Saturation and Wilson line distribution

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We introduce a Wilson line distribution function $\overline{W}_\tau(v)$ to study gluon saturation at small Feynman x_F, or large $\tau=\ln(1/x_F)$. This new distribution can be obtained from the distribution $W_\tau(\alpha)$ of the color glass condensate model and the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) renormalization group equation. $\overline{W}_\tau(v)$ is physically more relevant, and mathematically simpler to deal with because of unitarity of the Wilson line v. A JIMWLK equation is derived for $\overline{W}_\tau(v)$; its properties are studied. These properties are used to complete Mueller's derivation of the JIMWLK equation, though for $\overline{W}_\tau(v)$ and not $W_\tau(\alpha)$. They are used to derive a generalized Balitsky-Kovchegov equation for higher multipole amplitudes. They are also used to compute the unintegrated gluon distribution at $x_F=0$, yielding a completely flat spectrum in transverse momentum squared $\mathbf{k}^2$, with a known height. This is similar but not identical to the mean field result at small $\mathbf{k}^2$.

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I. INTRODUCTION

Soft gluons are produced by radiation from more energetic partons. Since the number of sources increases at small Feynman x_F, the soft gluon density $x_F G$ per unit rapidity interval increases with $\tau=\ln(1/x_F)$. In fact, both the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [1] and the Balitskiĭ-Fadin-Kuraev-Lipatov (BFKL) equation [2] predict a growth so fast that the unitarity bound τ^2 is violated. To restore unitarity, a new mechanism is required to slow down the growth toward $x_F=0$ [3]. The momentum Q_s at which this mechanism sets in is known as the *saturation momentum*.

The phenomenological implications for the presence of a saturation momentum have been discussed in many papers [4], but it is not yet clear whether saturation has been observed experimentally. On the theoretical side, the important thing to note for our present discussion is that soft gluons can be treated as a classical color potential $\alpha(\vec{x})$, because of its large density at small x_F. In this background, energetic partons interact with soft gluons through their Wilson lines. It is this interaction that is responsible for the saturation process.

With the replacement of soft gluons by a classical Yang-Mills potential, the growth of soft gluon density is determined by the τ dependence of the distribution function $W_\tau(\alpha)$. This dependence is given by a renormalization group equation known as the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) equation [5].

In this paper we introduce and study a distribution $\overline{W}_\tau(v)$ for the Wilson line v. We will show that this distribution can be obtained from $W_\tau(\alpha)$, and that it still satisfies the JIMWLK equation. Compared to $W_\tau(\alpha)$, it has the advantage of being more directly physically relevant, because many physical processes can be described in terms of Wilson lines or dipole amplitudes. Moreover, the Wilson lines v are unitary matrices, living on the color group manifold, which is compact. This compactness brings with it a number of mathematical advantages not shared by $W_\tau(\alpha)$, whose argument α lives in a noncompact linear space.

$\overline{W}_\tau(v)$ can be used to derive a number of properties not easily obtainable directly from $W_\tau(\alpha)$. We will use these properties to complete Mueller's proof of the JIMWLK equation [6], although the proof is valid for $\overline{W}_\tau(v)$ and not for $W_\tau(\alpha)$. They will be used to derive a generalized Balitsky-Kovchegov (BK) equation for multipole amplitudes. We find that once the nonlinear BK equation for the dipole amplitude is solved, all subsequent multipole amplitudes can be obtained by solving only linear equations. This means that saturation of the dipole amplitude automatically leads to saturation of higher multipole amplitudes.

$\overline{W}_\tau(v)$ will also be used to compute the asymptotic behavior of unintegrated gluon distribution at $x_F=0$. We get a flat distribution in the gluon transverse momentum $\mathbf{k}^2$, with a height given by Eq. (6.7). This is to be contrasted with the mean field result (2.16) which yields a logarithmic dependence on $\mathbf{k}^2$ with an undetermined normalization. Preasymptotic corrections will also be briefly discussed.

In the next section we start with a short review of saturation, the JIMWLK equation, the related BK equation, and the BFKL equation, as well as some of their solutions. In Sec.

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III, the Wilson line distribution $\overline{W}_\tau(v)$ is introduced, together with some mathematical preliminaries. The JIMWLK equation for $\overline{W}_\tau(v)$ is derived, and its properties studied. In Sec. IV, the missing steps of Mueller's derivation of the JIMWLK equation are supplied. In Sec. V, the infrared divergence encountered in the JIMWLK equation is discussed. It is shown that certain multipole functions which we call multipole traces are free of these divergences. A generalized BK equation is derived to describe the evolution of the multipole amplitudes. In Sec. VI, we discuss the property and implications of the asymptotic solution of $\overline{W}_\tau(v)$ at $x_F=0$. In particular, the unintegrated momentum spectrum of the gluon density is derived. Correction to the asymptotic limit when $x_F\neq 0$ is briefly discussed. Certain mathematical details can be found in Appendixes A and B at the end.

After this paper was submitted for publication, we were informed of an interesting recent paper [7] in which $\overline{W}_\tau(v)$ was also introduced and its JIMWLK equation derived. It went on to give a random-walk interpretation of the JIMWLK equations, with τ playing the role of time, v the position in the group manifold, and α the velocity.

II. A BRIEF REVIEW

The number of gluons emitted by a valence quark, per unit rapidity $\tau=\ln(1/x_F)$ and per unit transverse momentum squared $\mathbf{k}^2$, is given in perturbation theory by $\alpha_s C_F/\pi\mathbf{k}^2$, where $\alpha_s=g^2/4\pi$ is the QCD fine structure constant. C_F is the Casimir number in the fundamental representation, being $(N_c^2-1)/2N_c$ for the gauge group $SU(N_c)$, and $\frac{1}{2}N_c$ for $U(N_c)$. A nucleus with atomic number A has AN_c valence quarks, so its unintegrated gluon distribution is $dN/d\tau d\mathbf{k}^2 \equiv d(x_F G)/d\mathbf{k}^2=\alpha_s C_F(AN_c)/\pi\mathbf{k}^2$.

When integrated, this formula encounters an infrared divergence at small $\mathbf{k}^2$, brought about by the long range gluon field of the unshielded valence quarks. However, quarks are confined inside color-singlet nucleons, so such a long range force is absent beyond the nucleon radius a. Thus $x_F G(x_F,Q^2)=[\alpha_s C_F(AN_c)/\pi]\ln(Q^2a^2)$.

In the central rapidity region where the Feynman x_F is small, the gluon density is much larger than the amount given by the perturbation formula. This is so because soft gluons can be radiated also from energetic gluons and sea quarks, not just from the valence quarks considered so far. According to the DGLAP [1] equation, the soft gluon density grows as $\exp(\kappa\sqrt{\tau})$ for some positive constant κ, and, according to the BFKL equation [2], it grows as $\exp[4\alpha_s\ln(2)N_c\tau/\pi]$. Both exceed the unitarity limit τ^2, so a new mechanism must kick in to dampen the growth and restore unitarity at small x_F [3]. This effect is known as *saturation*.

Saturation is thought to arise from a nonlinear mechanism which occurs when gluons are sufficiently dense to interact among themselves [8,9]. The number of gluons per unit rapidity interval is $x_F G$. In a nucleus of radius R_A, the transverse area per gluon is therefore $\pi R_A^2/x_F G(x_F,Q^2)$. The average color-charge squared of a gluon is $N_c/(N_c^2-1)$, their interaction strength is $\sim\alpha_s/\pi$, and their natural size is $\sim 1/Q$. Hence the cross section for two gluons to interact can be estimated to be $(\alpha_s/\pi)[N_c/(N_c^2-1)](\pi/Q^2)$. If the cross section is larger than the transverse area per gluon, then interaction will take place to set off the nonlinear mechanism. The onset therefore occurs at a momentum scale Q_s such that

$$\pi R_A^2/x_F G(x_F,Q_s^2)=c(\alpha_s/\pi)[N_c/(N_c^2-1)](\pi/Q_s^2), \tag{2.1}$$

or, equivalently,

$$Q_s^2=c\,\frac{\alpha_s N_c}{N_c^2-1}\,\frac{x_F G(x_F,Q_s^2)}{\pi R_A^2}. \tag{2.2}$$

A constant c has been inserted to account for the qualitative nature of this argument. Even when Q_s^2 is obtained from a detailed calculation, the constant c is still somewhat ambiguous because transition into saturation does not occur sharply. Thus one finds a number of c's used in the literature. For example, $c=1$ in [10], $c=\pi^2$ in [9] when estimated from the mean field approximation on the large $\mathbf{k}^2$ side, and $c=16\pi^2 c_1$ when estimated on the small $\mathbf{k}^2$ side, where c_1 is some unknown constant. And $c=\pi$ in [11].

Q_s can also be defined through the unintegrated gluon density $d(x_G G)/d\mathbf{k}^2$. When we reduce $\mathbf{k}^2$ from infinity, this density increases until a point $\mathbf{k}^2=Q_s^2$ when gluons become sufficiently dense to set off the nonlinear mechanism. From there on we enter a saturation region with much slower growth. However, this definition is also ambiguous unless the slowdown occurs fairly sharply, which turns out to be the case at $x_F=0$. As we shall see in Sec. VI A, at $x_F=0$, the saturation region is large and the unintegrated spectrum $d(x_F G)/d\mathbf{k}^2$ in this region is absolutely flat in $\mathbf{k}^2$. This then allows c to be determined unambiguously to be $c=8\pi^3$.

Using the BFKL solution for $x_F G$ as a qualitative estimate, and assuming that $x_F G$ is proportional to A, we see from Eq. (2.2) that Q_s^2 grows with a power of $1/x_F$ and $A^{1/3}$, making it large for large nucleus or small x_F. Equation (2.2) also implies that the gluon number per unit transverse area at saturation is $\sim Q_s^2(x)/\alpha_s$.

The large number of gluons present at saturation allows them to be treated as a classical Yang-Mills (YM) potential $\alpha^a(\vec{x})$. The superscript a is the color index, and $\vec{x}=(x^-,\mathbf{x})$ are the light cone (LC) coordinates, defined for a hadron moving along the $+z$ direction to be $x^\pm=(t\pm z)/\sqrt{2}$ and $\mathbf{x}=(x^1,x^2)$. It is also convenient to introduce the spacetime rapidity variable $y=\ln(x^-P^+)$, where P^+ is the $+$ component of the hadron momentum, and the gluon potential $\alpha_y^a(\mathbf{x})=x^-\alpha(x^-,\mathbf{x})$.

For a fast moving hadron (or nucleus), Lorentz contraction forces $\alpha(\vec{x})$ to be concentrated around $x^-=0$, and time dilation makes it effectively (LC) time (x^+) independent. The soft gluons are produced by partons within the hadron, so one can assume $\alpha_y^a(\mathbf{x})=0$ for $y>\tau$ [9].

Energetic partons, whether in the same hadron or not, interact with the soft gluons through the Wilson line factors[1]

$$
\begin{aligned}
v^\dagger(\mathbf{x}) &= P\exp\left(+ig\int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x})t_a\right)\\
&= P\exp\left(+ig\int_{-\infty}^{\tau} dy\ \alpha_y^a(\mathbf{x})t_a\right),\\
v(\mathbf{x}) &= \tilde{P}\exp\left(-ig\int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x})t_a\right)\\
&= \tilde{P}\exp\left(-ig\int_{-\infty}^{\tau} dy\ \alpha_y^a(\mathbf{x})t_a\right),\\
V^\dagger(\mathbf{x}) &= P\exp\left(+ig\int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x})T_a\right)\\
&= P\exp\left(+ig\int_{-\infty}^{\tau} dy\ \alpha_y^a(\mathbf{x})T_a\right),\\
V(\mathbf{x}) &= \tilde{P}\exp\left(-ig\int_{-\infty}^{\infty} dx^- \alpha^a(\vec{x})T_a\right)\\
&= \tilde{P}\exp\left(-ig\int_{-\infty}^{\tau} dy\ \alpha_y^a(\mathbf{x})T_a\right),
\end{aligned}
\tag{2.3}
$$

where P and $\tilde{P}$ indicate, respectively, path ordering and antipath ordering. The first two expressions describe the propagation of quarks and antiquarks, respectively, through the dense background of the soft gluons, and the last two expressions describe the propagation of gluons. These Wilson lines play a central role in the rest of the paper.

In this representation of soft gluons by a classical background field, the gluon distribution is determined by the distribution $W_\tau(\alpha)$ of the YM potential. W_τ depends on τ because the number of sources available to emit soft gluons increases at small x_F.

