$N=2$ Wess-Zumino model on the $d=2$ Euclidean lattice

Kazuo Fujikawa
Department of Physics, University of Tokyo, Bunkyo-ku, Tokyo 113, Japan
(Received 9 August 2002; published 29 October 2002)

We examine the $N=2$ Wess-Zumino model defined on the $d=2$ Euclidean lattice in connection with a restoration of the Leibniz rule in the limit $a\to 0$ in perturbatively finite theory. We are interested in the difference between the Wilson and Ginsparg-Wilson fermions and in the effects of extra interactions introduced by an analysis of Nicolai mapping. As for the Wilson fermion, it induces a linear divergence to individual tadpole diagrams in the limit $a\to 0$, which is absent in the Ginsparg-Wilson fermion. This divergence suggests that a careful choice of lattice regularization is required in a reliable numerical simulation. As for the effects of the extra couplings introduced by an analysis of Nicolai mapping, the extra couplings do not completely remedy the supersymmetry breaking in correlation functions induced by the failure of the Leibniz rule in perturbation theory, though those couplings ensure the vanishing of vacuum energy arising from disconnected diagrams. Supersymmetry in correlation functions is recovered in the limit $a\to 0$ *with or without* those extra couplings. In the context of lattice theory, it may be properly said that supersymmetry does not improve ultraviolet properties but rather it is well accommodated in theories with good ultraviolet properties.

DOI: 10.1103/PhysRevD.66.074510 PACS number(s): 11.15.Ha, 11.10.Gh

I. INTRODUCTION

It is important to define the supersymmetric Wess-Zumino model [1] on the lattice in such a way that the nonrenormalization theorem [2,3] and consequently the absence of quadratic divergence is preserved. Since the absence of quadratic divergence arises from a subtle cancellation of bosonic and fermionic contributions, we have to ensure the precise (not approximate) Bose-Fermi cancellation. This task is not easy if one recalls that the Leibniz rule is generally broken on the lattice [4]. See Refs. [5–14] for the analyses of related issues.

Recently, it was suggested [15] that a perturbatively finite theory, if latticized, could preserve supersymmetry to all orders in perturbation theory in the sense that the supersymmetry breaking terms induced by the failure of the Leibniz rule become irrelevant in the limit $a\to 0$. It was demonstrated in Ref. [15] that this is in fact realized if one first renders the 4-dimensional Wess-Zumino model finite by applying the higher derivative regularization. A non-perturbative confirmation of this proposal has not been given yet, but we believe that a perturbative confirmation of the absence of quadratic divergence is a prerequisite for the nonperturbative analysis.[1] In Ref. [15], the Ginsparg-Wilson fermion [17–19] was utilized, which has a nice chiral property but at the same time introduces certain subtle aspects to the analysis [14].

A 2-dimensional reduction of the Wess-Zumino model, which exhibits $N=2$ supersymmetry, is finite perturbatively with qualifications to be specified later, and thus it provides a good testing ground of the suggestion made in Ref. [15], though the crucial issue of quadratic divergence cannot be studied in this model. In the present paper, we examine the $N=2$ Wess-Zumino model on the $d=2$ Euclidean lattice perturbatively and clarify several basic issues involved which may become relevant in an actual numerical simulation. First, we examine the use of the Wilson fermion instead of the Ginsparg-Wilson fermion since the Wilson fermion is much easier to handle numerically. However, the Wilson fermion induces a strong chiral symmetry breaking and thus it is important to see if this introduces any new aspect into the problem. The suggestion in Ref. [15] is based on making *all* the Feynman diagrams finite, namely, the cancellation of divergences among Feynman diagrams is not sufficient in general. If one applies this criterion to the present problem, we encounter one-loop level divergences in some of the individual Feynman diagrams for correlation functions though those divergences cancel among bosonic and fermionic contributions. (In disconnected vacuum diagrams, two-loop diagrams contain divergences.) In particular, the Wilson fermion introduces a linear divergence to individual tadpole diagrams in $d=2$, which is absent in the Ginsparg-Wilson fermion. The presence of linear divergences suggests that the lattice regularization is not arbitrary but needs to be a "well-behaved" one, which ensures the precise cancellation of these linear divergences among diagrams for $a\neq 0$, such as in a formulation with precise lattice supersymmetry in the free part of the Lagrangian.[2] If all the Feynman diagrams should be absolutely convergent, the latticization would enjoy more freedom to recover supersymmetry in the limit $a\to 0$.

The second issue analyzed is the role played by extra couplings introduced by an analysis of Nicolai mapping [5]. The Nicolai mapping in the present context suggests an appearance of exta interactions [7,11,12] which vanish if the Leibniz rule is satisfied on the lattice. Also these extra terms spoil the naive hypercubic symmetry on the lattice. Because

[1] It should be noted that the lattice in this context is introduced not to control the divergences but to make numerical and other nonperturbative analyses possible.

[2] The actual numerical analysis, for example, is simplest in the simplest form of the Lagrangian, but due care is required to ensure supersymmetry in the limit $a\to 0$.

of these novel features of the extra interactions, one may hope that these extra terms might remedy the failure of the Leibniz rule appearing in the remaining conventional interaction terms. We analyze this issue in the framework of perturbation theory. Our result shows that these extra terms do not completely remedy supersymmetry in correlation functions, which is broken by the failure of the Leibniz rule, at least in weak coupling perturbation theory, though those terms ensure the vanishing of vacuum energy arising from disconnected diagrams. As far as the correlation functions are concerned, supersymmetry is recovered in the limit $a \to 0$ *with or without* those extra couplings.

II. $N=2$ WESS-ZUMINO MODEL AND NICOLAI MAPPING

We start with a Lagrangian defined in terms of the Wilson fermion on the $d=2$ Euclidean lattice

$$\begin{aligned}\mathcal{L}=&\bar{\psi}(D_{(1)}+D_{(2)})\psi-m\bar{\psi}\psi-2g\bar{\psi}(P_+\phi P_++P_-\phi^\star P_-)\psi\\&-\phi^\star D_{(1)}^\dagger D_{(1)}\phi+F^\star F-m[F\phi+(F\phi)^\star]\\&-g[F\phi^2+(F\phi^2)^\star]+FD_{(2)}\phi+(FD_{(2)}\phi)^\star\\&+g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star\end{aligned} \tag{2.1}$$

where ψ is a two-dimensional Dirac spinor. Since we are interested in the $d=2$ model as a toy model for 4-dimensional theory, we choose the superpotential to be a specific form

$$W'(\phi)=m\phi+g\phi^2. \tag{2.2}$$

Here we defined

$$D_{(1)}\psi(x)\equiv\gamma^\mu\nabla_\mu^S\psi(x)=\gamma^\mu\frac{1}{2a}(\psi(x+a\hat{\mu})-\psi(x-a\hat{\mu})), \tag{2.3}$$

$$\begin{aligned}D_{(2)}\psi(x)&\equiv\sum_\mu\nabla_\mu^A\psi(x)\\&=\sum_\mu\frac{1}{2a}(\psi(x+a\hat{\mu})+\psi(x-a\hat{\mu})-2\psi(x)).\end{aligned}$$

We note the important property $\Sigma_x f(x)(\nabla_\mu^S g)(x) =-\Sigma_x(\nabla_\mu^S f)(x)g(x)$. Our Euclidean γ matrix convention is

$$(\gamma^\mu)^\dagger=\gamma^\mu,\quad \gamma_5^\dagger=\gamma_5,\quad P_\pm=\frac{1}{2}(1\pm\gamma_5). \tag{2.4}$$

When we have the operator $D_{(1)}^\dagger D_{(1)}$ in the bosonic sector, we adopt the convention to discard the 2×2 unit matrix. The terms

$$\mathcal{L}_{kin}=\bar{\psi}D_{(1)}\psi-\phi^\star D_{(1)}^\dagger D_{(1)}\phi+F^\star F \tag{2.5}$$

stand for the kinetic (Kahler) terms. The last two terms in Eq. (2.1) are the extra terms introduced by an argument of Nicolai mapping [7], while other terms are the naive lattice translation of the continuum theory except for the Wilson term and its superpartners. Namely, the terms

$$\mathcal{L}_W=\bar{\psi}D_{(2)}\psi+FD_{(2)}\phi+(FD_{(2)}\phi)^\star \tag{2.6}$$

stand for a naive supersymmetrization of the Wilson term, which induce a hard breaking of continuum chiral symmetry.

