Infrared behavior of the running coupling constant and bound states in QCD

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(Received 18 February 2002; published 16 October 2002)

The perturbative expression of the running strong coupling constant $\alpha_s(Q^2)$ has an unphysical singularity for $Q^2=\Lambda^2_{\rm QCD}$. Various modification have been proposed for the infrared region. The effect of some of such proposals on the quark-antiquark spectrum is tested on a Bethe-Salpeter (second order) formalism which was successfully applied in previous papers to an overall evaluation of the spectrum in the light-light, light-heavy and heavy-heavy sectors (the only serious discrepancy with data being for the light pseudoscalar meson masses). In this paper only the $c\bar{c}$, $b\bar{b}$ and $q\bar{q}$ ($q=u$ or d) cases are considered and fine structure is neglected. It is found that in the $b\bar{b}$ and $c\bar{c}$ cases the results are little sensitive to the specific choice. In the light-light case the Dokshitzer *et al.* prescription is again essentially equivalent to the truncation prescription used in the previous calculation and it is consistent with the same *a priori* fixing of the light quark masses on the typical current values $m_u=m_d=10$ MeV (only the pion mass resulting completely out of scale at about 500 MeV). With the Shirkov-Solovtsov prescription, on the contrary, a reasonable agreement with the data is obtained only at the price of using a phenomenological momentum dependent effective mass for the quark. The use of such an effective mass amounts to a correction of the free quark propagator. It is remarkable that this also has the effect of bringing the pion mass into the correct range.

DOI: 10.1103/PhysRevD.66.074008 PACS number(s): 12.38.Aw, 11.10.St, 12.38.Lg, 12.39.Ki

I. INTRODUCTION

In perturbation theory the running coupling constant in QCD is usually written up to one loop as

$$\alpha_s(Q^2)=\frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} \tag{1}$$

or up to two loops as

$$\alpha_s(Q^2)=\frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} \times\left[1+\frac{2\beta_1}{\beta_0^2}\frac{\ln(\ln(Q^2/\Lambda^2))}{\ln(Q^2/\Lambda^2)}\right], \tag{2}$$

Q being the relevant energy scale, $\beta_0=11-\frac{2}{3}n_f$, $\beta_1=51-\frac{19}{3}n_f$, and n_f the number of flavors with masses smaller than Q.

Such expressions have been largely tested in large Q processes and are normally used to relate data obtained at different Q using the appropriate number of "active" flavors n_f and different values of Λ in the ranges between the various quark thresholds.

Both expressions, however, become singular and completely inadequate as Q^2 approaches Λ^2. Therefore they must be somewhat modified in the infrared region.

Various proposals have been made in this direction starting perhaps with the work of Ref. [1]. The most naive assumption consists in cutting the curve (1) at a certain maximum value $\alpha_s(0)=\overline{\alpha_s}$ to be treated as a mere phenomenological parameter (truncation prescription). Alternatively, on the basis of general analyticity arguments, Shirkov and Solovtsov [2] replace Eq. (1) with

$$\alpha_s(Q^2)=\frac{4\pi}{\beta_0}\left(\frac{1}{\ln(Q^2/\Lambda^2)}+\frac{\Lambda^2}{\Lambda^2-Q^2}\right). \tag{3}$$

This remains regular for $Q^2=\Lambda^2$ and has a finite Λ independent limit, $\alpha_s(0)=4\pi/\beta_0$, for $Q^2\to 0$. Finally, inspired also by phenomenological concerns, Dokshitzer *et al.* [3] write

$$\alpha_s(Q^2)=\frac{\sin(\pi\mathcal{P})}{\pi\mathcal{P}}\alpha_s^0(Q^2), \tag{4}$$

where $\alpha_s^0(Q^2)$ is the perturbative running coupling constant as given by Eq. (1) and $\mathcal{P}=d/d[\ln(Q^2/\Lambda^2)]$ is a derivative acting on $\alpha_s^0(Q^2)$. The various curves are reported in Fig. 1.

