Conformal higher spin symmetries of 4D massless supermultiplets and $osp(L,2M)$ invariant equations in generalized (super)space

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Realization of the conformal higher spin symmetry on the 4D massless field supermultiplets is given. The self-conjugated supermultiplets, including the linearized $\mathcal{N}=4$ super Yang-Mills theory, are considered in some detail. Duality between nonunitary field-theoretical representations and the unitary doubleton-type representations of the 4D conformal algebra $su(2,2)$ is formulated in terms of a Bogolyubov transform. The set of 4D massless fields of all spins is shown to form a representation of $sp(8)$. The results obtained are extended to the generalized superspace invariant under $osp(L,2M)$ supersymmetries. A world line particle interpretation of the free higher spin theories in the $osp(2\mathcal{N},2M)$ invariant (super)space is given. Compatible with unitarity, free equations of motion in the $osp(L,2M)$ invariant (super)space are formulated. A conjecture about the chain of $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d \rightarrow \mathrm{AdS}_d/\mathrm{CFT}_{d-1} \rightarrow \cdots$ (where CFT indicates conformal field theory) dualities in the higher spin gauge theories is proposed.

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I. INTRODUCTION

AdS conformal field theory (CFT) correspondence [1–5] relates theories of gravity in the $(d+1)$-dimensional anti–de Sitter space AdS_{d+1} to conformal theories in d-dimensional (conformal) boundary space. Elementary fields in the bulk are related to the currents in boundary theory associated with nonlinear colorless combinations of the elementary boundary fields.

From the $d=4$ example it is known [6,7] that gauge theories of massless fields of all spins $0 \leqslant s \leqslant \infty$ admit a consistent formulation in AdS_4 (see [8,9] for more details and references on the higher spin gauge theories). The cosmological constant $\Lambda = -\lambda^2$ should necessarily be nonzero in the interacting higher spin gauge theories because it appears in negative powers in the interaction terms that contain higher derivatives of the higher spin gauge fields. This property is in agreement with the fact that higher spin gauge fields do not admit consistent interactions with gravity in the flat background [10].

Since the nonlinear higher spin gauge theory contains gravity and is formulated in AdS space-time, an interesting question is what is its AdS/CFT dual. It was recently conjectured [11,12] that the boundary theories dual to the AdS_{d+1} higher spin gauge theories are free conformal theories. These theories exhibit infinite-dimensional symmetries which are expected to be isomorphic to the AdS_{d+1} higher spin gauge symmetries. This conjecture is in agreement with the results of [13] where the conserved higher spin currents in d-dimensional free scalar field theory were shown to be in one-to-one correspondence with the set of one-forms associated with the totally symmetric higher spin gauge fields. The AdS/CFT regime associated with the higher spin gauge theories was conjectured [11,12] to correspond to the limit $g^2 n \rightarrow 0$. It is therefore opposite to the regime $g^2 n \rightarrow \infty$ underlying the standard AdS/CFT correspondence [1], which relates strongly coupled boundary theory to the classical regime of the bulk theory.

To test the AdS/CFT correspondence for the higher spin gauge theories it is instructive to realize the higher spin symmetries of the bulk higher spin gauge theories in AdS_{d+1} as higher spin conformal symmetries of the free conformal fields in d dimensions. In a recent paper [14] this problem was solved for the case of $\mathrm{AdS}_4/\mathrm{CFT}_3$. In particular, it was shown in [14] that 3D conformal matter fields are naturally described in terms of a certain Fock module F over the star product algebra identified [15] with the AdS_4 higher spin algebra [16,17]. The results of [14] confirmed the conjecture of Fradkin and Linetsky [18] that 3D conformal higher spin algebras are isomorphic to the AdS_4 higher spin algebras. The nonunitary Fock module F was interpreted in [14] as the field-theoretical dual of the unitary singleton module over $sp(4|\mathbf{R})$.

One of the aims of this paper is to extend the results of [14] to $\mathrm{AdS}_5/\mathrm{CFT}_4$ higher spin correspondence, which case is of most interest from the string theory perspective. We present a realization of the 4D conformal higher spin supermultiplets in terms of the field-theoretical Fock modules (fiber bundles) dual to the unitary doubleton [19] representations of $su(2,2)$. The conformal equations of motion for a 4D massless supermultiplet are formulated in the "unfolded" form of the covariant constancy conditions that makes the infinite-dimensional 4D conformal higher spin symmetries manifest. We compare the results obtained with the conjecture on the structure of 4D conformal higher spin symmetries made by Fradkin and Linetsky [20,21] in their analysis of 4D nonunitary higher spin conformal theories that generalize C^2 gravity, arriving at somewhat different conclusions. Also, the results obtained are compared with the conjecture of the recent paper [22] and the results of the forthcoming papers [23,24] on the (unitary) interacting higher spin theories in AdS_5 (i.e., those referred to in the $\mathrm{AdS}_5/\mathrm{CFT}_4$ higher spin correspondence).

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We show that the fundamental 4D conformal higher spin algebras are the infinite-dimensional algebras called $hu(m,n|8)$ in [25]. Here n and m refer to the spin 1 Yang-Mills symmetries $u(m)\oplus u(n)$ while the label 8 refers to the eight spinor generating elements of the higher spin star product algebra. Let us recall the definition of $hu(m,n|8)$. Consider the algebra of $(m+n)\times(m+n)$ matrices

$$\begin{pmatrix} A(a,b) & B(a,b) \\ C(a,b) & D(a,b) \end{pmatrix} \tag{1.1}$$

with the even functions (polynomials) of the auxiliary spinor variables $a_{\hat\alpha}$ and $b^{\hat\alpha}$ ($\hat\alpha,\hat\beta=1$–4) in the diagonal $m\times m$ block $A(a,b)$ and the $n\times n$ block $D(a,b)$,

$$A(-a,-b)=A(a,b),\quad D(-a,-b)=D(a,b), \tag{1.2}$$

and odd functions in the off-diagonal $m\times n$ block $B(a,b)$ and $n\times m$ block $C(a,b)$,

$$B(-a,-b)=-B(a,b),$$
$$C(-a,-b)=-C(a,b). \tag{1.3}$$

Consider the associative algebra of matrices of the form (1.1) with the associative star product law for the functions of the spinor variables $a_{\hat\alpha}$ and $b^{\hat\beta}$ defined as

$$\begin{aligned}(f*g)(a,b)&=\frac{1}{(\pi)^8}\int d^4u d^4v d^4s d^4t f(a+u,b+t)\\ &\quad\times g(a+s,b+v)\exp 2(s_{\hat\alpha}t^{\hat\alpha}-u_{\hat\alpha}v^{\hat\alpha})\\ &=e^{(\partial^2/\partial s_{\hat\alpha}\partial t^{\hat\alpha}-\partial^2/\partial u^{\hat\alpha}\partial v_{\hat\alpha})/2}f(a+s,b+u)\\ &\quad\times g(a+v,b+t)|_{s=t=u=v=0}.\end{aligned} \tag{1.4}$$

It is well known that this star product gives rise to the commutation relations

$$[a_{\hat\alpha},b^{\hat\beta}]_*=\delta_{\hat\alpha}^{\hat\beta},\quad [a_{\hat\alpha},a_{\hat\beta}]_*=0,\quad [b^{\hat\alpha},b^{\hat\beta}]_*=0 \tag{1.5}$$

with $[f,g]_*=f*g-g*f$. The associative star product algebra with eight generating elements $a_{\hat\alpha}$ and $b^{\hat\beta}$ is called Weyl algebra A_4 (i.e., A_l for l pairs of oscillators.) The particular star product realization of the algebra of oscillators we use describes the totally symmetric (i.e., Weyl) ordering. The matrices (1.1) result from the truncation of $A_4\otimes Mat_{m+n}$ by the parity conditions (1.2) and (1.3). Let us now treat this algebra as $\mathbf{Z}_2$-graded algebra with even elements in the blocks A and D and odd in B and C, i.e.,

$$\pi(A)=\pi(D)=0,\qquad \pi(B)=\pi(C)=1. \tag{1.6}$$

The Lie superalgebra $hgl(m,n|8;\mathbf{C})$ is the algebra of the same matrices with the product law defined via the graded commutator

$$[f,g]_\pm=f*g-(-1)^{\pi(f)\pi(g)}g*f. \tag{1.7}$$

Note that the $\mathbf{Z}_2$ grading (1.6) in $hgl(m,n|8;\mathbf{C})$ is in accordance with the standard relationship between spin and statistics once $a_{\hat\alpha}$ and $b^{\hat\beta}$ are interpreted as spinors.

The algebra $hu(m,n|8)$ is a particular real form of $hgl(m,n|8;\mathbf{C})$ defined so that the finite-dimensional subalgebra of $hu(m,n|8)$ identified as the spin 1 Yang-Mills algebra, which is spanned by the elements A and D independent of the spinor elements $a_{\hat\alpha}$ and $b^{\hat\beta}$, is the compact algebra $u(m)\oplus u(n)$. The explicit form of the reality conditions imposed to extract $hu(m,n|8)$ [25] is given in Sec. IV C of this paper.

This construction is a straightforward extension of the 3D conformal $\sim$AdS$_4$ higher spin algebras $hu(m,n|4)$ via doubling of the spinor generating elements. It is in accordance with the conjecture of [26] that higher spin algebras in any dimension are built in terms of the star product algebras with spinor generating elements. The definition of $hu(m,n|2p)$ is analogous.

The Lie algebra gl_4 is spanned by the bilinears

$$T_{\hat\alpha}{}^{\hat\beta}=a_{\hat\alpha}b^{\hat\beta}\equiv\frac{1}{2}(a_{\hat\alpha}*b^{\hat\beta}+b^{\hat\beta}*a_{\hat\alpha})I, \tag{1.8}$$

where I is the unit element of the matrix part of $hu(m,n|8)$. The central element is

$$N_0=a_{\hat\alpha}b^{\hat\alpha}\equiv\frac{1}{2}(a_{\hat\alpha}*b^{\hat\alpha}+b^{\hat\alpha}*a_{\hat\alpha})I. \tag{1.9}$$

The traceless part

$$t_{\hat\alpha}{}^{\hat\beta}=\left(a_{\hat\alpha}b^{\hat\beta}-\frac{1}{4}\delta_{\hat\alpha}^{\hat\beta}N_0\right)I \tag{1.10}$$

spans sl_4. The $su(2,2)$ real form of $sl_4(\mathbf{C})$ results from the reality conditions

$$\overline{a_{\hat\alpha}}=b^{\hat\beta}C_{\hat\beta\hat\alpha},\qquad \overline{b^{\hat\alpha}}=C^{\hat\alpha\hat\beta}a_{\hat\beta}, \tag{1.11}$$

where the overbar denotes complex conjugation while $C_{\hat\alpha\hat\beta}=-C_{\hat\beta\hat\alpha}$ and $C^{\hat\alpha\hat\beta}=-C^{\hat\beta\hat\alpha}$ are some real antisymmetric matrices satisfying

$$C_{\hat\alpha\hat\gamma}C^{\hat\beta\hat\gamma}=\delta_{\hat\alpha}^{\hat\beta}. \tag{1.12}$$

In order to incorporate supersymmetry one introduces the Clifford elements ϕ_i and their complex conjugates $\overline{\phi}^j$ ($i,j=1$–$\mathcal{N}$) satisfying the commutation relations

$$\{\phi_i,\phi_j\}_*=0,\qquad \{\overline{\phi}^i,\overline{\phi}^j\}_*=0,\qquad \{\phi_i,\overline{\phi}^j\}_*=\delta_i^j \tag{1.13}$$

with respect to the Clifford star product

$$\begin{aligned}(f*g)(\phi,\overline{\phi})&=2^{\mathcal{N}}\int d^{\mathcal{N}}\psi d^{\mathcal{N}}\overline{\psi}d^{\mathcal{N}}\chi d^{\mathcal{N}}\overline{\chi}f(\phi+\psi,\overline{\phi}+\overline{\chi})\\ &\quad\times g(\phi+\chi,\overline{\phi}+\overline{\psi})\exp 2(\psi_i\overline{\psi}^i-\chi_i\overline{\chi}^i)\end{aligned} \tag{1.14}$$

with anticommuting ϕ_i, $\overline{\phi}^i$, ψ_i, $\overline{\psi}^i$, χ_i, and $\overline{\chi}^i$.

The superalgebra $u(2,2|\mathcal{N})$ is spanned by the $u(2,2)$ generators (1.8) along with the supergenerators

$$Q^i_{\hat{\alpha}} = a_{\hat{\alpha}}\overline{\phi}^i, \qquad \overline{Q}^{\hat{\beta}}_i = b^{\hat{\beta}}\phi_i \tag{1.15}$$

and $u(\mathcal{N})$ generators

$$T_i{}^j = \phi_i\overline{\phi}^j. \tag{1.16}$$

The central element $N_{\mathcal{N}}$ of $u(2,2|\mathcal{N})$ is

$$N_{\mathcal{N}} = a_{\hat{\alpha}}b^{\hat{\alpha}} - \phi_i\overline{\phi}^i. \tag{1.17}$$

For $\mathcal{N}\neq 4$, $su(2,2|\mathcal{N}) = u(2,2|\mathcal{N})/N_{\mathcal{N}}$. The case of $\mathcal{N}=4$ is special because $N_{\mathcal{N}}$, which acts as the unit operator on the oscillators, has a trivial supertrace thus generating an additional ideal in $su(2,2|\mathcal{N})$. The corresponding simple quotient algebra is called $psu(2,2|4)$.

A natural higher spin extension of $su(2,2|\mathcal{N})$ is associated with the star product algebra of even functions of superoscillators

$$f(-a,-b;-\phi,-\overline{\phi}) = f(a,b;\phi,\overline{\phi}). \tag{1.18}$$

Since the Clifford algebra with $2\mathcal{N}$ generating elements is isomorphic to $Mat_{2^{\mathcal{N}}}$, one finds that the appropriate real form of the infinite-dimensional Lie superalgebra defined this way is isomorphic to $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$. Note that for $\mathcal{N}=4$ this gives rise to $hu(8,8|8)$. For $\mathcal{N}=0$ the Clifford algebra is one dimensional and, therefore, $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|2p)$ at $\mathcal{N}=0$ is identified with $hu(1,0|2p)$. The restriction of $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ to a particular supermultiplet gives rise to a smaller higher spin algebra we shall call $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$. α is a number characterizing a supermultiplet. The case of $\alpha=0$ will be shown to correspond to the self-conjugated supermultiplets. [Note that the algebra $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ was called $shsc(4|\mathcal{N})$ in [20].] An exciting possibility discussed at the end of this paper is that, once there exists a phase with the whole symmetry $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ unbroken, it may imply an infinite chain of the generalized AdS/CFT correspondences

$$\cdots \mathrm{AdS}^{p+1}/\mathrm{CFT}^{p} \rightarrow \mathrm{AdS}^{p}/\mathrm{CFT}^{p-1} \rightarrow \mathrm{AdS}^{p-1}/\mathrm{CFT}^{p-2}\cdots, \tag{1.19}$$

resulting in a surprising generalized space-time dimension democracy in the higher spin theories. [Abusing notation, we use the abbreviation AdS^p for the generalized $\frac{1}{2}p(p+1)$-dimensional space-time defined in Sec. IX.] The algebras $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ associated with the usual lower spin supermultiplets and AdS/CFT dualities are argued to result from some kind of spontaneous breakdown of the symmetries $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$.

The key idea of our approach is that the dynamics of the 4D higher spin massless multiplets admits a formulation in terms of certain Fock modules over $hu(m,n|8)$ analogously to what was shown previously for $d=2$ in [27] and for $d=3$ in [14]. Such a formulation makes the higher spin symmetries of the conformal systems manifest. The field theory formalism we work with operates with modules dual to the doubleton modules used for the description of the unitary representations associated with the one-particle states of the same system [19]. (Note that these Fock modules are somewhat reminiscent of the modules introduced for the description of noncommutative solitons in string theory [28].)

In addition to the $su(2,2|\mathcal{N})$ generators, the algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ contains the bilinear generators

$$U_{\hat{\alpha}\hat{\beta}} = a_{\hat{\alpha}}a_{\hat{\beta}}, \qquad V^{\hat{\alpha}\hat{\beta}} = b^{\hat{\alpha}}b^{\hat{\beta}}, \tag{1.20}$$

$$U_{ij} = \phi_i\phi_j, \qquad \overline{V}^{ij} = \overline{\phi}^i\overline{\phi}^j \tag{1.21}$$

and supergenerators

$$R_{\hat{\alpha}i} = a_{\hat{\alpha}}\phi_i, \qquad \overline{R}^{\hat{\beta}i} = b^{\hat{\beta}}\overline{\phi}^i, \tag{1.22}$$

which extend $u(2,2;\mathcal{N})$ to $osp(2\mathcal{N},8)$. [Recall that one can define $osp(p,2q)$ as the superalgebra spanned by various bilinears built from p fermionic oscillators and q pairs of bosonic oscillators; see, e.g., [29] for more details on the oscillator realizations of simple superalgebras.] $u(2,2;\mathcal{N})$ is spanned by the bilinears in oscillators that commute to the operator $N_{\mathcal{N}}$, i.e., $u(2,2;\mathcal{N})$ is the centralizer[1] of $N_{\mathcal{N}}$ in $osp(2\mathcal{N},8)$. An important consequence of this simple fact is that

$$su(2,2;\mathcal{N}) \subset osp(2\mathcal{N};8) \subset hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8). \tag{1.23}$$

As a result, once the higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ is shown to admit a realization on the conformal supermultiplets of massless fields, it follows that the same is true for its finite-dimensional subalgebra $osp(2\mathcal{N};8)$. Indeed, we shall show explicitly how the $osp(2\mathcal{N};8)$ transformations link together different massless (super)fields, requiring infinite sets of massless supermultiplets to be involved. This result is the field-theoretical counterpart of the fact that the singleton representation of $osp(2\mathcal{N};8)$ decomposes into all doubleton representations of $su(2,2;\mathcal{N})$. Note that the field-theoretical realization of $osp(2\mathcal{N};8)$ will be shown to be local.

This result confirms the conjecture of [30,26] that the algebras $osp(L,2^p)$ may play a distinguished role in the higher spin gauge theories in higher dimensions. More generally, it was first suggested in [31] that algebras of this class result from the supersymmetrization of conformal and AdS space-time symmetry algebras. In [32] a contraction of $osp(1,32)$ was applied for the description of 11-dimensional superalgebra. Somewhat later it was found that the algebras $osp(L,2M)$ (in most cases with $M=2^q$) and their contractions appear naturally in the context of M-theory dualities and brane charges [33–37]. One of the messages of this paper is that these symmetries can be unbroken in the phase in which all higher spin fields are massless. An immediate speculation is that not only do massive higher spin modes in

[1] I am grateful to M. Günaydin for drawing my attention to this fact.

fundamental strings result from some spontaneous breaking of the higher spin symmetries, but also branes are built from the higher spin gauge fields.

This raises the important question of what are the higher dimensional geometry and dynamics that support $osp(L,2^p)$ symmetries. Generally, there is no genuine reason to believe that a higher dimensional geometry should necessarily be Riemannian and, in particular, that the bosonic coordinates are necessarily Lorentz vectors. We shall call this presently dominating belief the "Minkowski track." An alternative option, which looks more natural from various points of view, is that higher dimensional bosonic and fermionic dimensions beyond $d=4$ may be associated with certain coset superspaces built from $osp(L,2M)$. We call this alternative the "symplectic track." An important advantage of this alternative is due to supersymmetry. Indeed, the main reason why supersymmetry singles out some particular dimensions in the Minkowski track is the mismatch between the numbers of bosonic and fermionic coordinates in higher dimensions as a result of the fact that the dimension of the spinor representations of the Lorentz algebra increases with the space-time dimension as $2^{[d/2]}$ while the dimensions of its tensor representations increase polynomially. Only for some lower dimensions $d\leqslant 11$, where the number of spinor coordinates is not too high due to some Majorana and/or Weyl conditions, can the matching be restored.

Some ideas on the possible structure of an alternative to Minkowski space-times have appeared in both the field-theoretical [38,30,39–46] and world particle dynamics [47–50] contexts. In particular, important algebraic and geometric insights most relevant to the subject of this paper were elaborated by Fronsdal in the pioneering work [30]. Further extensions with higher rank tensor coordinates were discussed in [51,52]. The nontrivial issue, however, is that it is not *a priori* clear whether a particular $osp(L,2M)$ invariant symplectic track equation allows for quantization compatible with unitarity for $M>2$. This point is tricky. On the one hand, a Lorentz invariant interval built from the "central charge coordinates" associated with $sp(2^p)$ has many timelike directions which, naively, would imply ghosts. On the other hand, it is well known [29] that $osp(L,2M)$ admits unitary lowest weight representations (by "lowest weight" we mean that it is a quotient of a Verma module), thus indicating that some of its quantum-mechanically consistent field-theoretical realizations have to exist.

Here is where the power of the "unfolded formulation" dynamics [53–55] plays a crucial role. Because this approach suggests a natural Bogolyubov transform duality between the field-theoretical unfolded equations and lowest weight unitary modules [14], which, in fact, implies quantization, it allows us to solve the problem by identifying the differential equations that give rise to the field-theoretical module dual to an appropriate unitary module. This is achieved by solving a certain cohomology problem. One of the central results of this paper consists of the explicit formulation of the $osp(L,2M/R)$ invariant equations of motion in the symplectic track space associated with the massless unitary lowest weight modules of $osp(L,2M/R)$ via a Bogolyubov duality transform. Let us note that for the particular case of $sp(8)$ two simple equations in the symplectic track space for scalar and svector (i.e., a vector of the symplectic algebra interpreted as a spinor in the Minkowski track) fields encode all massless equations in the usual 4D Minkowski space. This opens an exciting new avenue to higher dimensional physics in the framework of the symplectic track. To put it briefly, the right geometry is going to be associated in all cases with symplectic twistors, while for some lower dimensions we happened to live in it turns out to be equivalent to the usual Minkowski geometry.

The rest of the paper is organized as follows. In Sec. II we summarize the general approach to unfolded dynamics with the emphasize on the cohomological interpretation of the dynamical fields and equations of motion. In Sec. III we identify the vacuum gravitational field and discuss the global higher spin symmetries. 4D free equations for massless fields of all spins in the unfolded form are studied in Sec. IV. In Sec. IV A we reformulate the free massless equations of motion for 4D massless fields of all spins in terms of flat sections of an appropriate Fock fiber bundle and identify various types of the 4D higher spin conformal algebras. A generic solution of these equations in flat space-time is presented in Sec. IV B. The reality conditions are defined in Sec. IV C. The reduction to self-conjugated supermultiplets based on a certain antiautomorphism and the corresponding reduced higher spin algebras are discussed in Sec. IV D. In Sec. V, we explain how the formulas for any global conformal higher spin symmetry transformation of the massless fields can be derived and present explicit formulas for the global $osp(2\mathcal{N},8)$ transformations. The duality between the field-theoretical Fock module and unitary [$sp(8)$ singleton] module is discussed in Sec. VI. The dynamics of the 4D conformal massless fields is reformulated in the $osp(2\mathcal{N},8)$ invariant (super)spaces in Sec. VII. We start in Sec. VII A with the example of the usual superspace. The unfolded equations compatible with unitarity in $sp(2M)$ invariant space-time are derived in Sec. VII B. The unfolded dynamics in the $osp(L,2M)$ invariant superspaces is formulated in Sec. VII C. Further extension of the equations to the infinite-dimensional higher spin superspace is given in Sec. VII D. The world line particle interpretation of the massless equations of motion obtained is discussed in Sec. VIII where some new twistorlike particle models are presented. The AdS/CFT correspondence in the framework of higher spin gauge theories is the subject of Sec. IX where, in particular, the possibility of an infinite chain of AdS/CFT dualities in higher spin gauge theories is discussed. Finally, Sec. X contains a summary of the main results of the paper and discussion of some perspectives.