The last two terms in Eq. (2.1) vanish if the Leibniz rule for ∇_μ^S should hold on the lattice, namely, if $\phi^2(x)(\nabla_\mu^S\phi)(x)=\frac{1}{3}(\nabla_\mu^S\phi^3)(x)$. This fact suggests that the extra terms might remedy the supersymmetry breaking induced by the failure of the Leibniz rule in the remaining interaction terms. These extra terms also break the (hyper) cubic symmetry on the lattice.

The elimination of the auxiliary fields F and $F^\star$ in the starting Lagrangian gives

$$\begin{aligned}\mathcal{L}=&\bar{\psi}(D_{(1)}+D_{(2)})\psi-\bar{\psi}(P_+W''+P_-(W'')^\star)\psi\\&-\phi^\star D_{(1)}^\dagger D_{(1)}\phi-(D_{(2)}\phi)^\star D_{(2)}\phi+(W')^\star D_{(2)}\phi\\&+W'(D_{(2)}\phi)^\star-(W')^\star W'+W'(\nabla_1^S+i\nabla_2^S)\phi\\&+(W'(\nabla_1^S+i\nabla_2^S)\phi)^\star\end{aligned} \tag{2.7}$$

with $W'=m\phi+g\phi^2$ by noting

$$\sum_x \phi(x)\nabla_\mu^S\phi(x)=-\sum_x \nabla_\mu^S\phi(x)\phi(x)=0. \tag{2.8}$$

This Lagrangian agrees with the one introduced in Refs. [7,11,12] by an analysis of Nicolai mapping.[3]

The Nicolai mapping here is defined by

$$\begin{aligned}\xi_1&=(\nabla_1^S+D_{(2)})A-U-\nabla_2^S B,\\ \xi_2&=(-\nabla_1^S+D_{(2)})B-V-\nabla_2^S A\end{aligned} \tag{2.9}$$

with

$$\phi(x)=\frac{1}{\sqrt{2}}(A+iB),\quad U=\frac{1}{\sqrt{2}}(W'+(W')^\star),$$
$$V=\frac{1}{\sqrt{2}i}(W'-(W')^\star). \tag{2.10}$$

[3]Recently, the Nicolai mapping was extended to the $d=2$ Wess-Zumino model defined in terms of Ginsparg-Wilson operators, which makes chiral symmetry manifest [16].

If one uses the specific representation of γ matrices

$$\gamma^1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \gamma^2=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\qquad \gamma_5=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} \tag{2.11}$$

the Jacobian for the transformation from (ξ_1,ξ_2) to (A,B) precisely agrees with the determinant of the fermion operator in Eq. (2.7). The bosonic part of the Lagrangian is then written as

$$\sum_x \mathcal{L}_{boson}(x)=-\sum_x \frac{1}{2}[\xi_1^2(x)+\xi_2^2(x)] \tag{2.12}$$

and the partition function is given by

$$Z=\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\mathcal{D}B\,\exp\left[\sum_x \mathcal{L}(x)\right] =\int \mathcal{D}\xi_1\mathcal{D}\xi_2\,\exp\left\{-\sum_x \frac{1}{2}[\xi_1^2(x)+\xi_2^2(x)]\right\} \tag{2.13}$$

if one imposes universal (periodic) boundary conditions both on fermionic and bosonic variables. The vanishing of the vacuum energy is thus ensured by the Nicolai mapping even for $a\neq 0$. The partition function is reduced to

$$\mathrm{Tr}(-1)^{\hat{F}}\,\exp[-\beta\hat{H}] \tag{2.14}$$

if the continuum limit $a\rightarrow 0$ is well-defined. The presence of the Nicolai mapping would then ensure the degeneracy of bosonic and fermionic spectra of $\hat{H}$ in the continuum limit.

III. SUPERSYMMETRY TRANSFORMATION

One may define a lattice supersymmetry transformation parametrized by a constant Dirac-type Grassmann parameter $\bar{\epsilon}$ by

$$\begin{aligned}&\delta\bar{\psi}=-\bar{\epsilon}[P_-\phi+P_+\phi^\star]D_{(1)}-\bar{\epsilon}[P_-F^\star+P_+F]\\ &\delta\psi=0\quad \delta\phi=\bar{\epsilon}P_+\psi,\quad \delta\phi^\star=\bar{\epsilon}P_-\psi\\ &\delta F=\bar{\epsilon}P_-D_{(1)}\psi,\quad \delta F^\star=\bar{\epsilon}P_+D_{(1)}\psi.\end{aligned} \tag{3.1}$$

The supersymmetry transformation parametrized by a constant Dirac-type Grassmann parameter ϵ, which is treated to be independent of $\bar{\epsilon}$, is given by

$$\begin{aligned}&\delta\psi=-D_{(1)}[P_-\phi+P_+\phi^\star]\epsilon-[P_-F^\star+P_+F]\epsilon\\ &\delta\bar{\psi}=0\quad \delta\phi=\bar{\psi}P_+\epsilon,\quad \delta\phi^\star=\bar{\psi}P_-\epsilon\quad \delta F=\bar{\psi}D_{(1)}P_-\epsilon,\\ &\delta F^\star=\bar{\psi}D_{(1)}P_+\epsilon.\end{aligned} \tag{3.2}$$

Here we note that

$$\begin{aligned}&D_{(1)}^\dagger=-D_{(1)},\quad D_{(1)}\gamma_5+\gamma_5D_{(1)}=0,\\ &D_{(2)}^\dagger=D_{(2)},\quad D_{(2)}\gamma_5-\gamma_5D_{(2)}=0.\end{aligned} \tag{3.3}$$

Under the transformation (3.1), it can be confirmed that the kinetic term

$$\begin{aligned}\int\ \delta\mathcal{L}_{kin}=&\int\{-\bar{\epsilon}[P_-\phi+P_+\phi^\star]D_{(1)}-\bar{\epsilon}[P_-F^\star\\ &+P_+F]\}D_{(1)}\psi+\int\ \phi^\star\partial_\mu\partial_\mu\bar{\epsilon}P_+\psi\\ &+\int\ \phi\partial_\mu\partial_\mu\bar{\epsilon}P_-\psi+\int\ F^\star\bar{\epsilon}P_-D_{(1)}\psi\\ &+\int\ F\bar{\epsilon}P_+D_{(1)}\psi=0\end{aligned} \tag{3.4}$$

is in fact invariant. The mass term

$$\mathcal{L}_{mass}=-m\bar{\psi}\psi-mF\phi-mF^\star\phi^\star \tag{3.5}$$

and the Wilson term

$$\mathcal{L}_W=\bar{\psi}D_{(2)}\psi+FD_{(2)}\phi+(FD_{(2)}\phi)^\star \tag{3.6}$$

are also confirmed to be invariant under the above supersymmetry transformation by using the relation $D_{(1)}D_{(2)}=D_{(2)}D_{(1)}$.

The variation of the (conventional) interaction terms

$$\mathcal{L}_{int}=-2g\bar{\psi}(P_+\phi P_++P_-\phi^\star P_-)\psi-g[F\phi^2+(F\phi^2)^\star] \tag{3.7}$$

is given by

$$\begin{aligned}\int\ \delta\mathcal{L}_{int}=g\int\ \bar{\epsilon}[&-2P_-(D_{(1)}\phi)\phi\psi-2P_+(D_{(1)}\phi^\star)\phi^\star\psi\\ &+P_-(D_{(1)}\phi^2)\psi+P_+(D_{(1)}(\phi^\star)^2)\psi]\end{aligned} \tag{3.8}$$

where we used the relation $\Sigma_x f(x)(D_{(1)}g)(x)=-\Sigma_x(D_{(1)}f)(x)g(x)$. This variation of the interaction terms would vanish if the difference operator $D_{(1)}$ should satisfy the Leibniz rule $2(D_{(1)}\phi)(x)\phi(x)=(D_{(1)}\phi^2)(x)$. (The terms quadratic in ψ vanish by themselves.)

As for the $U(1)\times U_R(1)$ charges (and holomorphicity) analogous to those in the 4-dimensional theory [9], we may assign

$$\begin{aligned}&\phi=(1,1),\quad F=(1,-1),\quad P_+\psi=(1,0),\\ &P_-\psi=(-1,0),\quad m=(-2,0),\quad g=(-3,-1)\end{aligned} \tag{3.9}$$

for the terms appearing in the conventional formulation. Even for $m=g=0$, the Wilson term $\mathcal{L}_W$ in the Lagrangian violates the $U(1)$ symmetry.

As for the extra terms introduced by an argument of Nicolai mapping

$$\mathcal{L}_{extra}=g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+(g\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star \tag{3.10}$$

which break hypercubic symmetry, they are not invariant under the above supersymmetry transformation. Since the lattice supersymmetry transformation we defined above preserves transformation properties under the hypercubic symmetry, the supersymmetry variation of these extra terms do not mix with the variation of other terms. Those extra terms also break $U_R(1)$ symmetry. We reiterate that these extra terms vanish if the Leibniz rule should hold on the lattice.