The resulting change of the distribution functional $W_\tau(\alpha)$ can be shown to satisfy the JIMWLK renormalization group equation [5]

$$
\frac{\partial W_\tau(\alpha)}{\partial \tau} = -HW_\tau(\alpha), \tag{2.4}
$$

where

$$
\begin{aligned}
H &= \frac{1}{\pi}\int d^2\mathbf{z}d^2\mathbf{x}d^2\mathbf{y}K(\mathbf{xy}|\mathbf{z})\mathcal{O}(\mathbf{xy}|\mathbf{z}),\\
\mathcal{O}(\mathbf{xy}|\mathbf{z}) &= \frac{\delta}{\delta\alpha_\tau^a(\mathbf{x})}[V^\dagger(\mathbf{x})-V^\dagger(\mathbf{z})]_{ac}\\
&\quad\times[V(\mathbf{y})-V(\mathbf{z})]_{cb}\frac{\delta}{\delta\alpha_\tau^b(\mathbf{y})},
\end{aligned}
\tag{2.5}
$$

and

$$
K(\mathbf{xy}|\mathbf{z}) = \frac{1}{4\pi^3}\frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}. \tag{2.6}
$$

A consequence of Eqs. (2.4) and (2.5) is that the normalization $\int\mathcal{D}[\alpha]W_\tau(\alpha)$ is independent of τ. We will normalize it to be 1, so that the average of any functional of α is given simply by $\langle F\rangle_\tau = \int\mathcal{D}[\alpha]F(\alpha)W_\tau(\alpha)$.

The functional derivatives $\delta/\delta\alpha_\tau^a$ of $V^\dagger$ and V in Eq. (2.5), and similarly those of $v^\dagger$ and v that we will encounter later, are

$$
\begin{aligned}
\frac{\delta V^\dagger(\mathbf{z})}{\delta\alpha_\tau^a(\mathbf{x})} &= igT_aV^\dagger(\mathbf{z})\delta(\mathbf{x}-\mathbf{z}),\\
\frac{\delta v^\dagger(\mathbf{z})}{\delta\alpha_\tau^a(\mathbf{x})} &= igt_av^\dagger(\mathbf{z})\delta(\mathbf{x}-\mathbf{z}),\\
\frac{\delta V(\mathbf{z})}{\delta\alpha_\tau^a(\mathbf{x})} &= -igV(\mathbf{z})T_a\delta(\mathbf{x}-\mathbf{z}),\\
\frac{\delta v(\mathbf{z})}{\delta\alpha_\tau^a(\mathbf{x})} &= -igV(\mathbf{z})t_a\delta(\mathbf{x}-\mathbf{z}).
\end{aligned}
\tag{2.7}
$$

For calculational simplicity it is useful to note that $V^\dagger(x)-V^\dagger(z)$ in Eq. (2.5) may be put in front of the operator $\delta/\delta\alpha_\tau^a(\mathbf{x})$. This follows from Eq. (2.7) and the observation that $(T_a)_{ac} = if_{aac} = 0$.

A particularly important physical quantity to study is the dipole amplitude [12]

$$
s_\tau(\mathbf{x},\mathbf{y}) = \frac{1}{N_c}\langle\mathrm{tr}[v^\dagger(\mathbf{x})v(\mathbf{y})]\rangle_\tau. \tag{2.8}
$$

At coincident points

$$
s_\tau(\mathbf{x},\mathbf{x}) = 1 \tag{2.9}
$$

because of unitarity of v. It can be shown from Eq. (2.4) that the dipole amplitude satisfies the Balitsky equation [13]

[1]We shall use upper-case letters to denote the adjoint representation and lower-case letters to denote the defining representation. In this notation, the generators in the defining representation will be denoted by t_a, and they will be normalized to be $\mathrm{tr}(t_at_b) = (1/2)\delta_{ab}$. The generators in the adjoint representation are denoted by T_a. They are related to the totally antisymmetric structure constants by $(T_a)_{bc} = if_{bac}$. Hence $(T_a)_{bc}$ is imaginary and totally antisymmetric in the three indices. Similarly, the quark and antiquark Wilson lines will be denoted by the lower-case letters $v^\dagger$ and v, and the gluon Wilson line will be denoted by the upper-case letters $V^\dagger$ and V. A slight drawback of this convention is that we are forced to denote the dipole amplitude (2.8) by a lower-case letter s_τ, whereas the usual notation for it is S_τ.

$$\frac{\partial s_\tau(\mathbf{x},\mathbf{y})}{\partial\tau} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}\cdot\Big\{s_\tau(\mathbf{x},\mathbf{y}) - \frac{1}{N_c^2}\langle \mathrm{tr}[v^\dagger(\mathbf{x})v(\mathbf{z})]\mathrm{tr}[v^\dagger(\mathbf{z})v(\mathbf{y})]\rangle_\tau\Big\}. \tag{2.10}$$

Note that the infrared divergence occuring at large $\mathbf{z}$ in Eq. (2.5) is absent in the Balitsky equation.

For large N_c, the last term factorizes and we arrive at the Kovchegov equation [14]

$$\frac{\partial s_\tau(\mathbf{x},\mathbf{y})}{\partial\tau} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}\cdot\{s_\tau(\mathbf{x},\mathbf{y}) - s_\tau(\mathbf{x},\mathbf{z})s_\tau(\mathbf{z},\mathbf{y})\}. \tag{2.11}$$

Since $s_\tau(\mathbf{x},\mathbf{x})=1$, the quantity $t_\tau(\mathbf{x},\mathbf{y})\equiv 1-s_\tau(\mathbf{x},\mathbf{y})$ is expected to be small when $\mathbf{x}\sim\mathbf{y}$. If we deal with soft gluons outside the saturation region, when $\alpha(\vec{x})$ is small, the Wilson lines v and $v^\dagger$ are close to 1 anyway, so we expect to be able to drop the quadratic term $t_\tau(\mathbf{x},\mathbf{z})t_\tau(\mathbf{z},\mathbf{y})$. In this way we get the dipole form of the BFKL equation [2]

$$\frac{\partial t_\tau(\mathbf{x},\mathbf{y})}{\partial\tau} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\mathbf{z}\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2}\cdot\{t_\tau(\mathbf{x},\mathbf{y}) - t_\tau(\mathbf{x},\mathbf{z}) - t_\tau(\mathbf{z},\mathbf{y})\}. \tag{2.12}$$

For $|\mathbf{x}-\mathbf{y}|^2\gg 1/Q_s^2$ and inside the saturation region, the strong classical YM potential α causes large and independent oscillations to both Wilson lines $v^\dagger$ and v. Consequently the dipole amplitude $s_\tau(\mathbf{x}-\mathbf{y})$ is expected to be small, thus enabling the nonlinear term in Eq. (2.11) to be dropped. The solution of the resulting linear equation can be shown to be [8]

$$s_\tau(\mathbf{x}-\mathbf{y}) = \exp\left[-\frac{\alpha_s N_c}{\pi}\int_{\tau_0}^{\tau} dy\,\ln[Q_s^2(y)\mathbf{x}^2]\right]s_{\tau_0}(\mathbf{x}-\mathbf{y}). \tag{2.13}$$

If $Q_s^2(y)=\exp[c\alpha_s N_c(y-\tau_0)/\pi]Q_s^2(\tau_0)$ [15], then

$$s_\tau(\mathbf{x}-\mathbf{y}) = \exp\left[-\frac{c}{2}\left(\frac{\alpha_s N_c}{\pi}\right)^2(\tau-\tau_0)^2\right]s_{\tau_0}(\mathbf{x}-\mathbf{y}), \tag{2.14}$$

provided τ_0 is chosen so that $Q_s^2(\tau_0)(\mathbf{x}-\mathbf{y})^2=1$. This condition implies $(\mathbf{x}-\mathbf{y})^2=Q_s^{-2}(\tau_0)\gg Q_s^{-2}(\tau)$, if $\tau\gg\tau_0$. The solution (2.14) then confirms the expectation that s_τ is small in that region.

To solve any of these evolution equations we need an initial condition at some $\tau=\tau_0$. For a large nucleus, and a τ_0 where the source is dominated by the valence quarks, the initial condition is provided by the McLerran-Venugopalan model [16], in which a Gaussian distribution is assumed for $W_{\tau_0}(\alpha)$. For large A and small $\alpha_s(Q_s^2)$, the Gaussian distribution can be shown to be a good approximation [17]. Saturation is now provided by the valence quarks alone, so A has to be very large and the resulting Q_s^2 is relatively small. The detail of the confinement which affects the small $\mathbf{k}^2$ region then becomes relatively important [18].

With a Gaussian distribution, the perturbative gluon distribution is modified by a gluon dipole factor [9]

$$S_{x^-}(\mathbf{r}) = \frac{1}{N_c^2-1}\langle \mathrm{Tr}\,U^\dagger(x^-,\mathbf{r})U(x^-,\mathbf{0})\rangle, \tag{2.15}$$

where $U^\dagger$ is equal to $V^\dagger$ in Eq. (2.3) with τ replaced by x^-, and $\mathbf{r}$ is the conjugate variable to the transverse momentum $\mathbf{k}$. This factor gives rise to the nonlinear effect that is responsible for saturation, with a saturation momentum Q_s given by Eq. (2.2).

For $\tau>\tau_0$, the $W_\tau(\alpha)$ determined by Eq. (2.4) no longer has a Gaussian distribution. Nevertheless, in a mean field approximation, the approximate solution is still Gaussian. The gluon distribution for small $\mathbf{k}^2$ is then given by [9]

$$\frac{d(x_F G)}{d^2\mathbf{k}} \simeq c_2\frac{N_c^2-1}{N_c}\frac{\pi R_A^2}{\alpha_s}\ln\frac{Q_s^2(\tau)}{\mathbf{k}^2} \qquad [\mathbf{k}^2\ll Q_s^2(\tau)] \tag{2.16}$$

for some constant c_2.

III. WILSON LINE DISTRIBUTION

We shall show later in this section that the distribution $W_\tau(\alpha)$ of the YM potential α leads to a distribution $\overline{W}_\tau(v)$ of the Wilson lines.

The notation might suggest that $\overline{W}_\tau(v)$ describes only the distribution of the anti-quark Wilson line v, but actually it provides a distribution for the Wilson lines of other partons as well. Since v is unitary, $v^\dagger=v^{-1}$, the variable $v^\dagger$ is a rational function of v_{ij}, so $\overline{W}_\tau(v)$ does provide the distribution for quark Wilson lines $v^\dagger$.

From the group-theoretical relation

$$t_a V^\dagger_{ac} = v^\dagger t_c v = V_{cb}t_b, \tag{3.1}$$

or, equivalently, the relation

$$V^\dagger_{ac} = 2\,\mathrm{tr}(v^\dagger t_c v t_a) = V_{ca}, \tag{3.2}$$

gluon Wilson lines can be expressed as a quark-antiquark pair of Wilson lines, so we can also compute the distribution of gluon Wilson lines from $\overline{W}_\tau(v)$.

Physical observables are often expressible in terms of Wilson lines, so it is clearly desirable to know their distributions directly. Moreover, the Wilson line v is unitary, which allows the theory of representation of the unitary group to be used for computations. For example, an orthonormal complete set of polynomials, given by the irreducible representations of the unitary group, exists on the group manifold. Therefore a harmonic analysis of $\overline{W}_\tau(v)$ and other physical quantities can be carried out to allow their integrals to be

computed. In contrast, $W_\tau(\alpha)$ is a function on a noncompact linear space of the Lie algebra, and the only functional that can be integrated in practice is the Gaussian.

We shall show that $\overline{W}_\tau(v)$ satisfies the same JIMWLK equation (2.4), but with H replaced by $\overline{H}$. The latter is obtained from H simply by replacing the α derivative by a differential operator in v. Therefore, $\overline{H}$ is still Hermitian and positive semidefinite.

Since the group manifold is compact, the spectrum and eigenfunctions of $\overline{H}$ are more manageable. For example, $\overline{W}_\tau(v)=1$ is a normalized eigenfunction of $\overline{H}$ with zero eigenvalue, whereas $W_\tau(\alpha)=1$ has a divergent integral in the α space.

Before showing how $\overline{W}_\tau(v)$ is obtained from $W_\tau(\alpha)$, let us first review some basic facts about integrations and orthonormal relations on a compact Lie group.