II. UNFOLDED DYNAMICS

As usual in the higher spin theory framework, we shall use the "unfolded formulation" approach [53–55] which allows one to reformulate any dynamical equations in the form

$$dw^A = F^A(w) \tag{2.1}$$

($d=dx^{\underline{n}}\partial/\partial x^{\underline{n}}$; underlined indices $\underline{m}$, $\underline{n}=0-d-1$ are used for the components of differential forms) with some set of differential forms w and a function $F^A(w)$ built from w with the help of the exterior product and satisfying the compatibility condition

$$F^B(w)\frac{\delta F^A(w)}{\delta w^B}=0. \tag{2.2}$$

In the linearized approximation, i.e., expanding near some particular solution w_0 of Eq. (2.1), one finds that nontrivial dynamical equations are associated with null vectors of the linearized part F_1 of F.

For example, consider the system of equations

$$\partial_{\underline{n}}C_{a_1\cdots a_n}(x)+h_{\underline{n}}{}^{b}C_{ba_1\cdots a_n}(x)=0, \tag{2.3}$$

$$dh^a=0 \tag{2.4}$$

with the set of 0-forms $C_{a_1\cdots a_n}$ with all $n=0,1,2,\ldots,\infty$ and the one-form $h^a=dx^{\underline{n}}h_{\underline{n}}{}^{a}$ ($a,b,\ldots=0\text{–}d-1$ are fiber vector indices). This system is obviously consistent in the sense of Eq. (2.2). Assuming that $h_{\underline{n}}{}^{a}$ is a nondegenerate matrix (in fact, the flat space-time frame), say, choosing $h_{\underline{n}}{}^{a}=\delta_{\underline{n}}{}^{a}$ as a particular solution of Eq. (2.4), one finds that the system is dynamically empty, just expressing the highest components $C_{a_1\cdots a_n}$ via the highest derivatives of C:

$$C_{a_1\cdots a_n}(x)=(-1)^n\partial_{a_1}\cdots\partial_{a_n}C(x). \tag{2.5}$$

However, once some of the components of $C_{a_1\cdots a_n}$ are missed in a way consistent with the compatibility condition (2.2), this will impose the differential restrictions on the "dynamical field" $C(x)$. In particular, this happens if the tensors are required to be traceless,

$$C^b{}_{ba_3\ldots a_n}=0. \tag{2.6}$$

In accordance with Eq. (2.5) this implies the Klein-Gordon equation

$$\Box C(x)=0 \tag{2.7}$$

and, in fact, no other independent conditions.

An important point is that any system of differential equations can be reformulated in the form (2.1) by virtue of introducing enough (usually, infinitely many) auxiliary fields. We call such a reformulation "unfolding." In many important cases the linearized equations have the form

$$(\mathcal{D}+\sigma_-+\sigma_+)C=0, \tag{2.8}$$

where C denotes some (usually infinite) set of fields (i.e., a section of some linear fiber bundle over the space-time with a fiber space V) and the operators $\mathcal{D}$ and $\sigma_\pm$ have the properties

$$(\sigma_\pm)^2=0,\quad \mathcal{D}^2+\{\sigma_-,\sigma_+\}=0,\quad \{\mathcal{D},\sigma_\pm\}=0. \tag{2.9}$$

It is assumed that only the operator $\mathcal{D}$ acts nontrivially on (differentiates) the space-time coordinates while $\sigma_\pm$ act in the fiber V. It is also assumed that there exists a gradation operator G such that

$$[G,\mathcal{D}]=0,\qquad [G,\sigma_\pm]=\pm\sigma_\pm, \tag{2.10}$$

G can be diagonalized in the fiber space V, and the spectrum of G in V is bounded from below. In the example above $\mathcal{D}=d$, $\sigma_+=0$, $\sigma_-(C)_{a_1\cdots a_n}=h^bC_{ba_1\cdots a_n}$. The gradation operator G counts a number of indices $G(C)_{a_1\cdots a_n}=nC_{a_1\cdots a_n}$.

The important observation is (see, e.g., [56]) that the nontrivial dynamical equations hidden in Eq. (2.8) are in one-to-one correspondence with the nontrivial cohomology classes of σ_-. For the case under consideration with C being a 0-form, the relevant cohomology group is $H^1(\sigma_-)$. For the more general situation with C being a p-form, the relevant cohomology group is $H^{p+1}(\sigma_-)$ (in a somewhat implicit form this analysis for the case of one-forms was applied in [57,26]).

Indeed, consider the decomposition of the space of fields C into the direct sum of eigenspaces of G. Let a field having the definite eigenvalue k of G be denoted $C|_k$, $k=0,1,2\ldots.$ Suppose that the dynamical content of Eqs. (2.8) with the eigenvalues $k\leqslant k_q$ is found. Applying the operator $\mathcal{D}+\sigma_+$ to the left hand side of Eqs. (2.8) at $k\leqslant k_q$ we obtain taking into account Eq. (2.9) that

$$\sigma_-((\mathcal{D}+\sigma_-+\sigma_+)(C)|_{k_q+1})=0. \tag{2.11}$$

Therefore $(\mathcal{D}+\sigma_-+\sigma_+)(C)|_{k_q+1}$ is σ_- closed. If the group $H^1(\sigma_-)$ is trivial in the grade k_q+1 sector, any solution of Eq. (2.11) can be written in the form $(\mathcal{D}+\sigma_-+\sigma_+)(C)|_{k_q+1}=\sigma_-(\widetilde{C}|_{k_q+2})$ for some field $\widetilde{C}|_{k_q+2}$. This, in turn, is equivalent to the statement that one can adjust $C|_{k_q+2}$ in such a way that $\widetilde{C}|_{k_q+2}=0$ or, equivalently, the part of Eq. (2.8) of the grade k_q+1, is some constraint that expresses $C|_{k_q+2}$ in terms of the derivatives of $C|_{k_q+1}$ (to say that this is a constraint we have used the assumption that the operator σ_- is algebraic in the space-time sense, i.e., it does not contain space-time derivatives). If $H^1(\sigma_-)$ is nontrivial, this means that Eq. (2.8) sends the corresponding cohomology class to zero and, therefore, not only expresses the field $C|_{k_q+2}$ in terms of derivatives of $C|_{k_q+1}$ but also imposes some additional differential conditions on $C|_{k_q+1}$. Thus, the nontrivial space-time differential equations described by Eq. (2.8) are classified by the cohomology group $H^1(\sigma_-)$.

The nontrivial dynamical fields are associated with $H^0(\sigma_-)$ which is always nonzero because it at least contains a nontrivial subspace of V of minimal grade. As follows from the $H^1(\sigma_-)$ analysis of the dynamical equations, all fields in $V/H^0(\sigma_-)$ are auxiliary, i.e., are expressed via the space-time derivatives of the dynamical fields by virtue of Eqs. (2.8).

For the scalar field example one finds [56] that $H^0(\sigma_-)$ is spanned by the linear space of the rank-zero tensors associated with the scalar field. For the case with the fiber V realized by all symmetric tensors, $H^1(\sigma_-)=0$ and, therefore, the corresponding system is dynamically empty. For the case of V spanned by traceless symmetric tensors, $H^1(\sigma_-)$ turns out to be one dimensional with the one-form representative

$$\kappa h_{\underline{n}}{}^a \tag{2.12}$$

taking values in the subspace of rank-1 tensors (i.e., vectors). Indeed, it is obvious that any element of the form (2.12) is σ_- closed. It is not σ_- exact because $h_{\underline{n}\,b}\neq h_{\underline{n}}{}^a C_{ab}$ with some symmetric traceless C_{ab}. As a result, the only nontrivial equation contained in Eq. (2.3) is its trace part at $n=1$, which is just the Klein-Gordon equation (2.7).

Let us note that the "unfolded equation" approach is to some extent analogous to the coordinate-free formulation of gravity by Penrose [58] and the concept of exact sets of fields (see [59] and references therein) in which the dynamical equations are required to express all space-time derivatives of the fields in terms of the fields themselves. The important difference between these two approaches is that "unfolded dynamics" operates in terms of differential forms, thus leaving room for gauge potentials and gauge symmetries that in most cases are crucial for the interaction problem. In some sense, the exact sets of fields formalism corresponds to the particular case of unfolded dynamics in which all fields are described as 0-forms.

III. VACUUM AND GLOBAL SYMMETRIES

Let us now consider the four-dimensional case introducing 4D index notation. We will use two pairs of two-component spinors a_α, b^α, $\tilde{a}_{\dot\alpha}$, and $\tilde{b}^{\dot\beta}$. The basis commutation relations become

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta, \quad [\tilde{a}_{\dot\gamma}, \tilde{b}^{\dot\beta}]_* = \delta_{\dot\gamma}^{\dot\beta}. \tag{3.1}$$

The 4D identification of the elements of $su(2,2)$ is as follows:

$$L_\alpha{}^\beta = a_\alpha b^\beta - \frac{1}{2}\delta_\alpha{}^\beta a_\gamma b^\gamma,$$

$$\bar{L}_{\dot\alpha}{}^{\dot\beta} = \tilde{a}_{\dot\alpha}\tilde{b}^{\dot\beta} - \frac{1}{2}\delta_{\dot\alpha}{}^{\dot\beta}\tilde{a}_{\dot\gamma}\tilde{b}^{\dot\gamma} \tag{3.2}$$

are Lorentz generators;

$$D = \frac{1}{2}(a_\alpha b^\alpha - \tilde{a}_{\dot\alpha}\tilde{b}^{\dot\alpha}) \tag{3.3}$$

is the dilatation generator;

$$P_\alpha{}^{\dot\beta} = a_\alpha \tilde{b}^{\dot\beta} \tag{3.4}$$

and

$$K_{\dot\alpha}{}^\beta = \tilde{a}_{\dot\alpha} b^\beta \tag{3.5}$$

are the generators of 4D translations and special conformal transformations, respectively. The complex conjugation rules

$$\overline{a_\alpha} = \tilde{b}_{\dot\alpha}, \quad \overline{b^\alpha} = \tilde{a}^{\dot\alpha},$$

$$\overline{\tilde{a}_{\dot\alpha}} = b_\alpha, \quad \overline{\tilde{b}^{\dot\alpha}} = a^\alpha \tag{3.6}$$

are in accordance with Eq. (1.11) with the antisymmetric matrix $C^{\hat\alpha\hat\beta}$ having nonzero components

$$C^{\alpha\dot\beta} = \varepsilon^{\alpha\dot\beta}, \qquad C^{\dot\gamma\beta} = \varepsilon^{\dot\gamma\beta}, \tag{3.7}$$

where $\varepsilon^{\alpha\beta}$ is the 2×2 antisymmetric matrix normalized to $\varepsilon^{12}=1$.

Let $\omega(a,b;\phi,\bar\phi|x)$ be a one-form taking values in the higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$, i.e., ω is the generating function of the conformal higher spin gauge fields

$$\omega(a,b;\phi,\bar\phi|x) = \sum_{m,n=0}^{\infty}\sum_{k,l=0}^{\mathcal{N}} \frac{1}{m!n!k!l!} \times \omega_{\hat\alpha_1\cdots\hat\alpha_m,}{}^{\hat\beta_1\cdots\hat\beta_n}{}_{i_1\cdots i_k}{}^{j_1\cdots j_l}(x) \times b^{\hat\alpha_1}\cdots b^{\hat\alpha_m} a_{\hat\beta_1}\cdots a_{\hat\beta_n}\phi^{i_1}\cdots\phi^{i_k}\bar\phi_{j_1}\cdots\bar\phi_{j_l}. \tag{3.8}$$

In the cases of interest the general equation (2.1) admits a solution with all fields equal to zero except for some one-forms ω_0 taking values in an appropriate Lie (super)algebra h [in the case under consideration $h=hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$]. Equation (2.1) then reduces to the zero-curvature equation on ω_0. To describe nontrivial space-time geometry one has to require h to contain an appropriate space-time symmetry algebra whose gauge fields identify with the background gravitational fields. In particular, the components of ω_0 in the sector of translations are identified with the gravitational frame field which is supposed to be nondegenerate. Let ω_0 be such a solution of the zero-curvature equation

$$d\omega_0 = \omega_0 \wedge * \omega_0. \tag{3.9}$$

Equation (3.9) is invariant under the gauge transformations

$$\delta\omega_0 = d\epsilon - [\omega_0, \epsilon]_*, \tag{3.10}$$

where $\epsilon(a,b;\phi,\bar\phi|x)$ is an infinitesimal symmetry parameter, which is a 0-form. Any vacuum solution ω_0 of Eq. (3.9) breaks the local higher spin symmetry to its stability subalgebra with the infinitesimal parameters $\epsilon_0(a,b;\phi,\bar\phi|x)$ satisfying the equation

$$d\epsilon_0 - [\omega_0, \epsilon_0]_* = 0. \tag{3.11}$$

Consistency of this equation is guaranteed by the zero-curvature equation (3.9).

Locally, Eq. (3.9) admits a pure gauge solution

$$\omega_0 = -g^{-1} * dg. \tag{3.12}$$

Here $g(a,b;\phi,\bar{\phi}|x)$ is some invertible element of the associative algebra, i.e., $g^{-1}*g=g*g^{-1}=1$. For ω_0 [Eq. (3.12)], one finds that the generic solution of Eq. (3.11) is

$$\epsilon_0(a,b;\phi,\bar{\phi}|x)=g^{-1}(a,b;\phi,\bar{\phi}|x) *\xi(a,b;\phi,\bar{\phi})*g(a,b;\phi,\bar{\phi}|x), \tag{3.13}$$

where $\xi(a,b;\phi,\bar{\phi})$ is an arbitrary x-independent element that plays the role of the "initial data" for Eq. (3.11).

$$\epsilon_0(a,b;\phi,\bar{\phi}|x)|_{x=x_0}=\xi(a,b;\phi,\bar{\phi}) \tag{3.14}$$

for such a point x_0 that $g(x_0)=1$. Since $[\epsilon_0^1,\epsilon_0^2]_*$ has the same form with $\xi^{12}=[\xi^1,\xi^2]_*$, it is clear that the global symmetry algebra is $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$.

As usual, the gravitational fields (i.e., frame and Lorentz connection) are associated with the generators of translations and Lorentz rotations in the Poincaré or AdS subalgebras of the conformal algebra. For AdS_4 one sets

$$\omega_0=\omega_0{}^{\alpha}{}_{\beta}(x)L_{\alpha}{}^{\beta}+\bar{\omega}_0{}^{\dot\alpha}{}_{\dot\beta}(x)\bar{L}_{\dot\alpha}{}^{\dot\beta} +h_0{}^{\alpha}{}_{\dot\beta}(x)(P_{\alpha}{}^{\dot\beta}+\lambda^2K^{\dot\beta}{}_{\alpha}), \tag{3.15}$$

where $-\lambda^2$ is the cosmological constant. The indices of $K_{\dot\beta\,\alpha}$ have been raised and lowered with the aid of the Lorentz invariant antisymmetric forms $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot\alpha\dot\beta}$ according to the rules

$$A^{\alpha}=\varepsilon^{\alpha\beta}A_{\beta}, \qquad A_{\beta}=\varepsilon_{\alpha\beta}A^{\alpha},$$
$$A^{\dot\alpha}=\varepsilon^{\dot\alpha\dot\beta}A_{\dot\beta}, \qquad A_{\dot\beta}=\varepsilon_{\dot\alpha\dot\beta}A^{\dot\alpha}, \tag{3.16}$$

which, as expected for the AdS_4 space having a dimensionful radius, breaks down the scaling symmetry of the ansatz (3.15). The condition that the ansatz (3.15) solves the zero-curvature equation (3.9) along with the condition that $h_0{}^{\alpha}{}_{\dot\beta}(x)$ is nondegenerate implies that $\omega_0{}^{\alpha}{}_{\beta}(x)$, $\bar{\omega}_0{}^{\dot\alpha}{}_{\dot\beta}(x)$, and $h_0{}^{\alpha}{}_{\dot\beta}(x)$ describe the AdS_4 Lorentz connection and the frame field, respectively. [Note that the generator $P_{\alpha}{}^{\dot\beta}+\lambda^2K^{\dot\beta}{}_{\alpha}$ describes the embedding of the AdS_4 translations into the conformal algebra $su(2,2)$.]

For the 4D flat Minkowski space one can choose

$$\omega_0=dx^{\underline{n}}\sigma_{\underline{n}}{}^{\alpha}{}_{\dot\beta}a_{\alpha}\tilde{b}^{\dot\beta}, \tag{3.17}$$

thus setting all fields equal to zero except for the flat space vierbein associated with the translation generator. Here $\sigma_{\underline{n}}{}^{\alpha\dot\beta}$ is the set of 2×2 Hermitian matrices normalized to

$$\sigma_{\underline{n}}{}^{\alpha\dot\beta}\sigma_{\underline{m}\,\alpha\dot\beta}=\eta_{\underline{nm}}, \qquad \sigma_{\underline{n}}{}^{\alpha\dot\beta}\sigma_{\underline{m}\,\gamma\dot\delta}\eta^{\underline{nm}}=\delta^{\alpha}_{\gamma}\delta^{\dot\beta}_{\dot\delta}, \tag{3.18}$$

where $\eta_{\underline{nm}}$ is the flat Minkowski metric tensor. The function g that gives rise to the flat gravitational field (3.17) is

$$g=\exp(-x^{\alpha}{}_{\dot\beta}a_{\alpha}\tilde{b}^{\dot\beta}), \tag{3.19}$$

where

$$x^{\alpha\dot\beta}=x^{\underline{n}}\sigma_{\underline{n}}{}^{\alpha\dot\beta}, \qquad x^{\underline{n}}=\sigma^{\underline{n}}{}_{\alpha\dot\beta}x^{\alpha\dot\beta}. \tag{3.20}$$

IV. 4D CONFORMAL FIELD EQUATIONS

As shown in [53,54], the equations of motion for massless fields in AdS_4 admit a formulation in terms of the generating function

$$C(y,\bar{y}|x)=\sum_{m,n=0}^{\infty}\frac{1}{m!n!}c_{\alpha_1\cdots\alpha_m,\dot\beta_1\cdots\dot\beta_n}(x) \times y^{\alpha_1}\cdots y^{\alpha_m}\bar{y}^{\dot\beta_1}\cdots\bar{y}^{\dot\beta_n} \tag{4.1}$$

with the auxiliary spinor variables y^{α} and $\bar{y}^{\dot\beta}$. $C(y,\bar{y}|x)$ is the generating function for all on-mass-shell nontrivial spin $s\geqslant1$ gauge invariant curvatures and matter fields of spin 0 and 1/2. Every spin s massless field appears in two copies because the generating function $C(y,\bar{y}|x)$ is complex. It forms the twisted adjoint representation of the algebra $hu(1,1|4)$. The associated covariant derivative reads

$$DC=dC-\omega*C+C*\tilde{\omega}, \tag{4.2}$$

where $\omega(y,\bar{y}|x)$ is the generating function for higher spin gauge fields taking values in $hu(1,1|4)$, $*$ denotes the Moyal star product induced by the Weyl (i.e., totally symmetric) ordering of the oscillators y^{α} and $\bar{y}^{\dot\alpha}$ with the basis commutation relations

$$[y^{\alpha},y^{\beta}]_*=2i\varepsilon^{\alpha\beta}, \qquad [\bar{y}^{\dot\alpha},\bar{y}^{\dot\beta}]_*=2i\varepsilon^{\dot\alpha\dot\beta}, \tag{4.3}$$

and the tilde denotes the involutive automorphism of the algebra[2] $\tilde{\omega}(y,\bar{y}|x)=\omega(-y,\bar{y}|x)$. Fluctuautions of the higher spin gauge fields are linked to the invariant field strengths by virtue of their own field equations [53,54]. The sector of higher spin gauge fields plays an important role in the analysis of higher spin interactions and Lagrangian higher spin dynamics, very much as the Lagrangian form of Maxwell theory is formulated in terms of potentials rather than field strengths. In this paper, however, we confine ourselves to consideration of free field equations formulated in terms of field strengths with $\omega=\omega_0$ being a fixed vacuum gravitational field taking values in the gravitational $sp(4|\mathbf{R})$ subalgebra of $hu(1,1|4)$ and satisfying zero-curvature equation for the higher spin algebra. Note that, as explained in the Introduction, $sp(4|\mathbf{R})$ belongs to the finite-dimensional subalgebra $osp(2,4)$ of $hu(1,1|4)$, i.e., the system under consideration exhibits $\mathcal{N}=2$ supersymmetry. [Also note that $osp(2,4)\oplus u(1)$ with the $u(1)$ factor associated with the unit

[2]The covariant derivative of the complex conjugated field $\bar{C}$ is analogous with the roles of dotted and undotted indices interchanged. Note that the twisted adjoint representation is most conveniently described with the help of Klein operators [54] (see also [8]).

element of the star product algebra is the maximal finite dimensional subalgebra of $hu(1,1|4)$.]

As shown in [53,54] the free equations of motion for massless fields of all spins have the form

$$D_0(C)=0 \qquad (4.4)$$

where D_0 is the covariant derivative (4.2) with respect to the vacuum field ω_0. Since Eqs. (3.9) and (4.4) are invariant under the gauge transformations (3.10) and

$$\delta C=\epsilon * C-C*\tilde{\epsilon}, \qquad (4.5)$$

from the general argument of Sec. III it follows that, for some fixed vacuum field ω_0 satisfying Eq. (3.9), Eq. (4.4) is invariant under global symmetry transformations with the parameters (3.13) that form the AdS_4 higher spin algebra $hu(1,1|4)$. This realization of the higher spin field equations therefore makes manifest the AdS_4 symmetry $sp(4|\mathbf{R})\subset hu(1,1|4)$, while the conformal symmetry $su(2,2)$ of the free massless equations remains hidden.

For the reader's convenience let us analyze the content of Eqs. (4.4) in somewhat more detail. Upon some rescaling of fields the free massless equations of motion for all spins in AdS_4 of [53,54] acquire the form

$$D_0^L C(y,\bar{y}|x)=-h_0{}^{\alpha\dot{\beta}}\left(\frac{1}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}}+\lambda^2 y_\alpha \bar{y}_{\dot{\beta}}\right)C(y,\bar{y}|x), \qquad (4.6)$$

where D_0^L is the background Lorentz covariant derivative

$$D_0^L=d-\left(\omega_0{}^\alpha{}_\beta(x)y^\beta\frac{\partial}{\partial y^\alpha}+\bar{\omega}_0{}^{\dot{\alpha}}{}_{\dot{\beta}}(x)\bar{y}^{\dot{\beta}}\frac{\partial}{\partial \bar{y}^{\dot{\alpha}}}\right). \qquad (4.7)$$

It gives a particular realization of Eq. (2.8) with

$$\mathcal{D}=D_0^L,\quad \sigma_-=h^{\alpha\dot{\beta}}\frac{1}{\partial y^\alpha\partial\bar{y}^{\dot{\beta}}},$$
$$\sigma_+=\lambda^2 h^{\alpha\dot{\beta}}y_\alpha\bar{y}_{\dot{\beta}}. \qquad (4.8)$$

The gradation operator is

$$G=\frac{1}{2}\left(y^\alpha\frac{\partial}{\partial y^\alpha}+\bar{y}^{\dot{\alpha}}\frac{\partial}{\partial\bar{y}^{\dot{\alpha}}}\right). \qquad (4.9)$$

Equation (4.6) decomposes into the infinite set of subsystems associated with the eigenvalues of the operator

$$\sigma=\frac{1}{2}\left(y^\alpha\frac{\partial}{\partial y^\alpha}-\bar{y}^{\dot{\alpha}}\frac{\partial}{\partial\bar{y}^{\dot{\alpha}}}\right) \qquad (4.10)$$

identified with spin

$$\sigma C(y,\bar{y}|x)=\pm s C(y,\bar{y}|x) \qquad (4.11)$$

(the fields associated with the eigenvalues that differ by sign are conjugated).