The above modified expressions have been applied to study various effects in which infrared behavior turns out to

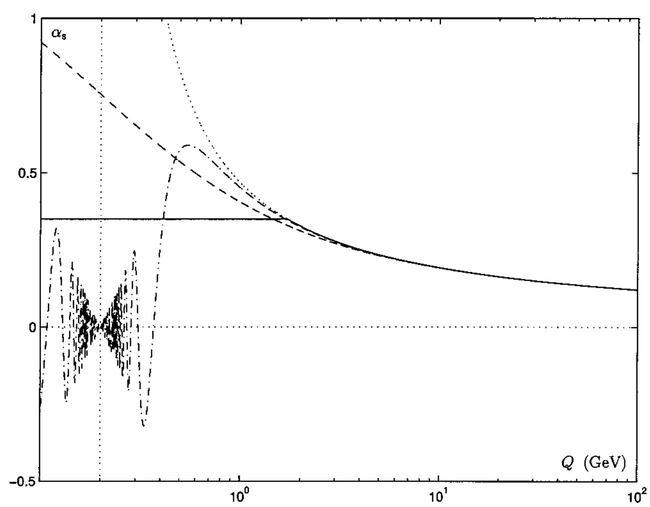

FIG. 1. Running coupling constant $\alpha_s(Q)$ on logarithmic scale. Truncation prescription (full line), Shirkov-Solovtsov prescription (dashed line), Dokshitzer *et al.* prescription (dot-dashed line).

be important. Electron-positron annihilation into hadrons, τ-lepton decay, lepton-hadron deep inelastic scattering, jet shapes, pion form factors etc. are of this type.

In the quark-antiquark bound state problem the variable Q^2 can be identified with the squared momentum transfer $\mathbf{Q}^2=(\mathbf{k}-\mathbf{k}')^2$ and formally the use of a running coupling constant amounts to include higher order terms in the perturbative part of the potential or the Bethe-Salpeter kernel. In this case all values of $\mathbf{Q}^2$ are involved and an infrared regularization becomes essential. Furthermore $\langle \mathbf{Q}^2 \rangle$ ranges typically between $(1\ \mathrm{GeV})^2$ and $(0.1\ \mathrm{GeV})^2$ for different quark masses and internal excitations and values of $\mathbf{Q}^2$ smaller than Λ^2 can be important. The specific infrared behavior is therefore expected to affect the spectrum and other properties of mesons.

The purpose of this paper is to test such effects in a particular formalism we have developed and used in previous papers.

II. FORMALISM

In Ref. [4] we obtained a good reproduction of the entire meson spectrum in terms of only four adjustable parameters, by solving numerically the eigenvalue equation for the squared mass operator

$$M^2=M_0^2+U \tag{5}$$

or the mass operator

$$M=M_0+V, \tag{6}$$

where $M_0=w_1+w_2=\sqrt{m_1^2+\mathbf{k}^2}+\sqrt{m_2^2+\mathbf{k}^2}$ is the kinetic term and U and V are complicated momentum dependent potentials. Up to the first order in the running coupling constant $\alpha_s(\mathbf{Q}^2)$ and in terms of the string tension σ, the "quadratic potential" U is given by

