.1
$R=m_s/(m_d-m_u)$	45.3	36.6	30.9	37.5
θ_3 (deg)	1.57	1.51	1.88	1.40
θ_8 (deg)	-0.56	-0.68	-0.94	-0.59
θ_0 (deg)	-18.6	-10.6	-8.7	-12.2

model. There are some disagreements between [25] and [29] on the size of the low energy constants, the latter quoting substantially smaller values. Taking this into account it is estimated that $|\kappa|\sim 10^{-2}$, thus leading to the estimate $|\delta_{\rm EM}F_\pi|\sim 0.1$ MeV which is within the experimental uncertainty in the value of F_{π^+}, and implies a correction to the π^0 width within the range mentioned above. An analysis of EM corrections recently carried out by Ananthanarayan and Moussallam [16] seems to give a slightly larger correction with definite sign, namely $\delta_{\rm EM}F_{\pi^0}\simeq 0.3$ MeV, while EM corrections to mixings are found to be much smaller. Their full analysis implies that the overall effect of EM corrections not taken into account in the current analysis amount to a reduction of the π^0 width by 0.6%. This result implies that the magnitude of those EM corrections turns out to be similar to the error induced by the uncertainty in the ratio R of quark masses, and smaller by about a factor of two than the natural size of NLO chiral corrections driven by the strange quark mass. The inclusion of the EM corrections from [16] leads in Fit No. 1 to $\Gamma_{\pi^0\to\gamma\gamma}=8.05$ eV.

Although the $\eta-\eta'$ complex is not the primary focus of this work, the analysis carried out illuminates crucial aspects of this system. At LO the description is rather poor, in particular because the two-photon widths depart quite substantially from the experimental world averages. Indeed, $\Gamma^{LO}_{\eta\to\gamma\gamma}=613$ eV versus the world average experimental value of 464 ± 45 eV, and $\Gamma^{LO}_{\eta'\to\gamma\gamma}=4.86$ keV vs 4.28 ± 0.34 keV. These latter disagreements are mostly due to the large $\eta-\eta'$ mixing angle that results from the LO mass formulas, and the fact that at LO all decay constants are set to be equal to F_π. At NLO the scheme that emerges is the one already found in other works [14], where the mixing angle of the pure $U(3)$ states is in the proximity of $-10°$ rather than the $-20°$ obtained in LO, and where the decay constant matrix can be parametrized by means of two angles and two decay constants [14,30]. Following the conventions and notation in [14], the present analysis gives (quantities are denoted in boldface not to be confused with quantities defined heretofore in the text): $\mathbf{F_0}\simeq 116$ MeV, $\mathbf{F_8}\simeq 122$ MeV, $\boldsymbol{\theta_8}\simeq -20°$ $\boldsymbol{\theta_0}=-2.5°$ to $0.5°$, and the angles $\boldsymbol{\theta_0}$ and $\boldsymbol{\theta_8}$ for the three NLO fits are respectively $(-0.9°,-19.9°)$, $(2.0°,-19.0°)$ and $(-2.5°,-21.5°)$. The difference $\boldsymbol{\theta_0}-\boldsymbol{\theta_8}$ turns out to be between 19° and 21°, to be compared with the 19° obtained in [14] (in a NLO estimate in that reference a value of 14° is obtained, which departs substantially from the one of the current analysis). There exist numerous studies of the $\eta-\eta'$ complex. It makes sense only to compare results with those using the two-angle scheme [31]. Although some of the analyses in these references are purely phenomenological, these results are in general in good agreement with the results obtained in this work.

The quark mass ratios obtained in the different fits deserve comment. The ratio $m_s/\hat{m}$ is in good agreement with the standard value 24.3 ± 1.2 obtained in SU(3) [32]; in fit No. 2 it is a couple of standard deviations smaller, most likely because the chiral logarithms included do not represent the full NNLO contributions. In all, this is not surprising as the assumption that the low energy constants that are subleading in $1/N_c$ can be disregarded is one of the important assumptions in the extraction of the standard ratios. However, a comment is in order: it is observed that using LO mass formulas in the NLO results—i.e., expressing quark masses in terms of meson masses squared in the NLO terms — leads to a fit that is less stable and generates large corrections to the quark mass ratios. The ratio R is smaller here than the standard value 42.3 ± 4.5, and this is simply because the corrections to Dashen's theorem have been included. The values for R in fits No. 1 and No. 3 are about one standard deviation smaller than the standard value, while in No. 2 the chiral logarithms involving the η' loops give a substantial reduction (when these are turned off R is similar to the result in the other fits). An interesting observation is that setting $m_u=0$ leads to an inconsistent fit and a value for the π^0 width of 8.5 eV.