The flat limit of the free equations of motion for the integer and half-integer spin massless fields of [53,54] has the form

$$dC(y,\bar{y}|x)+dx^{\underline{n}}\sigma_{\underline{n}}{}^{\alpha\dot{\beta}}\frac{1}{\partial y^\alpha\partial\bar{y}^{\dot{\beta}}}C(y,\bar{y}|x)=0, \qquad (4.12)$$

which provides a particular realization of Eq. (2.8) with

$$\mathcal{D}=d,\quad \sigma_-=dx^{\underline{n}}\sigma_{\underline{n}}{}^{\alpha\dot{\beta}}\frac{1}{\partial y^\alpha\partial\bar{y}^{\dot{\beta}}},\quad \sigma^+=0. \qquad (4.13)$$

Let us note that the fact that the free equations of motion of 4D massless fields in the flat space admit reformulation in the form (4.12) was also observed in [60].

The dynamical fields associated with $H^0(\sigma_-)$ identify with the lowest degree eigenspaces of G for various eigenvalues of σ. These are analytic fields $C(y,0|x)$ and their conjugates $C(0,\bar{y}|x)$. Some standard examples are provided with spin 0:

$$C(0,0|x)=c(x), \qquad (4.14)$$

spin 1/2

$$C(y,0|x)=y^\alpha c_\alpha(x),\quad C(0,\bar{y}|x)=\bar{y}^{\dot{\alpha}}\bar{c}_{\dot{\alpha}}(x), \qquad (4.15)$$

spin 1

$$C(y,0|x)=y^\alpha y^\beta c_{\alpha\beta}(x),$$
$$C(0,\bar{y}|x)=\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}\bar{c}_{\dot{\alpha}\dot{\beta}}(x), \qquad (4.16)$$

spin 3/2

$$C(y,0|x)=y^{\alpha_1}y^{\alpha_2}y^{\alpha_3}c_{\alpha_1\alpha_2\alpha_3}(x),$$
$$C(0,\bar{y}|x)=\bar{y}^{\dot{\alpha}_1}\bar{y}^{\dot{\alpha}_2}\bar{y}^{\dot{\alpha}_3}\bar{c}_{\dot{\alpha}_1\dot{\alpha}_2\dot{\alpha}_3}(x), \qquad (4.17)$$

and spin 2

$$C(y,0|x)=y^{\alpha_1}\cdots y^{\alpha_4}c_{\alpha_1\cdots\alpha_4}(x),$$
$$C(0,\bar{y}|x)=\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_4}\bar{c}_{\dot{\alpha}_1\cdots\dot{\alpha}_4}(x). \qquad (4.18)$$

All fields $C(y,\bar{y}|x)$ starting with spin 1 are associated with the appropriate field strengths, namely, with the Maxwell field strength, gravitino curvature, and Weyl tensor for spins 1, 3/2 and 2, respectively.

The analytic fields $C(y,0|x)$ and their conjugates $C(0,\bar{y}|x)$ are subject to the dynamical spin s massless equations [54] associated with $H^1(\sigma_-)$. Using the properties of two-component spinors it is elementary to prove that the representatives of $H^1(\sigma_-)$ are

$$y_\alpha h^{\alpha\dot{\beta}}E_{\dot{\beta}}(y),\quad \bar{y}_{\dot{\beta}}h^{\alpha\dot{\beta}}\bar{E}_\alpha(\bar{y}),\quad y_\alpha\bar{y}_{\dot{\beta}}h^{\alpha\dot{\beta}}\kappa, \qquad (4.19)$$

where the 0-forms $E_{\dot{\beta}}(y)$ and $\bar{E}_\alpha(\bar{y})$ are, respectively, analytic and antianalytic while κ is a constant. The cohomology class parametrized by κ corresponds to the $s=0$ massless equation, while the cohomology classes parametrized by $E_{\dot\beta}(y)$ and $\bar{E}_{\alpha}(\bar{y})$ are responsible for the field equations for spin $s>0$ massless fields. Note that the cohomology group $H^1(\sigma_-)$ is the same for the flat and AdS_4 cases. The explicit form of the flat space dynamical massless equations resulting from Eq. (4.12) is

$$\frac{\partial}{\partial y^{\alpha}}\frac{\partial}{\partial x_{\alpha}{}^{\dot\beta}}C(y,0|x)=0,\quad \frac{\partial}{\partial \bar{y}^{\dot\beta}}\frac{\partial}{\partial x^{\alpha}{}_{\dot\beta}}C(0,\bar{y}|x)=0,\quad (s\neq 0),$$

$$0=\frac{\partial^2}{\partial y^{\alpha}\partial \bar{y}_{\dot\beta}}\frac{\partial}{\partial x_{\alpha}{}^{\dot\beta}}C(y,\bar{y}|x)|_{y=\bar{y}=0}\rightarrow \partial_{\underline{n}}\partial^{\underline{n}}C(0,0|x)=0,$$
$$(s=0). \tag{4.20}$$

All other equations in (4.12) express the nonanalytic components of the fields $C(y,\bar{y}|x)$ via higher space-time derivatives of the dynamical massless fields $C(0,\bar{y}|x)$ and $C(y,0|x)$ or reduce to identities expressing some compatibility conditions. Therefore, the nonanalytic components in $C(y,\bar{y}|x)$ are auxiliary fields (in both the flat and AdS_4 cases).

A. Fock space realization

The formulation of [53,54] with the 0-form $C(y,\bar{y}|x)$ taking values in the twisted adjoint representation of the AdS_4 higher spin algebra made the symmetry $hu(1,1|4)$ manifest. Let us now show that the same equation (4.12) admits a realization in the Fock space that makes the higher spin conformal symmetries of the system manifest.

Let us introduce the Fock vacuum $|0\rangle\langle 0|$ defined by the relations

$$a_{\alpha}*|0\rangle\langle 0|=0,\quad \tilde{b}^{\dot\beta}*|0\rangle\langle 0|=0,\quad \phi_i*|0\rangle\langle 0|=0. \tag{4.21}$$

It can be realized as the element of the star product algebra

$$|0\rangle\langle 0|=2^{4-\mathcal{N}}\exp 2(\tilde{a}_{\dot\alpha}\tilde{b}^{\dot\alpha}-a_{\alpha}b^{\alpha}+\phi_i\bar{\phi}^i), \tag{4.22}$$

which also satisfies

$$|0\rangle\langle 0|*\tilde{a}^{\dot\alpha}=0,\quad |0\rangle\langle 0|*b_{\alpha}=0,\quad |0\rangle\langle 0|*\bar{\phi}^i=0. \tag{4.23}$$

As a result, the vacuum is bi-Lorentz invariant

$$L_{\alpha}{}^{\beta}*|0\rangle\langle 0|=0,\quad \bar{L}_{\dot\alpha}{}^{\dot\beta}*|0\rangle\langle 0|=0, \tag{4.24}$$

$$|0\rangle\langle 0|*L_{\alpha}{}^{\beta}=0,\quad |0\rangle\langle 0|*\bar{L}_{\dot\alpha}{}^{\dot\beta}=0, \tag{4.25}$$

bi-$su(\mathcal{N})$ invariant

$$T_i{}^j*|0\rangle\langle 0|=|0\rangle\langle 0|*T_i{}^j=\tfrac{1}{2}\,\delta_i^j|0\rangle\langle 0|, \tag{4.26}$$

and has conformal weight 1

$$D*|0\rangle\langle 0|=|0\rangle\langle 0|*D=|0\rangle\langle 0|. \tag{4.27}$$

Also, it is left Poincaré invariant

$$P_{\alpha}{}^{\dot\beta}*|0\rangle\langle 0|=0 \tag{4.28}$$

and supersymmetric

$$Q^i_{\alpha}*|0\rangle\langle 0|=0,\quad \bar{Q}_i^{\dot\beta}*|0\rangle\langle 0|=0. \tag{4.29}$$

Note that $|0\rangle\langle 0|$ is a projector

$$|0\rangle\langle 0|*|0\rangle\langle 0|=|0\rangle\langle 0| \tag{4.30}$$

and space-time constant

$$d|0\rangle\langle 0|=0. \tag{4.31}$$

Let us now consider the left module over the algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ spanned by the states

$$|\Phi(\tilde{a},b,\bar{\phi}|x)\rangle=C(\tilde{a},b,\bar{\phi}|x)*|0\rangle\langle 0|, \tag{4.32}$$

where

$$C(\tilde{a},b,\bar{\phi}|x)=\sum_{m,n,k=0}^{\infty}\frac{1}{m!n!k!}c_{\beta_1\cdots\beta_m}{}^{\dot\alpha_1\cdots\dot\alpha_n}{}_{j_1\cdots j_k}(x)$$
$$\times\tilde{a}_{\dot\alpha_1}\cdots\tilde{a}_{\dot\alpha_n}b^{\beta_1}\cdots b^{\beta_m}\bar{\phi}^{j_1}\cdots\bar{\phi}^{j_k}. \tag{4.33}$$

Note that

$$C(\tilde{a},b,\bar{\phi}|x)*|0\rangle\langle 0|$$
$$=C(2\tilde{a},2b,2\bar{\phi}|x)2^{4-\mathcal{N}}\exp 2(\tilde{a}_{\dot\alpha}\tilde{b}^{\dot\alpha}-a_{\alpha}b^{\alpha}+\phi_i\bar{\phi}^i). \tag{4.34}$$

The system of equations

$$d|\Phi\rangle-\omega_0*|\Phi\rangle=0 \tag{4.35}$$

concisely encodes all 4D massless field equations provided that Eq. (3.9), which guarantees the formal consistency of Eq. (4.35), is true. Indeed, the choice of ω_0 in the form (3.17) makes Eq. (4.35) equivalent to Eq. (4.12) upon the identification of b^{α} with y^{α} and $\tilde{a}_{\dot\beta}$ with $\bar{y}_{\dot\beta}$ [for every $su(\mathcal{N})$ tensor structure]. Analogously, choosing ω_0 in the form Eq. (3.15), one finds that Eq. (4.35) describes massless fields in AdS_4. Let us note that the equations on the component fields are Lorentz and scale invariant due to the Lorentz invariance (4.24) and definite scaling (4.27) of the vacuum $|0\rangle\langle 0|$. The dynamical components identify with the holomorphic and antiholomorphic parts

$$c_{\beta_1\cdots\beta_n\, i_1\cdots i_k}(x)=\frac{\partial^n}{\partial b^{\beta_1}\cdots\partial b^{\beta_n}}\frac{\partial^k}{\partial\bar{\phi}^{i_1}\cdots\partial\bar{\phi}^{i_k}}$$
$$\times C(0,b,\bar{\phi}|x)|_{b^{\alpha}=\bar{\phi}^j=0}, \tag{4.36}$$

$$c^{\dot\alpha_1\cdots\dot\alpha_m}{}_{i_1\cdots i_k}(x)=\frac{\partial^m}{\partial\tilde a_{\dot\alpha_1}\cdots\partial\tilde a_{\dot\alpha_m}}\frac{\partial^k}{\partial\bar\phi^{i_1}\cdots\partial\bar\phi^{i_k}}\times C(\tilde a,0,\bar\phi|x)|_{\tilde a_{\dot\alpha}=\bar\phi^j=0}. \tag{4.37}$$

Recall that Eq. (4.35) imposes the dynamical massless equations of motion on the components (4.36) and (4.37) and expresses all other components in $C(\tilde a_{\dot\alpha},b^\beta,\bar\phi|x)$ via their derivatives according to Eq. (4.12) rewritten in the form

$$\frac{\partial^2}{\partial b^\alpha\partial\tilde a_{\dot\beta}}C(\tilde a,b,\bar\phi|x)=\sigma^{\underline n}{}_\alpha{}^{\dot\beta}\partial_{\underline n}C(\tilde a,b,\bar\phi|x), \tag{4.38}$$

or, equivalently,

$$\frac{\partial^2}{\partial b^\alpha\partial\tilde a_{\dot\beta}}C(\tilde a,b,\bar\phi|x)=-\frac{\partial}{\partial x^\alpha{}_{\dot\beta}}C(\tilde a,b,\bar\phi|x). \tag{4.39}$$

As discussed in more detail in Sec. V the system of massless equations in the form (4.35) is manifestly invariant under the higher spin global symmetry $hu(1,1|8)$. Note that the formulation we use is in a certain sense dual to the usual construction of induced representations [61]. The difference is that the module we use is realized in the auxiliary Fock space, while the space-time dependence is reconstructed by virtue of the dynamical equation (4.35) that links the dependence on the space-time coordinates to the dependence on the auxiliary coordinates. This module is induced from the vacuum annihilated by the translation generator $P_\alpha{}^{\dot\beta}$ that acts on the auxiliary spinor coordinates, while in the construction of [61] the vacuum state is assumed to be annihilated by the generators $K^{\underline n}$ of the special conformal transformations acting directly on the dynamical relativistic fields. [Let us stress that this is not just a matter of notation since $P_{\underline n}$ is eventually identified with the $\partial_{\underline n}$ by virtue of Eq. (4.35).]

Because $N_{\mathcal N}$ commutes to the generators of $su(2,2|\mathcal N)$, the Fock module F of $su(2,2|\mathcal N)$ decomposes into submodules F_α of $su(2,2|\mathcal N)$ classified by eigenvalues of $N_{\mathcal N}$, i.e., spanned by the vectors satisfying

$$N_{\mathcal N}*|\Phi\rangle=\alpha|\Phi\rangle. \tag{4.40}$$

According to the definition (1.17), the vacuum has a definite eigenvalue of $N_{\mathcal N}$

$$N_{\mathcal N}*|0\rangle\langle 0|=-\frac{\mathcal N}{2}. \tag{4.41}$$

Because

$$[N_{\mathcal N},f]_*=(N_b+N_{\bar\phi}-N_a-N_\phi)f, \tag{4.42}$$

where

$$N_a=a_{\hat\alpha}\frac{\partial}{\partial a_{\hat\alpha}},\quad N_b=b^{\hat\alpha}\frac{\partial}{\partial b^{\hat\alpha}}, \tag{4.43}$$

$$N_\phi=\phi_i\frac{\partial}{\partial\phi_i},\quad N_{\bar\phi}=\bar\phi^j\frac{\partial}{\partial\bar\phi^j}, \tag{4.44}$$

the eigenvalue in Eq. (4.40) takes values

$$\alpha=m-\frac{\mathcal N}{2},\quad m\in\mathbf Z, \tag{4.45}$$

i.e., α is an arbitrary half integer for odd $\mathcal N$ and an arbitrary integer for even $\mathcal N$.

From Eq. (4.42) it follows that the fields contained in F_α are $C(\tilde a_{\dot\alpha},b^\beta,\bar\phi|x)$ with

$$(N_a-N_b-N_{\bar\phi}+m)C(\tilde a_{\dot\alpha},b^\beta,\bar\phi|x)=0. \tag{4.46}$$

From Eq. (4.11) it follows that the relationship between the number of inner indices and the spin s of a field in the supermultiplet is

$$s=\tfrac12|N_{\bar\phi}-m|. \tag{4.47}$$

For definiteness, let m be some nonnegative integer. Then the following dynamical massless fields appear in the multiplet:

$$\begin{gathered}c_{\alpha_1\cdots\alpha_m(x)},\quad c_{\alpha_1\cdots\alpha_{m-1},\,i}(x),\ \dots,\\ c_{\alpha_1\cdots\alpha_{m-k},\,i_1\cdots i_k}(x),\ \dots,\\ c_{i_1\cdots i_m}(x),\ \dots,\quad c^{\dot\beta_1\cdots\dot\beta_k}{}_{,\,i_1\cdots i_{m+k}}(x),\ \dots,\\ c^{\dot\beta_1\cdots\dot\beta_{\mathcal N-m}}{}_{,\,i_1\cdots i_{\mathcal N}}(x).\end{gathered} \tag{4.48}$$

The modules F_α describe various supermultiplets of $su(2,2|\mathcal N)$ with the type of the conformal supermultiplet characterized by α. The most interesting case is $\alpha=0$. According to Eq. (4.45) $\alpha=0$ requires $\mathcal N$ to be even. Let us show that the $\alpha=0$ supermultiplets are self-conjugated conformal supermultiplets. These include the $\mathcal N=2$ hypermultiplet and $\mathcal N=4$ Yang-Mills supermultiplet.

From Eq. (4.45) it follows that $\alpha=0$ implies $m=\mathcal N/2$ and, therefore, the set of dynamical massless fields in the supermultiplet contains

$$\begin{gathered}c_{\alpha_1\cdots\alpha_{\mathcal N/2}}(x),\quad c_{\alpha_1\cdots\alpha_{\mathcal N/2-1},\,i}(x),\ \dots,\\ c_{\alpha_1\cdots\alpha_{\mathcal N/2-k},\,i_1\cdots i_k}(x),\ \dots,c_{i_1\cdots i_{\mathcal N/2}}(x),\end{gathered} \tag{4.49}$$

along with

$$\begin{gathered}c^{\dot\beta_1}{}_{,\,i_1\cdots i_{\mathcal N/2+1}}(x),\ \dots,\quad c^{\dot\beta_1\cdots\dot\beta_k}{}_{,\,i_1\cdots i_{\mathcal N/2+k}}(x),\ \dots,\\ c^{\dot\beta_1\cdots\dot\beta_{\mathcal N/2}}{}_{,\,i_1\cdots i_{\mathcal N}}(x).\end{gathered} \tag{4.50}$$

In particular, for the case $\mathcal N=0$ we obtain a single scalar field. For $\mathcal N=2$ the hypermultiplet appears:

$$c_\alpha(x),\quad c_i(x),\quad c^{\dot\beta}{}_{,\,ij}. \tag{4.51}$$

For $\mathcal N=4$ we find the $\mathcal N=4$ Yang-Mills multiplet:

$$c_{\alpha\beta}(x),\quad c_{\alpha\,,\,i}(x),\quad c_{ij}(x),$$
$$c^{\dot\alpha}{}_{,\,ijk}(x),\quad c^{\dot\alpha\dot\beta}{}_{,\,ijkl}(x). \tag{4.52}$$

The algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ contains the infinite-dimensional subalgebra $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ which is the centralizer of $N_{\mathcal{N}}$ in $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$, i.e., $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ is spanned by the elements $f\in hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ that commute with $N_{\mathcal{N}}$

$$[N_{\mathcal{N}},f]_*=0. \tag{4.53}$$

This is equivalent to

$$(N_a+N_\phi)f=(N_b+N_{\bar\phi})f. \tag{4.54}$$

Because of Eq. (4.53), the algebra $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ is not simple, containing ideals I_α spanned by elements of the form $h=(N_{\mathcal{N}}-\alpha)*f$, $[f,N_{\mathcal{N}}]_*=0$. Now we observe that the operator $N_{\mathcal{N}}-\alpha$ is trivial on the module F_α. Therefore, F_α forms a module over the quotient algebra $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)=cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)/I_\alpha$. Thus, different α correspond to different subsectors (quotients) of $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ associated with different supermultiplets.

Let us note that in [20] the algebra $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ was called $shsc^\infty(4|\mathcal{N})$, while the algebra $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ was called $shsc^0_\alpha(4|\mathcal{N})$. It was argued in [21] that it is the algebra $cu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ that plays the role of the 4D higher spin conformal algebra, while the algebra $shsc^0_\alpha(4|\mathcal{N})$ is unlikely to allow consistent conformal higher spin interactions. The conclusions of the present paper are somewhat opposite. We will argue that consistent conformal theories exhibiting the higher spin conformal symmetries may correspond to the simple (modulo the trivial center associated with the unit element) algebras $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ or $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ and their further simple reductions of orthogonal or symplectic type (see subsection IV D). Note that in [23] it is shown that the $\mathcal{N}=0$ algebra $hu_0(1,0|8)$ admits consistent cubic higher spin interactions in AdS_5. An interesting problem for the future is to extend the proposed conformal form of massless field equations to the case with dynamical conformal higher spin gauge fields (one-forms) included. Taking into account that the conformal higher spin gauge theory framework allows for off-mass-shell formulation of higher spin constraints for higher spin gauge fields [21,62], such an extension is expected to be of crucial importance for the construction of nonlinear off-mass-shell higher spin dynamics.

Finally, let us note that it is straightforward to introduce color indices by allowing the Fock vacuum to be the column

$$|\Phi\rangle=\begin{pmatrix}E^p(\tilde a,b|x)*|0\rangle\langle 0|\\ O^r(\tilde a,b|x)*|0\rangle\langle 0|\end{pmatrix}, \tag{4.55}$$

where $E^p(\tilde a,b)$ and $O^r(\tilde a,b)$ are, respectively, even and odd functions of the spinor variables $\tilde a_{\dot\alpha}$ and b^α

$$E^p(-\tilde a,-b|x)=E^p(\tilde a,b|x),$$

$$O^r(-\tilde a,-b|x)=-O^r(\tilde a,b|x) \tag{4.56}$$

and $p=1-m$, $r=1-n$. The algebra $hu(m,n|8)$ realized by the matrices (1.1) acts naturally on such a column. It is clear that the fermionic Fock states due to the Clifford variables ϕ_i and $\bar\phi^i$ give rise to a particular realization of this construction. Most of the content of this paper applies equally well to both constructions. We will mainly use the Clifford realization because, although it is less general, it has larger supersymmetries explicit. Note that the algebras $hu(m,n|8)$ are not supersymmetric for generic m and n (i.e., they do not contain the usual supersymmetry algebras as finite-dimensional subalgebras). They are $\mathcal{N}=1$ conformal supersymmetric, however, for the case $m=n$ and acquire more supersymmetries when $m=n$ are multiples of 2^q. The superalgebras $hu(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$ and their orthogonal and symplectic reductions $ho(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$ and $husp(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$ act on the set of n copies of $\mathcal{N}$-extended conformal supersymmetry multiplets. In this notation it is the $n\to\infty$ limit that plays a crucial role in the string theory AdS/CFT correspondence [1,3–5]. (For more detail on the properties of $hu(m,n|8)$ we refer the reader to [25]. See also Sec. IV D.)

B. Generic solution

Once the massless equations are reformulated in the form (4.35) and the vacuum background field ω_0 is represented in the pure gauge form (3.12), the generic solution of the massless equations acquires the form

$$|\Phi(\tilde a,b,\bar\phi|x)\rangle=g^{-1}(\tilde a,b;\phi,\bar\phi|x)*|\Phi_0(\tilde a,b,\bar\phi)\rangle, \tag{4.57}$$

where $|\Phi_0(\tilde a,b,\bar\phi)\rangle=|\Phi_0(\tilde a,b,\bar\phi|x_0)\rangle$ at such a point x_0 that $g(x_0)=1$. For the gauge function g [Eq. (3.19)] one obtains with the help of Eq. (4.34) the general solution in the form

$$\begin{aligned}C(\tilde a,b,\bar\phi|x)&=\frac{1}{(2\pi)^2}\int d^2\tilde s\,d^2\tilde t\,C_0(\tilde a+\tilde s,b^\alpha+x^\alpha{}_{\dot\beta}\tilde t^{\dot\beta},\bar\phi)\\&\quad\times\exp\tilde s_{\dot\alpha}\tilde t^{\dot\alpha}\\&=\exp\left(-x^\alpha{}_{\dot\beta}\frac{\partial^2}{\partial b^\alpha\partial\tilde a_{\dot\beta}}\right)C_0(\tilde a,b,\bar\phi).\end{aligned} \tag{4.58}$$

Here $C_0(\tilde a,b,\bar\phi)$ is an arbitrary function of the variables $\tilde a_{\dot\alpha}$, b^α, and $\bar\phi^i$. It provides "initial data" for the problem. Choosing $C_0(\tilde a,b,\bar\phi)$ in the form

$$C_0(\tilde a,b,\bar\phi)=c_0(\bar\phi)\exp(b^\alpha\eta_\alpha+\bar\eta^{\dot\beta}\tilde a_{\dot\beta}), \tag{4.59}$$

where η_α and $\bar\eta^{\dot\beta}$ are (commuting) spinor parameters, one obtains the plane wave solution

$$C(\tilde a,b,\bar\phi|x)=c_0(\bar\phi)\exp(b^\alpha\eta_\alpha+\bar\eta^{\dot\beta}\tilde a_{\dot\beta}-\bar\eta^{\dot\beta}x^\alpha{}_{\dot\beta}\eta_\alpha) \tag{4.60}$$

with the lightlike wave vector

$$k_{\alpha\dot{\beta}}=\eta_\alpha\overline{\eta}_{\dot{\beta}}. \tag{4.61}$$

Let us note that our approach exhibits deep similarity with the twistor theory [63,64,59]. The conformal spinors $a_{\hat{\alpha}}$ and $b^{\hat{\beta}}$, which play a key role in the construction as the generating elements of the star product algebra, are analogous to the quantum twistors of [64]. An important difference, however, is that we do not assume that $x_{\hat{\alpha}}{}^{\hat{\beta}}$ maps one pair of twistors to another. In our construction x space is treated as the base manifold while the spinor variable generate the Fock space fiber. At the first stage the field variables (sections of the vector fiber bundle) are arbitrary functions of the variables $x_{\hat{\alpha}}{}^{\hat{\beta}}$, $a_{\hat{\alpha}}$, and $b^{\hat{\beta}}$ so that there is no direct relationship between the two sectors. They are linked to each other by the equations of motion (4.35) which imply that solutions of the massless equations are flat sections of the Fock fiber bundle over space-time. This allows one to solve the field equations using star product techniques as explained in this section, thus providing a counterpart of the twistor contour integral formulas. Typical twistor combinations of the coordinates and spinors [such as e.g., the combination $x^\alpha{}_{\dot{\beta}}\widetilde{t}^{\dot{\beta}}$ in Eq. (4.58)] then appear as a result of insertion of the gauge function g [Eq. (3.19)] that reproduces Cartesian coordinates in the flat space. Another difference mentioned at the end of Sec. II is due to systematic use of the language of x space differential forms in our approach. In fact, this allows us to handle higher spin gauge symmetries in a systematic way that is of key importance for the analysis of interactions.