A. Inner products on the color group

For the sake of definiteness we shall assume the color group to be $U(n)$, although a similar analysis can be carried out for $SU(n)$. The n^2-dimensional group $U(n)$ will be parametrized by the n^2 matrix elements $v_{ij}(1\leqslant i,j\leqslant n)$ in the defining representation. Unitarity equates $v^\dagger$ to v^{-1}, so v^*_{ij} is to be regarded as a dependent variable, given as a rational function of v_{kl} via this relation.

The invariant volume element on $U(n)$ will be denoted by $d_H[v]$. This Haar measure is a left and right invariant n^2-form, satisfying

$$d_H[v]=d_H[v_0v]=d_H[vv_0] \tag{3.3}$$

for any constant $v_0\in U(n)$. It is positive, and it shall be normalized to $\int d_H[v]=1$.

The Haar measure $d_H[v]$ is proportional but not equal to the product measure $d[v]$, obtained by taking the exterior product of the n^2 1-forms dv_{ij}. They differ by a Jacobian $J(v)$, so

$$d_H[v]=J(v)d[v]. \tag{3.4}$$

To get an idea how this comes about, consider a change dv in the vicinity of a group element $v\in U(n)$. Then $v^{-1}dv$ constitutes a change around the identity, so it can be parametrized in the form $-it_a(d\eta_a)$. The volume element at the identity is proportional to the exterior product of the $n^2 d\eta_a$'s, or, equivalently, the exterior product of $2i\mathrm{tr}(v^{-1}dv)$. If we want the volume element to be left and right invariant, as in Eq. (3.3), this expression should be taken as the volume element at any point $v\in U(n)$. The presence of v^{-1} in this expression is the origin of the Jacobian $J(v)$ in Eq. (3.4). When n is odd, $d_H[v]$ has a very simple analytical form, given by Eq. (A1) in Appendix A.

Let $f(v)$ and $g(v)$ be two functions on the group manifold. Their inner product is defined to be

$$\langle f(v)|g(v)\rangle=\int d_H[v]f^*(v)g(v). \tag{3.5}$$

It will be shown later that the operator $\overline{\mathcal{D}}=v_{ij}(\partial/\partial v_{ij})$ is Hermitian with respect to this inner product. Since $\overline{\mathcal{D}}v_{kl}=v_{kl}$, the eigenfunctions of $\overline{\mathcal{D}}$ are monomials of v, whose eigenvalues are the degrees of the monomials. The inner product of two eigenfunctions of a Hermitian operator is zero if their eigenvalues are different. Hence if $M_k(v)$ denotes a monomial of degree k and $M'_l(v)$ a monomial of degree $l\neq k$, then

$$\langle M_k(v)|M'_l(v)\rangle=0 \quad (k\neq l). \tag{3.6}$$

This result can be stated in another way. An integral $\int d_H[v]B(v,v^\dagger)$ is nonzero only when the number of v's in B is equal to its number of $v^\dagger$'s. We shall refer to this later as the *matching rule*. It is one of the main tools for our later calculations.

To compute inner products when $k=l$, we resort to the theory of representation of the $U(n)$ group, which asserts that if $D^\lambda_{CD}(v)$ is the matrix element of an irreducible representation λ, then

$$\int d_H[v]D^{*\lambda'}_{AB}(v)D^\lambda_{CD}(v)=\frac{1}{N(\lambda)}\delta_{\lambda\lambda'}\delta_{AC}\delta_{BD}, \tag{3.7}$$

where $N[\lambda]$ is the dimension of the irreducible representation λ.

For example, the defining representation v is irreducible and has dimension N_c; hence

$$\langle v_{ij}|v_{kl}\rangle=\frac{1}{N_c}\delta_{ik}\delta_{jl}. \tag{3.8}$$

The orthonormal relation (3.7) can be used to compute inner products of any two monomials in the following way. First, apply Young's idempotent operators of k boxes to decompose $M_k(v)$ into a linear combination of irreducible representations $D^\lambda(v)$. Similarly $M'_l(v)$ is decomposed into a linear combination of $D^{\lambda'}(v)$. Then Eq. (3.7) can be used to calculate $\langle M_k(v)|M'_l(v)\rangle$. Clearly it can also be used to calculate integrals $\int d_H[v]B(v,v^\dagger)$.

B. Measures and functionals on $\mathcal{A}$ and $\mathcal{U}$

To apply these properties to the physical problem on hand, we need to generalize them to the case when v depends on the transverse position $\mathbf{x}$. We shall denote the color group containing $v(\mathbf{x})$ as $U(n)_\mathbf{x}$, and $\mathcal{U}\equiv\Pi_\mathbf{x}U(n)_\mathbf{x}$.

The product over $\mathbf{x}$ is to be interpreted in the following way. Cover the transverse $\mathbf{x}$ plane by a square lattice with a lattice constant a. The product $\mathbf{x}$ is to be taken over all lattice points within the Lorentz-contracted nucleus of transverse radius R_A. The same convention will be applied to sums over $\mathbf{x}$.

The measure on $\mathcal{U}$ is defined to be

$$\mathcal{D}_H[v]=\prod_\mathbf{x} d_H[v(\mathbf{x})], \tag{3.9}$$

where $d_H[v(\mathbf{x})]$ is the Haar measure on $U(n)_\mathbf{x}$. Using Eq. (3.4) and denoting $\Pi_\mathbf{x} J(v(\mathbf{x}))$ by $\mathcal{J}(v)$, we get

$$\mathcal{D}_H[v]=\mathcal{J}(v)\mathcal{D}[v],$$
$$\mathcal{D}[v]\equiv\prod_\mathbf{x} d[v(\mathbf{x})]. \tag{3.10}$$

The measure $\mathcal{D}[\alpha]$ on the Lie algebra $\mathcal{A}$ of YM potentials can be defined in the following way. Divide the y axis into intervals of size ϵ. Since we are interested in $\alpha^a_y(\mathbf{x})$ only for $y\leqslant\tau$, the appropriate measure is

$$\mathcal{D}[\alpha]=\mathcal{D}_\tau[\alpha]\mathcal{D}_<[\alpha], \tag{3.11}$$

where

$$D_\tau[\alpha]\equiv\prod_{a,\mathbf{x}} d\alpha^a_\tau(\mathbf{x}),$$
$$D_<[\alpha]\equiv c_3\prod_{y\leqslant\tau-\epsilon}\prod_{a,\mathbf{x}} d\alpha^a_y(\mathbf{x}), \tag{3.12}$$

with a normalization constant c_3 to be chosen later. In the same vein, the Wilson line $v(\mathbf{x})$ of Eq. (2.3) can be factorized into

$$v(\mathbf{x})=v_<(\mathbf{x})v_\tau(\mathbf{x}), \tag{3.13}$$

with

$$v_<(\mathbf{x})=\tilde{P}\exp\left(-ig\int_{-\infty}^{\tau-\epsilon}dy\,\alpha^a_y(\mathbf{x})t_a\right),$$
$$v_\tau(\mathbf{x})=\exp[-ig\alpha^a_\tau(\mathbf{x})t_a\epsilon]. \tag{3.14}$$

A change of $\alpha^a_\tau(\mathbf{x})$ of amount $d\alpha^a_\tau(\mathbf{x})$ causes a change of $v_\tau(\mathbf{x})$ of amount $dv_\tau(\mathbf{x})=-ig\epsilon v_\tau(\mathbf{x})t_a d\alpha^a_\tau(\mathbf{x})$, and hence a change in $v(\mathbf{x})$ by the amount $dv(\mathbf{x})=-ig\epsilon v(\mathbf{x})t_a d\alpha^a_\tau(\mathbf{x})$. Or $v^{-1}dv(\mathbf{x})=-ig\epsilon t_a d\alpha^a_\tau$. In light of the remark below Eq. (3.4), we can now choose the constant c_3 in Eq. (3.12) so that

$$\mathcal{D}_H[v]=\mathcal{D}_\tau[\alpha]. \tag{3.15}$$

We are now ready to discuss how $\overline{W}_\tau(v)$ can be obtained from $W_\tau(\alpha)$.

A functional of $\alpha_y(\mathbf{x})$ for $y\leqslant\tau$ can be folded into a functional of $v(\mathbf{x})$ using the formula

$$\overline{F}(v)=\int F(\alpha)\delta(v-u)\frac{1}{\mathcal{J}(u)}\mathcal{D}[\alpha]$$
$$=\int F(\alpha)\delta(v-u)\mathcal{D}[u]\mathcal{D}_<[\alpha], \tag{3.16}$$

where

$$u(\mathbf{x})=\tilde{P}\exp\left(-ig\int_{-\infty}^{\tau}dy\,\alpha^a_y(\mathbf{x})t_a\right), \tag{3.17}$$

and

$$\delta(v-u)\equiv\prod_\mathbf{x}\prod_{i,j=1}^{n}\delta[v(\mathbf{x})_{ij}-u(\mathbf{x})_{ij}], \tag{3.18}$$

so that $\int f(u)\delta(v-u)\mathcal{D}[u]=f(v)$ for any functional $f(u)$. The second equality of (3.16) comes from Eqs. (3.10), (3.12), and (3.15).

It follows from Eq. (3.16) that

$$\int\overline{F}(v)\mathcal{D}_H[v]=\int\mathcal{D}_H[v]\mathcal{D}[\alpha]\delta(v-u)F[\alpha]/\mathcal{J}(u)$$
$$=\int F(\alpha)\mathcal{D}[\alpha], \tag{3.19}$$

so $\overline{W}_\tau(v)$ is normalized if $W_\tau(\alpha)$ is.

It also follows from Eq. (3.16) by an integration by parts that the transform of $\delta F(\alpha)/\delta\alpha^a_\tau(\mathbf{x})$ is $\hat{\mathcal{D}}_a(\mathbf{x})\overline{F}(v)$, where

$$\hat{\mathcal{D}}_a(\mathbf{x})\equiv igv_{ij}(\mathbf{x})(t_a)_{jk}\frac{\delta}{\delta v_{ik}(\mathbf{x})}$$
$$=ig\,\mathrm{tr}\left[v(\mathbf{x})t_a\frac{\delta}{\delta v^T(\mathbf{x})}\right], \tag{3.20}$$

and the functional derivative in v is defined so that

$$\frac{\delta v_{pq}(\mathbf{y})}{\delta v_{ij}(\mathbf{x})}=\delta_{pi}\delta_{qj}\delta(\mathbf{x}-\mathbf{y}). \tag{3.21}$$

The inner product between two functionals of v is defined to be

$$\langle\overline{f}(v)|\overline{g}(v)\rangle=\int\overline{f}^*(v)\overline{g}(v)\mathcal{D}_H[v]. \tag{3.22}$$

It can be shown (see Appendix A) that $\hat{\mathcal{D}}_a(\mathbf{x})$ is anti-Hermitian with respect to this inner product. In particular,

$$\hat{\mathcal{D}}_0(\mathbf{x})=igv_{ij}(\mathbf{x})[\delta/\delta v_{ij}(\mathbf{x})]/\sqrt{2N_c}$$

is anti-Hermitian, so the operator $\overline{\mathcal{D}}$ defined below Eq. (3.5) is Hermitian, as previously claimed.

Instead of $\hat{\mathcal{D}}_a(\mathbf{x})$ in Eq. (3.20), it is more convenient to deal with the matrix operators $\mathcal{D}(\mathbf{x})$ and $\mathcal{D}'(\mathbf{x})$, whose (mn) matrix elements are defined to be

$$\mathcal{D}_{mn}(\mathbf{x})\equiv\frac{2}{ig}(t_a)_{mn}\hat{D}_a(\mathbf{x})=v_{in}(\mathbf{x})\frac{\delta}{\delta v_{im}(\mathbf{x})},$$
$$\mathcal{D}'_{mn}(\mathbf{x})\equiv v_{mi}(\mathbf{x})\frac{\delta}{\delta v_{ni}(\mathbf{x})}. \tag{3.23}$$

It can be checked that these two are related by

$$v(\mathbf{x})\mathcal{D}(\mathbf{x})v^\dagger_\mathbf{x}(\mathbf{x})=\mathcal{D}'(\mathbf{x}), \tag{3.24}$$

where the subscript $\mathbf{x}$ in $v^\dagger_{\mathbf{x}}(\mathbf{x})$ indicates that this $v^\dagger(\mathbf{x})$ should not be differentiated by the $\delta/\delta v(\mathbf{x})$ in $\mathcal{D}(\mathbf{x})$. In other words, in component forms, Eq. (3.24) reads $v_{im}v^\dagger_{nj}\mathcal{D}_{mn}=\mathcal{D}'_{ij}$.