IV. FEYNMAN RULES FOR PERTURBATIVE CALCULATIONS

It is interesting to examine to what extent the extra terms introduced by an argument of Nicolai mapping [7,11,12], namely the last two terms in $\mathcal{L}$ (2.1), are essential to maintain supersymmetry in perturbation theory. Although the extra terms, which break hypercubic symmetry, do not mix with other terms under the lattice supsersymmetry transformation in Eq. (3.1) as we already explained, the lattice supersymmetry transformation is not unique and thus we cannot *a priori* exclude the possible cancellation of supersymmetry breaking effects among the interaction terms.[4] Our assumption is that perturbative calculations are universal at least for a small coupling constant and independent of the specific definitions of lattice supersymmetry transformation. One starts with the free part of the Lagrangian

$$\mathcal{L}_0=\overline{\psi}(D_{(1)}+D_{(2)})\psi-\phi^\star D_{(1)}^\dagger D_{(1)}\phi+F^\star F-m\overline{\psi}\psi$$
$$-m[F\phi+(F\phi)^\star]+FD_{(2)}\phi+(FD_{(2)}\phi)^\star. \tag{4.1}$$

When we have the operator $D_{(1)}^\dagger D_{(1)}$ in the bosonic sector, we adopt the convention to discard the 2×2 unit matrix.

The propagators are given by

$$\langle\phi\phi^\star\rangle=\frac{1}{D_{(1)}^\dagger D_{(1)}+(-m+D_{(2)})^2},$$

$$\langle FF^\star\rangle=(-)\frac{D_{(1)}^\dagger D_{(1)}}{D_{(1)}^\dagger D_{(1)}+(-m+D_{(2)})^2},$$

$$\langle F\phi\rangle=\langle F^\dagger\phi^\dagger\rangle=(-)\frac{(-m+D_{(2)})}{D_{(1)}^\dagger D_{(1)}+(-m+D_{(2)})^2},$$

$$\langle\psi\overline{\psi}\rangle=\frac{-1}{D_{(1)}+D_{(2)}-m}=(-)\frac{-D_{(1)}+D_{(2)}-m}{D_{(1)}^\dagger D_{(1)}+(-m+D_{(2)})^2}, \tag{4.2}$$

and other propagators vanish. Here we have $D_{(1)}^\dagger=-D_{(1)}$. In the momentum representation, we have

$$\langle\phi\phi^\star\rangle=\frac{a^2}{(\sin ak_\mu)^2+(a\mathcal{M})^2},$$

$$\langle FF^\star\rangle=(-)\frac{(\sin ak_\mu)^2}{(\sin ak_\mu)^2+(a\mathcal{M})^2},$$

$$\langle F\phi\rangle=\langle F^\star\phi^\star\rangle=\frac{a(a\mathcal{M})}{(\sin ak_\mu)^2+(a\mathcal{M})^2}, \tag{4.3}$$

$$\langle\psi\overline{\psi}\rangle=\frac{ai\gamma^\mu\sin ak_\mu+a(a\mathcal{M})}{(\sin ak_\mu)^2+(a\mathcal{M})^2},$$

where

$$\mathcal{M}(ak_\mu)\equiv\sum_\mu\frac{1}{a}(1-\cos ak_\mu)+m. \tag{4.4}$$

The interaction terms for perturbative calculations are given by

$$\mathcal{L}_{int}=-2g\overline{\psi}(P_+\phi P_++P_-\phi^\star P_-)\psi-g[F\phi^2+(F\phi^2)^\star]$$
$$+g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star. \tag{4.5}$$

If one sets one of ϕ in the extra interaction terms (i.e., in the last two terms in $\mathcal{L}_{int}$) to be a constant

$$\phi(x)=\phi_0=\text{const} \tag{4.6}$$

then the extra terms go away by noting $\Sigma_x\phi(x)\nabla_\mu^S\phi(x)=0$. This means that the effects of the extra terms introduced by the Nicolai mapping do not appear in the one-loop diagrams if one sets the momenta of external lines at 0. In other words, only those diagrams where the momenta of external lines cannot be set to be zero are affected by those extra terms.

V. LOWER ORDER DIAGRAMS

We now examine several lower order diagrams for correlation functions in perturbation theory.[5] The theory in $d=2$ becomes more convergent in higher order diagrams, and one can confirm that the possible supersymmetry breaking effects in higher order diagrams become irrelevant in the limit $a\rightarrow 0$, provided that one-loop subdiagrams are properly treated. The one-loop diagrams are thus crucial in the analysis of supersymmetry. As for the disconnected vacuum diagrams, they shall be later analyzed separately.

[4]In fact, an exact Ward identity which is regarded as a part of supersymmetry is known to exist in the Lagrangian defined by the Nicolai mapping [11,12]. We shall analyze this identity later.

[5]A perturbative analysis of the 4-dimensional Wess-Zumino model with the Wilson fermion was performed in [8]. In the present $d=2$ model, the perturbation is based on $g/m\ll 1$.

A. Tadpole diagrams

The one-loop tadpole diagrams for the scalar ϕ consist of two diagrams; the first one is a scalar loop and the second is a fermion loop. The scalar loop contribution is given by

$$-2g\phi\langle F\phi\rangle = -2g\phi\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{a(a\mathcal{M})}{(\sin ak_\mu)^2+(a\mathcal{M})^2} = -2g\phi\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left(\frac{1}{a}\right)\frac{(a\mathcal{M}(k_\mu))}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2} \tag{5.1}$$

where we rescaled the integration variable in the last expression as

$$ak_\mu \rightarrow k_\mu \tag{5.2}$$

and defined

$$a\mathcal{M}(k_\mu)=\sum_\mu (1-\cos k_\mu)+am. \tag{5.3}$$

In the limit $a\rightarrow 0$ this diagram diverges as $\sim 1/a$, namely, linearly divergent. This divergence is worse than the divergence in continuum theory (and also in the lattice theory with the Ginsparg-Wilson fermion[6]), which is logarithmic.

The fermion loop contribution is given by

$$2g\phi \mathrm{Tr}\, P_+\langle\psi\bar{\psi}\rangle = 2g\phi\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\mathrm{Tr}\, P_+\frac{ai\gamma^\mu \sin ak_\mu + a(a\mathcal{M})}{(\sin ak_\mu)^2+(a\mathcal{M})^2} = 2g\phi\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left(\frac{1}{a}\right)\frac{(a\mathcal{M}(k_\mu))}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2} \tag{5.4}$$

which is precisely canceled by the scalar contribution (5.1) for a finite a. However, each diagram is linearly divergent due to the strong chiral symmetry breaking by the Wilson term. In a numerical simulation, one would need to choose the free part of the Lagrangian to be lattice supersymmetric, as in the present formulation, so that the cancellation of linear divergence is exact.[7] Alternatively, one may introduce an auxiliary regularization such as higher derivative regularization to make each diagram convergent and thus less sensitive to the parameter a in the limit $a\rightarrow 0$, as we discussed in 4-dimensional theory [15].

B. Induced ϕ^2 coupling

We have contributions from a scalar loop and a fermion loop. The scalar loop contribution is given by

$$\frac{1}{2!}(2g)^2\phi\langle F\phi\rangle\langle F\phi\rangle\phi = 2g^2\phi^2\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{a(a\mathcal{M})}{\sin^2(ak_\mu+ap_\mu)+(a\mathcal{M})^2}\frac{a(a\mathcal{M})}{(\sin ak_\mu)^2+(a\mathcal{M})^2} = 2g^2\phi(-p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{(a\mathcal{M}(k_\mu+ap_\mu))}{\sin^2(ak_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\frac{(a\mathcal{M}(k_\mu))}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2} \tag{5.5}$$

which approaches a constant for $a\rightarrow 0$. This behavior is consistent with the continuum behavior, but the difference is that all the momentum regions, not only the infrared region, contribute to the integral. This is an effect of the chiral symmetry breaking by the Wilson term.

The fermion loop contribution gives

$$\frac{-1}{2!}(2g)^2\mathrm{Tr}\,\phi\langle P_+\psi\bar{\psi}P_+\rangle\langle P_+\psi\bar{\psi}P_+\rangle\phi = -2g^2\phi(-p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{(a\mathcal{M}(k_\mu+ap_\mu))}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\times\frac{(a\mathcal{M}(k_\mu))}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2} \tag{5.6}$$

[6]The power counting with the Ginsparg-Wilson fermion is identical to that of continuum theory [14].