$$\begin{aligned}\langle \mathbf{k}|U|\mathbf{k}'\rangle = &\sqrt{\frac{(w_1+w_2)(w_1'+w_2')}{w_1w_2w_1'w_2'}}\Bigg\{\frac{4}{3}\frac{\alpha_s(\mathbf{Q}^2)}{\pi^2}\Bigg[-\frac{1}{\mathbf{Q}^2}\left(q_{10}q_{20}+\mathbf{q}^2-\frac{(\mathbf{Q}\cdot\mathbf{q})^2}{\mathbf{Q}^2}\right)+\frac{i}{2\mathbf{Q}^2}\mathbf{k}\times\mathbf{k}'\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+\frac{1}{2\mathbf{Q}^2}[q_{20}(\boldsymbol{\alpha}_1\cdot\mathbf{Q})\\ &-q_{10}(\boldsymbol{\alpha}_2\cdot\mathbf{Q})]+\frac{1}{6}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\frac{1}{4}\left(\frac{1}{3}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-\frac{(\mathbf{Q}\cdot\boldsymbol{\sigma}_1)(\mathbf{Q}\cdot\boldsymbol{\sigma}_2)}{\mathbf{Q}^2}\right)+\frac{1}{4\mathbf{Q}^2}(\boldsymbol{\alpha}_1\cdot\mathbf{Q})(\boldsymbol{\alpha}_2\cdot\mathbf{Q})\Bigg]\\ &+\int\frac{d^3\mathbf{r}}{(2\pi)^3}e^{i\mathbf{Q}\cdot\mathbf{r}}J^{\mathrm{inst}}(\mathbf{r},\mathbf{q},q_{10},q_{20})\Bigg\},\end{aligned} \tag{7}$$

with

$$\begin{aligned}J^{\mathrm{inst}}(\mathbf{r},\mathbf{q},q_{10},q_{20})=&\frac{\sigma r}{q_{10}+q_{20}}\left[q_{20}^2\sqrt{q_{10}^2-\mathbf{q}_{\mathrm{T}}^2}+q_{10}^2\sqrt{q_{20}^2-\mathbf{q}_{\mathrm{T}}^2}+\frac{q_{10}^2q_{20}^2}{|\mathbf{q}_{\mathrm{T}}|}\left(\arcsin\frac{|\mathbf{q}_{\mathrm{T}}|}{q_{10}}+\arcsin\frac{|\mathbf{q}_{\mathrm{T}}|}{q_{20}}\right)\right]-\frac{\sigma}{r}\Bigg[\frac{q_{20}}{\sqrt{q_{10}^2-\mathbf{q}_{\mathrm{T}}^2}}[\mathbf{r}\times\mathbf{q}\cdot\boldsymbol{\sigma}_1\\ &+iq_{10}(\mathbf{r}\cdot\boldsymbol{\alpha}_1)]+\frac{q_{10}}{\sqrt{q_{20}^2-\mathbf{q}_{\mathrm{T}}^2}}[\mathbf{r}\times\mathbf{q}\cdot\boldsymbol{\sigma}_2-iq_{20}(\mathbf{r}\cdot\boldsymbol{\alpha}_2)]\Bigg]\end{aligned} \tag{8}$$

and $\alpha_j^k=\gamma_j^0\gamma_j^k$, $\sigma_j^k=(i/4)\varepsilon^{knm}[\gamma_j^n,\gamma_j^m]$, $q_{j0}=(w_j+w_j')/2$, $j=1,2$, $\mathbf{Q}=\mathbf{k}-\mathbf{k}'$, $\mathbf{q}=(\mathbf{k}+\mathbf{k}')/2$, $q_{\mathrm{T}}^h=(\delta^{hk}-\hat{r}^h\hat{r}^k)q^k$, and $\hat{\mathbf{r}}=\mathbf{r}/r$.

The above expression was obtained by reduction of a Bethe-Salpeter like equation which was obtained in Ref. [5] from first principles QCD under the only assumption that the logarithm of the Wilson loop correlator W could be written as the sum of its perturbative expression and an area term

$$i\ln W=i(\ln W)_{\mathrm{pert}}+\sigma S \tag{9}$$

(advantage was taken in the derivation of an appropriate Feynman-Schwinger representation of the quark propagator in an external field).

An expression for $\langle \mathbf{k}|V|\mathbf{k}'\rangle$ can be obtained by a direct comparison of Eq. (6) with Eq. (5). Neglecting terms in V^2 (which is consistent with the other approximations), this amounts simply to changing the kinematical factor in front of the right-hand side of Eq. (7). Properly one should divide $\langle \mathbf{k}|U|\mathbf{k}'\rangle$ by $w_1+w_2+w_1'+w_2'$; in practice the simpler replacement $\sqrt{(w_1+w_2)(w_1'+w_2')/w_1w_2w_1'w_2'}\to 1/2\sqrt{w_1w_2w_1'w_2'}$ is essentially equivalent.

