# Exact Foldy-Wouthuysen transformation for real spin-0 particle in curved space

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Up to now, the only known exact Foldy-Wouthuysen transformation (FWT) in curved space is that concerning Dirac particles coupled to static spacetime metrics. Here we construct the exact FWT related to a real spin-0 particle for the aforementioned spacetimes. This exact transformation exists independently of the value of the coupling between the scalar field and gravity. Moreover, the gravitational Darwin term written for the conformal coupling is one-third of the corresponding term in the fermionic case. There are some arguments in the literature that seem to favor the choice $\lambda=\frac{1}{6}$. We rehearse a number of claims of these works.

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The Colella-Overhauser-Werner (COW) experiment [1] as well as the Bonse-Wroblewski [2] one have not only shed new light on the physical phenomena in which gravitational and quantum effects are interwoven, they have also shown that the aforementioned phenomena are no more beyond our reach. The theoretical analysis concerning these experiments consisted simply in inserting the Newtonian gravitational potential into the Schrödinger equation. To improve their analysis we need to learn certainly how to obtain an adequate interpretation for relativistic wave equations in curved space. In other words, we have to acquaint ourselves with the issue of the gravitational effects on quantum mechanical systems. This can be done by constructing the Foldy-Wouthuysen transformation (FWT) [3,4]—the keystone of relativistic quantum mechanics—for both bosons and fermions coupled to the spacetime metric. However, there are very few known problems in flat space that admit an exact FWT [5–7]. In curved space the situation is quite dramatic since up to now the only known exact FWT is that related to Dirac particles coupled to a static spacetime metric [8].

Here we address ourselves to the problem of finding the exact FWT for a real spin-0 particle coupled to the static metrics

$$ds^2=V^2dt^2-W^2d\mathbf{x}^2, \tag{1}$$

where $V=V(\mathbf{x})$ and $W=W(\mathbf{x})$. For the sake of clarification concerning the interpretation of the relativistic single particle wave mechanics for spin-0 boson, we reproduce a remark made by Feshbach and Villars [9] in the late 1950s: "Although it is well known that the Dirac equation gives within proper limits a relativistic wave-mechanical description of a single electron, we find in the literature the (incorrect!) statement that an analogous formalism does not exist for charged spin-0 particles."

By the middle of the 1970s, Guertin [10] constructed the generalized FWT for any $2(2J+1)$-component Poincaré-invariant Hamiltonian theory that describes free massive spin $-J$ particles and that is subject to the conditions: (a) every observable is either Hermitian or pseudo-Hermitian and (b) the theory is invariant under certain discrete symmetries.

In our convention the signature is $(+---)$. The curvature tensor is defined by $R^{\alpha}{}_{\beta\gamma\delta}=-\partial_\delta\Gamma^{\alpha}{}_{\beta\gamma}+\cdots$, the Ricci tensor by $R_{\mu\nu}=R^{\alpha}{}_{\mu\nu\alpha}$, and the curvature scalar by $R=g^{\mu\nu}R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor. Natural units are used throughout.

Currently, we do not have a standard theory of massive spinless bosons in curved space. That is not the case as far as the Dirac fermions are concerned. Therefore our first task is to find out how the Klein-Gordon (KG) equation should be written in the general case of a spacetime with nonvanishing curvature. Let us then start with the following scalar field equation

$$(\Box+m^2+\lambda R)\phi=0, \tag{2}$$

which is obtained from the action

$$S=\int\frac{1}{2}\sqrt{-g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-(m^2+\lambda R)\phi^2]d^4x. \tag{3}$$

Note that the coupling between the real scalar field $\phi$ and the gravitational field represented by the term $\lambda R\phi^2$, where $\lambda$ is a numerical factor and $R$ is the Ricci scalar, is included as the only possible local scalar coupling of this sort [11]. Here

$$\Box\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu=\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu).$$

The coupling constant $\lambda$, of course, can have any real value. This raises a delicate question: Which value of $\lambda$ should we single out? There are some arguments in the literature that seem to favor the choice $\lambda=\frac{1}{6}$. We rehearse a number of claims of these works: (i) the equation for the massless scalar field is conformally invariant [11–13]; (ii) under the assumption that (a) the scalar field satisfies Eq. (2), and (b) the field $\phi$ does not violate the equivalence principle, the coupling constant is forced to assume the value 1/6 [14,15]; (iii) the minimal coupling leads to a tachyonic behavior whereas the conformal one ($\lambda=1/6$) has a correct quasiclassical limit [16].

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There are other reasons (see, e.g., [17] and references therein) that perhaps may justify the presence of the nonminimal term in Eq. (3).

Here we examine the problem in the context of the exact FWT transformations for spin-0 particles. Let us then concentrate our attention on the curved spacetimes described by Eq. (1). The Ricci scalar related to this metric is given by

$$R=\frac{2}{W^4}(\nabla W)^2-\frac{2}{VW^3}\nabla V\cdot\nabla W-\frac{2}{VW^2}\nabla^2 V-\frac{4}{W^3}\nabla^2 W. \tag{4}$$

Inserting Eq. (4) into Eq. (2), we promptly obtain

$$\ddot{\phi}-F^2\nabla^2\phi-F^2\nabla\ln(VW^3)\cdot\nabla\phi+m^2V^2\phi+\lambda RV^2\phi=0, \tag{5}$$

where $F^2\equiv V^2/W^2$. Here the differentiation with respect to time is denoted by dots.