[7]If all the Feynman diagrams should be absolutely convergent, one would enjoy more freedom in choosing lattice regularization. I thank H. Kawai and T. Onogi for a helpful comment on this point.

which is precisely canceled by the scalar contribution (5.5) even for a finite lattice spacing a.

Each Feynman diagram which gives induced couplings higher powers in ϕ such as ϕ^3 is reduced to the continuum result in the limit $a\to 0$. Those couplings in any case cancel among the scalar and fermion contributions even for finite a. The nonrenormalization of the superpotential in this sense is thus maintained in the one-loop level, and in higher-loop levels in the limit $a\to 0$.

C. Self-energy corrections

The simplest self-energy correction is that to the auxiliary fields F and $F^\star$. The one-loop correction is given by

$$\begin{aligned}\frac{1}{2!}2g^2F\langle\phi^2(\phi^\star)^2\rangle F^\star &= 2g^2FF^\star\langle\phi\phi^\star\rangle\langle\phi\phi^\star\rangle\\ &= 2g^2FF^\star\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\frac{a^2}{\sin^2(ak_\mu+ap_\mu)+(a\mathcal{M})^2}\frac{a^2}{(\sin ak_\mu)^2+(a\mathcal{M})^2}\\ &= 2g^2F(p_\mu)F^\star(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{\sin^2k_\mu+(a\mathcal{M}(k_\mu))^2}\frac{a^2}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}.\end{aligned} \tag{5.7}$$

This integral vanishes for $a\to 0$ if one keeps the integration domain *outside*

$$|k_\mu|<\delta \quad \text{for all} \quad \mu \tag{5.8}$$

for arbitrarily small but finite δ and for fixed p_μ. The integration inside the above domain gives a finite continuum result if one notes

$$(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2\simeq k_\mu^2+\left(\frac{1}{2}k_\mu^2+am\right)^2\simeq k_\mu^2+amk_\mu^2+(am)^2 \tag{5.9}$$

inside the above domain. A rescaling of k_μ back to the original momentum variables $k_\mu\to ak_\mu$ gives the continuum result in the limit $a\to 0$.

The fermion self-energy correction is given by

$$\begin{aligned}&\frac{1}{2!}(2g)^2\bar{\psi}(P_+\phi P_++P_-\phi^\star P_-)\psi\bar{\psi}(P_+\phi P_++P_-\phi^\star P_-)\psi\\ &\to(2g)^2[\bar{\psi}(P_+\phi P_+\langle\psi\bar{\psi}\rangle P_-\phi^\star P_-)\psi+\bar{\psi}(P_-\phi^\star P_-)\langle\psi\bar{\psi}\rangle(P_+\phi P_+)\psi]\\ &\to(2g)^2[\bar{\psi}P_+\langle\psi\bar{\psi}\rangle P_-\psi\langle\phi\phi^\star\rangle+\bar{\psi}P_-\langle\psi\bar{\psi}\rangle P_+\psi\langle\phi\phi^\star\rangle]\\ &=(2g)^2\bar{\psi}(p_\mu)\int_{-\pi/a}^{\pi/a}\frac{d^2k}{(2\pi)^2}\left[\frac{ai\gamma^\mu\sin(ap_\mu+ak_\mu)}{\sin^2(ap_\mu+ak_\mu)+(a\mathcal{M})^2}\frac{a^2}{(\sin ak_\mu)^2+(a\mathcal{M})^2}\right]\psi(p_\mu)\\ &=(2g)^2\bar{\psi}(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{ai\gamma^\mu\sin(ap_\mu+k_\mu)}{\sin^2(ap_\mu+k_\mu)+(a\mathcal{M})^2}\frac{1}{(\sin k_\mu)^2+(a\mathcal{M})^2}\right]\psi(p_\mu).\end{aligned} \tag{5.10}$$

This integral vanishes if one sets $p_\mu=0$, which means that the fermion mass receives no quantum correction when renormalized at vanishing momentum, despite the chiral symmetry breaking by the Wilson term.[8] This integral also vanishes for the domain outside Eq. (5.8) in the limit $a\to 0$, and the integral is reduced to the continuum result for the domain inside Eq. (5.8) in the limit $a\to 0$ for fixed p_μ.

To analyze the wave function renormalization, we consider the case with an infinitesimal p_μ but the lattice spacing a kept fixed.[9] Namely,

---

[8]This vanishing mass correction arises from the differences of the Feynman rules in the present model and QCD. Also, all the higher loop corrections are reduced to the (supersymmetric) continuum results in the limit $a\to 0$ in the present model. The Wilson term does not always imply the mass shift.

[9]It should be noted that we assume a small coupling $g/m\ll 1$ and infinitesimally small external momentum p_μ but otherwise make no assumption about the lattice spacing a.

$$|p_\mu| \ll m, \quad 1/a. \tag{5.11}$$

We thus expand

$$\begin{aligned}\sin(ap_\mu+k_\mu)\simeq\sin k_\mu+ap_\mu\cos k_\mu,\quad a\mathcal{M}(k_\mu+ap_\mu)\simeq a\mathcal{M}(k_\mu)+ap_\mu\sin k_\mu,\\ \sin^2(ap_\mu+k_\mu)+(a\mathcal{M})^2\simeq\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2+ap_\mu\sin 2k_\mu+2a\mathcal{M}(k_\mu)ap_\mu\sin k_\mu.\end{aligned} \tag{5.12}$$

The integral in this expression is given by

$$\begin{aligned}&a^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left\{\frac{\sum_\mu i\gamma^\mu p_\mu\cos k_\mu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}-\frac{\sum_\mu i\gamma^\mu\sin k_\mu[p_\nu\sin 2k_\nu+2(a\mathcal{M}(k))p_\nu\sin k_\nu]}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^3}\right\}\\ &=a^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left\{\frac{\sum_\mu i\gamma^\mu p_\mu\cos k_\mu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}+\frac{1}{2}\sum_\mu i\gamma^\mu\sin k_\mu\sum_\nu p_\nu\frac{\partial}{\partial k_\nu}\frac{1}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}\right\}\\ &=\frac{1}{2}a^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left\{\frac{\sum_\mu i\gamma^\mu p_\mu\cos k_\mu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}\right\}.\end{aligned} \tag{5.13}$$

By noting the symmetry under $k_1\leftrightarrow k_2$, we thus have the wave function renormalization for the fermion

$$2g^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{a^2\frac{1}{2}\sum_\nu\cos k_\nu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}\right]\overline{\psi}(p_\mu)i\gamma^\mu p_\mu\psi(p_\mu) \tag{5.14}$$

which disagrees with the finite renormalization factor for the fields F and $F^\star$ at $p_\mu=0$ in Eq. (5.7) for a finite a,

$$\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2}{[(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2]^2}-\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\frac{1}{2}\sum_\nu\cos k_\nu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}=\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\frac{1}{2}\sum_\nu(1-\cos k_\nu)}{[(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2]^2}>0 \tag{5.15}$$

for $a\neq 0$, though this difference vanishes in the limit $a\to 0$. This shows that the finite wave function renormalization factor *breaks* supersymmetry for $a\neq 0$.

We next examine the self-energy corrections to the scalar field ϕ. The contribution from a scalar loop diagram in the *conventional* interaction terms gives

$$\begin{aligned}\frac{1}{2!}g^2[F\phi^2+(F\phi^2)^\star][F\phi^2+(F\phi^2)^\star]&\to g^2\langle(F\phi^2)^\star F\phi^2\rangle\\ &=4g^2\phi^\star\langle F^\star F\rangle\langle\phi^\star\phi\rangle\phi\\ &=-4g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\\ &\quad\times\frac{(\sin k_\mu)^2}{(\sin k_\mu)^2+(a\mathcal{M})^2}\frac{1}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}.\end{aligned} \tag{5.16}$$

The one-loop fermion contribution is given by

$$
\begin{aligned}
&\frac{1}{2!}(2g)^2\overline{\psi}(P_+\phi P_+ + P_-\phi^\star P_-)\psi\overline{\psi}(P_+\phi P_+ + P_-\phi^\star P_-)\psi\\
&\quad\to(2g)^2\overline{\psi}P_-\phi^\star P_-\psi\overline{\psi}P_+\phi P_+\psi\\
&\quad\to-(2g)^2\phi^\star\ \mathrm{Tr}[\langle P_+\psi\overline{\psi}P_-\rangle\langle P_-\psi\overline{\psi}P_+\rangle]\phi\\
&\quad=-(2g)^2\phi^\star\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\mathrm{Tr}\left[\frac{i\gamma^\mu\sin(k_\mu+ap_\mu)}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\frac{i\gamma^\mu\sin k_\mu}{(\sin k_\mu)^2+(a\mathcal{M})^2}\right]\phi\\
&\quad=4g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{\sin(k_\mu+ap_\mu)\sin k_\mu}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\frac{1}{(\sin k_\mu)^2+(a\mathcal{M})^2}\right].
\end{aligned}
\tag{5.17}
$$

The sum of these two contributions gives rise to

$$
4g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{(\sin(k_\mu+ap_\mu)-\sin k_\mu)\sin k_\mu}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\frac{1}{(\sin k_\mu)^2+(a\mathcal{M})^2}\right]
\tag{5.18}
$$

which vanishes for $p_\mu=0$. This means that the mass correction to the scalar particles exactly vanishes in the one-loop level. However, each term logarithmically diverges in the limit $a\to0$, which suggests that the choice of the free part of the Lagrangian should be at least invariant under the lattice supersymmetry transformation to ensure the divergence cancellation, such as in the present formulation. This integral vanishes for $a\to0$ for the domain outside Eq. (5.8) and for fixed p_μ. For the domain inside Eq. (5.8) and for fixed p_μ, the integral is reduced to the continuum result in the limit $a\to0$.