The interest of the more conventional Eq. (6) is that it makes more immediate a comparison with ordinary potential approaches and the consideration of the non-relativistic limit. In particular V coincides with the Cornell potential $\langle \mathbf{k}|V|\mathbf{k}'\rangle=\langle \mathbf{k}|(-\frac{4}{3}\alpha_s/r+\sigma r)|\mathbf{k}'\rangle$ in the static limit and with the potential obtained in [6] when the first relativistic corrections are included. In this paper, however, we shall refer only to Eq. (5), as more directly related to the original Bethe-Salpeter (BS) equation.

The method used in [4] consists in solving first the eigenvalue equation for M in the static limit of V by the Rayleigh-Ritz method (using a harmonic oscillator basis); then in evaluating the quantity $\langle M^2\rangle$ (or $\langle M\rangle$) for the resulting zero

TABLE I. $b\bar{b}$. $n_f=4$, $\Lambda=0.2$ GeV. (a) $m_b=4.763$ GeV, $\sigma=0.2$ GeV^2, $\alpha_s(0)=0.35$, Ref. [4]. (b) $m_b=4.898$ GeV, $\sigma=0.18$ GeV^2, Shirkov-Solovtsov $\alpha_s(Q^2)$. (c) $m_b=4.7605$ GeV, $\sigma=0.18$ GeV^2, Dokshitzer *et al.* $\alpha_s(Q^2)$.

State		Experiment (GeV)		(a) (GeV)	(b) (GeV)	(c) (GeV)
1^1S_0				9.374	9.374	9.375
1^3S_1	$\Upsilon(1S)$	9.46037 ± 0.00021		9.460	9.460	9.460
2^1S_0				9.975	9.988	9.983
2^3S_1	$\Upsilon(2S)$	10.02330 ± 0.00031		10.010	10.023	10.017
3^1S_0				10.322	10.342	10.328
3^3S_1	$\Upsilon(3S)$	10.3553 ± 0.0005		10.348	10.368	10.352
4^1S_0				10.598	10.618	10.391
4^3S_1	$\Upsilon(4S)$	10.5800 ± 0.0035		10.620	10.639	10.612
5^1S_0				10.837	10.854	10.816
5^3S_1	$\Upsilon(10860)$	10.865 ± 0.008		10.857	10.872	10.834
6^1S_0				11.060	11.070	11.026
6^3S_1	$\Upsilon(11020)$	11.019 ± 0.008		11.079	11.089	11.044
1^1P_1				9.908	9.918	9.914
1^3P_2	$\chi_{b2}(1P)$	9.9132 ± 0.0006				
1^3P_1	$\chi_{b1}(1P)$	9.8919 ± 0.0007	9.900	9.908	9.920	9.917
1^3P_0	$\chi_{b0}(1P)$	9.8598 ± 0.0013				
2^1P_1				10.260	10.279	10.269
2^3P_2	$\chi_{b2}(2P)$	10.2685 ± 0.0004				
2^3P_1	$\chi_{b1}(2P)$	10.2552 ± 0.0005	10.260	10.260	10.280	10.271
2^3P_0	$\chi_{b0}(2P)$	10.2321 ± 0.0006				

order eigenfunctions [7].

Actually in [4] we neglected the complicated spin orbit and tensorial terms occurring in Eq. (7) and took into account only the hyperfine term in $\frac{1}{6}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. We used the expression (1) for the running coupling constant with $n_f=4$ and $\Lambda=0.2$ GeV frozen at $\bar{\alpha}_s=0.35$; we have also taken $\sigma=0.2$ GeV^2, $m_u=m_d=10$ MeV,[1] $m_c=1.394$ GeV, $m_b=4.763$ GeV. The quantities Λ and $m_u=m_d$ were fixed *a priori* from high energy data; $\bar{\alpha}_s$, σ, m_c and m_b were adjusted on the ground $c\bar{c}$ and $b\bar{b}$ states, the $J/\psi-\eta_c$ splitting and the Regge trajectory slope.