When operated on v_{pq} and $v^\dagger_{pq}=v^{-1}_{pq}$, they yield

$$\begin{aligned}
\mathcal{D}_{mn}(\mathbf{x})v_{pq}(\mathbf{y})&=+\delta(\mathbf{x}-\mathbf{y})\,\delta_{mq}v_{pn}(\mathbf{y}),\\
\mathcal{D}_{mn}(\mathbf{x})v^\dagger_{pq}(\mathbf{y})&=-\delta(\mathbf{x}-\mathbf{y})\,\delta_{np}v^\dagger_{mq}(\mathbf{y}),\\
\mathcal{D}'_{mn}(\mathbf{x})v_{pq}(\mathbf{y})&=+\delta(\mathbf{x}-\mathbf{y})\,\delta_{np}v_{mq}(\mathbf{y}),\\
\mathcal{D}'_{mn}(\mathbf{x})v^\dagger_{pq}(\mathbf{y})&=-\delta(\mathbf{x}-\mathbf{y})\,\delta_{mq}v^\dagger_{pn}(\mathbf{y}).
\end{aligned}\tag{3.25}$$

These formulas give rise to the following formulas which are very useful in practical calculations. When $\mathcal{D}$ or $\mathcal{D}'$ operates on a v or $v^\dagger$ in a trace, we have

$$\begin{aligned}
\mathcal{D}(\mathbf{x})\mathrm{tr}[Av(\mathbf{y})]&=+\delta(\mathbf{x}-\mathbf{y})[Av(\mathbf{y})],\\
\mathcal{D}(\mathbf{x})\mathrm{tr}[v^\dagger(\mathbf{y})A]&=-\delta(\mathbf{x}-\mathbf{y})[v^\dagger(\mathbf{y})A],\\
\mathcal{D}'(\mathbf{x})\mathrm{tr}[v(\mathbf{y})A]&=+\delta(\mathbf{x}-\mathbf{y})[v(\mathbf{y})A],\\
\mathcal{D}'(\mathbf{x})\mathrm{tr}[Av^\dagger(\mathbf{y})]&=-\delta(\mathbf{x}-\mathbf{y})[Av^\dagger(\mathbf{y})].
\end{aligned}\tag{3.26}$$

In other words, when the trace is written in a certain order, the operators $D(\mathbf{x})$ and $D'(\mathbf{x})$ simply remove the trace, and append to it the factor $\pm\delta(\mathbf{x}-\mathbf{y})$. When $\mathcal{D}$ or $\mathcal{D}'$ operates on a v or $v^\dagger$ in the same trace, we get

$$\begin{aligned}
\mathrm{tr}[A\mathcal{D}(\mathbf{x})Bv(\mathbf{y})]&=+\delta(\mathbf{x}-\mathbf{y})\mathrm{tr}[A]\mathrm{tr}[Bv(\mathbf{y})],\\
\mathrm{tr}[A\mathcal{D}(\mathbf{x})Bv^\dagger(\mathbf{y})]&=-\delta(\mathbf{x}-\mathbf{y})\mathrm{tr}[B]\mathrm{tr}[v^\dagger(\mathbf{y})A],\\
\mathrm{tr}[A\mathcal{D}'(\mathbf{x})Bv(\mathbf{y})]&=+\delta(\mathbf{x}-\mathbf{y})\mathrm{tr}[B]\mathrm{tr}[v(\mathbf{y})A],\\
\mathrm{tr}[A\mathcal{D}'(\mathbf{x})Bv^\dagger(\mathbf{y})]&=-\delta(\mathbf{x}-\mathbf{y})\mathrm{tr}[A]\mathrm{tr}[Bv^\dagger(\mathbf{y})].
\end{aligned}\tag{3.27}$$

In other words, the single trace is broken up into a product of two traces. The matrices A,B in these equations are constant matrices.

C. $\overline{W}_\tau(v)$ and its JIMWLK equation

Using Eq. (3.16), the distribution function $W_\tau(\alpha)$ can be folded into the distribution function $\overline{W}_\tau(v)$ of Wilson lines. Since $\delta/\delta\alpha^a_\tau(\mathbf{x})$ is transformed into $\hat{D}_a(\mathbf{x})$ of Eq. (3.20), the JIMWLK equation (2.4) for $W_\tau(\alpha)$ is now changed into a JIMWLK equation for $\overline{W}_\tau(v)$:

$$\frac{\partial\overline{W}_\tau(v)}{\partial\tau}=-\overline{H}\overline{W}_\tau(v),\tag{3.28}$$

where

$$\begin{aligned}
\overline{H}&=\frac{1}{\pi}\int d^2\mathbf{z}d^2\mathbf{x}d^2\mathbf{y}K(\mathbf{xy}|\mathbf{z})\overline{\mathcal{O}}(\mathbf{xy}|\mathbf{z}),\\
\overline{\mathcal{O}}(\mathbf{xy}|\mathbf{z})&=\hat{D}_a(\mathbf{x})[V^\dagger(\mathbf{x})-V^\dagger(\mathbf{z})]_{ac}[V(\mathbf{y})\\
&\quad-V(\mathbf{z})]_{cb}\hat{D}_b(\mathbf{y}).
\end{aligned}\tag{3.29}$$

Like H, $\overline{H}$ is also Hermitian and positive semidefinite.

Using Eqs. (3.23) and (3.2), the operator $\overline{\mathcal{O}}(\mathbf{xy}|\mathbf{z})$ can be written in a form more convenient for practical calculations. Remember for this purpose the remark after Eq. (2.7) that the factor $[V^\dagger(\mathbf{x})-V^\dagger(z)]_{ac}$ in Eq. (2.5) can be written to the left of the differential operator $\delta/\delta\alpha^a_\tau(\mathbf{x})$. In terms of Eq. (3.29), this means to the left of $\hat{\mathcal{D}}_a(\mathbf{x})$. In what follows we shall use $\overline{\mathcal{O}}(\mathbf{xy}|\mathbf{z})$ of Eq. (3.29) in this form.

Using Eqs. (3.23), (3.24), and (3.2), and this remark above, we have

$$\begin{aligned}
V_{cb}(\mathbf{z})\hat{\mathcal{D}}_b(\mathbf{y})&=ig\,\mathrm{tr}[v^\dagger(\mathbf{z})t_cv(\mathbf{z})\mathcal{D}(\mathbf{y})],\\
\hat{\mathcal{D}}_a(\mathbf{x})V^\dagger(\mathbf{z})&=ig\,\mathrm{tr}[v^\dagger(\mathbf{z})t_cv(\mathbf{z})\mathcal{D}(\mathbf{x})],\\
V_{cb}(\mathbf{y})\hat{\mathcal{D}}_b(\mathbf{y})&=ig\,\mathrm{tr}[v^\dagger(\mathbf{y})t_cv(\mathbf{y})\mathcal{D}(\mathbf{y})]\\
&=ig\,\mathrm{tr}[t_c\mathcal{D}'(\mathbf{y})],\\
\hat{\mathcal{D}}_a(\mathbf{x})V^\dagger(\mathbf{x})&=ig\,\mathrm{tr}[v^\dagger(\mathbf{x})t_cv(\mathbf{x})\mathcal{D}(\mathbf{x})]\\
&=ig\,\mathrm{tr}[t_c\mathcal{D}'(\mathbf{x})].
\end{aligned}\tag{3.30}$$

From these relations, and the identity $\mathrm{tr}(t_cA)\mathrm{tr}(t_cB)=\frac{1}{2}\mathrm{tr}(AB)$, we get

$$\begin{aligned}
\overline{\mathcal{O}}(\mathbf{xy}|\mathbf{z})&=\overline{\mathcal{O}}_{xy}+\overline{\mathcal{O}}_{xz}+\overline{\mathcal{O}}_{zy}+\overline{\mathcal{O}}_{zz},\\
\overline{\mathcal{O}}_{xy}&=-\frac{1}{2}g^2\mathrm{tr}[\mathcal{D}'(\mathbf{x})\mathcal{D}'(\mathbf{y})],\\
\overline{\mathcal{O}}_{xz}&=+\frac{1}{2}g^2\mathrm{tr}[\mathcal{D}'(\mathbf{x})v(\mathbf{z})\mathcal{D}(\mathbf{y})v^\dagger_{\mathbf{y}}(\mathbf{z})],\\
\overline{\mathcal{O}}_{yz}&=+\frac{1}{2}g^2\mathrm{tr}[v(\mathbf{z})\mathcal{D}(\mathbf{x})v^\dagger_{\mathbf{x}}(\mathbf{z})\mathcal{D}'(\mathbf{y})],\\
\overline{\mathcal{O}}_{zz}&=-\frac{1}{2}g^2\mathrm{tr}[v(\mathbf{z})\mathcal{D}(\mathbf{x})v^\dagger_{\mathbf{x}}(\mathbf{z})v(\mathbf{z})\mathcal{D}(\mathbf{y})v^\dagger_{\mathbf{y}}(\mathbf{z})]\\
&=-\frac{1}{2}g^2\mathrm{tr}[\mathcal{D}(\mathbf{x})\mathcal{D}(\mathbf{y})].
\end{aligned}\tag{3.31}$$

Assuming $\overline{W}_\tau(v)$ to be normalized, $\int\mathcal{D}_H[v]\overline{W}_\tau(v)=1$, the average of any functional $B(v,v^\dagger)$ is equal to

$$\langle B\rangle_\tau=\int\mathcal{D}_H[v]B(v,v^\dagger)\overline{W}_\tau(v).\tag{3.32}$$

If B and $\overline{W}_\tau$ are both monomial functionals of v and $v^\dagger$, this functional integral factorizes into a product of integrals on the group $U(n)$, each of which can be computed using Eqs. (3.6) and (3.7). In particular, the functional integral is non-

zero only when the number of v's and $v^\dagger$'s in $B\overline{W}_\tau$ is the same for every transverse position **x**. This is the functional form of the *matching rule* previously considered.

IV. MUELLER'S DERIVATION OF THE JIMWLK EQUATION

In a recent paper [6], Mueller proposed a simple derivation of the JIMWLK equation in the following way. He showed that if $W_\tau(\alpha)$ is equal to the dipole functional $v^\dagger(\mathbf{s})_{ij}v(\mathbf{t})_{kl}$, then the Feynman diagrams for $\partial W_\tau(\alpha)/\partial\tau$ can be written in the form $-HW_\tau(\alpha)$, with H given by Eq. (2.5). He then stated that the same is true if $W_\tau(\alpha)$ is equal to any multipole functional $v^\dagger(\mathbf{s}_1)_{i_1j_1}v^\dagger(\mathbf{s}_2)_{i_2j_2}\cdots v(\mathbf{t}_1)_{k_1l_1}v(\mathbf{t}_2)_{k_2l_2}\cdots$, and hence it is likely that the JIMWLK equation for $W_\tau(\alpha)$ is also valid.

In this section we shall supply the missing steps of this proof. This consists of filling in the detailed arguments for the multiple functionals, and then showing that they lead to the JIMWLK equation for $\overline{W}_\tau(v)$. We do not know whether the JIMWLK equation for $W_\tau(\alpha)$ follows or not. However, for all our applications, a JIMWLK equation for $\overline{W}_\tau(v)$ is sufficient, so it really does not matter whether the equation for $W_\tau(\alpha)$ can be proven this way or not.

Instead of starting from the Feynman diagrams to derive the evolution equation, as is done in Ref. [6], we find it easier to do everything in reverse. That is, we start from the JIMWLK equation and show that they lead to the correct set of Feynman graphs. This inverse approach makes it more manageable to deal with the complicated multipole functionals. Actually, simplification already occurs at the dipole level: a necessary cancellation in the original derivation is avoided altogether in this way.

The Wilson line $v^\dagger(\mathbf{s})$ for a quark is drawn in Fig. 1 as a left-pointing arrow, and the Wilson line $v(\mathbf{t})$ for an antiquark is drawn as a right-pointing arrow. Time x^- is drawn to increase from right to left; multiplication of color matrices from left to right should be carried out against the arrow of the fermion lines.

The short vertical bars in the middle of the lines [labeled the interaction point (IP)] indicate the light cone longitudinal position $x^-=0$ where interaction between the multipole and the pancake nucleus takes place. Since $\alpha(\vec{x})$ is concentrated around $x^-=0$, we may regard v and $v^\dagger$ to be located at the IP.

The operation of $\hat{D}_a(\mathbf{x})$ or $\delta/\delta\alpha^a_\tau(\mathbf{x})$ on the Wilson lines is given in Eq. (2.7). This operation can be represented graphically by putting a vertex to the left (the larger x^- side) of the IP, both for the quark Wilson line $v^\dagger$ and the antiquark Wilson line v. The vertex for a quark is igt_a, and the vertex for an antiquark is $-igt_a$.