For an infinitesimal p_μ, we have

$$
\begin{aligned}
&4g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left\{\frac{\sum_\mu ap_\mu\cos k_\mu\sin k_\mu}{(\sin k_\mu)^2+(a\mathcal{M})^2}\sum_\nu ap_\nu\frac{\partial}{\partial k_\nu}\left[\frac{1}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\right]-\frac{\frac{1}{2}\sum_\mu[(ap_\mu)^2\sin^2k_\mu]}{[(\sin k_\mu)^2+(a\mathcal{M})^2]^2}\right\}\\
&=4g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left\{\sum_\mu ap_\mu\cos k_\mu\sin k_\mu\frac{1}{2}\sum_\nu ap_\nu\frac{\partial}{\partial k_\nu}\frac{1}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}\right.\\
&\quad\left.-\frac{\frac{1}{2}\sum_\mu[(ap_\mu)^2\sin^2 k_\mu]}{[(\sin k_\mu)^2+(a\mathcal{M})^2]^2}\right\}\\
&=-2g^2\phi^\star(p_\mu)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}a^2\sum_\mu[p_\mu^2\cos 2k_\mu+p_\mu^2\sin^2 k_\mu]\frac{1}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}\\
&=-2g^2\phi^\star(p_\mu)p_\mu^2\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}a^2\frac{1}{2}\sum_\mu\left[\cos 2k_\mu+\frac{1}{2}(1-\cos 2k_\mu)\right]\frac{1}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}.
\end{aligned}
\tag{5.19}
$$

This deviates from the renormalization of F and $F^\star$ for finite a,

$$
\begin{aligned}
&\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2}{[\sin^2 k_\mu+(a\mathcal{M}(k_\mu))^2]^2}-\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\frac{1}{2}\sum_\nu\left[\cos 2k_\nu+\frac{1}{2}(1-\cos 2k_\mu)\right]}{[\sin^2 k_\mu+(a\mathcal{M}(k))^2]^2}\\
&=\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\frac{1}{4}\sum_\nu(1-\cos 2k_\nu)}{[\sin^2 k_\mu+(a\mathcal{M}(k_\mu))^2]^2}>0
\end{aligned}
\tag{5.20}
$$

though this difference vanishes in the limit $a\to0$.

D. Self-energy corrections induced by extra couplings

Finally, we examine the effects of the extra couplings in Eq. (4.5) introduced by an argument of Nicolai mapping on the self-energy of scalar particles. This is given by

$$\begin{aligned}
&\frac{1}{2!}[g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star][g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star]\\
&\to g^2[(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star][\phi^2(\nabla_1^S+i\nabla_2^S)\phi]\\
&\to 2g^2((\nabla_1^S+i\nabla_2^S)\phi)^\star\langle\phi^\star\phi\rangle\langle\phi^\star\phi\rangle(\nabla_1^S+i\nabla_2^S)\phi+4g^2((\nabla_1^S+i\nabla_2^S)\phi)^\star\langle\phi^\star\phi\rangle\langle\phi^\star(\nabla_1^S+i\nabla_2^S)\phi\rangle\phi\\
&\quad+4g^2\phi^\star\langle((\nabla_1^S+i\nabla_2^S)\phi)^\star\phi\rangle\langle\phi^\star\phi\rangle(\nabla_1^S+i\nabla_2^S)\phi+4g^2\phi^\star\langle((\nabla_1^S+i\nabla_2^S)\phi)^\star\phi\rangle\langle\phi^\star(\nabla_1^S+i\nabla_2^S)\phi\rangle\phi\\
&\quad+4g^2\phi^\star\langle((\nabla_1^S+i\nabla_2^S)\phi)^\star(\nabla_1^S+i\nabla_2^S)\phi\rangle\langle\phi^\star\phi\rangle\phi.
\end{aligned}\tag{5.21}$$

The first term in Eq. (5.21) gives

$$\begin{aligned}
&2g^2((\nabla_1^S+i\nabla_2^S)\phi(p_\mu))^\star(\nabla_1^S+i\nabla_2^S)\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\,\frac{a^2}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\\
&=2g^2\phi(p_\mu)^\star\left(\frac{\sin ap_\mu}{a}\right)^2\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\,\frac{a^2}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}.
\end{aligned}\tag{5.22}$$

This gives for an infinitesimal p_μ

$$2g^2\phi(p_\mu)^\star p_\mu^2\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a^2}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}.\tag{5.23}$$

The second term gives

$$4g^2\left(\left(i\frac{\sin ap_1}{a}-\frac{\sin ap_2}{a}\right)\phi(p_\mu)\right)^\star\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\,\frac{-i\sin(k_1+ap_1)+\sin(k_2+ap_2)}{\sin^2(k_\mu+ap_\mu)+(a\mathcal{M}(k_\mu+ap_\mu))^2}\tag{5.24}$$

which gives for an infinitesimal p_μ

$$\begin{aligned}
&4g^2(-ip_1-p_2)\phi(p_\mu)^\star\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\left[\frac{-iap_1\cos k_1+ap_2\cos k_2}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}+(-i\sin k_1+\sin k_2)\right.\\
&\quad\left.\times\sum_\nu ap_\nu\frac{\partial}{\partial k_\nu}\,\frac{1}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\right]\\
&=4g^2(-ip_1-p_2)\phi(p_\mu)^\star\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\frac{1}{2}\left[\frac{-iap_1\cos k_1+ap_2\cos k_2}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\right]\\
&=g^2(-ip_1-p_2)\phi(p_\mu)^\star\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\left[\frac{(-iap_1+ap_2)\sum_\nu\cos k_\nu}{\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2}\right]\\
&=g^2(-p_\mu^2)\phi(p_\mu)^\star\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a^2\sum_\nu\cos k_\nu}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}.
\end{aligned}\tag{5.25}$$

The third term gives the complex conjugate of the second term, which agrees with the second term itself. Thus we have altogether

$$-2g^2\phi(p_\mu)^\star p_\mu^2\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\sum_\nu \cos k_\nu}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}. \tag{5.26}$$

The fourth and the fifth terms give precisely the negative of the contributions of the conventional interactions to the self-energy of scalar particles, Eqs. (5.16) and (5.17), which we have already evaluated.

If we collect all the terms arising from the extra couplings together, we obtain

$$2g^2\phi(p_\mu)^\star p_\mu^2\phi(p_\mu)\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2\left[1-\sum_\nu \cos k_\nu+\frac{1}{2}\sum_\nu \cos 2k_\nu+\frac{1}{4}\sum_\nu (1-\cos 2k_\nu)\right]}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2} \tag{5.27}$$

which vanishes in the limit $a\to 0$, as it should be since the conventional interaction already ensures supersymmetry in the limit $a\to 0$. These terms do not help the wave function renormalization factor of ϕ agree with that of either F or ψ. The breaking of supersymmetry in the wave function renormalization factors persist for $a\neq 0$ even if one includes the effects of the extra couplings induced by an analysis of Nicolai mapping.