Note that our approach can be used in any other coordinate system by choosing other forms of g. Provided that the higher spin symmetry algebra contains conformal subalgebra (as is the case in this paper), analogously to the twistor theory, it works for any conformally flat geometry because conformally flat gravitational fields satisfy the zero curvature equations of the conformal algebra. For example, it can be applied to the AdS_4 space. The generic solution of the massless field equations in AdS_4 was found by a similar method in [65,8].

C. Reality conditions

So far we have considered complex fields. The conjugated multiplet is described by the right module formed by the states

$$\langle\Psi(\widetilde{a},b^\beta,\phi_j|x)|=|\overline{0}\rangle\langle\overline{0}|*G(b,\widetilde{a},\phi_j|x), \tag{4.62}$$

where the vacuum $|\overline{0}\rangle\langle\overline{0}|$ is defined by the conditions[3]

$$|\overline{0}\rangle\langle\overline{0}|*a_\alpha=0,\quad |\overline{0}\rangle\langle\overline{0}|*\widetilde{b}^{\dot{\beta}}=0,\quad |\overline{0}\rangle\langle\overline{0}|*\overline{\phi}^j=0, \tag{4.63}$$

i.e.,

$$|\overline{0}\rangle\langle\overline{0}|=2^{4-\mathcal{N}}\exp 2(a_\alpha b^\alpha-\widetilde{a}_{\dot{\alpha}}\widetilde{b}^{\dot{\alpha}}+\phi_i\overline{\phi}^i). \tag{4.64}$$

In components,

$$G(b,\widetilde{a},\phi|x)=\sum_{m,n=0}^{\infty}\sum_{k=0}^{\mathcal{N}}\frac{1}{m!n!k!}g_{\alpha_1\cdots\alpha_n}{}^{\dot{\beta}_1\cdots\dot{\beta}_m j_1\cdots j_k}(x)\times b^{\alpha_1}\cdots b^{\alpha_n}\widetilde{a}_{\dot{\beta}_1}\cdots\widetilde{a}_{\dot{\beta}_m}\phi_{j_1}\cdots\phi_{j_k}. \tag{4.65}$$

Analogously, one can consider the row representation of $hu(m,n|8)$.

The dynamical equation for $\langle\Psi|$ is

$$d\langle\Psi|+\langle\Psi|*\omega_0=0. \tag{4.66}$$

To impose the reality conditions let us define the involution † by the relations

$$(a_\alpha)^\dagger=i\widetilde{b}_{\dot{\alpha}},\quad (b^\alpha)^\dagger=i\widetilde{a}^{\dot{\alpha}},$$

$$(\widetilde{a}_{\dot{\alpha}})^\dagger=ib_\alpha,\quad (\widetilde{b}_{\dot{\alpha}})^\dagger=ia_\alpha, \tag{4.67}$$

$$(\phi_i)^\dagger=\overline{\phi}^i,\quad (\overline{\phi}^i)^\dagger=\phi_i. \tag{4.68}$$

Since an involution is defined to reverse the order of product factors

$$(f*g)^\dagger=g^\dagger*f^\dagger \tag{4.69}$$

and conjugate complex numbers

$$(\mu f)^\dagger=\overline{\mu}f^\dagger,\quad \mu\in\mathbf{C}, \tag{4.70}$$

one can see that † leaves invariant the defining relations (1.5) and (1.13) of the star product algebra and has the involutive property $(\dagger)^2=Id$. By Eq. (4.69) the action of † extends to an arbitrary element f of the star product algebra. Since the star product we use corresponds to the totally (anti)symmetric (i.e., Weyl) ordering of the product factors, the result is simply

$$(f(a,\widetilde{a},b,\widetilde{b};\phi,\overline{\phi}))^\dagger=\overline{f}^r(i\widetilde{b},ib,i\widetilde{a},ia;\overline{\phi},\phi), \tag{4.71}$$

where f^r implies reversal of the order of the Grassmann factors ϕ and $\overline{\phi}$, i.e., $f^r=(-1)^{n(n-1)}/2f$ if f is an order-n polynomial in ϕ and $\overline{\phi}$. One can check directly with the formulas (1.4) and (1.14) that Eq. (4.71) defines an involution of the star product algebra.

Let us note that in the general case of $hu(m,n|8)$ the involution † is defined by Eq. (4.67) along with the usual Hermitian conjugation in the matrix sector. The column (4.55) is mapped to the appropriate conjugated row vector

$$\langle\Psi|=(|\overline{0}\rangle\langle\overline{0}|*\overline{E}_p(\widetilde{a},b|x),\quad |\overline{0}\rangle\langle\overline{0}|*\overline{O}_r(\widetilde{a},b|x)). \tag{4.72}$$

The reality conditions on the elements of the higher spin algebra have to be imposed in a way consistent with the form of the zero-curvature equations (3.9). This is equivalent to

[3]Let us note that the vacua $|0\rangle\langle 0|$ and $|\overline{0}\rangle\langle\overline{0}|$ belong to algebraically distinct sectors of the star product algebra: the computation of $|0\rangle\langle 0|*|\overline{0}\rangle\langle\overline{0}|$ leads to a divergency.

singling out a real form of the higher spin Lie superalgebra. With the help of any involution † this is achieved by imposing the reality conditions

$$f^{\dagger}=-i^{\pi(f)}f \tag{4.73}$$

[$\pi(f)=0$ or 1]. This condition defines the real higher spin algebra $hu(m,n|2M)$ for M pairs of oscillators. For the Clifford realization of the matrix part one arrives at the real algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$.

Let us stress that the condition (4.73) extracts a real form of the Lie superalgebra built from the star product algebra but not of the associative star product algebra itself. The situation is very much the same as for the Lie algebra $u(n)$ singled out from the complex Lie algebra of $n\times n$ matrices by the condition (4.73) ($\pi=0$ for the purely bosonic case) with † identified with the Hermitian conjugation. Anti-Hermitian matrices form the Lie algebra but not an associative algebra. In fact, the relevance of the reality conditions of the form (4.73) is closely related to this matrix example because it guarantees that the spin 1 (i.e., purely Yang-Mills) part of the higher spin algebras is compact. More generally, these reality conditions guarantee that the higher spin symmetry admits appropriate unitary highest weight representations (see Sec. VI). Note that in the sector of the conformal algebra $su(2,2)$ the reality condition (4.73) is equivalent to (1.11).

Now one observes that

$$(|0\rangle\langle 0|)^{\dagger}=|\overline{0}\rangle\langle\overline{0}|. \tag{4.74}$$

Imposing a reality condition analogous to Eq. (4.73) on the conformal matter modules

$$(|\Phi\rangle)^{\dagger}=-i^{\pi(\Phi)}\langle\Psi| \tag{4.75}$$

equivalent to

$$C^{\dagger}=-i^{\pi(C)}G, \tag{4.76}$$

one finds by Eq. (4.70) that the matter fields $g_{\alpha_1\cdots\alpha_n}{}^{\dot\beta_1\cdots\dot\beta_m\, j_1\cdots j_k}(x)$ are complex conjugated to $c^{\beta_1\cdots\beta_m}{}_{\dot\alpha_1\cdots\dot\alpha_n\, j_1\cdots j_k}(x)$ up to some sign factors originating from the factors of i and the reversal of the order of Grassmann factors in the definition of † [Eq. (4.71)]. For example, for the scalars we have $g(x)=-\overline{c}(x)$, for the spin 1 field strengths $\overline{(g_{\alpha\beta})}=c_{\dot\alpha\dot\beta}$, etc.

Let us note that the operator $N_{\mathcal{N}}$ is self-conjugate

$$N_{\mathcal{N}}^{\dagger}=N_{\mathcal{N}}. \tag{4.77}$$

As a result, if $|\Phi\rangle$ satisfies Eq. (4.40) the conjugated module satisfies

$$\langle\Psi|*(N_{\mathcal{N}}-\alpha)=0 \tag{4.78}$$

with the same real α.

D. Antiautomorphism reduction and self-conjugated supermultiplets

The algebras $hu(m,n|2p)$ were shown [25] to admit truncations of the orthogonal and symplectic types $ho(m,n|2p)$ and $husp(m,n|2p)$ singled out by the appropriate antiautomorphisms of the underlying associative algebra. Let us recall some definitions.

Let B be some algebra with the (not necessarily associative) product law $\diamondsuit$. A linear invertible map τ of B onto itself is called automorphism if $\tau(a\diamondsuit b)=\tau(a)\diamondsuit\tau(b)$ (i.e., τ is an isomorphism of the algebra to itself). A useful fact is that the subset of elements $a\in B$ satisfying

$$\tau(a)=a \tag{4.79}$$

spans a subalgebra $B_\tau\subset B$. It is customary in physical applications to use this property to obtain reductions. In particular, applying the boson-fermion automorphism which changes the sign of the fermion fields, one obtains reduction to the bosonic sector. Another example is provided by the operation $\tau(a)=-a^t$ of the Lie algebra $gl(n)$ (t implies transposition). The condition (4.79) then singles out the orthogonal subalgebra $o(n)\subset gl(n)$.

A linear invertible map ρ of an algebra onto itself is called antiautomorphism if it reverses the order of product factors

$$\rho(a\diamondsuit b)=\rho(b)\diamondsuit\rho(a). \tag{4.80}$$

One example is provided by the transposition of matrices. More generally, let $A=Mat_M(\mathbf{C})$ be the algebra of $M\times M$ matrices over the field of complex numbers, with elements $a^i{}_j$ ($i,j=1$–M) and the product law

$$(a\circ b)^i{}_j=a^i{}_k b^k{}_j. \tag{4.81}$$

Let η^{ij} be a nondegenerate bilinear form with the inverse η_{ij}, i.e.,

$$\eta^{ik}\eta_{kj}=\delta^i_j. \tag{4.82}$$

It is elementary to see that the mapping

$$\rho_\eta(a)^i{}_j=\eta^{ik}a^l{}_k\eta_{lj} \tag{4.83}$$

is an antiautomorphism of $Mat_M(\mathbf{C})$. If the bilinear form η^{ij} is either symmetric

$$\eta_S^{ij}=\eta_S^{ji} \tag{4.84}$$

or antisymmetric

$$\eta_A^{ij}=-\eta_A^{ji}, \tag{4.85}$$

the antiautomorphism ρ_η is involutive, i.e., $\rho_\eta^2=Id$. One can extend the action of ρ to rows and columns in the standard way by raising and lowering indices with the aid of the bilinear form η^{ij} and its inverse.

The star product algebra admits the antiautomorphism defined by the relations

$$\rho(a_{\hat\alpha})=ia_{\hat\alpha},\quad \rho(b^{\hat\alpha})=ib^{\hat\alpha}, \tag{4.86}$$

$$\rho(\phi_i)=\phi_i\,,\quad \rho(\overline{\phi}^j)=\overline{\phi}^j. \tag{4.87}$$

This definition is consistent with the property (4.80) and the basis commutation relations (1.13) and (1.5). For the generic element of the star product algebra we have

$$\rho(f(a,\tilde{a},b,\tilde{b};\phi,\overline{\phi}))=f^r(ia,i\tilde{a},ib,i\tilde{b};\phi,\overline{\phi}). \tag{4.88}$$

Because the product law in a Lie superalgebra has definite symmetry properties, any antiautomorphism ρ of an associative algebra A that respects the $\mathbf{Z}_2$ grading used to define the Lie superalgebra l_A by Eq. (1.7) induces an automorphism of τ_ρ of l_A according to

$$\tau_\rho(f)=-(i)^{\pi(f)}\rho(f). \tag{4.89}$$

As a result, any antiautomorphism ρ of the associative algebra A allows one to single out a subalgebra of l_A by imposing the condition (4.79):

$$f=-(i)^{\pi(f)}\rho(f). \tag{4.90}$$

For example, for $A=Mat_M(\mathbf{C})$, $l_A=gl_M(\mathbf{C})$. The subalgebras of gl_M singled out by the condition (4.90) with $\tau_S=-\rho_S$ and $\tau_A=-\rho_A$ are $o(M|\mathbf{C})$ and $sp(M|\mathbf{C})$, respectively, because the condition (4.90) just implies that the form η^{ij} is invariant. Note that analogously, one can define involutions via nondegenerate Hermitian forms. If † is such an involution of $Mat_M(\mathbf{C})$ defined via a positive-definite Hermitian form, the resulting Lie algebra is $u(M)$.

The algebras $ho(m,n|2p)$ and $husp(m,n|2p)$ [25] are real Lie superalgebras satisfying the reality conditions (4.73) and the reduction condition (4.90) with the antiautomorphism ρ defined by the relations (4.86) along with the definition (4.83) for the action on the matrix indices with some $(m+n)\times(m+n)$ bilinear form η^{ij} that is block diagonal in the basis (1.1) and is either symmetric $\eta^{ij}=\eta^{ij}_S$ or antisymmetric $\eta^{ij}=\eta^{ij}_A$. For η^{ij}_S and η^{ij}_A we arrive, respectively, at the algebras $ho(m,n|2p)$ and $husp(m,n|2p)$ with the spin 1 Yang-Mills subalgebras $o(m)\oplus o(n)$ and $usp(m)\oplus usp(n)$ in the sector of elements independent of the spinor oscillators.

For the particular case of the algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ with the Clifford star product realization of the matrix part, the antiautomorphism ρ is defined in Eq. (4.87). As argued in [25] this antiautomorphism is diagonal in the basis (1.1) for even $\mathcal{N}$ and off diagonal for odd $\mathcal{N}$. To see this one can check that the element

$$K=\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix} \tag{4.91}$$

identifies in terms of the Clifford algebra with the element Γ that is the product of all Clifford generating elements (in the basis with the diagonal symmetric form in the defining Clifford relations) so that $\Gamma^2=I$, $\{\Gamma,\phi_i\}=0$, $\{\Gamma,\overline{\phi}^i\}=0$. Then one observes that

$$\rho(\Gamma)=(-1)^{\mathcal{N}}\Gamma. \tag{4.92}$$

Therefore we confine ourselves to the case of even $\mathcal{N}$. In fact, this case is most interesting because it admits the self-conjugated supermultiplets.[4]

Following the analysis of [25] one can check that the algebras extracted by the condition (4.90) for $\mathcal{N}=4p$ and $\mathcal{N}=4p+2$ are isomorphic to

$$ho(2^{4p-1},2^{4p-1}|8)\quad \text{for } \mathcal{N}=4p \tag{4.93}$$

and

$$husp(2^{4p+1},2^{4p+1}|8)\quad \text{for } \mathcal{N}=4p+2. \tag{4.94}$$

In particular, for $\mathcal{N}=2$ and $\mathcal{N}=4$ we get $husp(2,2|8)$ and $ho(8,8|8)$, respectively. Let us stress that the elements of the $su(2,2)$ algebra (1.10), (1.15), (1.16) all satisfy Eq. (4.90) and, thus belong to the truncated superalgebras $ho(2^{4p-1},2^{4p-1}|8)$ and $husp(2^{4p+1},2^{4p+1}|8)$. The same is true for the algebra $osp(2\mathcal{N},8)$ spanned by various bilinears of the superoscillators.

One observes that

$$\rho(N_{\mathcal{N}})=-N_{\mathcal{N}}. \tag{4.95}$$

This means that the reduction (4.90) is possible for the algebras $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ if and only if $\alpha=0$. We call the algebras resulting from the reduction of $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ by the antiautomorphism ρ as $ho_0(2^{4p-1},2^{4p-1}|8)$ for $\mathcal{N}=4p$ and $husp_0(2^{4p+1},2^{4p+1}|8)$ for $\mathcal{N}=4p+2$. The algebra $ho_0(8,8|8)$ is the minimal higher spin conformal symmetry algebra associated with the linearized $\mathcal{N}=4$ Yang-Mills supermultiplet, while the algebra $hu(2,2|8)$ is the minimal higher spin conformal algebra associated with the 4D $\mathcal{N}=2$ massless hypermultiplet. The minimal purely bosonic 4D conformal higher spin algebra associated with the spin 0 4D massless scalar field is $ho_0(1,0|8)$. This algebra was recently discussed by Sezgin and Sundell [22] in the context of the AdS_5 higher spin gauge theory [these authors denoted this algebra $hs(2,2)$]. Note that the higher spin gauge algebra of AdS_5 higher spin gauge theory dual to the $\mathcal{N}=4$ Super Yang-Mills (SYM) theory is $ho_0(8,8|8)$.

In the matter sector we define

$$\begin{aligned}\rho(|\Phi\rangle)&=\rho(C*|0\rangle\langle 0|)\\ &=\frac{1}{\mathcal{N}!}\varepsilon^{j_1\cdots j_{\mathcal{N}}}|\overline{0}\rangle\langle\overline{0}|*\phi_{j_1}*\cdots*\phi_{j_{\mathcal{N}}}*\rho(C),\end{aligned} \tag{4.96}$$

[4] Note that to make ρ diagonal for the case of odd $\mathcal{N}$ one can modify its definition in a way that breaks the $su(\mathcal{N})$ algebra to at least $su(\mathcal{N}-1)$. To this end it is enough to modify Eq. (4.87) to $\rho(\phi_1)=\overline{\phi}^1$, $\rho(\overline{\phi}^1)=\phi_1$, leaving the definition of ρ for ϕ_j and $\overline{\phi}^j$ with $j>1$ intact. This will bring an additional sign factor into Eq. (4.92).

$$\rho(\langle\Psi|)=\rho(|\bar{0}\rangle\langle\bar{0}|*G)$$
$$=\frac{1}{\mathcal{N}!}\varepsilon_{i_1\cdots i_{\mathcal{N}}}\rho(G)*\overline{\phi}^{i_1}*\cdots*\overline{\phi}^{i_{\mathcal{N}}}*|0\rangle\langle 0| \tag{4.97}$$

to make Eq. (4.87) consistent with Eqs. (4.21) and (4.63). Now we can impose the reduction condition on the matter fields

$$\rho(|\Phi\rangle)=-i^{\pi(\Phi)}\langle\Psi|, \tag{4.98}$$

which is consistent with Eq. (4.90). Along with the fact that $\langle\Psi|$ describes the conjugated fields subject to the Hermiticity condition (4.73) this imposes the reality conditions on the left module $|\Phi\rangle$

$$\rho(|\Phi\rangle)=(|\Phi\rangle)^{\dagger}. \tag{4.99}$$

For the self-conjugated supermultiplets with $\alpha=0$ this imposes the reality conditions on the fields of the same multiplet. In terms of components this implies that

$$\overline{c}_{\beta_1\cdots\beta_m\dot{\alpha}_1\cdots\dot{\alpha}_n}{}^{j_1\cdots j_k}(x)$$
$$=\frac{1}{(\mathcal{N}-k)!}\varepsilon^{j_1\cdots j_k i_{\mathcal{N}-k}\cdots i_1}c_{\alpha_1\cdots\alpha_n\dot{\beta}_1\cdots\dot{\beta}_m\, i_{\mathcal{N}-k}\cdots i_1}(x). \tag{4.100}$$

In particular, for the $\mathcal{N}=4$ multiplet we have

$$\overline{c}_{\alpha\beta}=\frac{1}{4!}\varepsilon^{ijkl}c_{\dot{\alpha}\dot{\beta}\,ijkl}, \tag{4.101}$$

$$\overline{c}_{\alpha}{}^{i}=\frac{1}{6}\varepsilon^{ijkl}c_{\dot{\alpha}\,jkl}, \tag{4.102}$$

$$\overline{c}^{ij}=\frac{1}{2}\varepsilon^{ijkl}c_{kl}. \tag{4.103}$$

The resulting set indeed corresponds to the real 4D $\mathcal{N}=4$ (SYM) supermultiplet with six real scalars, four Majorana spinors, and one spin 1 field strength.

The special property of the self-conjugated supermultiplets therefore is that the antiautomorphism ρ transforms them to themselves. In other words, they are self-conjugated with respect to the combined action of the conjugation † and the antiautomorphism ρ. The infinite-dimensional superalgebras $ho_0(2^{4p-1},2^{4p-1}|8)$ for $\mathcal{N}=4p$ and $husp_0(2^{4p+1},2^{4p+1}|8)$ for $\mathcal{N}=4p+2$ are therefore shown to be algebras of conformal higher spin symmetries acting on the self-conjugated supermultiplets. Finally, let us note that the whole construction extends trivially to the case with n supermultiplets described by the algebras $hu(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$ and their further reductions $ho(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$, $husp(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$ and $hu_0(n2^{\mathcal{N}-1},n2^{\mathcal{N}-1}|8)$, $ho_0(n2^{4p-1},n2^{4p-1}|8)$, $husp_0(n2^{4p+1},n2^{4p+1}|8)$ (the latter algebras are assumed to be defined as before as the quotients of the centralizer of $N_{\mathcal{N}}$).

V. 4D CONFORMAL HIGHER SPIN SYMMETRIES

The system of equations (3.9),(4.35) is invariant under the infinite-dimensional local conformal higher spin symmetries (3.10) and

$$\delta|\Phi\rangle=\epsilon*|\Phi\rangle. \tag{5.1}$$

The reduction condition (4.98) reduces the higher spin algebra to the subalgebra (4.93) or (4.94) with the symmetry parameters $\epsilon(a,b;\phi,\overline{\phi}|x)$ satisfying the condition (4.90).

Once a particular vacuum solution ω_0 is fixed, the local higher spin symmetry (5.1) breaks down to the global higher spin symmetry (3.13). Therefore the system (4.35) is invariant under the infinite-dimensional algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ of the global 4D conformal higher spin symmetries

$$\delta|\Phi\rangle=\epsilon_0*|\Phi\rangle, \tag{5.2}$$

where ϵ_0 satisfies Eq. (3.11) with the flat connection (3.17). After the higher components in $C(\tilde{a},b,\overline{\phi}|x)$ are expressed via the higher space-time derivatives of the dynamical massless fields according to Eq. (4.38), this implies invariance of the 4D massless equations for all spins (4.20) under global conformal higher spin symmetries. Thus, the fact that massless equations are reformulated in the form of the flatness conditions (4.35) supplemented with the zero-curvature equation (3.9) makes higher spin conformal symmetries of these equations manifest. Note that because of Eq. (4.38) and the quantum-mechanical nonlocality of the star product (1.4), the higher degree of $\epsilon_0(a,b|x)$ as a polynomial of a and b is, the higher space-time derivatives appear in the transformation law. This is a particular manifestation of the well known fact that the higher spin symmetries mix higher derivatives of the dynamical fields.