Using the remark following Eq. (2.7), the operator $\mathcal{O}(\mathbf{xy}|\mathbf{z})$ in Eq. (2.5) can be written in the form

$$\mathcal{O}=[V^\dagger(\mathbf{x})-V^\dagger(\mathbf{z})]_{ac}\frac{\delta}{\delta\alpha^a_\tau(\mathbf{x})}\cdot[V(\mathbf{y})$$

$$-V(\mathbf{z})]_{cb}\frac{\delta}{\delta\alpha^b_\tau(\mathbf{y})}$$

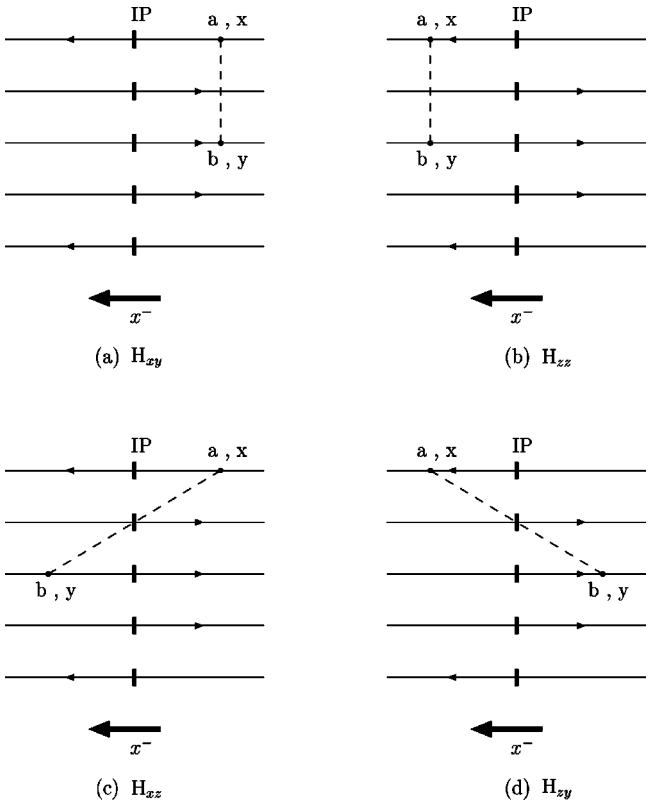

FIG. 1. Diagrams representing the result of $H=H_{xy}+H_{zz}+H_{xz}+H_{zy}$, or $\overline{H}=\overline{H}_{xy}+\overline{H}_{zz}+\overline{H}_{xz}+\overline{H}_{zy}$, operating on a multipole functional with two antiquark Wilson lines $v^\dagger$ (left-pointing arrows) and three quark Wilson lines v (right-pointing arrows). The dashed lines are the gluon propagators. See the text in Sec. IV. for further explanation.

$$\equiv\mathcal{O}_{\mathbf{xy}}+\mathcal{O}_{\mathbf{xz}}+\mathcal{O}_{\mathbf{zy}}+\mathcal{O}_{\mathbf{zz}}. \tag{4.1}$$

Using Eq. (3.1) and the unitarity relation $vv^\dagger=v^\dagger v=1$, we see that the effect of $[V(\mathbf{y})]_{cb}[\delta/\delta\alpha^b_\tau(\mathbf{y})]$, operating on a Wilson line, is to place a vertex to the right (the small x^- side) of the IP on the Wilson line. Similarly, the effect of $[V^\dagger(\mathbf{x})]_{ac}[\delta/\delta\alpha^a_\tau(\mathbf{x})]$, operating on a Wilson line, is also to place a vertex to the right (the small x^- side) of the IP on the Wilson line.

We are now ready to see what happens when

$$H=\int d^2\mathbf{z}d^2\mathbf{x}d^2\mathbf{y}K(\mathbf{xy}|\mathbf{z})$$

$$\times(\mathcal{O}_{\mathbf{xy}}+\mathcal{O}_{\mathbf{xz}}+\mathcal{O}_{\mathbf{zy}}+\mathcal{O}_{\mathbf{zz}})$$

$$\equiv H_{\mathbf{xy}}+H_{\mathbf{xz}}+H_{\mathbf{zy}}+H_{\mathbf{zz}} \tag{4.2}$$

operates on a multiple functional, i.e., a collection of p quark and q antiquark Wilson lines.

$H_{\mathbf{xy}}$ puts a vertex **y** to the right of the IP on a Wilson line, and a vertex **x** also to the right of the IP of the same or a different Wilson line. These two vertices are then connected by the "gluon propagator" $\int d^2\mathbf{z}K(\mathbf{xy}|\mathbf{z})\,\delta_{ab}$, where a,b are the color indices at the two vertices. The gluon propagator is shown in Fig. 1(a) with a dashed line. This operation is to be

applied to every pair of Wilson lines, including the possibility of applying to the same line twice.

Similarly, since $V^\dagger(\mathbf{z})V(\mathbf{z})=1$, the effect of $H_{\mathbf{zz}}$ is to put a vertex $\mathbf{y}$ to the left of the IP of a Wilson line, and another vertex $\mathbf{x}$ to the left of the IP of the same or another Wilson line. The two vertices are then linked by the same gluon propagator. This is shown in Fig. 1(b).

In both of these cases, the two vertices are both to the same side of the IP. This is not the case with the other two terms.

$H_{\mathbf{xz}}$ puts a vertex $\mathbf{y}$ with color b to the left of the IP, and a vertex $\mathbf{x}$ with color a to the right of the IP. These two vertices are linked by a gluon propagator $\int d^2\mathbf{z}[V(z)]_{ab}K(\mathbf{xy}|\mathbf{z})$. This is shown in Fig. 1(c).

There is potentially another contribution to $H_{\mathbf{xz}}$ when $V^\dagger_{ac}(x)\,\delta/\delta\alpha_a(x)$ operates on $V_{cb}(z)$. However, this term is proportional to $\delta(\mathbf{x}-\mathbf{z})V^\dagger_{ac}(x)V_{cd}(z)(T_a)_{db}$, which is proportional to $(T_a)_{ab}=0$, so that the term is actually absent.

Similarly, $H_{\mathbf{zy}}$ puts a vertex $\mathbf{y}$ of color b to the right of the IP, and a vertex $\mathbf{x}$ with color a to the left of the IP. These two vertices are linked by the gluon propagator $\int d^2\mathbf{z}[V^\dagger(z)]_{ab}K(\mathbf{xy}|\mathbf{z})$. This is shown in Fig. 1(d).

Appropriate signs and Dirac δ functions on the transverse coordinates must also be inserted.

One might be bothered that the gluon propagators in the four diagrams appear to be different. Fortunately this is only superficial. To see why they are actually the same, first note that T_a is imaginary so V is real. Hence the gluon Wilson line $[V^\dagger(\mathbf{z})]_{ab}$ in the propagator in Fig. 1(d) can be written as $[V(\mathbf{z})]_{ba}$. If we compare this with that of Fig. 1(c), we see that these two are the same, both equal to $[V(\mathbf{z})]_{a_1a_2}$, where a_1 is the color index before the interaction point, and a_2 is the color index after. Since $\alpha^a(x^-,\mathbf{z})$ is concentrated near $x^-=0$, we may write, in both cases, the gluon Wilson line as

$$[V(\mathbf{z})]_{a_1a_2}=\tilde{P}\exp\left[-ig\int_{x_1^-}^{x_2^-}dx^-\alpha^a(x^-,\mathbf{z})T_a\right], \quad (4.3)$$

where (a_1,x_1^-) is the interaction vertex to the right of the IP, and (a_2,x_2^-) is the one to the left of the IP. For the first two diagrams, we may replace δ_{ab} by the same expression (4.3), because in that case both x_1^- and x_2^- are to the same side of the IP; hence $\alpha^a(x^-,\mathbf{z})=0$ throughout the integration interval, so $[V(\mathbf{z})]_{ab}=\delta_{ab}$.

These four types of Feynman graphs are precisely those needed for the development $\partial W_\tau(\alpha)/\partial\tau$ [6]. Therefore the JIMWLK equation is satisfied whenever $W_\tau(\alpha)$ is given by a multipole function, namely, a monomial functional of v and $v^\dagger$. Since polynomial functionals on the group manifold form an orthonormal complete set, the JIMWLK equation (3.28) must be true in general. This completes Mueller's proof for $\overline{W}_\tau(v)$. However, since $W_\tau(\alpha)$ depends on many more variables $\alpha_a(x^-,\mathbf{x})$ than does $v(\mathbf{x})$, it does not necessarily follow from this argument that Eq. (2.4) for $W_\tau(\alpha)$ is true.

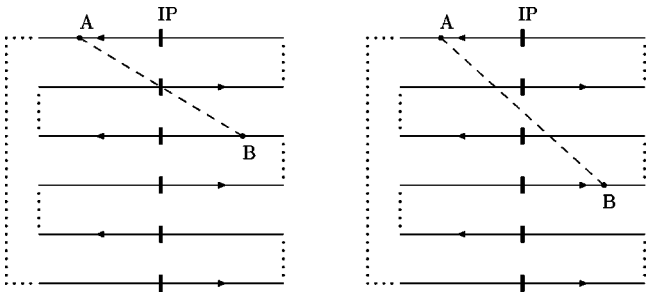

FIG. 2. A multiple trace with $k=3$. The two diagrams have identical color structure, so their two gluon propagators can be combined into an infrared finite expression.

V. INFRARED DIVERGENCE AND THE GENERALIZED BK EQUATION

The kernel $K(\mathbf{xy}|\mathbf{z})$ of the JIMWLK equation, given in Eq. (2.6), goes as $1/\mathbf{z}^2$ for large $|\mathbf{z}|$. This causes a log divergence in $HW_\tau(\alpha)$ of Eq. (2.5), and $\overline{HW}_\tau(v)$ of Eq. (3.29). On the other hand, the dipole amplitude (6.1) satisfies the Balitsky equation (2.10), whose kernel goes as $1/(\mathbf{z}^2)^2$ for large $|\mathbf{z}|$, so infrared divergence is absent in that case. In this section, we shall use the Feynman diagrams derived in the last section to show that the multipole traces defined below are equally free of infrared divergence.

Multipole traces are defined by

$$m(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k)\equiv\frac{1}{N_c}\mathrm{tr}[v^\dagger(\mathbf{s}_1)v(\mathbf{t}_1)\cdots v^\dagger(\mathbf{s}_k)v(\mathbf{t}_k)]. \quad (5.1)$$

We will call the functional average of a multiple trace, $\langle m_k(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k)\rangle$, a *multiple amplitude*. When $k=1$, this reduces to the dipole amplitude (2.8).

The diagram for a multiple trace is shown in Fig. 2. As far as color-matrix multiplication is concerned, these $2k$ lines should be considered to be joined together at their ends to form a single big loop with the arrows all pointing the same way. The joins are indicated by dotted lines in the figure. Note that if the amplitude does not have the form displayed in Eq. (5.1), with $v^\dagger$ and v occurring alternately inside a trace, such a big loop cannot be formed and the argument below will not be valid.

Consider the two gluon propagators shown in Fig. 2. The color structures are identical, but the two terms differ by a sign because the vertex B is on a quark line in one diagram and on an antiquark line in the other diagram. Their gluon propagators may therefore be combined into $K(\mathbf{xx}|\mathbf{z})-K(\mathbf{xy}|\mathbf{z})$. For large $|\mathbf{z}|$, this is proportional to $\mathbf{z}\cdot(\mathbf{x}-\mathbf{y})/(\mathbf{z}^2)^2=O(1/|\mathbf{z}|^3)$; hence the infrared divergence is absent. As a matter of fact, we can also combine diagrams with vertex A similarly shifted. The four K thus combined actually die down like $1/(\mathbf{z}^2)^2$ for large $|\mathbf{z}|$.

It is clear that all the diagrams for the multipole function can be paired up in a way similar to Fig. 2, thus eliminating all infrared divergences. Moreover, since the combination is

obtained by combining two diagrams with vertex B in different positions, but the same vertex A, finiteness persists for products of multipole traces.

In the rest of this section, we shall derive a generalization of the Balitsky-Kovchegov equation to multipole amplitudes. We will see that the kernel of the equation actually behaves as $1/(\mathbf{z}^2)^2$ for large $|\mathbf{z}|$, just like the kernel for the Balitsky equation.