VI. LOW-ENERGY EFFECTIVE ACTION WITH ONE-LOOP CORRECTIONS

The low-energy effective action which includes the one-loop quantum corrections is written in a momentum representation as

$$\begin{aligned}\mathcal{L}_{eff}&=(1+z_\psi)\bar{\psi}i\not{p}\psi-m\bar{\psi}\psi-(1+z_\phi)\phi^\star p_\mu^2\phi-\frac{m^2}{1+z_F}\phi^\star\phi+\cdots\\&=(1+z_\psi)\left[\bar{\psi}i\not{p}\psi-\frac{m}{1+z_\psi}\bar{\psi}\psi\right]-(1+z_\phi)\left[\phi^\star p_\mu^2\phi+\frac{m^2}{(1+z_F)(1+z_\phi)}\phi^\star\phi\right]+\cdots\end{aligned} \tag{6.1}$$

after the elimination of the auxiliary fields F and $F^\star$. Here we defined the finite wave function renormalization factors [see Eqs. (5.7), (5.14) and (5.19)]

$$\begin{aligned}&z_F=2g^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2}{[(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2]^2},\quad z_\psi=2g^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{a^2\frac{1}{2}\sum_\nu \cos k_\nu}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}\right],\\&z_\phi=2g^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\left[\frac{a^2\frac{1}{2}\sum_\nu \cos 2k_\nu+\frac{1}{4}\sum_\nu (1-\cos 2k_\nu)}{[\sin^2(k_\mu)+(a\mathcal{M}(k))^2]^2}\right].\end{aligned} \tag{6.2}$$

Supersymmetry suggests the uniform wave function renormalization

$$1+z_\psi=1+z_\phi \tag{6.3}$$

and the degeneracy of the mass parameter

$$\frac{m^2}{(1+z_\psi)^2}=\frac{m^2}{(1+z_F)(1+z_\phi)} \tag{6.4}$$

or in the accuracy of one-loop correction

$$2z_\psi=(z_F+z_\phi). \tag{6.5}$$

If one includes the contributions from the extra couplings, these conditions are replaced by

$$z_\psi=z_\phi+z_{extra},\quad 2z_\psi=(z_F+z_\phi+z_{extra}) \tag{6.6}$$

with [see Eq. (5.27)]

$$z_{extra}=2g^2\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\,\frac{a^2\left[-1+\sum_\nu \cos k_\nu-\frac{1}{2}\sum_\nu \cos 2k_\nu-\frac{1}{4}\sum_\nu (1-\cos 2k_\nu)\right]}{[\sin^2(k_\mu)+(a\mathcal{M}(k_\mu))^2]^2}$$
$$=2z_\psi-z_F-z_\phi. \tag{6.7}$$

It is interesting that the degeneracy of the mass parameter, namely, the second condition in Eq. (6.6), is satisfied even for finite a in the presence of the extra couplings. However, the uniform wave function renormalization condition

$$z_\psi=z_F \tag{6.8}$$

is still broken since $z_\psi<z_F$ for finite a. The supersymmetry is thus broken for $a\neq 0$ even with the extra couplings induced by the Nicolai mapping.

In the continuum limit $a\to 0$, we have

$$z_\psi=z_F=z_\phi,\quad z_{extra}=0 \tag{6.9}$$

and the supersymmetry is recovered, *with or without* the extra couplings. This conclusion is valid up to any finite order in perturbation theory.

VII. CHECK OF WARD IDENTITY

The Nicolai mapping suggests that the Lagrangian is written as

$$\mathcal{L}=\bar\psi^\alpha(x)\frac{\partial\xi_\alpha(x)}{\partial A^\beta(y)}\psi^\beta(y)-\frac{1}{2}\sum_\alpha (\xi_\alpha(x))^2 \tag{7.1}$$

where[10]

$$\{A_\alpha\}=(A,B), \tag{7.2}$$

and thus we have the relation

$$-\langle\psi^\alpha(x)\bar\psi^\beta(y)\rangle=\int\mathcal{D}\xi\frac{\partial A_\alpha(x)}{\partial\xi_\beta(y)}\times\exp\left[-\sum_x\frac{1}{2}\sum_\alpha(\xi_\alpha(x))^2\right] \tag{7.3}$$

which is equal to

$$\langle A_\alpha(x)\xi_\beta(y)\rangle=\int\mathcal{D}\xi\frac{\partial A_\alpha(x)}{\partial\xi_\beta(y)}\times\exp\left[-\sum_x\frac{1}{2}\sum_\alpha(\xi_\alpha(x))^2\right] \tag{7.4}$$

as can be confirmed by expanding $A_\alpha(x)$ formally in powers of $\xi_\kappa(z)$. These relations give rise to the identity [11,12]

$$\langle\psi^\alpha(x)\bar\psi^\beta(y)\rangle+\langle A_\alpha(x)\xi_\beta(y)\rangle=0. \tag{7.5}$$

We check this identity for a small momentum region. The fermion propagator with one-loop quantum corrections is given by

$$\langle\psi\bar\psi\rangle=\frac{1}{-i(1+z_\psi)\not p+m}=\frac{1}{m}+\frac{i(1+z_\psi)\not p}{m^2}+O(p_\mu^2) \tag{7.6}$$

in the low-energy limit $|\not p/m|\ll 1$ but with fixed a.

We next note

$$U=mA+\frac{g}{\sqrt{2}}(A^2-B^2),\quad V=mB+\frac{2g}{\sqrt{2}}AB. \tag{7.7}$$

We evaluate

$$\langle\xi_1(x)A(y)\rangle=(\nabla_1^S+D_{(2)})\langle A(x)A(y)\rangle-m\langle A(x)A(y)\rangle-\frac{g}{\sqrt{2}}\langle(A^2-B^2)(x)A(y)\rangle-\nabla_2^S\langle B(x)A(y)\rangle \tag{7.8}$$

with the interaction terms

$$\mathcal{L}_{int}=-\frac{g}{\sqrt{2}}((m-D_{(2)})A)(A^2-B^2)-\frac{2g}{\sqrt{2}}(m-D_{(2)})B)\times(AB)-\frac{g}{4}[(A^2-B^2)^2+4(AB)^2]+\frac{g}{\sqrt{2}}(\nabla_1^SA-\nabla_2^SB)(A^2-B^2)+\frac{2g}{\sqrt{2}}(-\nabla_1^SB-\nabla_2^SA)\times(AB)-\frac{2g}{\sqrt{2}}\bar\psi(A+iB\gamma_5)\psi. \tag{7.9}$$

We first have

[10]We identify the spinor index of the Dirac fermion with the flavor index of the scalar particle, an apparently Lorentz noninvariant operation.

$$(\nabla_1^S + D_{(2)})\langle A(x)A(y)\rangle - m\langle A(x)A(y)\rangle = \frac{-ip_1 - m}{(1+z_\phi)p_\mu^2 + \frac{m^2}{(1+z_F)}} = -\frac{(1+z_F)}{m} - \frac{ip_1(1+z_F)}{m^2} + O(p_\mu^2). \tag{7.10}$$

We next evaluate

$$-\frac{g}{\sqrt{2}}\langle (A^2 - B^2)(x)A(y)\rangle. \tag{7.11}$$

The contributions from the conventional interaction terms give in momentum representation

$$2g^2\left\{\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2}{(\sin k_\mu)^2 + (a\mathcal{M}(k_\mu))^2}\frac{1}{(\sin k_\mu + ap_\mu)^2 + (a\mathcal{M}(k_\mu + ap_\mu))^2}\right\}\frac{a^2\mathcal{M}(ap_\mu)}{(\sin ap_\mu)^2 + (a\mathcal{M}(ap_\mu))^2} \tag{7.12}$$

which gives for small p_μ

$$\frac{z_F}{m}. \tag{7.13}$$

The contributions from the extra couplings give in momentum representation

$$\begin{aligned} &2g^2\left\{\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a^2}{(\sin k_\mu)^2 + (a\mathcal{M}(k_\mu))^2}\frac{1}{(\sin k_\mu + ap_\mu)^2 + (a\mathcal{M}(k_\mu + ap_\mu))^2}\right\}\frac{ai\,\sin ap_1}{(\sin ap_\mu)^2 + (a\mathcal{M}(ap_\mu))^2} \\ &-4g^2\left\{\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{a}{(\sin k_\mu)^2 + (a\mathcal{M}(k_\mu))^2}\frac{i(\sin k_1 + ap_1)}{(\sin k_\mu + ap_\mu)^2 + (a\mathcal{M}(k_\mu + ap_\mu))^2}\right\}\frac{a^2}{(\sin ap_\mu)^2 + (a\mathcal{M}(ap_\mu))^2} \end{aligned} \tag{7.14}$$

which gives for small p_μ

$$(z_F - z_\psi)\frac{ip_1}{m^2}. \tag{7.15}$$

This term vanishes for $a \to 0$.