In this paper, to test the sensitivity of the results to the infrared behavior of $\alpha_s(\mathbf{Q}^2)$, we have performed the same calculation for the $b\bar{b}$, $c\bar{c}$ and $q\bar{q}$ ($q=u$ or d) systems using Eqs. (3) and (4), with an appropriate redefinition of the adjustable parameters, as reported in the following section.

Notice that in the second order formalism developed in [5] an uncolored full quark propagator $H_2(p)$ occurs in the BS equation. In the reduction of the BS equation to Eqs. (5)–(8) two approximations are involved. The first is the so called instantaneous approximation for the kernel $I(Q,p,p')$; the second consists in replacing $H_2(p)$ by the free propagator $i/(p^2-m^2)$ where one has to set $p=(M_B/2\pm k_0,\pm\mathbf{k})$ in the c.m. frame (the upper and lower signs refer to the quark and the antiquark respectively). In principle however $H_2(p)$ should be given by the solution of a very complicated Dyson-Schwinger (DS) equation involving the same kernel $I(Q,p,p')$. If, consistently with what we have done for $\langle\mathbf{k}|U|\mathbf{k}'\rangle$, we neglected the spin dependent terms even in the DS equation we could more sensibly write $H_2(p)=i/[p^2-m^2+\Gamma(p)]$. In this case, since $I(Q,p,p')$ is given in [4,5] in terms of c.m. variables and is not formally covariant, $\Gamma(p)$ must depend separately on p_0^2 and $\mathbf{p}^2=\mathbf{k}^2$. Then the pole of $H_2(p)$ defined by

$$p_0^2-\mathbf{k}^2-m^2+\Gamma(p_0^2,\mathbf{k}^2)=0 \tag{10}$$

could be written

$$p_0^2=m_{\mathrm{eff}}^2(|\mathbf{k}|)+\mathbf{k}^2 \tag{11}$$

where $m_{\mathrm{eff}}^2(|\mathbf{k}|)$ would be a $|\mathbf{k}|$ dependent expression that is expected to approach the current m^2 for $|\mathbf{k}|\to0$ and to increase toward a kind of constituent m'^2 as $|\mathbf{k}|$ increases.[2]

[1] Notice that the results are very little sensitive to the precise values of m_u and m_d if these are small.

[2] Take into account that from the virial theorem for a linear potential in the extreme relativistic approximation we have $\langle|\mathbf{k}|\rangle=\sigma\langle r\rangle$ and a large $|\mathbf{k}|$ corresponds to a peripheral interaction or, which is the same, to a small $|\mathbf{Q}|$.

TABLE II. $c\bar{c}$. $n_f=4$, $\Lambda=0.2$ GeV. (a) $m_c=1.394$ GeV, $\sigma=0.2$ GeV^2, $\alpha_s(0)=0.35$, Ref. [4]. (b) $m_c=1.545$ GeV, $\sigma=0.18$ GeV^2, Shirkov-Solovtsov $\alpha_s(Q^2)$. (c) $m_c=1.383$ GeV, $\sigma=0.18$ GeV^2, Dokshitzer *et al.* $\alpha_s(Q^2)$.