The explicit form of the transformations can be obtained by the substitution of Eq. (4.38) into Eq. (5.2). In practice, it is most convenient to evaluate the higher spin conformal transformations for the generating parameter

$$\xi(a,\tilde{a},b,\tilde{b},\phi,\overline{\phi};h,\tilde{h},j,\tilde{j},\eta,\overline{\eta})$$
$$=\xi\exp(h^{\alpha}a_{\alpha}+\tilde{h}^{\dot{\alpha}}\tilde{a}_{\dot{\alpha}}+j_{\beta}b^{\beta}+\tilde{j}_{\dot{\alpha}}\tilde{b}^{\dot{\alpha}}+\phi_i\overline{\eta}^i+\eta_i\overline{\phi}^i), \tag{5.3}$$

where ξ is an infinitesimal parameter. The polynomial symmetry parameters can be obtained via differentiation of $\xi(a,\tilde{a},b,\tilde{b},\phi,\overline{\phi};h,\tilde{h},j,\tilde{j},\eta,\overline{\eta})$ with respect to the commuting "sources" h^{α}, $\tilde{h}^{\dot{\alpha}}$ j_{α}, $\tilde{j}_{\dot{\alpha}}$ and anticommuting "sources" $\overline{\eta}^i$, η_i. For the case of flat space, using Eqs. (3.13),(3.19) and the star product (1.4), we obtain upon evaluation of elementary Gaussian integrals

$$\epsilon_0(a,\tilde{a},b,\tilde{b},\phi,\bar{\phi};h,\tilde{h},j,\tilde{j},\eta,\bar{\eta}|x)$$
$$=\xi\exp(h^{\alpha}a_{\alpha}+\tilde{h}^{\dot{\alpha}}\tilde{a}_{\dot{\alpha}}+j_{\beta}b^{\beta}+\tilde{j}_{\dot{\alpha}}\tilde{b}^{\dot{\alpha}}+\phi_i\bar{\eta}^i$$
$$+\eta_i\bar{\phi}^i+j_{\alpha}x^{\alpha}{}_{\dot{\beta}}\tilde{b}^{\dot{\beta}}-a_{\alpha}x^{\alpha}{}_{\dot{\beta}}\bar{h}^{\dot{\beta}}). \tag{5.4}$$

Substitution of ϵ_0 into Eq. (5.2) gives the global higher spin conformal symmetry transformations induced by the parameter (5.3):

$$\delta|\Phi(\tilde{a},b,\bar{\phi}|x)\rangle=\delta C(\tilde{a},b,\bar{\phi}|x)*|0\rangle\langle 0|, \tag{5.5}$$

where

$$\delta C(\tilde{a},b,\bar{\phi}|x)$$
$$=\xi\exp\Big(\tilde{h}^{\dot{\alpha}}\tilde{a}_{\dot{\alpha}}+j_{\beta}b^{\beta}+\eta_i\bar{\phi}^i$$
$$-j_{\alpha}x^{\alpha}{}_{\dot{\beta}}\tilde{h}^{\dot{\beta}}-\frac{1}{2}\tilde{j}_{\dot{\alpha}}\tilde{h}^{\dot{\alpha}}+\frac{1}{2}j_{\alpha}h^{\alpha}-\frac{1}{2}\eta_i\bar{\eta}^i\Big)$$
$$\times C(\tilde{a}_{\dot{\alpha}}-\tilde{j}_{\dot{\alpha}}-j_{\beta}x^{\beta}{}_{\dot{\alpha}},b^{\alpha}+h^{\alpha}-x^{\alpha}{}_{\dot{\beta}}\tilde{h}^{\dot{\beta}},\bar{\phi}^i-\bar{\eta}^i|x). \tag{5.6}$$

Such a compact form of the higher spin conformal transformations is a result of the reformulation of the dynamical equations in the unfolded form of the covariant constancy conditions, i.e., in terms of a flat section of the Fock fiber bundle. Differentiating with respect to the sources one derives explicit expressions for the particular global higher spin conformal transformations.

For at most quadratic conformal supergenerators acting on $C(\tilde{a},b,\bar{\phi}|x)$ one obtains with the help of Eq. (4.38)

$$P_{\alpha}{}^{\dot{\beta}}=\frac{\partial}{\partial x^{\alpha}{}_{\dot{\beta}}},\qquad P_{\underline{n}}=\sigma_{\underline{n}}^{\alpha\dot{\beta}}P_{\alpha\dot{\beta}}=\frac{\partial}{\partial x^{\underline{n}}}, \tag{5.7}$$

$$D=1+x^{\underline{n}}\frac{\partial}{\partial x^{\underline{n}}}+\frac{1}{2}\left(\tilde{a}_{\dot{\alpha}}\frac{\partial}{\partial\tilde{a}_{\dot{\alpha}}}+b^{\alpha}\frac{\partial}{\partial b^{\alpha}}\right), \tag{5.8}$$

$$K_{\dot{\alpha}}{}^{\beta}=\tilde{a}_{\dot{\alpha}}b^{\beta}-x^{\beta}{}_{\dot{\delta}}\tilde{a}_{\dot{\alpha}}\frac{\partial}{\partial\tilde{a}_{\dot{\delta}}}-x^{\gamma}{}_{\dot{\alpha}}\frac{\partial}{\partial b^{\gamma}}b^{\beta}-x^{\gamma}{}_{\dot{\alpha}}x^{\beta}{}_{\dot{\delta}}\frac{\partial}{\partial x^{\gamma}{}_{\dot{\delta}}}, \tag{5.9}$$

$$L_{\alpha}{}^{\beta}=b^{\beta}\frac{\partial}{\partial b^{\alpha}}+x^{\beta}{}_{\dot{\alpha}}\frac{\partial}{\partial x^{\alpha}{}_{\dot{\alpha}}}-\frac{1}{2}\delta^{\beta}_{\alpha}\left(b^{\gamma}\frac{\partial}{\partial b^{\gamma}}+x^{\gamma}{}_{\dot{\alpha}}\frac{\partial}{\partial x^{\gamma}{}_{\dot{\alpha}}}\right), \tag{5.10}$$

$$\bar{L}_{\dot{\alpha}}{}^{\dot{\beta}}=-\tilde{a}_{\dot{\alpha}}\frac{\partial}{\partial\tilde{a}_{\dot{\beta}}}-x^{\gamma}{}_{\dot{\alpha}}\frac{\partial}{\partial x^{\gamma}{}_{\dot{\beta}}}+\frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\left(\tilde{a}_{\dot{\delta}}\frac{\partial}{\partial\tilde{a}_{\dot{\delta}}}+x^{\gamma}{}_{\dot{\delta}}\frac{\partial}{\partial x^{\gamma}_{\dot{\delta}}}\right), \tag{5.11}$$

$$T^{j}{}_{i}=\frac{1}{2}\delta^{j}_{i}-\bar{\phi}^{j}\frac{\partial}{\partial\bar{\phi}^{i}}, \tag{5.12}$$

$$Q_{\alpha}{}^{i}=\bar{\phi}^{i}\frac{\partial}{\partial b^{\alpha}}, \tag{5.13}$$

$$Q_{\dot{\alpha}}{}^{i}=\bar{\phi}^{i}\left(\tilde{a}_{\dot{\alpha}}-x^{\beta}{}_{\dot{\alpha}}\frac{\partial}{\partial b^{\beta}}\right), \tag{5.14}$$

$$\bar{Q}_{i}{}^{\alpha}=\frac{\partial}{\partial\bar{\phi}^{i}}\left(b^{\alpha}-x^{\alpha}{}_{\dot{\beta}}\frac{\partial}{\partial\tilde{a}_{\dot{\beta}}}\right), \tag{5.15}$$

$$\bar{Q}_{i}{}^{\dot{\alpha}}=-\frac{\partial}{\partial\bar{\phi}^{i}}\frac{\partial}{\partial\tilde{a}_{\dot{\alpha}}}. \tag{5.16}$$

Here the x-independent supercharges (5.13) and (5.16) correspond to Q supersymmetry while the x-dependent supercharges (5.14) and (5.15) correspond to S-supersymmetry.

F is a module over the algebra $osp(2\mathcal{N},8)$ which, together with the $u(1)$ algebra generated by the unit element of the star product algebra, forms a maximal finite-dimensional subalgebra of the higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$. Equation (4.35) contains the equations for all supermultiplets. The $osp(2\mathcal{N},8)$ invariance links together all free 4D conformal supermultiplets. The explicit transformation laws derived from Eq. (5.6) are

$$U_{\alpha\beta}=\frac{\partial^2}{\partial b^{\alpha}\partial b^{\beta}}, \tag{5.17}$$

$$U_{\alpha\dot{\beta}}=\left(\tilde{a}_{\dot{\beta}}-x^{\gamma}{}_{\dot{\beta}}\frac{\partial}{\partial b^{\gamma}}\right)\frac{\partial}{\partial b^{\alpha}}, \tag{5.18}$$

$$U_{\dot{\alpha}\dot{\beta}}=\left(\tilde{a}_{\dot{\alpha}}-x^{\gamma}{}_{\dot{\alpha}}\frac{\partial}{\partial b^{\gamma}}\right)\left(\tilde{a}_{\dot{\beta}}-x^{\alpha}{}_{\dot{\beta}}\frac{\partial}{\partial b^{\alpha}}\right), \tag{5.19}$$

$$V^{\dot{\alpha}\dot{\beta}}=\frac{\partial^2}{\partial\tilde{a}_{\dot{\alpha}}\partial\tilde{a}_{\dot{\beta}}}, \tag{5.20}$$

$$V^{\alpha\dot{\alpha}}=-\left(b^{\alpha}-x^{\alpha}{}_{\dot{\beta}}\frac{\partial}{\partial\tilde{a}_{\dot{\beta}}}\right)\frac{\partial}{\partial\tilde{a}_{\dot{\alpha}}}, \tag{5.21}$$

$$V^{\alpha\beta}=\left(b^{\alpha}-x^{\alpha}{}_{\dot{\alpha}}\frac{\partial}{\partial\tilde{a}_{\dot{\alpha}}}\right)\left(b^{\beta}-x^{\beta}{}_{\dot{\beta}}\frac{\partial}{\partial\tilde{a}_{\dot{\beta}}}\right), \tag{5.22}$$

$$U_{ij}=\frac{\partial^2}{\partial\overline{\phi}^i\partial\overline{\phi}^j},\qquad U^{ij}=\overline{\phi}^i\overline{\phi}^j, \tag{5.23}$$

$$R_{\alpha i}=\frac{\partial^2}{\partial b^\alpha\partial\overline{\phi}^i}, \tag{5.24}$$

$$R_{\dot\alpha i}=\left(\tilde a_{\dot\alpha}-x^{\gamma}{}_{\dot\alpha}\frac{\partial}{\partial b^\gamma}\right)\frac{\partial}{\partial\overline{\phi}^i}, \tag{5.25}$$

$$R^{\alpha i}=\left(b^\alpha-x^{\alpha}{}_{\dot\beta}\frac{\partial}{\partial\tilde a_{\dot\beta}}\right)\overline{\phi}^i, \tag{5.26}$$

$$R^{\dot\alpha i}=-\frac{\partial}{\partial\tilde a_{\dot\alpha}}\overline{\phi}^i. \tag{5.27}$$

To obtain the variation $\delta C(\tilde a,b,\overline{\phi}|x)$, one has to apply these generators to $C(\tilde a,b,\overline{\phi}|x)$. Application of the formulas (4.36) and (4.37) to $\delta C(\tilde a,b,\overline{\phi}|x)$ then gives the variation of the particular dynamical higher spin fields. The rule is that, whenever the second derivative $(\partial^2/\partial b^\alpha\partial\tilde a_{\dot\alpha})(C)$ appears, it has to be replaced by the space-time derivative ∂_n according to Eq. (4.38). As a result, a parameter of the higher spin conformal transformation $\epsilon(a,b;\phi,\overline{\phi}|x)$ polynomial in $\tilde a$ and b generates a local transformation of a dynamical field with a finite number of derivatives. In particular, the usual $su(2,2;\mathcal{N})$ conformal transformations and their extension to the $osp(2\mathcal{N},8)$ transformations contain at most the first space-time derivatives of the dynamical fields. Thus, $osp(2\mathcal{N},8)$ is shown to act by local transformations on the massless fields of all spins in four dimensions. That $osp(2\mathcal{N},8)$ must act on the 4D massless fields was emphasized by Fronsdal [30]. The reformulation of the higher spin dynamics in terms of the flat sections of the Fock fiber bundle allows us to derive simple and manifestly local explicit formulas (5.17)–(5.27).

Analogously, one can derive from Eq. (5.6) the transformation laws for the higher spin gauge symmetries associated with the whole infinite-dimensional superalgebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$. Note that the specific form of the dependence on the space-time coordinates $x^{\alpha}{}_{\dot\beta}$ originates from the choice of the gauge function (3.19). The approach we use is applicable to any other coordinate system and conformally flat background (for example, AdS_4). Also, let us note that it is straightforward to realize $osp(L,8)$ supersymmetry with odd L by starting with the Clifford algebra with an odd number of generating elements. The reason we mostly focused on the case $L=2\mathcal{N}$ was that we started with $su(2,2;\mathcal{N})$. For general L the maximal conformal embedding is $su(2,2;1/2[L])\subset osp(L,8)$.

VI. UNFOLDED FIELD THEORY AND QUANTIZATION

The formulation of the higher spin dynamics proposed in this paper operates in terms of the Fock module F over $su(2,2)$ induced from the vacuum (4.22). This Fock module is analogous to the Fock module S over $su(2,2)$ that contains all irreducible 4D massless unitary representations of the conformal algebra called doubletons in [19]. In fact, S is the so-called singleton module over $sp(8)$ that decomposes into irreducible doubleton modules over $su(2,2)$. The difference is that the $sp(8)$ singleton module S is unitary while the Fock module F is not. That there exists a mapping between the doubleton and field-theoretical representations of the conformal (or AdS) algebra was originally shown in [66]. The goal of this section is to demonstrate that, analogously to the 3D case considered in [14], in our approach the duality between the two pictures has the simple interpretation of a certain Bogolyubov transform. Remarkably, this form of duality is coordinate independent. The coordinate dependence results from the gauge choice (3.12) that fixes a particular form of the background gravitational field.

That the module (4.32) is nonunitary is obvious from the fact that, as a result of the Lorentz invariance of the vacuum $|0\rangle\langle 0|$, the set of component fields (4.33) decomposes into the infinite sum of finite-dimensional representations of the noncompact 4D Lorentz algebra $o(3,1)$. (Recall that noncompact semisimple Lie algebras do not admit finite-dimensional unitary representations.) Also, this is in agreement with the fact that the conjugated vacuum $|\overline{0}\rangle\langle\overline{0}|$ [Eq. (4.64)] is different from $|0\rangle\langle 0|$.

The unitary Fock module S over $sp(8)\supset su(2,2)$ is built in terms of the oscillators

$$[e_{\nu A},e_{\mu B}]_*=0,\quad [f^\nu_A,f^\mu_B]_*=0,\ [e_{\nu A},f^\mu_B]_*=\delta^\mu_\nu\kappa_{AB}, \tag{6.1}$$

where $\mu,\nu=1,2;\ A,B=1,2$, and $\kappa_{11}=1,\ \kappa_{22}=-1,\ \kappa_{12}=\kappa_{21}=0$. The oscillators obey the Hermiticity conditions

$$(e_{\nu A})^\dagger=f^\nu_A. \tag{6.2}$$

The unitary Fock vacuum $|0_u\rangle\langle 0_u|$ is defined as

$$e_{\nu 1}*|0_u\rangle\langle 0_u|=0,\quad f^{\mu}{}_2*|0_u\rangle\langle 0_u|=0,\ |0_u\rangle\langle 0_u|*f^\nu_1=0,\ |0_u\rangle\langle 0_u|*e_{\mu 2}=0. \tag{6.3}$$

The compact subalgebra $u(2)\oplus u(2)$ of $u(2,2)$ is spanned by

$$\tau_{A\nu}{}^\mu=e_{A\nu}f^\mu_A\ \ (A=1,2 \text{ no summation over } A). \tag{6.4}$$

Noncompact generators of $su(2,2)$ are

$$t^{-\nu}_\mu=e_{1\mu}f^\nu_2,\qquad t^{+\nu}_\mu=e_{2\mu}f^\nu_1. \tag{6.5}$$

(Recall that we use the Weyl star product notation, i.e., all bilinears listed above are elements of the star product algebra.) The superextension is trivially achieved by requiring

$$\phi_i*|0_u\rangle\langle 0_u|=0,\quad |0_u\rangle\langle 0_u|*\overline{\phi}^j=0. \tag{6.6}$$

The relationship between the two sets of oscillators is

$$e_{1,1}=\frac{1}{\sqrt{2}}(a_1+i\tilde{a}_{\dot 2}),\quad e_{2,1}=\frac{1}{\sqrt{2}}(\tilde{a}_{\dot 1}+ia_2),$$

$$e_{1,2}=\frac{1}{\sqrt{2}}(a_1-i\tilde{a}_{\dot 2}),\quad e_{2,2}=\frac{1}{\sqrt{2}}(\tilde{a}_{\dot 1}-ia_2),\tag{6.7}$$

$$f^1{}_1=\frac{1}{\sqrt{2}}(b_2+i\tilde{b}_{\dot 1}),\quad f^2{}_1=\frac{1}{\sqrt{2}}(\tilde{b}_{\dot 2}+ib_1),$$

$$f^1{}_2=\frac{1}{\sqrt{2}}(-b_2+i\tilde{b}_{\dot 1}),\quad f^2{}_2=\frac{1}{\sqrt{2}}(-\tilde{b}_{\dot 2}+ib_1).\tag{6.8}$$

The unitary Fock vacuum is realized in terms of the star product algebra (1.4) as

$$|0_u\rangle\langle 0_u|=2^{4-\mathcal{N}}\exp 2(-e_{1\nu}f_1^\nu-e_{2\nu}f_2^\nu+\phi_i\overline{\phi^i}).\tag{6.9}$$

The unitary left and right Fock modules S and $\overline{S}$ built from the vacuum $|0_u\rangle\langle 0_u|$ are identified with the direct sum of all superdoubleton representations of $su(2,2)$ and their conjugates. As in the nonunitary case, the irreducible components are singled out by the condition (4.40). In the unitary basis, N_0 has the form

$$N_0=e_{\nu A}f^\nu{}_B\kappa^{AB}.\tag{6.10}$$

The Fock space S forms a unitary module over $sp(8)$ called a singleton. It contains two irreducible components spanned by even and odd functions, respectively.

The dependence on the space-time coordinates of the elements of the field $|\Phi(x)\rangle$ is determined completely by Eq. (4.35) in terms of its value at any fixed point x_0. This means that the module $|\Phi(x_0)\rangle$ contains the complete information on the on-mass-shell dynamics of the 4D conformal fields. Analogously, the doubleton module contains complete information on the (on-mass-shell) quantum states of the corresponding free field theory. Let us note that the two types of module have different gradations associated with the respective definitions of the creation and annihilation oscillators. In the unitary case the gradation is induced by the AdS energy operator which, together with the maximal compact subalgebra, spans the grade zero subalgebra. In the field-theoretical case the gradation is induced by the $o(1,1)$ dilatation generator, which together with the Lorentz algebra spans the (noncompact) grade zero subalgebra.

We conclude that there is a natural duality between the field-theoretical module F used in the unfolded formulation of the conformal dynamics and the unitary module S. This duality has the simple form of the Bogolyubov transform (6.7),(6.8). As a result, although unitary inequivalent, the module associated with the classical and quantum pictures become equivalent upon complexification. The important consequence of this fact is that the values of the Casimir operators of the symmetry algebras in the two pictures coincide. Indeed, the values of the Casimir operators in the corresponding irreducible representations [e.g., of $sp(8)$ in F or S] are determined by the fact of the realization of the algebras in terms of oscillators rather than the particular conditions (6.3) or (4.21) on the vacuum state. The duality map between the field-theoretical picture and the unitary picture is essentially the quantization procedure. The two modules are unitary inequivalent because the respective classes of functions associated with solutions of the field equations are different. We believe that this phenomenon is quite general, i.e., the unfolded reformulation of dynamical systems in the form of some flatness (i.e., covariant constancy and/or zero-curvature) conditions will make the duality between the classical and quantum descriptions of the dynamical systems manifest for the general case. Hopefully, the Bogolyubov transform duality between the classical and quantum field theory descriptions can eventually shed some more light on the nature of quantization and the origin of quantum mechanics.

The classical-quantum duality of the unfolded formulation of field-theoretical equations allows a simple criterion for the compatibility of a field-theoretical system with consistent quantization; namely, if a nonunitary module that appears in the unfolded description of some classical dynamics admits a dual unitary module with the same number of states (i.e., generated with the same number of oscillators) we interpret this as an indication that the dynamical system under consideration admits a consistent quantization. Since every dynamical system admits some unfolded formulation, this provides us with a rather general criterion. Moreover, this technique can be used in the opposite direction to derive field-theoretical differential equations compatible with unitarity such as those associated with the cohomology group $H^1(\sigma_-)$ of the unfolded systems that admit consistent quantization. We now apply this idea to the derivation of the compatible with unitarity $sp(2M)$ invariant equations in generalized space-times.

VII. CONFORMAL DYNAMICS IN $osp(L,2M)$ SUPERSPACE

The unfolded formulation of the field-theoretical dynamical systems allows one to extend the equations to superspace and spaces with additional coordinates in a rather straightforward way. In this section we apply this formalism to the 4D $\mathcal{N}$-extended superspace and to superspaces with "central charge coordinates" in four and higher dimensions. As a result, we shall be able to formulate appropriate equations of motion in the generalized (super)spaces. The manifest Bogolyubov transform duality between the field-theoretical picture and the singleton pictures will guarantee that the proposed equations in generalized space-times correspond to the unitary quantum picture.

The main idea is simple. In Sec. IV we showed that the dynamics of 4D free massless fields is described in terms of the generating function $|\Phi(\tilde{a},b,\overline{\phi}|x)\rangle$ satisfying Eq. (4.35). Equation (4.35) can be interpreted in two ways. The σ_- picture used in Sec. II implies that Eq. (4.35) imposes Eqs. (4.20) associated with the first cohomology group $H^1(\sigma_-)$ on the dynamical fields associated with the cohomology group $H^0(\sigma_-)$. All other (auxiliary) components in $|\Phi(\tilde{a},b,\bar{\phi}|x)\rangle$ are expressed via space-time derivatives of the dynamical fields by virtue of Eq. (4.39). The d picture used in Sec. IV B implies that Eq. (4.35) allows one to reconstruct the x dependence of $|\Phi(\tilde{a},b,\bar{\phi}|x)\rangle$ in terms of the ''initial data'' $|\Phi(\tilde{a},b,\bar{\phi}|x_0)\rangle$ taken at some particular point of space-time x_0. The d picture is local.