These calculations show that the Ward identity [11,12]

$$\langle\psi\bar{\psi}\rangle_{11} + \langle\xi_1(x)A(y)\rangle = 0 \tag{7.16}$$

is precisely satisfied up to the order $O(p_\mu^2)$,

$$\langle\psi\bar{\psi}\rangle = \frac{1}{-i(1+z_\psi)\not{p} + m} = \frac{1}{m} + \frac{i(1+z_\psi)\not{p}}{m^2} + O(p_\mu^2),$$

$$\begin{aligned} \langle\xi_1(x)A(y)\rangle &= (\nabla_1^S + D_{(2)})\langle A(x)A(y)\rangle - m\langle A(x)A(y)\rangle - \frac{g}{\sqrt{2}}\langle (A^2 - B^2)(x)A(y)\rangle - \nabla_2^S\langle B(x)A(y)\rangle¥ \\ &= -\frac{(1+z_F)}{m} - \frac{ip_1(1+z_F)}{m^2} + \frac{z_F}{m} + (z_F - z_\psi)\frac{ip_1}{m^2} + O(p_\mu^2), \\ &= -\frac{1}{m} - \frac{i(1+z_\psi)p_1}{m^2} + O(p_\mu^2) \end{aligned} \tag{7.17}$$

even for $z_F \neq z_\psi$ at $a \neq 0$, if one recalls $(\not{p})_{11} = p_1$. But in any case the correction terms induced by the extra interactions vanish, $z_F - z_\psi \to 0$, in the limit $a \to 0$. Consequently, the Ward identity in the limit $a \to 0$ is not modified by the extra interactions introduced by the Nicolai mapping. This is consistent with the numerical findings of the behavior of various exact and not-exact Ward identities, which appear to be equally valid numerically in the limit $a\to 0$ [12].

VIII. DISCONNECTED VACUUM DIAGRAMS

Contributions to the vacuum energy cancel exactly for the free part of the Lagrangian due to the precise lattice supersymmetry. The lowest nontrivial contributions from interaction terms arise in the two-loop level. The (conventional) two-loop scalar contribution is given by

$$\begin{aligned}\frac{1}{2!}g^2[F\phi^2+(F\phi^2)^\star][F\phi^2+(F\phi^2)^\star]&\to g^2\langle (F\phi^2)^\star F\phi^2\rangle\\ &=2g^2\langle F^\star F\rangle\langle\phi^\star\phi\rangle\langle\phi^\star\phi\rangle\\ &=-2g^2\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{(\sin p_\mu)^2+(a\mathcal{M}(p_\mu))^2}\\ &\quad\times\frac{(\sin k_\mu)^2}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2}\frac{1}{\sin^2(k_\mu+p_\mu)+(a\mathcal{M}(k_\mu+p_\mu))^2}.\end{aligned}\tag{8.1}$$

The two-loop contribution which contains a fermion loop is given by

$$\begin{aligned}&\frac{1}{2!}(2g)^2\overline{\psi}(P_+\phi P_+ + P_-\phi^\star P_-)\psi\overline{\psi}(P_+\phi P_+ + P_-\phi^\star P_-)\psi\\ &\to(2g)^2\overline{\psi}P_-\phi^\star P_-\psi\overline{\psi}P_+\phi P_+\psi\\ &\to-(2g)^2\langle\phi^\star\phi\rangle\mathrm{Tr}[\langle P_+\psi\overline{\psi}P_-\rangle\langle P_-\psi\overline{\psi}P_+\rangle]\\ &=-(2g)^2\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{(\sin p_\mu)^2+(a\mathcal{M}(p_\mu))^2}\\ &\quad\times\mathrm{Tr}\left[\frac{i\gamma^\mu\,\sin(k_\mu+p_\mu)}{\sin^2(k_\mu+p_\mu)+(a\mathcal{M}(k_\mu+p_\mu))^2}\frac{i\gamma^\mu\,\sin k_\mu}{(\sin k_\mu)^2+(a\mathcal{M}(p_\mu))^2}\right]\phi\\ &=4g^2\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{(\sin p_\mu)^2+(a\mathcal{M}(p_\mu))^2}\left[\frac{\sin(k_\mu+p_\mu)\sin k_\mu}{\sin^2(k_\mu+p_\mu)+(a\mathcal{M}(k_\mu+p_\mu))^2}\frac{1}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2}\right].\end{aligned}\tag{8.2}$$

In the limit $a\to 0$, these two integrals contain logarithmically divergent infrared sigularities in the rescaled variables p_μ and k_μ. These divergences, which agree with the divergences in continuum theory, precisely cancel each other. However, the remaining finite parts of these two integrals do not quite cancel each other even in the limit $a\to 0$ and thus lead to the nonvanishing vacuum energy. This is a result of supersymmetry breaking by the failure of the Leibniz rule. This complication arises since the vacuum diagrams are not finite even in $d=2$ (in fact contain logarithmic overlap-divergence) and all the momentum regions contribute to these vacuum diagrams.

A way to remove these finite contributions in the limit $a\to 0$ (without relying on the extra couplings) is to apply a higher derivative regularization [15] which amounts to the replacement of all the terms in the free part of the Lagrangian (4.1) as

$$\mathcal{L}_0=\overline{\psi}(D_{(1)}+D_{(2)})R\psi-\phi^\star D_{(1)}^\dagger D_{(1)}R\phi+F^\star RF-m\overline{\psi}R\psi-m[FR\phi+(FR\phi)^\star]+FD_{(2)}R\phi+(FD_{(2)}R\phi)^\star\tag{8.3}$$

where R is the higher derivative regulator

$$R=\frac{D_{(1)}^\dagger D_{(1)}+D_{(2)}^2+M^2}{M^2}\tag{8.4}$$

with a new fixed mass scale M. This regularization preserves the lattice supersymmetry (3.1) and (3.2) in the free part of the Lagrangian. By this way, all the nonvanishing contributions are limited to the momentum regions $p_\mu^2\leqslant M^2$, and the vacuum diagrams completely cancel in the limit $a\to 0$.

It is interesting to see how the extra couplings help to remove the vacuum energy even for $a\neq 0$. This is given by

$$\frac{1}{2!}[g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star][g\phi^2(\nabla_1^S+i\nabla_2^S)\phi+g(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star]$$
$$\to g^2\langle[(\phi^2(\nabla_1^S+i\nabla_2^S)\phi)^\star][\phi^2(\nabla_1^S+i\nabla_2^S)\phi]\rangle$$
$$=4g^2\langle((\nabla_1^S+i\nabla_2^S)\phi)^\star\phi\rangle\langle\phi^\star(\nabla_1^S+i\nabla_2^S)\phi\rangle\langle\phi^\star\phi\rangle+2g^2\langle((\nabla_1^S+i\nabla_2^S)\phi)^\star(\nabla_1^S+i\nabla_2^S)\phi\rangle\langle\phi^\star\phi\rangle\langle\phi^\star\phi\rangle$$
$$=-4g^2\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{(\sin p_\mu)^2+(a\mathcal{M}(p_\mu))^2}\left[\frac{\sin(k_\mu+p_\mu)\sin k_\mu}{\sin^2(k_\mu+p_\mu)+(a\mathcal{M}(k_\mu+p_\mu))^2}\frac{1}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2}\right]$$
$$+2g^2\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\frac{1}{(\sin p_\mu)^2+(a\mathcal{M}(p_\mu))^2}\frac{(\sin k_\mu)^2}{(\sin k_\mu)^2+(a\mathcal{M}(k_\mu))^2}\frac{1}{\sin^2(k_\mu+p_\mu)+(a\mathcal{M}(k_\mu+p_\mu))^2}. \tag{8.5}$$

We thus confirm that the vacuum energies (8.1), (8.2) and (8.5) put together completely cancel for finite a, and this is a nice aspect of the analysis based on the Nicolai mapping.

IX. DISCUSSION AND CONCLUSION

We have examined the $N=2$ Wess-Zumino model on $d=2$ Euclidean lattice in connection with a restoration of the Leibniz rule in the limit $a\to 0$. In particular, we examined the Wilson fermion instead of the Ginsparg-Wilson fermion. We also examined the effects of extra couplings introduced by an analysis of Nicolai mapping.

As for the Wilson fermion, it introduces linear and logarithmic divergences in some of the individual Feynman diagrams, though those divergences precisely cancel among Feynman diagrams for correlation functions in the formulation which ensures supersymmetry for the free part of the Lagrangian. In the general analysis of the Leibniz rule in Ref. [15], each Feynman diagram was made finite to ensure the Leibniz rule in the limit $a\to 0$. In such a case, the lattice regularization would enjoy more freedom since it is introduced just to allow the numerical and other nonperturbative analyses, and the lattice artifact is safely removed in the limit $a\to 0$. Each Feynman diagram in the $N=2$ Wess-Zumino model in $d=2$ which was analyzed here, however, is not finite in general in particular with the Wilson fermion, and the precise cancellation of these divergences for *finite a* is important.