V. CONCLUSIONS

The decay rate for $\pi^0\to\gamma\gamma$ has been calculated within a combined chiral and $1/N_c$ expansion. At leading order in the expansion, the isospin-breaking induced mixing of the pure $U(3)$ states increases the size of $\Gamma_{\pi^0\to\gamma\gamma}$ from the value 7.725 eV predicted by the lowest order chiral anomaly by more than 4.5%. This effect is largely due to the fact that the contributions from mixing with the π_8 and π_0 add constructively, and are of similar magnitude. However, at LO the resulting $\eta,\eta'\to\gamma\gamma$ widths are found to be too large, and in general the fit is quite poor. There is a clear need then for the NLO calculation, both in order to improve the results in the $\eta-\eta'$ sector and to test the stability of the enhancement of $\Gamma_{\pi^0\to\gamma\gamma}$ observed at LO. The NLO calculation reveals that the LO result for $\Gamma_{\pi^0\to\gamma\gamma}$ is quite robust, being modified by the NLO corrections by less than 1%. This stability is, however, nontrivial. Indeed, as already noted, at NLO the mixing angles are substantially affected—the mixing angles ϵ and $\tilde{\epsilon}$ are reduced by 10 to 30% (a more dramatic reduction of roughly 50% results for the mixing angle θ_0). The π^0 width, however, is only slightly affected because the effects that ultimately determine the corrections to the amplitude are in the decay constant matrix $F_{a\bar{a}}$ shown in Eq. (17). This matrix is affected by the mixing, and also receives NLO corrections that reside in F_{ab}, and apparently the NLO modifications to the mixings are partially compensated by the latter corrections in the case of the entries relevant to the π^0.

The primary source of theoretical uncertainty in the present calculation of $\Gamma_{\pi^0\to\gamma\gamma}$ resides in the value of R, which has an uncertainty of about 15%. Using the empirical formula resulting from the results in Table II, $\Gamma_{\pi^0\to\gamma\gamma}\sim(7.725+14.1/R)$ eV, the uncertainty in R translates into an error in the π^0 width of 0.6%. Other sources of uncertainty are the NNLO corrections, of which the chiral logs are an example, and which should be expected to be in the range of 0.2–0.3%, and also EM corrections which according to a straightforward order of magnitude analysis are in similar range (more precisely, according to [16], they should give a -0.6% reduction of the width). Considering these uncertainties in quadrature, the theoretical uncertainty in the prediction of $\Gamma_{\pi^0\to\gamma\gamma}$ turns out to be about 1%. Note that there is an overall uncertainty in $\Gamma_{\pi^0\to\gamma\gamma}$ due to the 0.3% error in F_{π^+}. It is noted that the result for the π^0 rate obtained here is in excellent agreement with the recent result of Ref. [16]. The fact that the theoretical prediction for $\Gamma_{\pi^0\to\gamma\gamma}$ shows a 4.5% enhancement that can be experimentally observed, and the fact that the experimental result with the smallest quoted error [4] lies more than three standard deviations below that prediction lend great significance to the upcoming PRIMEX measurement.

It is evident from the above analysis that no predictions for the η and η' two-photon widths can be made. Rather, these quantities are inputs, and their precise values do not affect in any dramatic way the π^0 width. In a more complete study, wherein the analysis is extended to additional processes such as $\eta\to\pi^+\pi^-\pi^0$, a more precise experimental knowledge of such widths would be necessary. A more extensive analysis would also illuminate the $1/N_c$ corrections encoded in the LECs Λ_1 and Λ_2 and help determine them more precisely.

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