Suppose now that we have a manifold $M^{p,q}$ with a larger set of p even and q odd coordinates $X^{\underline{A}}$ that contains the original 4D coordinates $x^{\underline{n}}$ as a subset, i.e., $X^A=(x^{\underline{n}},y^{\underline{\nu}})$, where $y^{\underline{\nu}}$ are additional coordinates. Let $\hat{d}$ be the de Rahm differential on $M^{p,q}$:

$$\hat{d}=dX^{\underline{A}}\frac{\partial}{\partial X^{\underline{A}}}=dx^{\underline{n}}\frac{\partial}{\partial x^{\underline{n}}}+dy^{\underline{\nu}}\frac{\partial}{\partial y^{\underline{\nu}}},\quad \hat{d}^2=0, \tag{7.1}$$

and $\hat{\omega}_0$ be a zero-curvature connection in the (appropriate fiber bundle over) $M^{p,q}$:

$$\hat{\omega}_0(a,b,\phi,\bar{\phi}|X)=dX^{\underline{A}}\hat{\omega}_{0\,\underline{A}}(a,b,\phi,\bar{\phi}|X),$$

$$d\hat{\omega}_0=\hat{\omega}_0 * \wedge\hat{\omega}_0, \tag{7.2}$$

such that its pullback to the original 4D space-time M^4 equals the 4D connection ω_0, i.e.,

$$\hat{\omega}_{0\,\underline{n}}(a,b,\phi,\bar{\phi}|X)=\omega_{0\underline{n}}(a,b,\phi,\bar{\phi}|x). \tag{7.3}$$

Replacing the 4D equation (4.35) with

$$\hat{d}|\Phi\rangle-\hat{\omega}_0 * |\Phi\rangle=0,\quad |\Phi\rangle=|\Phi(\tilde{a},b,\bar{\phi}|x,y)\rangle, \tag{7.4}$$

one observes that the extended system is formally consistent, while its restriction to M^4 coincides with the original system (4.35). As a result, it turns out that the system (7.4) is equivalent to the original 4D system (4.35) at least locally in the additional coordinates. Indeed, as is obvious in the $\hat{d}$ picture, the equations in (7.4) different from those in (4.35) just reconstruct the dependence of $|\Phi(\tilde{a},b,\bar{\phi}|x,y)\rangle$ on the additional coordinates $y^{\underline{\nu}}$ of the 4D field $|\Phi(\tilde{a},b,\bar{\phi}|x,y_0)\rangle$ for some y_0 (e.g., $y_0=0$). Let us note that to link the global symmetries associated with the Lie superalgebra in which $\hat{\omega}_0$ takes its values to the symmetries of the extended space $M^{p,q}$, one has to find such an extension of the space-time that a frame field in the generalized space-time is invertible. In the σ_- picture this means that the cohomology group $H^r(\sigma_-)$ is small enough. An important example of the application of the proposed scheme is the usual superspace. An additional simplification here is due to the fact that the extension along supercoordinates is always global because superfields are polynomial in the odd coordinates.

The extension of the unfolded dynamical equations discussed in this section has some similarity to the ''group manifold approach'' developed in the context of supersymmetry and supergravity (see [67] and references therein). As we shall see, the maximal natural extension of the space-time corresponds to the situation when coordinates of the extended space are associated with all generators of the gauge Lie superalgebra that underlies the unfolded formulation.

A. Superspace

As a useful illustration let us embed the 4D dynamics of massless fields into superspace. We introduce anticommuting coordinates $\theta^{\underline{\alpha}}_{\underline{i}}$ and $\bar{\theta}^{\underline{j}}_{\underline{\dot{\beta}}}$ associated with the Q supersymmetry supergenerators Q^i_α and $\bar{Q}^{\dot{\beta}}_j$, so that $X=(x,\theta,\bar{\theta})$ (to simplify formulas, in the rest of this section we shall not distinguish between the underlined and fiber indices). The vacuum connection one-form satisfying the zero-curvature equation (3.9) can be chosen in the form

$$\hat{\omega}_0=\left(dx^{\alpha}{}_{\dot{\beta}}+\frac{1}{2}[(1+\gamma)d\theta^{\alpha}_i\bar{\theta}^i_{\dot{\beta}}+(1-\gamma)d\bar{\theta}^i_{\dot{\beta}}\theta^{\alpha}_i]\right)a_{\alpha}\tilde{b}^{\dot{\beta}}$$
$$+d\bar{\theta}^i_{\dot{\beta}}\tilde{b}^{\dot{\beta}}\phi_i+d\theta^{\alpha}_i a_{\alpha}\bar{\phi}^i, \tag{7.5}$$

where γ is an arbitrary parameter. Spinor differentials $d\theta^{\alpha}_i$ and $d\bar{\theta}^i_{\dot{\beta}}$ are required to commute with each other but anticommute with $dx^{\alpha}{}_{\dot{\beta}}$, ϕ_i, $\bar{\phi}^i$, and the supercoordinates $\theta^{\alpha}{}_i,\bar{\theta}_{\dot{\beta}}{}^j$. $\hat{\omega}_0$ admits the pure gauge representation $\hat{\omega}_0=-g^{-1}*dg$, with the gauge function g of the form

$$g=\exp-\left[\left(x^{\alpha}{}_{\dot{\beta}}+\frac{1}{2}\gamma\theta^{\alpha}{}_i\bar{\theta}^i_{\dot{\beta}}\right)a_{\alpha}\tilde{b}^{\dot{\beta}}+\bar{\theta}^i_{\dot{\beta}}\tilde{b}^{\dot{\beta}}\phi_i+\theta^{\alpha}_i a_{\alpha}\bar{\phi}^i\right]. \tag{7.6}$$

The dependence on the supercoordinates is reconstructed by the formula (4.57) in terms of the initial data fixed at any point in superspace.

The superfield equations of motion have the form (7.4). The superspace formulation, however, does not have the decomposition (2.8). Instead it has the $\mathbf{Z}\times\mathbf{Z}$ grading

$$(\hat{d}+\sigma_{--}+\sigma_{-0}+\sigma_{0-})|\Phi\rangle=0 \tag{7.7}$$

associated separately with the elements a_α and $\tilde{b}^{\dot{\beta}}$. This does not affect the interpretation of the dynamical superfields as representatives of the zeroth cohomology group $H^0(\sigma_{--},\sigma_{-0},\sigma_{0-})$ with the cohomologies of σ_{-0} and σ_{0-} computed on the subspace of σ_{--} closed 0-forms on which σ_{-0} and σ_{0-} anticommute to zero. As a result, the dynamical superfields can be identified with $|\Phi(0,b,0|X)\rangle$ and with the field $|\Phi(\tilde{a},0,\bar{\phi}|X)\rangle$ of maximal degree $\mathcal{N}$ in $\bar{\phi}$. Thus, as expected, the free field dynamics is described by general superfields carrying external dotted or undotted spinor indices (contracted with b^α or $\tilde{a}_{\dot{\beta}}$) that characterize the spin of the supermultiplet. Superfields of this type were used in [68] for the description of on-mass-shell massless supermultiplets in terms of field strengths. To extend our formalism to the off-mass-shell description of massless supermultiplets [69,70] one has to introduce higher spin superconnections.

The cohomological identification of the dynamical superspace equations is less straightforward, however, in view of

Eq. (7.7), although the main idea is still the same: the superspace equations are identified with the null vectors of the operator $\sigma_{--}+\sigma_{-0}+\sigma_{0-}$. One complication might be that, as is typical for the superspace approach, it may not always be possible to distinguish between dynamical equations and constraints in the absence of a clear σ_- cohomological interpretation of the dynamical equations. We hope to come back to the analysis of this interesting issue elsewhere.

B. $sp(2M)$ covariant space-time

As shown in Sec. V, the set of 4D conformal equations for all spins is invariant under the $sp(8)$ symmetry that extends the 4D conformal symmetry $su(2,2)$. This raises the problem of an appropriate extension of the space-time that would allow $sp(8)$ symmetry in a natural way. The question of possible extensions of the space-time beyond the traditional Minkowski-Riemann extension to higher dimension has been addressed by many authors (see, e.g., [30,38–52]). In particular, a very interesting option comes from the Jordan algebras [39,40]. However, to the best of our knowledge, no dynamical analysis of possible equations has been done so far. One important and difficult issue to be addressed in such an analysis is whether the proposed equations give rise to consistent quantum mechanics, and, in particular, allow one to get rid of negative norm states.

More specifically, the analysis of $sp(8)$ invariant extended space-time was originally undertaken by Fronsdal in [30] just in the context of a unified description of 4D massless higher spins. It was argued in [30] that the simplest appropriate extension of the usual space-time is a certain $sp(8)$ invariant ten-dimensional space. As shown in this section, our approach leads to the same conclusion. The new result will consist of the formulation of local covariant field equations compatible with unitarity in this generalized space.

The unfolded formulation of the dynamical equations in the form of covariant constancy conditions is ideal for the analysis of this kind of question for several reasons.

It allows one to define an appropriately extended space-time in a natural way via the (locally equivalent) extension of the known conformal 4D equations of motion.

It suggests that the resulting equations are compatible with unitarity once there is Bogolyubov transform duality with some unitary module.

Starting from the infinite unfolded system of $sp(8)$ invariant equations of motion (7.4) we identify the finite system of $sp(8)$ invariant dynamical differential equations as the σ_- cohomology $H^1(\sigma_-)$. Being equivalent to the original 4D conformal unfolded system of equations, the resulting $sp(8)$ invariant differential equations inherit all its properties such as symmetries and compatibility with unitarity.

The approach we use is applicable to any algebra $sp(2M)$. We therefore consider the general case. In this subsection we suppress the dependence on the Clifford elements $\overline{\phi}^i$ and ϕ_j which are inert in our consideration of the purely bosonic space. They will play a role in the superspace consideration of the next subsection.

Let us introduce the oscillators

$$[\alpha_{\hat\alpha},\beta^{\hat\beta}]_*=\delta_{\hat\alpha}^{\hat\beta},\quad [\alpha_{\hat\alpha},\alpha_{\hat\beta}]_*=0,\quad [\beta^{\hat\alpha},\beta^{\hat\beta}]_*=0. \tag{7.8}$$

We still use the Weyl star product (1.4) for the oscillators $\alpha_{\hat\alpha}$ and $\beta^{\hat\beta}$ instead of $a_{\hat\alpha}$ and $b^{\hat\beta}$ but now we allow the indices $\hat\alpha$ and $\hat\beta$ to range from 1 to M where M is an arbitrary positive integer. [The normalization factor in Eq. (1.4) has to be changed appropriately: $\pi^8\to\pi^{2M}$.]

The generators of $sp(2M)$ are spanned by various bilinears built from the oscillators $\alpha_{\hat\alpha}$ and $\beta^{\hat\beta}$:

$$T_{\hat\alpha}{}^{\hat\beta}=\alpha_{\hat\alpha}\beta^{\hat\beta},\quad P_{\hat\alpha\hat\beta}=\alpha_{\hat\alpha}\alpha_{\hat\beta},\quad K^{\hat\alpha\hat\beta}=\beta^{\hat\alpha}\beta^{\hat\beta}. \tag{7.9}$$

We interpret the generators $P_{\hat\alpha\hat\beta}$ and $K^{\hat\alpha\hat\beta}$ as $sp(2M)$ "translations" and "special conformal transformations," respectively. The $gl(M)$ generators $T_{\hat\alpha}{}^{\hat\beta}$ decompose into the $sl(M)$ "Lorentz" and $o(1,1)$ "dilatation" generators

$$L_{\hat\alpha}{}^{\hat\beta}=\alpha_{\hat\alpha}\beta^{\hat\beta}-\frac{1}{M}\delta_{\hat\alpha}{}^{\hat\beta}\alpha_{\hat\gamma}\beta^{\hat\gamma}, \tag{7.10}$$

$$D=\frac{1}{2}\alpha_{\hat\alpha}\beta^{\hat\alpha}. \tag{7.11}$$

Note that D is the gradation operator

$$[D,P_{\hat\alpha\hat\beta}]_*=-P_{\hat\alpha\hat\beta},\quad [D,K^{\hat\alpha\hat\beta}]_*=K^{\hat\alpha\hat\beta},\quad [D,L_{\hat\alpha}{}^{\hat\beta}]_*=0. \tag{7.12}$$

$P_{\hat\alpha\hat\beta}$ and $K^{\hat\alpha\hat\beta}$ generate Abelian subalgebras

$$[K^{\hat\alpha\hat\beta},K^{\hat\gamma\hat\delta}]_*=0,\quad [P_{\hat\alpha\hat\beta},P_{\hat\gamma\hat\delta}]_*=0. \tag{7.13}$$

Together with $sp(2M)$ "Lorentz rotations," $sp(2M)$ "translations" span the $sp(2M)$ "Poincaré subalgebra"

$$[L_{\hat\alpha}{}^{\hat\beta},P_{\hat\gamma\hat\delta}]_*=-\delta_{\hat\gamma}^{\hat\beta}P_{\hat\alpha\hat\delta}-\delta_{\hat\delta}^{\hat\beta}P_{\hat\alpha\hat\gamma}+\frac{2}{M}\delta_{\hat\alpha}^{\hat\beta}P_{\hat\gamma\hat\delta}. \tag{7.14}$$

Analogously,

$$[L_{\hat\alpha}{}^{\hat\beta},K^{\hat\gamma\hat\delta}]_*=\delta_{\hat\alpha}^{\hat\gamma}K^{\hat\beta\hat\delta}+\delta_{\hat\alpha}^{\hat\delta}K^{\hat\beta\hat\gamma}-\frac{2}{M}\delta_{\hat\alpha}^{\hat\beta}K^{\hat\gamma\hat\delta}. \tag{7.15}$$

The superextension to $osp(1,2M)$ is achieved by adding the supergenerators

$$Q_{\hat\alpha}=\alpha_{\hat\alpha},\quad S^{\hat\beta}=\beta^{\hat\beta}. \tag{7.16}$$

According to Eq. (7.9), we have

$$T_{\hat\alpha}{}^{\hat\beta}\equiv L_{\hat\alpha}{}^{\hat\beta}+\frac{1}{M}\delta_{\hat\alpha}^{\hat\beta}D=\frac{1}{2}\{Q_{\hat\alpha},S^{\hat\beta}\}_*, \tag{7.17}$$

$$P_{\hat\alpha\hat\beta}=\frac{1}{2}\{Q_{\hat\alpha},Q_{\hat\beta}\}_*,\quad K^{\hat\alpha\hat\beta}=\frac{1}{2}\{S^{\hat\alpha},S^{\hat\beta}\}. \tag{7.18}$$

To compare with the 4D case, let us note that the operators $\beta^{\hat{\alpha}}$ and $\alpha_{\hat{\alpha}}$ are to be identified with the pairs $\tilde{a}_{\dot{\alpha}}, b^{\beta}$ and a_{α}, $\tilde{b}^{\dot{\beta}}$, respectively. The 4D notation used so far was convenient in the $su(2,2)$ framework because of the simple form of the operator N_0 singling out $su(2,2)$ as its centralizer in $sp(8)$. Since N_0 does not play a role in the manifestly $sp(2M)$ invariant setting, it is now more convenient to have a simple form of the gradation operator D.

The Hermiticity conditions are introduced via the involution † as in Sec. IV C with

$$\alpha^{\dagger}_{\hat{\alpha}} = iC_{\hat{\alpha}}{}^{\hat{\beta}}\alpha_{\hat{\beta}}, \quad (\beta^{\hat{\alpha}})^{\dagger} = iC_{\hat{\beta}}{}^{\hat{\alpha}}\beta^{\hat{\beta}}, \tag{7.19}$$

where $C_{\hat{\alpha}}{}^{\hat{\beta}}$ is some real involutive matrix (i.e., $C^2 = Id$). In particular, one can fix $C_{\hat{\alpha}}{}^{\hat{\beta}} = \delta_{\hat{\alpha}}{}^{\hat{\beta}}$ that makes all the $sp(2M)$ generators manifestly real. For even M we shall sometimes use another form of $C_{\hat{\alpha}}{}^{\hat{\beta}}$ analogous to the 4D decomposition of a real four-component Majorana spinor into two pairs of mutually conjugated complex two-component spinors; namely, we decompose $\alpha_{\hat{\alpha}}$ and $\beta^{\hat{\beta}}$ into two pairs of mutually conjugated oscillators α_{α}, $\overline{\alpha}_{\dot{\alpha}}$ and β^{α}, $\overline{\beta}^{\dot{\alpha}}$ with $\alpha, \dot{\alpha} = 1 - M/2$.

By analogy with the usual Minkowski space-time we introduce $\frac{1}{2}M(M+1)$ coordinates $X^{\underline{\hat{\alpha}}\underline{\hat{\beta}}} = X^{\underline{\hat{\beta}}\underline{\hat{\alpha}}}$, the de Rahm differential

$$\hat{d} = dX^{\underline{\hat{\alpha}}\underline{\hat{\beta}}}\frac{\partial}{\partial X^{\underline{\hat{\alpha}}\underline{\hat{\beta}}}}, \quad \hat{d}^2 = 0, \tag{7.20}$$

and the flat frame

$$\hat{\omega}_0 = dX^{\underline{\hat{\alpha}}\underline{\hat{\beta}}} h_{\underline{\hat{\alpha}}\underline{\hat{\beta}}}{}^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}, \tag{7.21}$$

where $h_{\underline{\hat{\alpha}}\underline{\hat{\beta}}}{}^{\hat{\alpha}\hat{\beta}}$ is some constant nondegenerate matrix so that

$$\hat{d}\hat{\omega}_0 = 0. \tag{7.22}$$

For example, one can set

$$h_{\underline{\hat{\alpha}}\underline{\hat{\beta}}}{}^{\hat{\alpha}\hat{\beta}} = \frac{1}{2}(\delta_{\underline{\hat{\alpha}}}{}^{\hat{\alpha}}\delta_{\underline{\hat{\beta}}}{}^{\hat{\beta}} + \delta_{\underline{\hat{\alpha}}}{}^{\hat{\beta}}\delta_{\underline{\hat{\beta}}}{}^{\hat{\alpha}}). \tag{7.23}$$

$\hat{\omega}_0$ satisfies the zero-curvature equation

$$\hat{d}\hat{\omega}_0 = \hat{\omega}_0 \wedge * \hat{\omega}_0 \tag{7.24}$$

because the $sp(2M)$ translations are commutative and, therefore, $\hat{\omega}_0 \wedge * \hat{\omega}_0 = 0$. The pure gauge representation (3.12) for $\hat{\omega}_0$ is given by

$$g = \exp - X^{\underline{\hat{\alpha}}\underline{\hat{\beta}}} h_{\underline{\hat{\alpha}}\underline{\hat{\beta}}}{}^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}. \tag{7.25}$$

For Eq. (7.23) we get

$$g = \exp - X^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}. \tag{7.26}$$

In terms of dotted and undotted indices (for even M), there are $M^2/4$ Hermitian coordinates $X^{\alpha\dot{\beta}}$ and $M(M+2)/4$ coordinates parametrized by the complex matrix $X^{\alpha\beta}$ and its complex conjugate $X^{\dot{\alpha}\dot{\beta}}$. For $M=2$ our approach is equivalent to the standard treatment of the 3D conformal theory with the conformal symmetry $sp(4) \sim o(3,2)$. Here $X^{\hat{\alpha}\hat{\beta}}$ parametrize the three real coordinates. Therefore the 3D approach of [14] was equivalent to a particular $M=2$ case of the general $sp(2M)$ invariant approach. For the case of $sp(8)$ (i.e., $M=4$), $X^{\alpha\dot{\beta}}$ identify with the usual space-time coordinates $x^{\alpha\dot{\beta}}$ while $X^{\alpha\beta}$ and $X^{\dot{\alpha}\dot{\beta}}$ parametrize six additional real coordinates $y^{\underline{\nu}}_{-}$. Altogether we have ten-dimensional extended space in accordance with the proposal of Fronsdal [30].

Let us now introduce the left Fock module

$$|\Phi(\beta|X)\rangle = C(\beta|X) * |0\rangle\langle 0| \tag{7.27}$$

with the vacuum state

$$|0\rangle\langle 0| = \exp - 2\alpha_{\hat{\alpha}}\beta^{\hat{\alpha}}, \tag{7.28}$$

satisfying

$$\alpha_{\hat{\alpha}} * |0\rangle\langle 0| = 0, \quad |0\rangle\langle 0| * \beta^{\hat{\alpha}} = 0, \ \ \hat{d}(|0\rangle\langle 0|) = 0. \tag{7.29}$$

The $sp(2M)$ unfolded equation is

$$(\hat{d} - \hat{\omega}_0) * |\Phi(\beta|X)\rangle = 0. \tag{7.30}$$

It is $sp(2M)$ [in fact, $osp(1,2M)$] invariant according to the general analysis of Sec. III. Moreover, this equation has the infinite-dimensional higher spin symmetry $hu(1,1|2M)$.

The duality with the unitary singleton module over $sp(2M)$ in the basis with the real matrix $C_{\hat{\alpha}}{}^{\hat{\beta}} = \delta^{\hat{\beta}}_{\hat{\alpha}}$ [Eq. (7.19)] is achieved by the Bogolyubov transform

$$\gamma^{-}_{\hat{\alpha}} = \frac{1}{\sqrt{2}}(\alpha_{\hat{\alpha}} + i\beta^{\hat{\alpha}}), \quad \gamma^{+\hat{\alpha}} = \frac{i}{\sqrt{2}}(\alpha_{\hat{\alpha}} - i\beta^{\hat{\alpha}}), \tag{7.31}$$

$$[\gamma^{-}_{\hat{\alpha}}, \gamma^{+\hat{\beta}}]_* = \delta^{\hat{\beta}}_{\hat{\alpha}}, \quad (\gamma^{+\hat{\alpha}})^{\dagger} = \gamma^{-}_{\hat{\alpha}}. \tag{7.32}$$

The unitary vacuum

$$|0_u\rangle\langle 0_u| = \exp - 2\gamma^{-}_{\hat{\alpha}}\gamma^{+\hat{\alpha}} \tag{7.33}$$

satisfies

$$\gamma^{-}_{\hat{\alpha}} * |0_u\rangle\langle 0_u| = 0, \quad |0_u\rangle\langle 0_u| * \gamma^{+\hat{\alpha}} = 0. \tag{7.34}$$

As a result of this duality, Eq. (7.30) is expected to admit consistent quantization.

Equation (7.30) has the form

$$\left(\frac{\partial}{\partial X^{\hat{\alpha}\hat{\beta}}} - \frac{\partial^2}{\partial\beta^{\hat{\alpha}}\partial\beta^{\hat{\beta}}}\right)C(\beta|X) = 0. \tag{7.35}$$

For the particular case of $sp(8)$, in the sector of ordinary coordinates $X^{\alpha\dot{\beta}}$ it reduces to the 4D conformal higher spin equations (4.35). Equation (7.35) has the form (2.8) with

$$\sigma_- = -dX^{\hat{\alpha}\hat{\beta}} \frac{\partial^2}{\partial \beta^{\hat{\alpha}} \partial \beta^{\hat{\beta}}}, \quad \sigma_+ = 0, \quad \mathcal{D} = \hat{d}. \tag{7.36}$$

Its content can therefore be analyzed in terms of the σ_- cohomology. The cohomology group $H^0(\sigma_-)$ is parametrized by the solutions of the equation $\sigma_-(C(\beta,X))=0$ which consist of a scalar function $c(X)$ and a linear function $c_{\hat{\alpha}}(X)\beta^{\hat{\alpha}}$. These are the dynamical fields of the $sp(2M)$ setup. We shall call $sp(2M)$ vectors $c^{\hat{\alpha}}(X)$ "svectors" to distinguish them from the vectors of the Minkowski space-time. Svectors are fermions (i.e., anticommuting fields that are spinors with respect to the usual space-time symmetry algebras). The scalar and svector form an irreducible supermultiplet of $osp(1,2M)$ dual to its unitary supersingleton representation.