As for the effects of the extra couplings introduced by an analysis of Nicolai mapping, which breaks hypercubic symmetry, these couplings do not completely remedy the breaking of supersymmetry induced by the failure of the Leibniz rule, though those extra couplings ensure the vanishing vacuum energy. We also illustrated how the Ward identity [11,12] is satisfied even if the uniform wave function renormalization, which is required by supersymmetry, is not satisfied for finite a.

For a minimal latticization of the Wess-Zumino model in $d=2$ which ensures lattice supersymmetry for the free part of the Lagrangian but without the extra couplings

$$\mathcal{L}=\bar\psi(D_{(1)}+D_{(2)})\psi-m\bar\psi\psi-2g\bar\psi(P_+\phi P_++P_-\phi^\star P_-)\psi$$
$$-\phi^\star D_{(1)}^\dagger D_{(1)}\phi+F^\star F-m[F\phi+(F\phi)^\star]$$
$$-g[F\phi^2+(F\phi^2)^\star]+FD_{(2)}\phi+(FD_{(2)}\phi)^\star, \tag{9.1}$$

we have illustrated that all the supersymmetry breaking terms in correlation functions induced by the failure of the Leibniz rule are irrelevant in the sense that those terms all vanish in the limit $a\to 0$. This is consistent with the general analysis of perturbatively finite theory on the lattice [15].

The lattice operation implies

$$(\nabla(fg))(x)=(\nabla f)(x)g(x)+f(x)(\nabla g)(x)+a(\nabla f)(x)(\nabla g)(x) \tag{9.2}$$

if one defines $(\nabla f)(x)=(f(x+a)-f(x))/a$, and thus the breaking of the Leibniz rule is of order $O(1)$ if the momentum carried by field variables is of order $O(1/a)$. To the extent that the derivative of field variables is required in supersymmetry to balance the dimensionality of bosonic and fermionic variables, the Leibniz rule is indispensable for the validity of supersymmetry. It is well known that supersymmetry improves ultraviolet properties of field theory. In the context of lattice theory, one may rather reverse the argument and one may even argue that the finite theory is *required* to accommodate supersymmetry in a consistent manner since the conventional definition of derivative

$$\frac{df(x)}{dx}=\lim_{a\to 0}\frac{f(x+a)-f(x)}{a} \tag{9.3}$$

which satisfies the Leibniz rule presumes that the momentum carried by the field variable $f(x)$ is negligibly small compared to $1/a$. This is realized in lattice theory only if the theory is finite at least in the perturbative sense.

In conclusion, our analysis is consistent with the past analyses of the $d=2$ Wess-Zumino model and we believe that our analysis gives an explanation why these past nonperturbative numerical analyses worked [11,12], in particular, both of the Ward identity which is expected to be exact on the lattice and those Ward identities which are expected to be broken by the lattice artifacts[11] [12]. A numerical calculation of the mass correction also appears to be consistent with the (continuum) perturbative estimate, as was noted in [12]. All the supersymmetry breaking effects in correlation functions induced by the failure of the Leibniz rule become irrelevant in the limit $a\to 0$ for a finite theory. The existence of the Nicolai mapping in the $d=2$ Wess-Zumino model is a nice property of a specific formulation of the specific model, such as ensuring the vanishing vacuum energy, but it is not a necessary condition for a consistent definition of supersymmetric models on the lattice in the limit $a\to 0$. The finiteness is a more universal condition which ensures supersymmetry in the limit $a\to 0$.

Finally, the analyses of other aspects of supersymmetry on the lattice, which were not discussed in the present paper, are found in Refs. [20–23].

Note added. Golterman and Petcher [24] analyzed related issues in the context of the $N=1$ Wess-Zumino model in $d=2$. I thank M. Golterman for calling the above work to my attention. S. Elitzur and A. Schwimmer [25] discussed a related problem in a Hamiltonian formalism. I thank A. Schwimmer for calling their work to my attention.

[11] I thank S. Catterall for a helpful communication related to this issue.

[1] J. Wess and B. Zumino, Phys. Lett. **49B**, 52 (1974).
[2] J. Iliopoulos and B. Zumino, Nucl. Phys. **B76**, 310 (1974).
[3] K. Fujikawa and W. Lang, Nucl. Phys. **B88**, 61 (1975); P. West, *ibid.* **B106**, 219 (1976); M.T. Grisaru, M. Rocek, and W. Siegel, *ibid.* **B159**, 429 (1979).
[4] P. Dondi and H. Nicolai, Nuovo Cimento A **41**, 1 (1977).
[5] H. Nicolai, Phys. Lett. **89B**, 341 (1980).
[6] S. Elitzur, E. Ravinovici, and A. Schwimmer, Phys. Lett. **119B**, 165 (1982); J. Bartels and J.B. Bronzan, Phys. Rev. D **28**, 818 (1983); S. Nojiri, Prog. Theor. Phys. **74**, 819 (1985); **74**, 1124 (1985).
[7] N. Sakai and M. Sakamoto, Nucl. Phys. **B229**, 173 (1983).
[8] J. Bartels and G. Kramer, Z. Phys. C **20**, 159 (1983).
[9] N. Seiberg, Phys. Lett. B **318**, 469 (1993).
[10] T. Aoyama and Y. Kikukawa, Phys. Rev. D **59**, 054507 (1999); H. So and N. Ukita, Phys. Lett. B **457**, 314 (1999); Nucl. Phys. B (Proc. Suppl.) **94**, 795 (2001); W. Bietenholz, Mod. Phys. Lett. A **14**, 51 (1999).
[11] M. Beccaria, G. Curci, and E. D'Ambrosio, Phys. Rev. D **58**, 065009 (1998).
[12] S. Catterall and S. Karamov, Phys. Rev. D **65**, 094501 (2002); Nucl. Phys. B (Proc. Suppl.) **106**, 935 (2002).
[13] M. Beccaria, M. Campostrini, and A. Feo, Nucl. Phys. B (Proc. Suppl.) **106**, 944 (2002).
[14] K. Fujikawa and M. Ishibashi, Nucl. Phys. **B622**, 115 (2002); Phys. Lett. B **528**, 295 (2002); K. Fujikawa, M. Ishibashi, and H. Suzuki, *ibid.* **538**, 197 (2002).
[15] K. Fujikawa, Nucl. Phys. **B636**, 80 (2002).
[16] Y. Kikukawa and Y. Nakayama, hep-lat/0207013.
[17] H. Neuberger, Phys. Lett. B **417**, 141 (1998); **427**, 353 (1998).
[18] K. Fujikawa, Nucl. Phys. **B589**, 487 (2000).
[19] F. Niedermayer, Nucl. Phys. B (Proc. Suppl.) **73**, 105 (1999); H. Neuberger, Annu. Rev. Nucl. Part. Sci. **51**, 23 (2001); M. Lüscher, "Chiral gauge theories revisited," hep-th/0102028; Y. Kikukawa, "Analytic progress on exact lattice chiral symmetry," hep-lat/0111035.
[20] G. Curci and G. Veneziano, Nucl. Phys. **B292**, 555 (1987).
[21] T. Banks and P. Windy, Nucl. Phys. **B198**, 226 (1982); I. Ichinose, Phys. Lett. **122B**, 68 (1983); R. Nakayama and Y. Okada, *ibid.* **134B**, 241 (1984); I. Montvay, "SYM on the lattice," hep-lat/9801023; I. Campos *et al.*, Eur. Phys. J. C **11**, 507 (1999); F. Farchioni *et al.*, *ibid.* **23**, 719 (2002).
[22] R. Narayanan and H. Neuberger, Nucl. Phys. **B443**, 305 (1995); S. Aoki, K. Nagai, and S.V. Zenkin, *ibid.* **B508**, 715 (1997); N. Maru and J. Nishimura, Int. J. Mod. Phys. A **13**, 2841 (1998); R.G. Edwards, U.M. Heller, and R. Narayanan, Chin. J. Phys. (Taipei) **38**, 594 (2000); D.B. Kaplan and M. Schmaltz, *ibid.* **38**, 543 (2000); G.T. Fleming, J.B. Kogut, and P.M. Vranas, Phys. Rev. D **64**, 034510 (2001); K. Itoh, M. Kato, H. So, H. Sawanaka, and N. Ukita, hep-lat/0112052.
[23] D.B. Kaplan, E. Katz, and M. Unsal, hep-lat/0206019.
[24] M. Golterman and D.N. Petcher, Nucl. Phys. **B319**, 307 (1989).
[25] S. Elitzur and A. Schwimmer, Nucl. Phys. **B226**, 109 (1983).