In order to bring the equation in hand to Schrödinger form we introduce the two-component formalism for the Klein-Gordon (KG) equation

$$\phi=\phi_1+\phi_2,\quad \frac{i}{m}\dot{\phi}=\phi_1-\phi_2.$$

Accordingly, the KG equation can be written in first-order form

$$i\dot{\Phi}=\mathcal{H}\Phi, \tag{6}$$

with the Hamiltonian given by

$$\mathcal{H}=\frac{m}{2}\xi^T-\xi\theta, \tag{7}$$

where

$$\Phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\qquad \xi=\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}$$

and the operator $\theta$ is defined by

$$\theta\equiv\frac{F^2}{2m}\nabla^2-\frac{F^2}{2m}\nabla\ln(VW)\cdot\nabla-\frac{m}{2}V^2-\frac{\lambda}{2m}V^2R.$$

Note that the matrix $\xi$ has the following algebraic properties

$$\xi^2=0,\quad \{\xi,\xi^T\}=4.$$

It is worth mentioning that the equations of motion derived from Eq. (6) are invariant under $\mathcal{H}\to-\mathcal{H}^*$ and $\phi_{1,2}\to\pm\phi_{2,1}$, which implies that in the two-component description of neutral spin-0 particles the particle and antiparticle may be identified since the gravitational interaction does not remove the particle-antiparticle degeneracy.

The operator $\theta$ is formally self-adjoint [18] with respect to an inner product provided the spatial integrations are carried out using the correct measure [19]

$$\langle\theta\rangle=\int\rho\, d^3\mathbf{x}\psi^\dagger\theta\psi, \tag{8}$$

where $\rho\equiv g^{00}\sqrt{-g}=W^3/V$.

However, it is more convenient to write the wave function so that $\theta$ is Hermitian with respect to the usual flat space measure. We do this by means of a transformation

$$\Phi\to\Phi'=f\Phi,\quad \theta'=f\theta f^{-1},\quad \text{and}\quad \mathcal{H}'=f\mathcal{H}f^{-1},$$

with $f\equiv\sqrt{\rho}=V^{-1/2}W^{3/2}$.

Therefore

$$\mathcal{H}'=\frac{m}{2}\xi^T-\xi\theta',$$

where

$$\theta'\Phi'=f\theta f^{-1}\Phi'.$$

Performing the computation, we then find that $\theta'$ can be written as

$$\theta'=-\frac{m}{2}V^2-\frac{1}{2m}F\hat{p}^2F+\frac{1}{8m}\nabla F\cdot\nabla F-\frac{1}{2m}\mathcal{D}_\lambda(V,W), \tag{9}$$

where $\hat{\mathbf{p}}=-i\nabla$ denotes the momentum operator and the last term becomes

$$\mathcal{D}_\lambda(V,W)\equiv\lambda\Bigg[\left(\frac{1}{2\lambda}-2\right)\frac{V}{W^2}\nabla^2V-2\frac{V}{W^3}\nabla V\cdot\nabla W+\left(\frac{1}{2\lambda}-4\right)\frac{V^2}{W^3}\nabla^2W+2\frac{V^2}{W^4}(\nabla W)^2\Bigg]. \tag{10}$$

The fascinating property of the transformed Hamiltonian $\mathcal{H}'$ is that its square,

$$\mathcal{H}'^{\Box}=-\frac{m}{2}\theta'\{\xi,\xi^T\}=-2m\theta' I, \tag{11}$$

where

$$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Note that formally

$$\sqrt{\mathcal{H}'^{\Box}}=(-2m\theta')^{1/2}I^{1/2}.$$

Since the square root of the $2\times2$ identity matrix is not unique the FWT transformation needs an extra diagonalizing transformation to the basis where positive and negative energy eigenstates are decoupled. This process can be made with the help of a nondegenerate matrix $U$ such that [20]

$$\mathcal{H}''\equiv(-2m\theta')^{1/2}U I^{1/2}U^{-1}$$
$$=(-2m\theta')^{1/2}\eta,$$

where

$$\eta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Accordingly, $\mathcal{H}\rightarrow\mathcal{H}''$ is the exact FWT for the KG equation in curved space.

Taking Eq. (9) into account, we arrive at the following expression for the Hamiltonian squared:

$$\mathcal{H}'^{\square}=m^2V^2+F\hat{p}^2F-\tfrac{1}{4}\nabla F\cdot\nabla F+\mathcal{D}_\lambda(V,W). \qquad (12)$$

The quasirelativistic Hamiltonian is simply obtained by assuming that $m$ is the dominating term. We thus arrive at

$$\mathcal{H}''\approx\Bigg\{mV+\frac{1}{4m}(W^{-1}\hat{p}^2F+F\hat{p}^2W^{-1}) -\frac{1}{8mV}\nabla F\cdot\nabla F+\frac{1}{2m}\mathcal{D}_\lambda(V,W)\Bigg\}\eta. \qquad (13)$$

Some comments are in order here.