We see that the number of dynamical fields in the $sp(8)$ invariant generalized space is much smaller than in the standard 4D approach. Instead of the infinite set of 4D massless fields of all spins we are left with only two $sp(8)$ fields, namely, the scalar $c(X)$ and svector $c_{\hat{\alpha}}(X)$ that form an irreducible supermultiplet of $osp(1,8)$. From this perspective, the situation in all generalized $sp(2M)$ invariant symplectic spaces is analogous to that of the 3D model of [14] containing the massless scalar and spinor as the only 3D conformal fields. The 4D fields now appear in the expansion of the scalar and svector in powers of the extra six coordinates,

$$c(X) = \sum_{m,n} c(x)_{\alpha_1\beta_1\cdots\alpha_n\beta_n, \dot{\alpha}_1\dot{\beta}_1\cdots\dot{\alpha}_m\dot{\beta}_m} \times X^{\alpha_1\beta_1}\cdots X^{\alpha_n\beta_n} X^{\dot{\alpha}_1\dot{\beta}_1}\cdots X^{\dot{\alpha}_m\dot{\beta}_m}, \tag{7.37}$$

$$c_{\hat{\gamma}}(X) = \sum_{m,n} c(x)_{\hat{\gamma}\,\alpha_1\beta_1\cdots\alpha_n\beta_n, \dot{\alpha}_1\dot{\beta}_1\cdots\dot{\alpha}_m\dot{\beta}_m} \times X^{\alpha_1\beta_1}\cdots X^{\alpha_n\beta_n} X^{\dot{\alpha}_1\dot{\beta}_1}\cdots X^{\dot{\alpha}_m\dot{\beta}_m}, \tag{7.38}$$

where $x^{\alpha\dot{\beta}} = X^{\alpha\dot{\beta}}$ are the 4D coordinates. It has been argued by Fronsdal [30] that such an expansion is appropriate for the description of the set of all 4D massless fields. Another important point discussed in [30] was that the analytic expansions in the extra coordinates in Eqs. (7.37) and (7.38) are complete in generalized symplectic spaces. Once this is true, the local equivalence of Eq. (7.30) to the original 4D system extends to the full (global) equivalence.

For $sp(2M)$ with $M>4$ the interpretation in terms of the Minkowski picture is less straightforward because the set of Hermitian coordinates $X^{\alpha\dot{\beta}}$ becomes larger than the usual set of Minkowski coordinates. To this end one has to identify the usual coordinates with the appropriate projection of $X^{\alpha\dot{\beta}}$ with the gamma matrices $\Gamma^n_{\alpha\dot{\beta}}$ that is possible for $M=2^p$. It is not at all clear, however, how important it is to describe $sp(2M)$ invariant phenomena in terms of Minkowski geometry beyond $d=4$. From this perspective, it looks as if the usual Minkowskian supergravity and superstring models in higher dimensions might be some very specific reductions of the new class of models in generalized $sp(2M)$ invariant space-times underlying the (generalized beyond $d=4$) higher spin dynamics.

Note that, geometrically, the generalized space-time considered in this section is the coset space P_M/SL_M, where P is the $Sp(2M)$ analogue of the Poincaré group with the generators $L_{\hat{\alpha}}{}^{\hat{\beta}}$ and $P_{\hat{\alpha}\hat{\beta}}$ while SL_M is the $Sp(2M)$ analogue of the Lorentz algebra with the generators $L_{\hat{\alpha}}{}^{\hat{\beta}}$ isomorphic to $sl_M(\mathbf{R})$. The $sp(2M)$ conformal transformations of the generalized symplectic space-time are realized by the following vector fields:

$$P_{\hat{\alpha}\hat{\beta}} = \frac{\partial}{\partial X^{\hat{\alpha}\hat{\beta}}}, \tag{7.39}$$

$$T_{\hat{\alpha}}{}^{\hat{\beta}} = 2X^{\hat{\beta}\hat{\gamma}} \frac{\partial}{\partial X^{\hat{\alpha}\hat{\gamma}}}, \tag{7.40}$$

$$K^{\hat{\alpha}\hat{\beta}} = 4X^{\hat{\alpha}\hat{\gamma}} X^{\hat{\beta}\hat{\eta}} \frac{\partial}{\partial X^{\hat{\gamma}\hat{\eta}}}. \tag{7.41}$$

To derive the independent equations on the dynamical conformal fields $c(X)$ and $c_{\hat{\alpha}}(X)$ in the $sp(2M)$ invariant conformal space, the cohomology group $H^1(\sigma_-)$ has to be studied for σ_- of the form (7.36). An elementary exercise with Young diagrams shows that $H^1(\sigma_-)$ is parametrized by one-forms that are either linear or bilinear in the oscillators,

$$dX^{\underline{\hat{\alpha}}\underline{\hat{\beta}}} h_{\underline{\hat{\alpha}}\underline{\hat{\beta}}}{}^{\hat{\alpha}\hat{\beta}} (F_{\hat{\alpha}\hat{\beta},\hat{\gamma}} \beta^{\hat{\gamma}} + B_{\hat{\alpha}\hat{\beta},\hat{\gamma}\hat{\delta}} \beta^{\hat{\gamma}} \beta^{\hat{\delta}}), \tag{7.42}$$

where $F_{\hat{\alpha}\hat{\beta},\hat{\gamma}}$ has the symmetry properties of the three-cell hook diagram, i.e.,

$$F_{\hat{\alpha}\hat{\beta},\hat{\gamma}} + F_{\hat{\alpha}\hat{\gamma},\hat{\beta}} + F_{\hat{\beta}\hat{\gamma},\hat{\alpha}} = 0, \quad F_{\hat{\alpha}\hat{\beta},\hat{\gamma}} = F_{\hat{\beta}\hat{\alpha},\hat{\gamma}}, \tag{7.43}$$

while $B_{\hat{\alpha}\hat{\beta},\hat{\gamma}\hat{\delta}}$ has the symmetry properties of the four-cell square diagram, i.e., it is symmetric within each pair of indices $\hat{\alpha},\hat{\beta}$ and $\hat{\gamma},\hat{\delta}$ and vanishes upon symmetrization over any three indices,

$$B_{\hat{\alpha}\hat{\beta},\hat{\gamma}\hat{\delta}} + B_{\hat{\alpha}\hat{\gamma},\hat{\beta}\hat{\delta}} + B_{\hat{\beta}\hat{\gamma},\hat{\alpha}\hat{\delta}} = 0. \tag{7.44}$$

Note that the trivial cohomology class of $H^1(\sigma_-)$ is parametrized by totally symmetric (i.e., one-row) diagrams of arbitrary length.

This structure of $H^1(\sigma_-)$ implies that the only nontrivial differential equations on the dynamical fields $c(X)$ and $c_{\hat{\alpha}}(X)$ hidden in the infinite system of equations (7.30) are

$$\left(\frac{\partial^2}{\partial X^{\hat{\alpha}\hat{\beta}} \partial X^{\hat{\gamma}\hat{\delta}}} - \frac{\partial^2}{\partial X^{\hat{\alpha}\hat{\gamma}} \partial X^{\hat{\beta}\hat{\delta}}} \right) c(X) = 0 \tag{7.45}$$

for the $sp(2M)$ scalar and

$$\frac{\partial}{\partial X^{\hat\alpha\hat\beta}}c_{\hat\gamma}(X)-\frac{\partial}{\partial X^{\hat\alpha\hat\gamma}}c_{\hat\beta}(X)=0 \tag{7.46}$$

for the $sp(2M)$ svector. Equations (7.45) and (7.46) are dynamically equivalent to the system of equations (7.30) and therefore inherit all symmetries of the latter. Note that in agreement with the analysis of [14], because antisymmetrization of any two-component indices $\hat\alpha$ and $\hat\beta$ is equivalent to their contraction with $\epsilon^{\alpha\beta}$, for the case of 3D conformal dynamics, Eqs. (7.45) and (7.46) coincide with the 3D massless Klein-Gordon and Dirac equations, respectively. From the 4D perspective the meaning of Eqs. (7.45) and (7.46) is twofold. They imply that the expansions (7.37) and (7.38) contain only totally symmetric multispinors and that the latter satisfy the 4D massless equations.

The infinitesimal global symmetry transformation that leaves Eqs. (7.45) and (7.46) invariant is given by the formula (5.2) with the global symmetry parameter ϵ_0 (3.13). Let us choose the symmetry generating parameter in Eq. (3.13) in the form

$$\xi(\alpha,\beta;h,j)=\xi\exp(h^{\hat\alpha}\alpha_{\hat\alpha}+j_{\hat\beta}\beta^{\hat\beta}), \tag{7.47}$$

where ξ is an infinitesimal parameter. The polynomial symmetry parameters can be obtained via differentiation of $\xi(\alpha,\beta;h,j)$ with respect to the commuting "sources" $h^{\hat\alpha}$ and $j_{\hat\alpha}$. Using Eqs. (3.13), (5.2) and the star product (1.4) we obtain upon evaluation of the elementary Gaussian integrals

$$\epsilon_0(\alpha,\beta;h,j|X)=\xi\exp(h^{\hat\alpha}\alpha_{\hat\alpha}+j_{\hat\beta}\beta^{\hat\beta}+2X^{\hat\alpha\hat\beta}j_{\hat\alpha}\alpha_{\hat\beta}). \tag{7.48}$$

Substitution of ϵ_0 into Eq. (5.2) gives the global higher spin conformal symmetry transformations induced by the parameter ξ (5.3)

$$\delta|\Phi(\beta,|X)\rangle=\delta C(\beta|X)*|0\rangle\langle 0|, \tag{7.49}$$

where

$$\delta C(\beta|X)=\xi\exp\left(j_{\hat\beta}\beta^{\hat\beta}+\frac{1}{2}j_{\hat\beta}h^{\hat\beta}+X^{\hat\alpha\hat\beta}j_{\hat\alpha}j_{\hat\beta}\right)\times C(\beta^{\hat\gamma}+h^{\hat\gamma}+2X^{\hat\gamma\hat\delta}j_{\hat\delta}|X). \tag{7.50}$$

Differentiating with respect to the sources one derives explicit expressions for the particular global higher spin conformal transformations.

The physical fields are

$$c(X)=C(0|X),\qquad c_{\hat\alpha}(X)=\frac{\partial}{\partial\beta^{\hat\alpha}}C(\beta|X)\big|_{\beta=0}. \tag{7.51}$$

All higher derivatives with respect to $\beta^{\hat\alpha}$ are expressed via the derivatives in $X^{\hat\alpha\hat\beta}$ by the equation (7.35). For example, for $c(X)$ we obtain

$$\delta c(X)=\xi\exp\left(\frac{1}{2}j_{\hat\beta}h^{\hat\beta}+X^{\hat\alpha\hat\beta}j_{\hat\alpha}j_{\hat\beta}\right)C(h^{\hat\gamma}+2X^{\hat\gamma\hat\delta}j_{\hat\delta}|X). \tag{7.52}$$

For at most quadratic supergenerators of $osp(1,2M)$ acting on $C(\beta|X)$ one finds

$$P_{\hat\alpha\hat\beta}=\frac{\partial}{\partial X^{\hat\alpha\hat\beta}}, \tag{7.53}$$

$$T_{\hat\alpha}{}^{\hat\beta}=\frac{1}{2}\delta_{\hat\alpha}^{\hat\beta}+\beta^{\hat\beta}\frac{\partial}{\partial\beta^{\hat\alpha}}+2X^{\hat\beta\hat\gamma}\frac{\partial}{\partial X^{\hat\alpha\hat\gamma}}, \tag{7.54}$$

$$K^{\hat\alpha\hat\beta}=\beta^{\hat\alpha}\beta^{\hat\beta}+2X^{\hat\alpha\hat\beta}+4X^{\hat\alpha\hat\gamma}X^{\hat\beta\hat\eta}\frac{\partial}{\partial X^{\hat\gamma\hat\eta}}+2X^{\hat\alpha\hat\gamma}\beta^{\hat\beta}\frac{\partial}{\partial\beta^{\hat\gamma}}+2X^{\hat\beta\hat\gamma}\beta^{\hat\alpha}\frac{\partial}{\partial\beta^{\hat\gamma}}, \tag{7.55}$$

$$Q_{\hat\alpha}=\frac{\partial}{\partial\beta^{\hat\alpha}}, \tag{7.56}$$

$$S^{\hat\alpha}=\beta^{\hat\alpha}+2X^{\hat\alpha\hat\beta}\frac{\partial}{\partial\beta^{\hat\beta}}. \tag{7.57}$$

From here one derives in particular that the fields $c(X)$ and $c_{\hat\alpha}(X)$ form a supermultiplet with respect to the Q supersymmetry transformation

$$\delta c(x)=\varepsilon^{\hat\alpha}c_{\hat\alpha}(x),\qquad \delta c_{\hat\alpha}(x)=\varepsilon^{\hat\beta}\frac{\partial}{\partial X^{\hat\alpha\hat\beta}}c(x), \tag{7.58}$$

where $\varepsilon^{\hat\alpha}$ is an X-independent global supersymmetry parameter. The S supersymmetry with a constant superparameter $\varepsilon_{\hat\alpha}$ has the form

$$\delta c(x)=2\varepsilon_{\hat\alpha}X^{\hat\alpha\hat\beta}c_{\hat\beta}(X),$$

$$\delta c_{\hat\alpha}(X)=2\varepsilon_{\hat\gamma}X^{\hat\gamma\hat\beta}\frac{\partial}{\partial X^{\hat\beta\hat\alpha}}c(X). \tag{7.59}$$

Note that the (symplectic) conformal transformations of the scalar field are described by the transformations (7.53)–(7.55) at $\beta^{\hat\beta}=0$. The T and K transformation law of the svector $c_{\hat\alpha}$ gets additional "spin" terms from the β-dependent part of the generators.

The sl_M generalized Lorentz transformations with the traceless infinitesimal parameter $\varepsilon_{\hat\beta}{}^{\hat\alpha}$, $\varepsilon_{\hat\alpha}{}^{\hat\alpha}=0$ act as follows:

$$\delta^{lor}c(X)=2\varepsilon_{\hat\beta}{}^{\hat\alpha}X^{\hat\beta\hat\gamma}\frac{\partial}{\partial X^{\hat\alpha\hat\gamma}}c(X), \tag{7.60}$$

$$\delta^{lor}c_{\hat\alpha}(X)=2\varepsilon_{\hat\beta}{}^{\hat\delta}X^{\hat\beta\hat\gamma}\frac{\partial}{\partial X^{\hat\delta\hat\gamma}}c_{\hat\alpha}(X)+\varepsilon_{\hat\alpha}{}^{\hat\beta}c_{\hat\beta}(X). \tag{7.61}$$

The dilatation transformations associated with the trace part $D=\frac{1}{2}T_{\hat\alpha}{}^{\hat\alpha}$ are

$$\delta^{dil}c(X)=\varepsilon X^{\hat\alpha\hat\gamma}\frac{\partial}{\partial X^{\hat\alpha\hat\gamma}}c(X)+\frac{M}{4}c(X), \tag{7.62}$$

$$\delta^{dil}c_{\hat\alpha}(X)=\varepsilon X^{\hat\beta\hat\gamma}\frac{\partial}{\partial X^{\hat\beta\hat\gamma}}c_{\hat\alpha}(X)+\left(\frac{M}{4}+\frac{1}{2}\right)c_{\hat\alpha}(X). \tag{7.63}$$

Since Eqs. (7.45) and (7.46) are derived from an unfolded system that admits a dual unitary formulation, they are expected to admit consistent quantization. In a separate publication [71], where the equations in generalized space-times are studied within the traditional field theoretical approach, we show that they indeed admit a consistent quantization. A nontrivial question for the future is the nature of a Lagrangian formulation that might lead to Eqs. (7.45) and (7.46). It is clear that in order to solve this problem some auxiliary fields have to be introduced in analogy with the Pauli-Fierz program [72] for the usual higher spin fields.

C. $osp(L,2M)$ superspace

To describe $osp(2\mathcal{N},2M)$ we reintroduce the Clifford elements ϕ_i and $\overline{\phi}^j$ and add the bosonic generators (1.16) and (1.21) along with the supergenerators

$$Q^i_{\hat\alpha}=\alpha_{\hat\alpha}\overline{\phi}^i, \quad Q_{\hat\alpha i}=\alpha_{\hat\alpha}\phi_i\,. \tag{7.64}$$

$$S^{\hat\alpha}{}_i=\beta^{\hat\alpha}\overline{\phi}^i, \quad S^{\hat\alpha i}=\beta^{\hat\alpha}\phi_i\,. \tag{7.65}$$

In particular, the following anticommutation relations are true:

$$\{Q^i_{\hat\alpha},Q_{\hat\beta j}\}=\delta^i_jP_{\hat\alpha\hat\beta}, \quad \{Q_{\hat\alpha i},Q_{\hat\beta j}\}=0, \quad \{Q^i_{\hat\alpha},Q^j_{\hat\beta}\}=0, \tag{7.66}$$

$$\{S^{\hat\alpha\, i},S^{\hat\beta}_j\}=\delta^i_jK^{\hat\alpha\hat\beta}, \quad \{S^{\hat\alpha\, i},S^{\hat\beta\, j}\}=0, \quad \{S^{\hat\alpha}_i,S^{\hat\beta}_j\}=0. \tag{7.67}$$

We introduce the Grassmann odd coordinates $\theta^{\hat\alpha}_i$ and $\theta^{\hat\alpha i}$ and differentials $d\theta^{\hat\alpha}_i$ and $d\theta^{\hat\alpha i}$ associated with the Q supergenerators. It is convenient to define the differentials $d\theta^{\hat\alpha}_i$ and $d\theta^{\hat\alpha i}$ to commute with each other but anticommute with $dX^{\hat\alpha\hat\beta}$ and the Grassmann coordinates $\theta^{\hat\alpha}_i$ and $\theta^{\hat\alpha i}$.

The vacuum 0-form is defined as

$$\hat\omega_0=\left(dX^{\hat\alpha\hat\beta}+\frac{1}{2}[(1+\gamma)d\theta^{\hat\alpha\, i}\theta^{\hat\beta}_i+(1-\gamma)d\theta^{\hat\alpha}_i\theta^{\hat\beta\, i}]\right)P_{\hat\alpha\hat\beta}+d\theta^{\hat\alpha\, i}Q_{\hat\alpha\, i}+d\theta^{\hat\alpha}_iQ^i_{\hat\alpha}. \tag{7.68}$$

The gauge function analogous to Eq. (7.65) is

$$g=\exp-\left[\left(X^{\hat\alpha\hat\beta}+\frac{1}{2}\gamma\theta^{\hat\alpha}{}_i\theta^{\hat\beta\, i}\right)\alpha_{\hat\alpha}\alpha_{\hat\beta}+\theta^{\hat\beta\, i}\alpha_{\hat\beta}\phi_i+\theta^{\hat\alpha}{}_i\alpha_{\hat\alpha}\overline{\phi}^i\right]. \tag{7.69}$$

The left Fock module $|\Phi(\beta,\overline{\phi}|X,\theta)\rangle$ satisfies the $osp(2\mathcal{N},2M)$ supersymmetric equations

$$(\hat d-\hat\omega_0)|\Phi(\beta,\overline{\phi}|X,\theta)\rangle=0. \tag{7.70}$$

Let us note that these formulas are trivially generalized to the case of $osp(L,2M)$ with odd L by writing

$$Q^i_{\hat\alpha}=\alpha_{\hat\alpha}\psi^i, \quad S^{j\hat\beta}=b^{\hat\alpha}\psi^j \tag{7.71}$$

with

$$\{\psi^i,\psi^j\}_*=\delta^{ij} \tag{7.72}$$

so that

$$\{Q^i_{\hat\alpha},Q^j_{\hat\beta}\}=\delta^{ij}P_{\hat\alpha\hat\beta}, \quad \{S^{\hat\alpha i},S^{\hat\beta j}\}=\delta^{ij}K^{\hat\alpha\hat\beta}, \tag{7.73}$$

and

$$\hat\omega_0=\left(dX^{\hat\alpha\hat\beta}+\frac{1}{2}d\theta^{\hat\alpha}_i\theta^{\hat\beta i}\right)P_{\hat\alpha\hat\beta}+d\theta^{\hat\alpha}_iQ^i_{\hat\alpha}, \tag{7.74}$$

$$g=\exp-(X^{\hat\alpha\hat\beta}\alpha_{\hat\alpha}\alpha_{\hat\beta}+\theta^{\hat\beta}_i\alpha_{\hat\beta}\psi^i). \tag{7.75}$$

Equation (7.70) still makes sense with the only comment that the Fock vacuum has to be defined in such a way that it is annihilated by the $\frac{1}{2}(L-1)$ annihilation Clifford elements and is an eigenvector of the central element $\psi_1\cdots\psi_L$.

D. Higher spin (super)space

One can further extend the base manifold description of the $osp(L,2M)$ conformal dynamics by introducing the higher spin coordinates $X^{\hat\alpha_1\cdots\hat\alpha_{2n}}$ and Grassmann odd supercoordinates $\theta_i^{\hat\alpha_1\cdots\hat\alpha_{2n+1}}$ associated with the mutually commuting higher spin generators

$$P_{\hat\alpha_1\ldots\hat\alpha_{2n}}=\alpha_{\hat\alpha_1}\cdots\alpha_{\hat\alpha_{2n}} \tag{7.76}$$

and supercharges

$$Q^i{}_{\hat\alpha_1\cdots\hat\alpha_{2n+1}}=\psi^i\alpha_{\hat\alpha_1}\cdots\alpha_{\hat\alpha_{2n+1}},\qquad \{\psi^i,\psi^j\}_*=\delta^{ij}, \tag{7.77}$$

which satisfy the higher spin super-Poincaré algebra with the nonzero relationships

$$\{Q^i{}_{\hat\alpha_1\cdots\hat\alpha_{2n+1}},Q^j{}_{\hat\beta_1\cdots\hat\beta_{2m+1}}\}=\delta^{ij}P_{\hat\alpha_1\ldots\hat\alpha_{2n+1}\hat\beta_1\cdots\hat\beta_{2m+1}}. \tag{7.78}$$

The zero-curvature vacuum one-form is

$$\hat{\omega}_0=\sum_n \left(\frac{1}{(2n)!}dX^{\hat{\alpha}_1\cdots\hat{\alpha}_{2n}}P_{\hat{\alpha}_1\cdots\hat{\alpha}_{2n}} + \frac{1}{(2n+1)!}d\theta_i^{\hat{\alpha}_1\cdots\hat{\alpha}_{2n+1}}Q^i_{\hat{\alpha}_1\cdots\hat{\alpha}_{2n+1}}\right) + \frac{1}{2}\sum_{q,p}\frac{1}{(2p+1)!(2q+1)!}P_{\hat{\alpha}_1\cdots\hat{\alpha}_{2p+1}\hat{\beta}_1\cdots\hat{\beta}_{2q+1}} \times d\theta_i^{\hat{\alpha}_1\cdots\hat{\alpha}_{2p+1}}\theta^{i\,\hat{\beta}_1\cdots\hat{\beta}_{2q+1}}. \tag{7.79}$$

Let us note that the higher spin (super)coordinates introduced here are to some extent reminiscent of the 4D higher spin coordinates discussed in [52], although the particular realization is different. Unfolded equations of the form (7.70) reconstruct the dependence on the higher spin coordinates in terms of the (usual) space-time derivatives of the massless higher spin fields. In principle, one can extend the formalism to the maximal case in which every element of the infinite-dimensional higher spin algebra [say, $hu(m,n|2M)$] has a coordinate counterpart. This is analogous to a description on the group manifold. Let us note that any further extension would imply a degenerate frame field and, therefore, does not lead to interesting equations. Equations with fewer coordinates corresponding to reductions to some coset spaces are possible, however. Let us note that the unfolded formulation in these smaller spaces is reminiscent of the group manifold approach [67].

VIII. WORLD LINE PARTICLE INTERPRETATION

Free field equations of motion in the unfolded form admit a natural interpretation in terms of a world line particle dynamics. The free field equation (4.35) is interpreted as an invariance condition

$$Q_0|\Phi\rangle=0 \tag{8.1}$$

with a Becchi-Rouet-Stora-Tyutin (BRST) operator built from some first class constraints. The zero-curvature condition (3.9) takes the form

$$Q_0^2=0. \tag{8.2}$$

To make contact with some world line particle dynamics one has to find a world line model that gives rise to an operator Q_0 associated with the unfolded equations under consideration. Usually it is a simple exercise.

The literature on world line (super)particle dynamics appearing after the classical work [73–76] is enormous. The twistor reformulation was initiated in [77,78] and further developed in [79–83,47,60]. The idea that additional (often called central charge) coordinates have to be introduced to extend the twistor approach beyond four dimensions was exploited in [47–50,84,85].