States		Experiment (MeV)	(a) (MeV)	(b) (MeV)	(c) (MeV)
1^1S_0	$\eta_c(1S)$	2979.8 ± 2.1	2982	2977	2982
1^3S_1	$J/\psi(1S)$	3096.88 ± 0.04	3097	3097	3097
$1\ \Delta SS$		117	115	119	116
2^1S_0	$\eta_c(2S)$	3594 ± 5	3575	3606	3573
2^3S_1	$\psi(2S)$	3686.00 ± 0.09	3642	3670	3636
$2\ \Delta SS$		92	67	64	63
3^1S_0			3974	4005	3950
3^3S_1	$\psi(4040)$	4040 ± 10	4025	4054	3998
4^1S_0			4298	4323	4252
4^3S_1	$\psi(4415)$	4415 ± 6	4341	4364	4291
1^1P_1			3529	3556	3528
1^3P_2	$\chi_{c2}(1P)$	3556.17 ± 0.13			
1^3P_1	$\chi_{c1}(1P)$	3510.53 ± 0.12 } 3525	3530	3561	3531
1^3P_0	$\chi_{c0}(1P)$	3415.1 ± 1.0			
2^1P_1			3925	3954	3904
2^3P			3927	3958	3906
1^1D_2			3813	3853	3811
1^3D_3					
1^3D_2	$\psi(3836)$	3836 ± 13	3813	3854	3811
1^3D_1	$\psi(3770)$	3769.9 ± 2.5			
2^1D_2			4149	4183	4121
2^3D_3					
2^3D_2			4149	4184	4121
2^3D_1	$\psi(4160)$	4159 ± 20			

Then, eventually, we should obtain again the operator M^2 as given by Eq. (5), but with $m_1=m_2$ replaced by $m_{\rm eff}^2(|\mathbf{k}|)$.

An actual evaluation of $\Gamma(p)$ by solving the DS equation even in the simplified form mentioned above is out of the scope of the present paper, and in the case of prescription (4), we have completely ignored the above complication using a fixed value for the quark masses. In the case of prescription (3) however, and for the light-light system alone, we have also tried to use a phenomenological expression for $m_{\rm eff}^2(|\mathbf{k}|)$ parametrized as a polynomial in $|\mathbf{k}|$ in the interval of interest.

Notice that the idea of a scale dependent effective mass, which is essentially equivalent to our dependence on k, has been considered by various authors from various point of view. In the framework of the Dyson-Schwinger equation see in particular [8] and references therein. In a renormalization group perspective see [9] (as well as [1]) and see [10] for numerical simulations.

III. RESULTS

In Tables I, II, and III we give the $b\bar{b}$, $c\bar{c}$, and $q\bar{q}$ ($q=u$ or d) quarkonium masses respectively obtained for the different running coupling constant prescriptions. In column (a) we report the results obtained in Ref. [4] for the truncated $\alpha_s(\mathbf{Q}^2)$, in column (b) those obtained by means of Eq. (3) proposed by Shirkov-Solovtsov and in column (c) those obtained by means of the $\alpha_s(\mathbf{Q}^2)$ of Eq. (4) proposed by Dokshitzer *et al.*

In columns (b) and (c) we used the same values $n_f=4$, $\Lambda=0.2$ GeV and $m_u=m_d=10$ MeV as in [4], but we have slightly redefined the adjustable parameters, taking $\sigma=0.18$ GeV^2 in both cases and $m_c=1.545$ GeV and $m_b=4.898$ GeV for prescription (3) [column (b)], $m_c=1.383$ GeV and $m_b=4.7605$ GeV for prescription (4) [column (c)].

Notice that, in spite of the reduced number of adjustable parameters, the spectra of bottonium and charmonium are not essentially modified by the new choice for $\alpha_s(\mathbf{Q}^2)$, with perhaps the exceptions of the highest $c\bar{c}$ states that are lower in the Dokshitzer *et al.* case. This indicates little sensitivity of such spectra to the infrared behavior of $\alpha_s(\mathbf{Q}^2)$.

The situation is completely different for the light-light spectrum of Table III. While in view of the experimental and theoretical uncertainties columns (a) and (c) can be considered not really distinguishable, the values reported in column

TABLE III. $q\bar{q}$ ($q=u$ or d), $m_{\mathrm{u,d}}=0.01$ GeV, $n_f=4$, $\Lambda=0.2$ GeV. (a) $\alpha_s(0)=0.35$, Ref. [4], $\sigma=0.2\ \mathrm{GeV}^2$. (b) Shirkov-Solovtsov $\alpha_s(Q^2)$, $\sigma=0.18\ \mathrm{GeV}^2$. (c) Dokshitzer *et al.* $\alpha_s(Q^2)$, $\sigma=0.18\ \mathrm{GeV}^2$.