Since $\bar{H}$ is Hermitian, the JIMWLK equation satisfied by $\bar{W}_\tau(v)$ is also satisfied by the multipole trace (5.1). We may now use Eqs. (3.26), (3.27), (3.29), and (3.31) to simplify $\bar{H}m(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k)$. The resulting equation is

$$\frac{\partial m(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k)}{\partial\tau}=-[H_{\mathbf{xy}}+H_{\mathbf{zz}}+H_{\mathbf{xz}}+H_{\mathbf{yz}}]m(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k), \tag{5.2}$$

where

$$\begin{aligned}
H_{\mathbf{xy}}m(\mathbf{s}_1\cdots\mathbf{t}_k)&=+\frac{1}{2}g^2N_c\sum_{i,j=1}^{k}\int d^2\mathbf{z}I_{ij}(\mathbf{z})m_{ij}^{aa}(\mathbf{s}_1\cdots\mathbf{t}_k),\\
H_{\mathbf{zz}}m(\mathbf{s}_1\cdots\mathbf{t}_k)&=+\frac{1}{2}g^2N_c\sum_{i,j=1}^{k}\int d^2\mathbf{z}I_{ij}(\mathbf{z})m_{ij}^{bb}(\mathbf{s}_1\cdots\mathbf{t}_k),\\
H_{\mathbf{xz}}m(\mathbf{s}_1\cdots\mathbf{t}_k)&=-\frac{1}{2}g^2N_c\sum_{i,j=1}^{k}\int d^2\mathbf{z}I_{ij}(\mathbf{z})m_{ij}^{ba}(\mathbf{s}_1\cdots\mathbf{t}_k),\\
H_{\mathbf{zy}}m(\mathbf{s}_1\cdots\mathbf{t}_k)&=-\frac{1}{2}g^2N_c\sum_{i,j=1}^{k}\int d^2\mathbf{z}I_{ij}(\mathbf{z})m_{ij}^{ab}(\mathbf{s}_1\cdots\mathbf{t}_k).
\end{aligned} \tag{5.3}$$

The kernel in these equations is

$$\begin{aligned}
I_{ij}(\mathbf{z})=&K(\mathbf{s}_i\mathbf{s}_j|\mathbf{z})-K(\mathbf{s}_i\mathbf{t}_j|\mathbf{z})-K(\mathbf{s}_j\mathbf{t}_i|\mathbf{z})\\
&+K(\mathbf{t}_i\mathbf{t}_j|\mathbf{z}),
\end{aligned} \tag{5.4}$$

which is $O[1/(\mathbf{z}^2)^2]$ for large $|\mathbf{z}|$. It is also symmetric in i and j: $I_{ij}(\mathbf{z})=I_{ji}(\mathbf{z})$. The meaning of the multipole traces in Eq. (5.2) will now be explained.

The argument $(\mathbf{s}_1\cdots\mathbf{t}_k)$ of the multiple trace in Eq. (5.1) is circular, because the trace is. We can consider the argument $\mathbf{s}_1$ to be behind the argument $\mathbf{t}_k$, forming a circle. For example,

$$\begin{aligned}
m(\mathbf{s}_1\mathbf{t}_1\cdots\mathbf{s}_k\mathbf{t}_k)&=m(\mathbf{t}_1\mathbf{s}_2\cdots\mathbf{t}_k\mathbf{s}_1)\\
&\equiv\frac{1}{N_c}\mathrm{tr}[v(\mathbf{t}_1)v^\dagger(\mathbf{s}_2)\cdots v(\mathbf{t}_k)v^\dagger(\mathbf{s}_1)].
\end{aligned} \tag{5.5}$$

In these formulas, v is always associated with a $\mathbf{t}_i$ and $v^\dagger$ is always associated with an $\mathbf{s}_i$.

The quantity $m_{ij}^{aa}(\mathbf{s}_1\cdots\mathbf{t}_k)$ appearing in the $H_{\mathbf{xy}}$ term in Eq. (5.3) is defined as follows. Put a vertical bar *after* $\mathbf{s}_i$, and another vertical bar *after* $\mathbf{s}_j$ in the circular argument $(\mathbf{s}_1\cdots\mathbf{t}_k)$. This pair of vertical bars separates the circular argument into two circular arguments. $m_{ij}^{ab}(\mathbf{s}_1\cdots\mathbf{t}_k)$ is defined to be the product of two multiple traces with these two circular argument. For example,

$$\begin{aligned}
m_{13}^{aa}(\mathbf{s}_1\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{s}_3\mathbf{t}_3)&\equiv m(\mathbf{s}_1|\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{s}_3|\mathbf{t}_3)\\
&\equiv m(\mathbf{t}_3\mathbf{s}_1)m(\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{s}_3).
\end{aligned} \tag{5.6}$$

The superscript a stands for "after." The superscript b to be found in the other three terms of Eq. (5.3) stands for "before." Each superscript together with its corresponding subscript tells us where the vertical bar is put. The pair (ai) means to put a vertical bar after $\mathbf{s}_i$. The pair (bj) tells us to put a vertical bar before $\mathbf{s}_j$. With this understanding the quantity $m_{ij}^{bb}(\mathbf{s}_1\cdots\mathbf{t}_k)$ can be defined similarly. For example,

$$\begin{aligned}
m_{13}^{bb}(\mathbf{s}_1\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{s}_3\mathbf{t}_3)&\equiv m(|\mathbf{s}_1\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2|\mathbf{s}_3\mathbf{t}_3)\\
&\equiv m(\mathbf{s}_1\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2)m(\mathbf{s}_3\mathbf{t}_3).
\end{aligned} \tag{5.7}$$

If we apply these recipes literally to the last two equations in (5.3) we will end up with something nonsensical, in that the numbers of v's and $v^\dagger$'s within each trace are not identical, and thus neither is a multipole trace. What we should do in these two cases is to insert a $v(\mathbf{z})$ into one trace and a $v^\dagger(\mathbf{z})$ into the other so that both become multiple functionals. For example,

$$\begin{aligned}
m_{13}^{ab}(\mathbf{s}_1\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{s}_3\mathbf{t}_3)&\equiv m(\mathbf{s}_1|\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2|\mathbf{s}_3\mathbf{t}_3)\\
&\equiv m(\mathbf{t}_1\mathbf{s}_2\mathbf{t}_2\mathbf{z})m(\mathbf{z}\mathbf{s}_3\mathbf{t}_3\mathbf{s}_1).
\end{aligned} \tag{5.8}$$

The $\mathbf{z}$ in the first factor is a $v^\dagger(\mathbf{z})$, and that in the second factor is a $v(\mathbf{z})$.

This completes the explanation of the symbols in Eq. (5.3), except for one last remark. If the two vertical bars are side by side, then the corresponding multiple trace m should be interpreted as $\mathrm{tr}(\mathbf{1})/N_c=1$.

When we take expectation values on both sides of Eq. (5.2), we get an equation for the multiple amplitudes. For large N_c, the expectation of products of traces factorizes into products of expectations of a trace. In this form Eqs. (5.2) and (5.3) generalize the Balitsky-Kovchegov equation to higher multiple amplitudes. We should interpret m in these equations as the average $\langle m\rangle$, and products of two m's as the product of the averages.

In the special case of a dipole, $k=1$, and hence $i=j=1$ in Eq. (5.3). In that case,

$$\begin{aligned}
m_{11}^{aa}&=m(\mathbf{s}_1||\mathbf{t}_1)=m(\mathbf{s}_1\mathbf{t}_1),\\
m_{11}^{bb}&=m(||\mathbf{s}_1\mathbf{t}_1)=m(\mathbf{s}_1\mathbf{t}_1),\\
m_{11}^{ba}&=m_{11}^{ab}=m(|\mathbf{s}_1|\mathbf{t}_1)=m(\mathbf{s}_1\mathbf{z})m(\mathbf{z}\mathbf{t}_1),
\end{aligned} \tag{5.9}$$

$$I_{11}(z)=\frac{1}{4\pi^3}\frac{(\mathbf{s}_1-\mathbf{t}_1)^2}{(\mathbf{s}_1-\mathbf{z})^2(\mathbf{t}_1-\mathbf{z})^2}.$$

Equation (5.2) coincides with the BK equation (2.11), as it should.

A very interesting fact emerges from these generalized BK equations for multipole amplitudes. If the multipole am-

plitudes for $k=1,2,\ldots,\ell-1$ are known, then the evolution equation determining the lth multipole is a *linear* inhomogeneous equation. Thus the only nonlinear equation one has to solve is the original BK equation for the dipole amplitude. It is well known that the nonlinearity of this equation leads to saturation. The linearity of the higher multipole amplitudes therefore means that saturation of the dipole amplitude automatically drives saturation of all higher multipole amplitudes.

If we require the solution of the JIMWLK equation (3.28) to be free of infrared singularitities, then presumably it will be made out of the multiple traces (5.1) and their products. In that case the difficult functional differential JIMWLK equation can be replaced by the more manageable set of generalized BK equations (5.2) and (5.3).

VI. SOLUTION OF THE JIMWLK EQUATION

Suppose we decompose $\overline{W}_\tau(v)$ into a linear combination of eigenfunctions $\overline{\phi}_\lambda(v)$ of $\overline{H}$. The operator $\overline{H}$ is Hermitian and positive semidefinite; hence the eigenvalues λ are real and non-negative. From Eq. (3.28), the τ dependence of $\overline{W}_\tau(v)$ is thus given by a linear combination of $\exp(-\lambda\tau)$. In the asymptotic limit $\tau\to\infty$, the lowest eigenvalue of $\overline{H}$ dominates.

The lowest eigenvalue is $\lambda=0$, and its normalized eigenfunction is $\overline{\phi}_0(v)=1$. This eigenfunction is normalized because $\int\mathcal{D}_H[v]=\Pi_{\mathbf{x}}\int d_H[v(\mathbf{x})]=1$.

In contrast, $\phi_0(\alpha)=1$ is also an eigenfunction of H with $\lambda=0$, but the integral of this eigenfunction is divergent because the linear space $\mathcal{A}$ is noncompact.

In the next subsection, we will discuss what happens in the asymtotic limit $\tau\to\infty$ when $\lambda=0$ dominates. In the subsequent subsection, we will look at $\overline{W}_\tau(v)$ and its applications for a smaller τ.

A. Asymptotic limit

When $\tau=\infty$, only $\overline{\phi}_0(v)$ contributes, so we can set $\overline{W}_\infty(v)=\overline{\phi}_0(v)=1$. Averages are then given by the integral $\langle B(v,v^\dagger)\rangle=\int\mathcal{D}_H[v]B(v,v^\dagger)$. From the matching rule, this integral is nonzero only when the number of v's in B exactly matches the number of $v^\dagger$'s at every transverse position $\mathbf{x}$.

For example, the dipole amplitude (2.8) is

$$s_\tau(\mathbf{x},\mathbf{y})=\frac{1}{N_c}\langle\mathrm{tr}[v^\dagger(\mathbf{x})v(\mathbf{y})]\rangle=\delta_{\mathbf{x},\mathbf{y}} \tag{6.1}$$

because $v^\dagger_{\mathbf{x}}v_{\mathbf{x}}=1$ and $\mathrm{tr}(1)=N_c$. This is consistent with Eq. (2.9), and also Eq. (2.14) at $\tau=\infty$. In other words, whenever the dipole has a finite size, the dense gluon will have such a strong absorption that the dipole amplitude always becomes zero.

A similar statement can be made about multipole amplitudes.

Let us now compute the gluon spectrum at $\tau=\infty$ by using the formula [9]

$$\frac{d(x_FG)}{d^2\mathbf{k}}=\frac{1}{4\pi^3}\langle F_a^{+i}(\vec{k})F_a^{+i}(-\vec{k})\rangle$$
$$=\frac{1}{4\pi^3N_c}\langle\mathrm{Tr}[F^{+i}(\vec{k})F^{+i}(-\vec{k})]\rangle, \tag{6.2}$$

where $\vec{k}=(k^+,\mathbf{k})$, and

$$F_a^{+i}(\vec{k})=\int d^3\mathbf{x}\exp(i\mathbf{k}\cdot\mathbf{x})F^{+i}(\vec{x}) \tag{6.3}$$

is the color electric field in the light cone gauge. In the approximation $F^{+i}(\vec{x})\simeq(i/g)\delta(x^-)V(\mathbf{x})\partial^iV^\dagger(\mathbf{x})$, which is supposed to be valid for a Lorentz-contracted pancake nucleon, the formula becomes

$$\frac{d(x_FG)}{d^2\mathbf{k}}=-\frac{1}{4\pi^3g^2N_c}\int d^2\mathbf{x}d^2\mathbf{y}\exp[i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})]$$
$$\times\langle C(\mathbf{x},\mathbf{y})\rangle,$$
$$C(\mathbf{x},\mathbf{y})=\mathrm{Tr}[V(\mathbf{x})V_i^\dagger(\mathbf{x})V(\mathbf{y})V_i^\dagger(\mathbf{y})], \tag{6.4}$$

where the subscript i represents a differentiation, namely, $V_i^\dagger\equiv\partial_iV^\dagger$.