(i) Notice the appearance of a Darwin-like term $1/2m\mathcal{D}_\lambda(V,W)$ in the quasirelativistic Hamiltonian (13). For $\lambda=1/6$ conformal invariance constrains the structure of the Darwin-like term to the form

$$\frac{1}{12mW}\nabla^2F. \qquad (14)$$

Therefore one obtains

$$\mathcal{H}''\approx\Bigg\{mV+\frac{1}{4m}(W^{-1}\hat{p}^2F+F\hat{p}^2W^{-1}) -\frac{1}{8mV}\nabla F\cdot\nabla F+\frac{1}{12mW}\nabla^2F\Bigg\}\eta. \qquad (15)$$

(ii) Equation (15) is identical to the spinless sector found by Obukhov [8] for the Dirac particle except for the Darwin term which is one-third of the corresponding term in the fermionic case [21].

(iii) The Darwin term (14) only exists in the context of the exact FWT if the interaction of the scalar field with gravity is of the conformal type $\lambda=1/6$, while for $\lambda\neq 1/6$ the Darwin term is more complicated.

Some remarks about (i) and (iii). It is claimed in the literature that Eq. (2) with $\lambda=1/6$ violates the equivalence principle and leads to the appearance of anomalous R-forces between two "scalar charged" particles [22]. Grib and Poberii [16] showed, however, that this is not the case. According to them the conformal coupling leads to a correct quasiclassical limit while the minimal one is responsible for a tachyonic behavior.

To conclude we shall prove that the conformal coupling does not violate the equivalence principle by making a comparison of the true gravitational coupling with the pure inertial case. To do that, we recall that far from the source the solution of the Einstein equation for a point particle of mass M located at $r=0$ is given by

$$g_{00}\approx 1-\frac{2MG}{r}, \qquad (16)$$

$$g_{11}=g_{22}=g_{33}\approx -1-\frac{2MG}{r}. \qquad (17)$$

From Eqs. (16) and (17) we get immediately

$$V\approx 1-\frac{MG}{r}, \quad W\approx 1+\frac{MG}{r}, \qquad (18)$$

$$\text{and} \quad F\approx 1-2\frac{MG}{r}. \qquad (19)$$

Inserting Eqs. (18) and (19) into Eq. (15) we obtain the nonrelativistic FW Hamiltonian, namely,

$$\mathcal{H}''=\left[m+m\,\mathbf{g}\cdot\mathbf{x}+\frac{\hat{\mathbf{p}}^2}{2m}+\frac{3}{2m}\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{x})\hat{\mathbf{p}}\right]\eta, \qquad (20)$$

where $\mathbf{g}=-GM\,\mathbf{r}/r^3$. On the other hand, in the case of the flat Minkowski space in accelerated frame,

$$V=1+\mathbf{a}\cdot\mathbf{x}, \quad W=1, \quad \text{and} \quad F=V,$$

one gets

$$\mathcal{H}''=\left[m+m\,\mathbf{a}\cdot\mathbf{x}+\frac{\hat{\mathbf{p}}^2}{2m}+\frac{1}{2m}\hat{\mathbf{p}}\cdot(\mathbf{a}\cdot\mathbf{x})\hat{\mathbf{p}}\right]\eta. \qquad (21)$$

In Eqs. (20) and (21) we have neglected the higher order relativistic and gravitational/inertial terms.

For the particle $m$ far away from the body $M$ one can neglect the terms $3/2m\hat{\mathbf{p}}\cdot(\mathbf{g}\cdot\mathbf{x})\hat{\mathbf{p}}$ and $1/2m\hat{\mathbf{p}}\cdot(\mathbf{a}\cdot\mathbf{x})\hat{\mathbf{p}}$ in Eqs. (20) and (21), respectively, since they are less than the kinetic term by a factor of $GM/r\sim 10^{-6}$ (for observations in the solar system) and much weaker by several orders than the leading and next to leading order terms linear in $m$. In Eq. (21) we are assuming that $\mathbf{a}$ is such that $|\mathbf{a}\cdot\mathbf{x}|\sim GM/r$. The Darwin term contributions in these expansions are zero in each case; in fact, in Eq. (20) we have $\nabla^2F=0$ (far away from the source and in the approximation considered) and in Eq. (21) for obvious reasons. Then, we come to the conclusion that the conformal coupling is in agreement with the equivalence principle.

Last but not least, we call attention to the fact that we are not claiming that the conformal coupling is the correct coupling for the various scalar particles. The question of which value(s) of $\lambda$ should constitute the correct coupling to gravity depends on the particular field theory used for the scalar field (see, e.g., [23] and references therein). Given the current theoretical situation it seems more of an experimental problem to identify which would be the correct $\lambda$ coupling(s) for the various scalar particles.

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