States		Experiment (MeV)		(a) (MeV)	(b) (MeV)	(c) (MeV)
1^1S_0	π^0	134.9764 ± 0.0006		479	–	575
	$\pi^\pm$	139.56995 ± 0.00035				
1^3S_1	$\rho(770)$	768.5 ± 0.6		846	423	904
$1\ \Delta SS$		630		367	–	329
2^1S_0	$\pi(1300)$	1300 ± 100		1326	952	1338
2^3S_1	$\rho(1450)$	1465 ± 25		1461	1128	1459
$2\ \Delta SS$		165		135	176	121
3^1S_0	$\pi(1800)$	1795 ± 10		1815	1485	1793
3^3S_1	$\rho(2150)$	2149 ± 17		1916	1600	1889
$3\ \Delta SS$		354		101	115	96
1^1P_1	$b_1(1235)$	1231 ± 10				
1^3P_2	$a_2(1320)$	1318.1 ± 0.7				
1^3P_1	$a_1(1260)$	1230 ± 40	1303	1333	1045	1365
1^3P_0	$a_0(1450)$	1450 ± 40				
1^1D_2	$\pi_2(1670)$	1670 ± 20				
1^3D_3	$\rho_3(1690)$	1691.1 ± 5				
1^3D_2				1701	1444	1715
1^3D_1	$\rho(1700)$	1700 ± 20				
1^1F_3						
1^3F_4	$a_4(2040)$	2037 ± 26				
1^3F_3	$X(2000)$			1990	1743	1985
1^3F_2						
1^1G_4						
1^3G_5	$\rho_5(2350)$	2330 ± 35				
1^3G_4				2238	1994	2214
1^3G_3	$\rho_3(2250)$					
1^1H_5						
1^3H_6	$a_6(2450)$	2450 ± 130				
1^3H_5				2460	2215	2416
1^3H_4						

(b) are definitely systematically too low (in particular $\langle M^2 \rangle < 0$ for the π meson).

Notice however that in the above reported calculations we have used for m_{u} and m_{d} the current mass value of 10 MeV. This amounts to assuming the difference between the current and the constituent masses to be essentially related to the kinematical relativistic correction (cf. [4]) or, which is the same, that the free quark propagator is a good approximation for the complete one in the BS equation. The inability of the formalism to reproduce a reasonable value for the π mass and all experience gained by the chiral symmetry problem ([8], see also [10] for lattice simulations) suggest that this should not be the case for the light-light systems.

For this reason we have repeated the calculation for choice (3) with various constituent values for the light quark masses.[3] Two sets of results are reported in columns (d) and (e) of Table IV. Notice that for $m_{\mathrm{u}}=m_{\mathrm{d}}=0.30$ GeV the situation is again very similar to those of column (a), Table III. Notice also however that as $m_{\mathrm{u,d}}$ increases, the bound state masses uniformly increase and that for low values of $m_{\mathrm{u,d}}$ the lowest bound state masses can be made to agree fairly well with the data; for high values the same occurs for higher

[3]On the use of effective masses for the quarks, outside the quark models, see three Refs. [1,8,10]; see also Refs. [9,11].

TABLE IV. $q\overline{q}$ ($q=u$ or d), $n_f=4$. Shirkov-Solovtsov $\alpha_s(Q^2)$. $\sigma=0.18$ GeV^2. $\Lambda=0.2$ GeV. (d) $m_{\mathrm{u,d}}=0.22$ GeV. (e) $m_{\mathrm{u,d}}=0.30$ GeV. (f) $m^2_{\mathrm{u,d}}=0.11\,k-0.025\,k^2+0.265\,k^4$.