To compute the average $\langle C(\mathbf{x},\mathbf{y})\rangle=\int\mathcal{D}_H[v]B(\mathbf{x},\mathbf{y})$, we need to use Eq. (3.2) to convert the gluon Wilson lines V and $V^\dagger$ to the quark and antiquark Wilson lines $v^\dagger$ and v. The result is

$$C(\mathbf{x},\mathbf{y})=2C_F\,\mathrm{tr}[v(\mathbf{y})v_i^\dagger(\mathbf{y})v(\mathbf{x})v_i^\dagger(\mathbf{x})$$
$$+v_i(\mathbf{y})v^\dagger(\mathbf{y})v_i(\mathbf{x})v^\dagger(\mathbf{x})]$$
$$+\mathrm{tr}[v_i(\mathbf{y})v^\dagger(\mathbf{y})]\mathrm{tr}[v(\mathbf{x})v_i^\dagger(\mathbf{x})]$$
$$+\mathrm{tr}[v(\mathbf{y})v_i^\dagger(\mathbf{y})]\mathrm{tr}[v_i(\mathbf{x})v^\dagger(\mathbf{x})]. \tag{6.5}$$

The functional integral $\langle C\rangle=\int\mathcal{D}_H[v]C$ is computed in Appendix B. The result is

$$\langle C(\mathbf{x},\mathbf{y})\rangle=-2\frac{N_c^2-1}{a^2}\delta_{\mathbf{x},\mathbf{y}}. \tag{6.6}$$

Changing the integral in Eq. (6.4) into a finite sum, and letting $\mathbf{r}=\mathbf{x}-\mathbf{y}$, we get

$$\frac{d(x_FG)}{d^2\mathbf{k}}=\frac{\pi R_A^2}{4\pi^3g^2N_c}\sum_{\mathbf{r}}a^2\exp(i\mathbf{k}\cdot\mathbf{r})\frac{2(N_c^2-1)}{a^2}\delta_{\mathbf{r},0}$$
$$=\frac{2\pi R_A^2(N_c^2-1)}{16\pi^4\alpha_sN_c}. \tag{6.7}$$

The unintegrated spectrum $d(x_FG)/d\mathbf{k}^2$ is therefore absolutely flat, up to the saturation momentum Q_s^2 which is in this case infinite. That is not unexpected at $\tau=\infty$.

The integrated cross section is then

$$x_F G = Q_s^2 \frac{\pi R_A^2 (N_c^2-1)}{8\pi^3 \alpha_s N_c}. \tag{6.8}$$

The spectrum in Eq. (6.7) differs from the mean field prediction (2.16), in that Eq. (6.7) is flat and Eq. (2.16) has a logarithmic dependence on $\mathbf{k}^2$. The integrated density $x_F G$ is, however, quite similar to the estimate given in Eq. (2.2); the only difference is a factor $c = 8\pi^3$.

B. Below the asymptotic limit

It is much more difficult to solve the JIMWLK equation for finite τ, because we know nothing about the other eigenfunctions and eigenvalues of $\bar{H}$. In this section, we will discuss an approximate correction to $\bar{W}_\tau(v)$ below the asymptotic limit, in a region where the Wilson lines are far apart.

To avoid the infrared divergence, the distribution functional $\bar{W}_\tau(v)$ will be assumed to depend on v and $v^\dagger$ only through the multipole traces (5.1), or products of them. From the results of the last subsection and the discussion in Sec. II, we know that for large τ the average of multipole traces (i.e., multipole amplitudes) are small if the Wilson lines in the multipoles are far apart. In fact, the higher the order of the multipole is, the smaller the amplitude is expected to be. Therefore it is reasonable to include only quadratic dependences of v and $v^\dagger$ in a first correction to the asymptotic limit, at least in the region when the Wilson lines are far apart. We will therefore assume

$$\bar{W}_\tau(v) = 1 + \sum_{\mathbf{s},\mathbf{t}} \mathrm{tr}[v^\dagger(\mathbf{s}) v(\mathbf{t})] b_\tau(\mathbf{t},\mathbf{s}). \tag{6.9}$$

The contribution $\mathrm{tr}[v^\dagger(\mathbf{s})v(\mathbf{s})] b_\tau(\mathbf{s},\mathbf{s}) = N_c b_\tau(\mathbf{s},\mathbf{s})$ may be absorbed into the v-independent term. We may therefore assume $b_\tau(\mathbf{s},\mathbf{s}) = 0$ without any loss of generality. In that case, using the matching rule and unitarity of v to do the functional integral, we see that $\bar{W}_\tau(v)$ is still normalized:

$$\int \bar{\mathcal{D}}_H[v] W_\tau(v) = 1. \tag{6.10}$$

To compute the dipole amplitude (2.8), we need the following integration formula, which can be obtained from the matching rule, Eq. (3.8), and the unitarity of v:

$$\int \mathcal{D}_H[v] \mathrm{tr}[v^\dagger(\mathbf{s})v(\mathbf{t})] \mathrm{tr}[v^\dagger(\mathbf{x})v(\mathbf{y})] = N_c^2 \delta_{\mathbf{s},\mathbf{t}} \delta_{\mathbf{x},\mathbf{y}} + \delta_{\mathbf{s},\mathbf{y}} \delta_{\mathbf{t},\mathbf{x}} - \delta_{\mathbf{s},\mathbf{y},\mathbf{t},\mathbf{x}}. \tag{6.11}$$

The last Kronecker δ is by definition nonzero only when the four arguments in its subscript are all equal.

We may now compute the dipole amplitude from Eq. (2.8) to be

$$s_\tau(\mathbf{x},\mathbf{y}) = \frac{1}{N_c} \int \mathcal{D}_H[v] \bar{W}_\tau(v) \mathrm{tr}[v^\dagger(\mathbf{x})v(\mathbf{y})] = \left[\delta_{\mathbf{x},\mathbf{y}} + \frac{1}{N_c} b_\tau(\mathbf{x},\mathbf{y}) \right]. \tag{6.12}$$

Since $b_\tau(\mathbf{x},\mathbf{x}) = 0$, we get $s_\tau(\mathbf{x},\mathbf{x}) = 1$, as it should be. If we stay away from $\mathbf{x} = \mathbf{y}$, then $b_\tau(\mathbf{x},\mathbf{y})/N_c$ is just the dipole amplitude $s_\tau(\mathbf{x},\mathbf{y})$. As such it should satisfy the BK equation. On the other hand, we should be able to get the equation of $b_\tau(\mathbf{x},\mathbf{y})$ directly from the JIMWLK equation (3.28) by using Eqs. (6.9), (3.31), and (3.25). The left hand side is proportional to $\mathrm{tr}[v^\dagger(\mathbf{s})v(\mathbf{t})]$, but the right hand side has two terms, one is proportional to $\mathrm{tr}[v^\dagger(\mathbf{s})v(\mathbf{t})]$, and the other it proportional to $\mathrm{tr}[v^\dagger(\mathbf{z})v(\mathbf{t})]\mathrm{tr}[v^\dagger(\mathbf{s})v(\mathbf{z})]$. If we drop this quartic term because it involves a higher order multipole function which is expected to be small, then $b_\tau(\mathbf{x},\mathbf{y})/N_c = s_\tau(\mathbf{x},\mathbf{y})$ simply satisfies the BK equation (2.11) with the quadratic term of s_τ dropped. This is justified when the dipole amplitude is small, which is the case when the two Wilson lines are far apart, as expected. The solution is given by Eq. (2.13).

We may use Eq. (6.9) to calculate the higher order amplitude. The result is a sum of $b_\tau/N_c = s_\tau$, one for each dipole pair inside the multipole.

VII. CONCLUSION

The density of the soft gluons is determined by the distribution $W_\tau(\alpha)$ of the classical Yang-Mills potential $\alpha(\vec{x})$. The interaction of fast partons with such a background is given by their Wilson lines. In this paper, we introduced the distribution $\bar{W}_\tau(v)$ of Wilson lines. It can be obtained from $W_\tau(\alpha)$, and it also satisfies the JIMWLK equation. We completed Mueller's derivation of the JIMWLK equation, although for $\bar{W}_\tau(v)$ and not for $W_\tau(\alpha)$. We derived a generalized BK equation for multipole amplitudes. We also used the normalizable property of $\bar{W}_\tau(v)$ to compute the properties of physical observables at $x_F = 0$. We obtained in this way an unintegrated gluon spectrum $d(x_F G)/d\mathbf{k}^2 = \pi R_A^2 (N_c^2-1)/8\pi^3 \alpha_s N_c$, independent of the transverse momentum $\mathbf{k}^2$ of the gluon. A correction to this asymptotic behavior was briefly discussed.

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APPENDIX A: MATHEMATICAL DETAILS

The Haar measure of $U(n)$ for odd n is given by

$$d_H[v] = c_4 \mathrm{tr}[v^{-1}dv \wedge v^{-1}dv \wedge \cdots \wedge v^{-1}dv], \tag{A1}$$

where c_4 is a normalization constant determined by $\int d_H[v]=1$, and the argument of tr consists of the exterior product of n^2 factors of $v^{-1}dv$. From the cyclic nature of the trace, and the antisymmetry character of the exterior product, it can be seen that $d_H[v]$ defined in this way is 0 when n is even. Hence we must confine ourselves to odd n if we want to use the expression (A1).

It is easy to check that Eq. (3.3) is satisfied for Eq. (A1).

We will show in two ways that $\hat{\mathcal{D}}_a(\mathbf{x})$ defined in Eq. (3.20) is anti-Hermitian. First, define the inner product of two functionals α by

$$\langle f|g\rangle_\alpha = \int \mathcal{D}[\alpha]\mathcal{D}[\alpha']f^*(\alpha)g(\alpha')\delta(u-u')/\mathcal{J}(u), \tag{A2}$$

where u is given by Eq. (3.17) and u' is given similarly in terms of α'. From Eqs. (3.10) and (3.15), we see that

$$\delta(u-u')/\mathcal{J}(u) = \prod_{a,\mathbf{x}} \delta(\alpha_\tau(\mathbf{x})-\alpha'_\tau(\mathbf{x})). \tag{A3}$$

Using integration by parts and assuming the resulting surface term to be zero, it is easy to see that $\delta/\delta\alpha^a_\tau(\tau,\mathbf{y})$ is anti-Hermitian with respect to this inner product.

If $\overline{f}(v)$ and $\overline{g}(v)$ are obtained, respectively, from $f(\alpha)$ and $g(\alpha)$ by Eq. (3.16), then the inner product between $\overline{f}(v)$ and $\overline{g}(v)$ defined by

$$\langle \overline{f}|\overline{g}\rangle_v \equiv \int \mathcal{D}_H[v]\overline{f}^*(v)g(v) \tag{A4}$$

is equal to the inner product $\langle f|g\rangle_\alpha$ defined in Eq. (A2). This is so because

$$\begin{aligned}\langle \overline{f}|\overline{g}\rangle_v &= \int \mathcal{D}_H[v]\mathcal{D}[\alpha]\mathcal{D}[\alpha']f^*(\alpha)\delta(v-u)g(\alpha)\delta(v-u')/\mathcal{J}(u)\mathcal{J}(u')\\ &= \int \mathcal{D}[\alpha]\mathcal{D}[\alpha']\delta(u-u')f^*(\alpha)g(\alpha')/J(u)\\ &= \langle f|g\rangle_\alpha .\end{aligned} \tag{A5}$$

We know that the transform of $\delta f(\alpha)/\delta\alpha^a_\tau(\mathbf{y})$ is $\hat{\mathcal{D}}_a(\mathbf{y})$. Since $\langle \overline{fg}\rangle_v = \langle f|g\rangle_\alpha$ and since $\delta/\delta\alpha^a_\tau(\mathbf{y})$ is anti-Hermitian with respect to $\langle f|g\rangle_\alpha$, it follows that $\hat{\mathcal{D}}_a(\mathbf{y})$ must also be anti-Hermitian with respect to $\langle \overline{f}|\overline{g}\rangle$.