States	(MeV)	Experiment		(d)	(e)	(f)
1^1S_0	π^0 $\pi^{\pm}$	134.9764 ± 0.0006 139.56995 ± 0.00035		26	473	124
1^3S_1	$\rho(770)$	768.5 ± 0.6		725	868	737
$1\Delta SS$		630		699	394	613
2^1S_0	$\pi(1300)$	1300 ± 100		1190	1326	1401
2^3S_1	$\rho(1450)$	1465 ± 25		1344	1468	1508
$2\Delta SS$		165		154	142	107
3^1S_0	$\pi(1800)$	1795 ± 10		1688	1806	1993
3^3S_1	$\rho(2150)$	2149 ± 17		1788	1900	2063
$3\Delta SS$		354		100	94	70
1^1P_1	$b_1(1235)$	1231 ± 10				
1^3P_2	$a_2(1320)$	1318.1 ± 0.7				
1^3P_1	$a_1(1260)$	1230 ± 40	1303	1243	1364	1319
1^3P_0	$a_0(1450)$	1450 ± 40				
1^1D_2	$\pi_2(1670)$	1670 ± 20				
1^3D_3	$\rho_3(1690)$	1691.1 ± 5				
1^3D_2				1603	1715	1741
1^3D_1	$\rho(1700)$	1700 ± 20				
1^1F_3						
$1\ ^3F_4$	$a_4(2040)$	2037 ± 26				
1^3F_3	$X(2000)$			1881	1979	2043
1^3F_2						
1^1G_4						
1^3G_5	$\rho_5(2350)$	2330 ± 35				
1^3G_4				2118	2209	2319
1^3G_3	$\rho_3(2250)$					
1^1H_5						
1^3H_6	$a_6(2450)$	2450 ± 130				
1^3H_5				2329	2415	2569
1^3H_4						

states.

According to the discussion given at the end of the preceding section this seems to suggest the use of a phenomenological k dependent effective mass.

In column (f) the results are reported for

$$m^2_{\mathrm{eff}}=0.11k-0.025k^2+0.265k^4, \tag{12}$$

k denoting the modulus of the quark momentum in the center of mass frame. In Eq. (12) the coefficients are chosen in order to obtain $m_{\mathrm{eff}}=0.22$, 0.28, 0.35 GeV for $k^2=0.26$, 0.41, 0.58 GeV^2 approximately corresponding to the $\langle k^2\rangle$ values for the $1S$, $1D$ and $1G$ states respectively. As can be seen the agreement with the data is much improved in this way and finally even a reasonable value for the π mass is obtained.

IV. CONCLUSIONS AND DISCUSSION

In conclusion the heavy quarkonium spectrum does not seem to be very sensitive to the specific infrared behavior of the running coupling constant as could be expected *a priori* since not too small values of Q^2 are implied in this case.

For the light quarkonium the Dokshitzer *et al.* prescription (4) does not essentially change the results in comparison with the truncation assumption adopted in [4]. Even this is not surprising since the average value of $\alpha_s(Q^2)$ for $Q^2<1$ GeV^2 is roughly equal in the two cases in spite of the very different appearance of the corresponding curves in Fig. 1.

On the contrary the situation changes drastically for prescription (3).

Actually such a prescription would be definitely ruled out, if we insisted on using a fixed current mass for the light quarks. This is clearly due to the marked increase of $\alpha_s(Q^2)$ for $Q^2 \to 0$. Notice however that in this case a much more reasonable agreement with the data can be obtained if one uses a phenomenological running effective mass of the type (12) and then even the π mass and the ρ-π separation turn out fair. Obviously a fine tuning of the coefficients in Eq. (12) could further improve the agreement with the data, but we feel that this would be meaningless in the present context.

Notice finally that by the use of prescription (4) we are left with only three adjustable parameters in the theory (m_c, m_b and σ), with prescription (3) and (12) on the contrary we are left only with the parameter σ in the potential, but we have introduced three additional adjustable parameters by Eq. (12).

ACKNOWLEDGMENTS

We gratefully acknowledge a discussion with Professor Shirkov who recalled our attention to the interest of the problem.

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