The second proof of the anti-Hermiticity of $\hat{\mathcal{D}}_a(\mathbf{y})$ makes use of the explicit formula (A1). It proceeds as follows.

Using the explicit formula (3.20) for $\hat{\mathcal{D}}_a(\mathbf{y})$, we must show that $\overline{\mathcal{D}}_a(\mathbf{y}) \equiv v_{ij}(\mathbf{y})(t_a)_{jk}\delta/\delta v_{ik}(\mathbf{y})$ is Hermitian with respect to the inner product (A4).

Let us first show that $\overline{\mathcal{D}}_a(\mathbf{y})$ is imaginary. Since $v^\dagger = v^{-1}$, it follows that

$$\frac{\delta}{\delta v^*_{ik}} = -v_{is}v_{rk}\frac{\delta}{\delta v_{rs}}. \tag{A6}$$

Using also the fact that t_a is Hermitian, then

$$\overline{\mathcal{D}}^*_a = v^*_{ij}(t^*_a)_{jk}\frac{\delta}{\delta v^*_{ik}} = -(vt_a)_{rs}\frac{\delta}{\delta v_{rs}} = -\overline{\mathcal{D}}_a . \tag{A7}$$

Now we use integration by parts to show that $\overline{\mathcal{D}}_a$ is antisymmetric with respect to the inner product $\langle f|g\rangle_v$. Remembering Eq. (3.10), integration by parts changes $\overline{\mathcal{D}}_a$ into

$$-\frac{1}{\mathcal{J}(v)}\frac{\delta}{\delta v_{ik}}(vt_a)_{ik}\mathcal{J}(v). \tag{A8}$$

This would indeed be equal to $-\overline{\mathcal{D}}_a$ if

$$\frac{\delta}{\delta v_{ik}}[(vt_a)_{ik}\mathcal{J}(v)] = 0. \tag{A9}$$

Since $\delta(d[v])/\delta v_{ik} = d[\delta v/\delta v_{ik}] = 0$, Eq. (A9) is true if

$$\frac{\delta[(vt_a)_{ik}\mathcal{D}_H[v]]}{\delta v_{ik}} = 0, \tag{A10}$$

which in turn is true if

$$n\,\mathrm{tr}(t_a)d_H[v] + (vt_a)_{ik}\frac{\partial d_H[v]}{\partial v_{ik}} = 0. \tag{A11}$$

Using Eq. (A1), we get

$$(vt_a)_{ik}\frac{\partial d_H[v]}{\partial v_{ik}} = -n^2\,\mathrm{tr}[t_a v^{-1}dv\wedge\cdots\wedge v^{-1}dv], \tag{A12}$$

namely, it is equal to $-n^2$ times a t_a inserted into the measure in Eq. (A1). If $a=0$, then t_0 is proportional to the identity matrix, so indeed Eq. (A11) is true. For $a>0$,

$\mathrm{tr}(t_a)=0$. Since $v^{-1}dv$ is a linear combination of the $U(n)$ generators t_b, we conclude that Eq. (A11) is valid if

$$\epsilon_{b_1 b_2 \cdots b_{n^2}} \mathrm{tr}[t_a t_{b_1} \cdots t_{b_{n^2}}]=0. \tag{A13}$$

This is indeed the case because $\mathrm{tr}(t_a)=0$ and $s \equiv \epsilon_{b_1 b_2 \cdots b_{n^2}} t_a t_{b_1} \cdots t_{b_{n^2}}$ is proportional to the identity matrix. This last statement can be proven as follows.

Let $v_0 \in U(n)$. Then $v_0^{-1} t_a v_0 = (V_0)_{ab} t_b$, where V_0 is the adjoint representation of v_0. Hence $v_0^{-1} s v_0 = \det(V_0) s$. Since the adjoint generator T_a has no diagonal matrix elements, it is traceless, so $\det(V_0)=1$. Therefore s commutes with every element v_0 of $U(n)$, so by Schur's lemma it must be proportional to the identity matrix.

We have thus shown that $\overline{\mathcal{D}_a}(\mathbf{y})$ is antisymmetric and imaginary. Hence it is Hermitian.

APPENDIX B: SATURATION CALCULATION

We want to compute $\langle C(\mathbf{xy})\rangle = \int \mathcal{D}_H[v] C(\mathbf{x},\mathbf{y})$ for the function $C(\mathbf{xy})$ given in Eq. (6.5). This function contains four terms. We will label them consecutively as C_1, C_2, C_3, and C_4.

We interpret the derivative $\partial_i v(\mathbf{x}) = v_i(\mathbf{x})$ on the lattice to be

$$v_i(\mathbf{x}) = \frac{1}{2a}[v(\mathbf{x}+\mathbf{a}_i) - v(\mathbf{x}-\mathbf{a}_i)], \tag{B1}$$

where $\mathbf{a}_i$ is a lattice vector along the ith direction. $v_i^\dagger(\mathbf{x})$ is defined similarly. The following computations make use of Eq. (3.6), $C_F = N_c/2$ for $U(n)$, the matching rule discussed at the end of Sec. III, and unitarity of v. There are also extra factors of 2 obtained by summing the two $\mathbf{a}_i$ directions. In this way we get

$$\begin{aligned}
\langle C_1\rangle &= \frac{2C_F}{4a^2}\langle \mathrm{tr}\{v(\mathbf{y})[v^\dagger(\mathbf{y}+\mathbf{a}_i) - v^\dagger(\mathbf{y}-\mathbf{a}_i)] \\
&\quad \times v(\mathbf{x})[v^\dagger(\mathbf{x}+\mathbf{a}_i) - v^\dagger(\mathbf{x}-\mathbf{a}_i)]\}\rangle \\
&= -\frac{2C_F N_c}{4a^2}[\delta_{\mathbf{x},\mathbf{y}-\mathbf{a}_i} + \delta_{\mathbf{x},\mathbf{y}+\mathbf{a}_i}] \\
&\simeq -\frac{2C_F N_c}{a^2}\delta_{\mathbf{x},\mathbf{y}} \\
&= -\frac{N_c^2}{a^2}\delta_{\mathbf{x},\mathbf{y}}, \\
\langle C_2\rangle &= \frac{2C_F}{4a^2}\langle \mathrm{tr}\rangle\{[v(\mathbf{y}+\mathbf{a}_i) - v(\mathbf{y}-\mathbf{a}_i)]v^\dagger(\mathbf{y}) \\
&\quad \times [v(\mathbf{x}+\mathbf{a}_i) - v(\mathbf{x}-\mathbf{a}_i)]v^\dagger(\mathbf{x})\} \\
&= \langle C_1\rangle, \\
\langle C_3\rangle &= \frac{1}{4a^2}\mathrm{tr}\{[v(\mathbf{y}+\mathbf{a}_i) - v(\mathbf{y}-\mathbf{a}_i)]v^\dagger(\mathbf{y})\} \\
&\quad \times \mathrm{tr}\{v(\mathbf{x})[v^\dagger(\mathbf{x}+\mathbf{a}_i) - v^\dagger(\mathbf{x}-\mathbf{a}_i)]\} \\
&= \frac{1}{a^2}\delta_{\mathbf{x},\mathbf{y}}, \\
\langle C_4\rangle &= \frac{1}{4a^2}\mathrm{tr}\{v(\mathbf{y})[v^\dagger(\mathbf{y}+\mathbf{a}_i) - v^\dagger(\mathbf{y}-\mathbf{a}_i)]\} \\
&\quad \times \mathrm{tr}\{[v(\mathbf{x}+\mathbf{a}_i) - v(\mathbf{x}-\mathbf{a}_i)v^\dagger(\mathbf{x})]\} \\
&= \frac{1}{a^2}\delta_{\mathbf{x},\mathbf{y}}.
\end{aligned} \tag{B2}$$

Thus

$$\begin{aligned}
\langle C(\mathbf{x},\mathbf{y})\rangle &= \langle C_1 + C_2 + C_3 + C_4\rangle \\
&= -2\frac{N_c^2-1}{a^2}\delta_{\mathbf{x},\mathbf{y}}.
\end{aligned} \tag{B3}$$

[1] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977); V.N. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. **15**, 438 (1972); Yu.L. Dokshitzer, Sov. Phys. JETP **46**, 641 (1977).

[2] V.S. Fadin, E.A. Kuraev, and L.N. Lipatov, Phys. Lett. **60B**, 50 (1975); E.A. Kuraev, L.N. Lipatov, and V.S. Fadin, Sov. Phys. JETP **44**, 443 (1976); **45**, 199 (1977); Ya.Ya. Balitsky and L.N. Lipatov, Sov. J. Nucl. Phys. **28**, 822 (1978).

[3] L.V. Gribov, E.M. Levin, and M.G. Ryskin, Phys. Rep. **100**, 1 (1983).

[4] M.A. Betemps, M.B. Gay Ducati, and M.V.T. Machado, hep-ph/0206218; J. Raufeisen, J. C. Peng, and G. C. Nayak, Phys. Rev. D **66**, 034024 (2002); J. Schaffner-Bielich, D. Kharzeev, L. McLerran, and R. Venugopalan, hep-ph/0108048; K. Golec-Biernat and M. Wüsthoff, Phys. Rev. D **60**, 114023 (1999); **59**, 014017 (1999); F. Gelis and A. Peshier, Nucl. Phys. **A697**, 879 (2002); F. Gelis and J. Jalilian-Marian, hep-ph/0205037; A. Dumitru and J. Jalilian-Marian, Phys. Rev. Lett. **89**, 022301 (2002); R. Baier, A.H. Mueller, D. Schiff, and D.T. Son, Phys. Lett. B **539**, 46 (2002); K.J. Eskola, K. Kajantie, P.V. Ruuskanen, and K. Tuominen, hep-ph/0204034; J. Jalilian-Marian and S. Jeon, Phys. Rev. C **65**, 065201 (2002); S. Munier, A.M. Stasto, and A.H. Mueller, Nucl. Phys. **B603**, 427 (2001); L. McLerran and J. Schaffner-Bielich, Phys. Lett. B **514**, 29 (2001); A.H. Mueller and G.P. Salam, Nucl. Phys. **B475**, 293 (1996).

[5] J. Jalilian-Marian, A. Kovner, L. McLerran, and H. Weigert, Phys. Rev. D **55**, 5414 (1997); J. Jalilian-Marian, A. Kovner, A. Leonidov, and H. Weigert, Nucl. Phys. **B504**, 415 (1997); Phys. Rev. D **59**, 014014 (1999); H. Weigert, Nucl. Phys. **A703**, 823 (2002); E. Iancu, A. Leonidiv, and L. McLerran,

ibid. **A692**, 583 (2001); Phys. Lett. B **510**, 133 (2001).
[6] A.H. Mueller, Phys. Lett. B **525**, 243 (2001).
[7] J.-P. Blaizot, E. Iancu, and H. Weigert, hep-ph/0206279.
[8] For a review, see A.H. Mueller, hep-ph/0111244, and references therein.
[9] E. Iancu, A. Leonidov, and L. McLerran, hep-ph/0202270, and references therein. See also Ref. [10].
[10] E. Iancu, hep-ph/0206073.
[11] E. Fereiro, E. Iancu, A. Leonidov, and L. McLerran, Nucl. Phys. **A703**, 489 (2002).
[12] A.H. Mueller, Nucl. Phys. **B335**, 115 (1990); Yu.V. Kovchegov and A.H. Mueller, *ibid.* **B529**, 451 (1998); A.H. Mueller, Nucl. Phys. **A654**, 37c (1999); hep-ph/9911289.
[13] I. Balitsky, Nucl. Phys. **B463**, 99 (1996); hep-ph/0101042; I.I. Balitsky and A.V. Belitsky, Nucl. Phys. **B629**, 290 (2002).
[14] Yu. Kovchegov, Phys. Rev. D **60**, 034008 (1999); **61**, 074018 (2000).
[15] A. Mueller, Nucl. Phys. **B558**, 285 (1999); E. Iancu and L. McLerran, Phys. Lett. B **510**, 145 (2001).
[16] L. McLerran and R. Venugopalan, Phys. Rev. D **49**, 2233 (1994); **49**, 3352 (1994); **50**, 2225 (1994).
[17] C.S. Lam and G. Mahlon, Phys. Rev. D **64**, 016004 (2001).
[18] C.S. Lam and G. Mahlon, Phys. Rev. D **61**, 014005 (2000); **62**, 114023 (2000).