The $sp(2M)$ invariant equation (7.35) can be obtained as a result of quantization of the following Lagrangian:

$$L=\dot{X}^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}+\alpha_{\hat{\alpha}}\dot{\beta}^{\hat{\alpha}}, \tag{8.3}$$

where the overdot denotes the derivative with respect to the world line parameter. Indeed, the primary constraints are

$$0=\chi_{\hat{\alpha}\hat{\beta}}=\pi_{\hat{\alpha}\hat{\beta}}-\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}} \tag{8.4}$$

and

$$0=\chi_{\hat{\alpha}}=\pi_{\hat{\alpha}}-\alpha_{\hat{\alpha}},\quad 0=\chi^{\hat{\alpha}}=\pi^{\hat{\alpha}}, \tag{8.5}$$

where $\pi_{\hat{\alpha}\hat{\beta}}$, $\pi_{\hat{\alpha}}$, and $\pi^{\hat{\alpha}}$ are momenta conjugated to $X^{\hat{\alpha}\hat{\beta}}$, $\beta^{\hat{\alpha}}$, and $\alpha_{\hat{\alpha}}$, respectively. The constraints (8.5) are second class. It is elementary to compute the corresponding Dirac brackets. The only important fact, however, is that within the set of variables $X^{\hat{\alpha}\hat{\beta}}$, $\pi_{\hat{\alpha}\hat{\beta}}$, $\beta^{\hat{\alpha}}$, and $\pi_{\hat{\alpha}}$ the Dirac brackets coincide with the Poisson ones,

$$\{X^{\hat{\alpha}\hat{\beta}},\pi_{\hat{\gamma}\hat{\delta}}\}=\frac{1}{2}(\delta^{\hat{\alpha}}_{\hat{\gamma}}\delta^{\hat{\beta}}_{\hat{\delta}}+\delta^{\hat{\beta}}_{\hat{\gamma}}\delta^{\hat{\alpha}}_{\hat{\delta}}),\quad \{\beta^{\hat{\alpha}},\pi_{\hat{\beta}}\}=\delta^{\hat{\alpha}}_{\hat{\beta}}. \tag{8.6}$$

This allows one to get rid of the variables $\alpha_{\hat{\alpha}}$ and $\pi^{\hat{\alpha}}$ by expressing them in terms of $X^{\hat{\alpha}\hat{\beta}}$, $\pi_{\hat{\alpha}\hat{\beta}}$, $\beta^{\hat{\alpha}}$, and $\pi_{\hat{\alpha}}$ with the help of the second-class constraints (8.5). The leftover constraints (8.4) acquire the form

$$0_0=\chi_{\hat{\alpha}\hat{\beta}}=\pi_{\hat{\alpha}\hat{\beta}}-\pi_{\hat{\alpha}}\pi_{\hat{\beta}}, \tag{8.7}$$

and are obviously first class. Interpreting the space-time differentials as ghost fields $c^{\hat{\alpha}\hat{\beta}}$ one arrives at the BRST operator

$$Q=c^{\hat{\alpha}\hat{\beta}}(\pi_{\hat{\alpha}\hat{\beta}}-\pi_{\hat{\alpha}}\pi_{\hat{\beta}}) \tag{8.8}$$

which, upon quantization, reproduces Eqs. (7.35) in the form (8.1).

The superextension is straightforward:

$$L=\dot{X}^{\hat{\alpha}\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}+\alpha_{\hat{\alpha}}\dot{\beta}^{\hat{\alpha}}-\overline{\phi^i}\,\dot{\phi}_i + \dot{\theta}^{\hat{\alpha}i}\left(\alpha_{\hat{\alpha}}\phi_i+\frac{1}{2}(1+\gamma)\theta_i^{\hat{\beta}}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}\right) + \dot{\overline{\theta}}{}_i^{\hat{\alpha}}\left(\alpha_{\hat{\alpha}}\overline{\phi^i}+\frac{1}{2}(1-\gamma)\theta^{\hat{\beta}i}\alpha_{\hat{\alpha}}\alpha_{\hat{\beta}}\right). \tag{8.9}$$

(The variables $\theta^{\hat{\beta}i}$, ϕ_i, and $\overline{\phi^i}$ are anticommuting and are assumed to have symmetric Poisson brackets {,} with their momenta.) Excluding by virtue of the second class constraints the variables $\alpha_{\hat{\alpha}}$, their conjugated momenta $\pi^{\hat{\alpha}}$, and the fermionic variables ϕ^i with their conjugated momenta, one is left with the conjugated pairs of variables ($X^{\hat{\alpha}\hat{\beta}}$, $\pi_{\hat{\alpha}\hat{\beta}}$),

$(\beta^{\hat{\alpha}},\ \pi_{\hat{\alpha}})$, $(\theta^{\hat{\alpha}i},\ \pi_{\hat{\alpha}i})$ $(\theta_i^{\hat{\alpha}},\ \pi^i_{\hat{\alpha}})$, and $(\overline{\phi^i},\ \pi_i)$ and the first class constraints (8.7) along with

$$\chi_{\hat{\alpha}i}=\pi_{\hat{\alpha}i}-\left(\alpha_{\hat{\alpha}}\pi_i+\frac{1}{2}(1+\gamma)\theta_i^{\hat{\beta}}\pi_{\hat{\alpha}}\pi_{\hat{\beta}}\right),$$

$$\chi^i_{\hat{\alpha}}=\pi^i_{\hat{\alpha}}-\left(\alpha_{\hat{\alpha}}\overline{\phi^i}+\frac{1}{2}(1-\gamma)\theta^{\hat{\beta}i}\pi_{\hat{\alpha}}\pi_{\hat{\beta}}\right). \tag{8.10}$$

Altogether, these first class constraints form a supersymmetry algebra with the only nonzero relation

$$\{\chi_{\hat{\alpha}i},\chi^j_{\hat{\beta}}\}=\delta^j_i\chi_{\hat{\alpha}\hat{\beta}}. \tag{8.11}$$

Quantum-mechanical models containing "central charge" coordinates associated with symplectic algebras, analogous to the coordinates $X^{\hat{\alpha}\hat{\beta}}$, were considered in [49,50,86]. However, to the best of our knowledge, the particular Lagrangians were different from those proposed above.

Analogously, one can consider the model with the Lagrangian

$$L=\dot{X}^{\alpha\dot{\beta}}\alpha_\alpha\alpha_{\dot{\beta}}+\alpha_\alpha\dot{\beta}^\alpha+\overline{\alpha_{\dot{\alpha}}}\dot{\overline{\beta^{\dot{\alpha}}}}-\overline{\phi^i}\dot{\phi}_i+\dot{\theta}^{\alpha i}\left(\alpha_\alpha\phi_i+\frac{1}{2}(1+\gamma)\theta_i^{\dot{\beta}}\alpha_\alpha\overline{\alpha_{\dot{\beta}}}\right)+\dot{\overline{\theta_i^{\dot{\alpha}}}}\left(\overline{\phi^i}\,\overline{\alpha_{\dot{\alpha}}}+\frac{1}{2}(1-\gamma)\theta^{\beta i}\overline{\alpha_{\dot{\alpha}}}\alpha_\beta\right) \tag{8.12}$$

(hopefully, the overdotted indices cause no confusion with the world line parameter derivative). In the 4D case this model gives rise to the 4D conformal equations of motion of Sec. IV A. The 4D Lagrangian (8.12) with $\gamma=0$ was introduced in [47] and was then shown to give rise to the massless equations in [60] [more precisely, the Lagrangians of [47,60] contained additional constraints giving rise to the irreducibility condition (4.40)]. The important difference from many other world line twistor Lagrangians is that no twistor relationship between the space-time coordinates and spinor variables is imposed; instead they are regarded as independent dynamical variables.

The generalization to the higher spin coordinates is described by the Lagrangian

$$L=\sum_n\frac{1}{(2n)!}\dot{X}^{\hat{\alpha}_1\cdots\hat{\alpha}_{2n}}\alpha_{\hat{\alpha}_1}\cdots\alpha_{\hat{\alpha}_{2n}}+\alpha_{\hat{\alpha}}\dot{\beta}^{\hat{\alpha}}-\overline{\phi^i}\dot{\phi}_i+\sum_n\frac{1}{(2n+1)!}(\dot{\theta}^{i\,\hat{\alpha}_1\cdots\hat{\alpha}_{2n+1}}\phi_i\alpha_{\hat{\alpha}_1}\cdots\alpha_{\hat{\alpha}_{2n+1}}+\dot{\theta}_i^{\hat{\alpha}_1\cdots\alpha_{2n+1}}\overline{\phi^i}\alpha_{\hat{\alpha}_1}\cdots\alpha_{\hat{\alpha}_{2n+1}})+\frac{1}{2}\sum_{q,p}\frac{1}{(2p+1)!(2q+1)!}\alpha_{\hat{\alpha}_1}\cdots\alpha_{\hat{\alpha}_{2p+1}}\alpha_{\hat{\beta}_1}\cdots\alpha_{\hat{\beta}_{2q+1}}\times[(1+\gamma)\dot{\theta}^{i\,\hat{\alpha}_1\cdots\hat{\alpha}_{2p+1}}\theta_i^{\hat{\beta}_1\cdots\hat{\beta}_{2q+1}}+(1-\gamma)\dot{\theta}_i^{\hat{\alpha}_1\cdots\hat{\alpha}_{2p+1}}\theta^{i\,\hat{\beta}_1\cdots\hat{\beta}_{2q+1}}]. \tag{8.13}$$

All world line particle Lagrangians discussed in this section have the general form

$$L=\dot{X}^{\underline{A}}\hat{\omega}_{0\underline{A}}(\alpha,\beta,\phi,\overline{\phi}|X)+\alpha_{\hat{\alpha}}\dot{\beta}^{\hat{\alpha}}-\overline{\phi^i}\dot{\phi}_i, \tag{8.14}$$

where $X^{\underline{A}}$ denotes the whole set of supercoordinates while $dX^{\underline{A}}\hat{\omega}_{0\underline{A}}(\alpha,\beta,\phi,\overline{\phi}|X)=\hat{\omega}_0(\alpha,\beta,\phi,\overline{\phi}|X)$ is some vacuum one-form satisfying the zero-curvature equation (3.9). Let us stress that Eq. (3.9) is supposed to be true in the quantum regime, i.e., with respect to the star product. In the classical approximation, the star product has to be replaced by the Poisson (in fact, Dirac) brackets, which usually makes sense for the BRST interpretation (8.2) of the vacuum condition (3.9) but not necessarily for the dynamical field equations in the essentially "quantum" form (8.1).

The constraints have the form

$$\chi_{\underline{A}}=\frac{\partial}{\partial X^{\underline{A}}}-\omega_{0\underline{A}}(X). \tag{8.15}$$

They are first class as a consequence of the flatness condition (3.9). We see that this construction indeed leads to the BRST realization of the linearized unfolded dynamics in the form (8.1), (8.2).

The Lagrangian (8.14) is universal in the sense that it gives rise to unfolded equations of the conformal higher spin fields interpreted as first class constraints independently of the particular form of the vacuum one-form $\hat{\omega}_0$ once it satisfies the zero-curvature equation (3.9). The ambiguity in $\hat{\omega}_0$ parametrizes the ambiguity in the choice of a particular geometry and/or coordinate system. For the particular case of conformal algebra, any conformally flat geometry is available. For example, AdS_4 geometry is described by the vacuum one-form (3.15). Note that it is well known that the zero-curvature (= left invariant Cartan) forms play the key role in the formulation of the (super)particle and brane dynamics because they possess the necessary global symmetries [namely, the symmetries (3.11)]. The fact that $\hat{\omega}_0$ satisfies the zero-curvature condition guarantees that the Lagrangian (8.14) has the necessary local symmetries (i.e., first class constraints). Note that some examples of the zero-curvature one-forms of $osp(1,2n)$ are given in [86].

Applying the Stokes theorem and using the zero-curvature condition for $\hat{\omega}_0$, the particle action (8.14) can be rewritten in topological string form as an integral over a two-dimensional surface bounded by a particle trajectory and parametrized by σ^l:

$$S=\int_{\Sigma^2}\Bigg[\hat{\omega}_0(\alpha,\beta,\phi,\overline{\phi}|X)*\wedge\hat{\omega}_0(\alpha,\beta,\phi,\overline{\phi}|X)$$

$$+d\alpha_{\hat{\alpha}}\wedge d\beta^{\hat{\alpha}}-d\overline{\phi}^i\wedge d\phi_i$$

$$+\left(d\alpha_{\hat{\alpha}}\frac{\partial}{\partial\alpha_{\hat{\alpha}}}+d\beta^{\hat{\alpha}}\frac{\partial}{\partial\beta^{\hat{\alpha}}}+d\phi_i\frac{\partial}{\partial\phi_i}+d\overline{\phi}^i\frac{\partial}{\partial\overline{\phi}^i}\right)$$

$$\wedge\hat{\omega}_0(\alpha,\beta,\phi,\overline{\phi}|X)\Bigg], \qquad (8.16)$$

where

$$\hat{\omega}_0(\alpha,\beta,\phi,\overline{\phi}|X)=d\sigma^l\frac{\partial X^{\underline{A}}}{\partial\sigma^l}\hat{\omega}_{0\,\underline{A}}(\alpha,\beta,\phi,\overline{\phi}|X), \qquad (8.17)$$

$$d\alpha_\alpha=d\sigma^l\frac{\partial\alpha_\alpha}{\partial\sigma^l},\quad d\beta^\alpha=d\sigma^l\frac{\partial\beta^\alpha}{\partial\sigma^l},$$

$$d\phi_i=d\sigma^l\frac{\partial\phi_i}{\partial\sigma^l},\quad d\overline{\phi}^i=d\sigma^l\frac{\partial\overline{\phi}^i}{\partial\sigma^l}. \qquad (8.18)$$

Keeping in mind that the theory of higher spin gauge fields is expected to be related to a symmetric phase of the superstring theory, let us speculate that this topological action can be related to superstring actions in the framework of some perturbative expansion relevant to the usual string picture, which, however, breaks down the manifestly topological form of the whole action defined in the generalized target superspace.

Note that the action (8.16) can be rewritten as

$$S=\int_{\Sigma^2}[w_0(\alpha,\beta,\phi,\overline{\phi}|X)*\wedge w_0(\alpha,\beta,\phi,\overline{\phi}|X)], \qquad (8.19)$$

where

$$w_0=\omega_0+d\beta^{\hat{\alpha}}\alpha_{\hat{\alpha}}-d\alpha_{\hat{\alpha}}\beta^{\hat{\alpha}}+d\phi_i\overline{\phi}^i-\phi_i d\overline{\phi}^i \qquad (8.20)$$

with the convention that the star product in Eq. (8.19) acts on the components of the differential form w_0 but not on the differentials $d\alpha_{\hat{\alpha}}$, $d\beta^{\hat{\alpha}}$, $d\phi_i$, and $d\overline{\phi}^i$.

A few comments are now in order.

It is important that the "quantization" is performed in such a way that equations like (7.35) contain differential rather than multiplication operators. This allows one to express all higher order polynomials in the twistor variables via higher space-time derivatives of the physical fields. Note that the "coordinate" and "momentum" representations are not equivalent in the framework of nonunitary modules underlying (classical) field theory dynamics. One way to see this is to observe that the dualization (Fourier transform) that interchanges twistors with their conjugate momenta interchanges the translations $P_{\hat{\alpha}\hat{\beta}}$ and the special conformal transformations $K^{\hat{\alpha}\hat{\beta}}$.

The conversion procedure applied in [50] to get rid of the complicated second class constraints in a particle-type twistor model based on the $osp(2,8)$ superalgebra led to first class constraints analogous to Eqs. (8.7) and (8.10) modulo exchange of the twistor variables with their momenta. It was concluded in [50] that the space of quantum states of this model consists of the massless fields of all spins (every spin appears in two copies), i.e., it is identical to the spectrum of massless higher spin fields associated with the simplest $\mathcal{N}=2$ supersymmetric conformal higher spin algebra $hu(1,1;8)$. Since the approach of [50] was insensitive to the difference between the twistor variables and their momenta, the one-to-one correspondence between the spectrum of 4D massless higher spin excitations found in [50] and in this paper is not accidental.

Beyond the linearized approximation the world line quantum-mechanical interpretation of the unfolded dynamics becomes less straightforward. Indeed, the interaction problem consists of searching for a consistent deformation of Eqs. (8.1) and (8.2) with nonlinear contributions to Eqs. (8.1) and (8.2) both from the dynamical gauge fields $\omega=\omega_0+\cdots$ and from the "matter sector" $|\Phi\rangle$. The modification due to the gauge fields admits interpretation in terms of a connection in the linear fiber bundle with the module F of quantum states $|\Phi\rangle$ as a fiber. The terms nonlinear in $|\Phi\rangle$ can, however, hardly be interpreted in the usual quantum-mechanical framework that respects the superposition principle. By relaxing the superposition principle, one arrives at the standard setting of free differential algebras (2.1), suggested originally in [54] for analysis of the higher spin problem. The world line particle models can be useful for a second quantized description of nonlinear higher spin dynamics in form analogous to the open string field theory functional of Witten [87]

$$S=\langle\overline{\Phi}|QA|\Phi\rangle+S^3, \qquad (8.21)$$

where A is some insertion needed to make the quadratic part well defined and S^3 is the interaction part to be determined.

IX. AdS/CFT CORRESPONDENCE

The classical result of Flato and Fronsdal [88] states that the tensor product of two singleton representations of $sp(4)$ amounts to the direct sum of all unitary representations of $sp(4)$ associated with the massless fields of all spins in AdS_4. Once the unfolded formulation of massless dynamics exhibits Bogolyubov duality with the unitary representations, there must be some field-theoretical dual version of the Flato-Fronsdal theorem. This was confirmed by analysis of the boundary current and bulk gauge field representations in [89]. It was also observed in [14] that for the 3D conformal theory there is one-to-one correspondence between the tensor product of 3D boundary fields and the set of AdS_4 bulk higher spin gauge fields (and, therefore, conserved higher spin currents of [13]). This statement is supposed to underly the AdS_4/CFT_3 duality in the framework of higher spin theories.

An AdS_5 analog of the Flato-Fronsdal theorem suggests [90,35] that the double tensor products of the doubleton representations contain all massless unitary representations of the AdS_5 algebra $o(4,2)\sim su(2,2)$. It is interesting to find a field-theoretical counterpart of this statement.

Consider first the self-conjugated massless supermultiplets with $\alpha=0$. The corresponding conformal higher spin gauge symmetry algebra $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ was argued in Sec. IV D to be spanned by the elements of the star product algebra (i.e., polynomials of oscillators) that commute to $N_{\mathcal{N}}$, are identified modulo $N_{\mathcal{N}}$, and satisfy the reality condition (4.73). On the other hand, the elements of the tensor product of the space of states satisfying (4.40) with its conjugate

$$E_{12}=|\Phi_1\rangle\otimes\langle\bar{\Phi}_2| \tag{9.1}$$

automatically satisfy these condition as a consequence of Eq. (4.40):

$$[N_{\mathcal{N}},E_{12}]_*=0, \qquad N_{\mathcal{N}}*E_{12}=0. \tag{9.2}$$

Also, it is consistent with the conditions (4.73), (4.90) after appropriate specification of the action of the involution and antiautomorphism on the tensor product symbol to compensate the insertions of the products of elements ϕ_i or $\bar{\phi}^j$ in Eqs. (4.96), (4.97).

The 4D conformal higher spin algebras $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ (being isomorphic to AdS_5 higher spin algebras) and their further orthogonal or symplectic subalgebras can be identified with the (sub)algebras of endomorphisms of the module F_0 spanned by the states satisfying Eq. (4.40) at $\alpha=0$. Discarding the (sometimes important) normalizability issues, it is a matter of basis choice to realize this algebra in terms of either elements (9.1) or polynomials of the star product algebra.[5] Therefore, the tensor product of the 4D matter multiplets has the same structure as the AdS_5 higher spin algebra $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ in which 5D higher spin gauge fields (equivalently, conserved currents [13]) take their values. This fact provides the field-theoretical counterpart of the statement on the structure of the tensor products of the unitary doubleton representations of [90,35]. The non-self-conjugated case is analogous except that the reduction condition (4.90) is inconsistent with the eigenvalues $\alpha\neq 0$ and, therefore, the subalgebras of symplectic and orthogonal types allowed for the self-conjugated case are not allowed for $\alpha\neq 0$. Note that it is also possible to relax the condition (4.90) in the self-conjugated case, that effectively leads to doubling of the self-conjugated multiplets.

[5]Note that the action of the operator (9.1) in F_0 is described by an infinite matrix having at most a finite number of nonzero elements, while the polynomial elements of the star product algebra have the Jacobi form with an infinite number of nonzero elements but at most a finite number of nonzero diagonals. This means that a polynomial in the star product algebra is described by an infinite sum in the basis (9.1).

Thus, the higher spin AdS/CFT correspondence suggests that the AdS_5 higher spin algebra associated with the boundary self-conjugated matter supermultiplets is one of the subalgebras (4.93) or (4.94). From the AdS_5 bulk perspective only the purely bosonic case $\mathcal{N}=0$ has been analyzed so far at the level of cubic Lagrangian interactions [23]. This analysis matches the consideration of the present paper since it was shown in [23] that the AdS_5 higher spin gauge fields associated with the algebras $hu_0(1,0|8)$ and $ho_0(1,0|8)$ allow consistent cubic interactions. In a forthcoming paper [24] we shall show that the same is true for the $\mathcal{N}=1$ supersymmetric case. In both of these cases the situation is relatively simple because the corresponding AdS_5 higher spin gauge fields correspond to the totally symmetric (spinor) tensor representations of AdS_5 algebra. The gauge field formalism for description of these fields suitable for the higher spin gauge problem in any dimension was elaborated in [57,26]. As shown in the recent publication [22] (see also [23]) for the bosonic case and in [91] for the fermionic case, the sets of gauge fields associated with the $\mathcal{N}=0$ and $\mathcal{N}=1$ AdS_5 higher spin algebras are just what is expected from the perspective of the approach of [57,26]; namely, the infinite-dimensional higher spin algebras decompose under the adjoint action of the AdS_5 subalgebra $o(4,2)\sim su(2,2)$ into an infinite sum of finite-dimensional representations associated with various two-row tensors or spinor tensors of $o(4,2)$ [23,91].

Starting from $\mathcal{N}=2$ representations of $o(4,2)$ with three rows appear, however. The simplest way to see this is to observe that, for increasing $\mathcal{N}$, the restriction $[N_{\mathcal{N}},f]_*=0$ on the types of representations of $su(2,2)$ contained in the star product element $f(a,b;\phi,\bar{\phi})$ becomes less and less restrictive, rather imposing some relationships between the types of $su(2,2)$ tensors and $u(\mathcal{N})$ tensors in the supermultiplet. One can see that three-row diagrams of $so(4,2)$ appear whenever the number of oscillators a and b in f can differ by 2, which is possible starting from $\mathcal{N}\geqslant 2$. As a result, the $\mathcal{N}\geqslant 2$ AdS_5 higher spin gauge theories based on the algebras $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ and their further reductions will contain some mixed symmetry gauge fields. Because the 5D massless little Wigner algebra is $o(3)$, in 5D flat space such fields are equivalent to the usual totally symmetric higher spin fields. This is not true, however, in the AdS_5 space where the systematics of the massless fields is different from the flat one [92]. In particular, to every two-row Young diagram of the maximal compact algebra $so(4)\sim su(2)\oplus su(2)$ corresponds a particular AdS_5 massless field. In the flat limit such fields decompose into a number of flat space massless fields, each equivalent (dual) to some totally symmetric field in the flat space. So far, no systematic approach to the mixed symmetry higher spin fields in the AdS space has been elaborated in the covariant approach underlying the unfolded dynamics, although considerable progress in the flat space was achieved in [93,94]. To extend the results of [23,24] to $\mathcal{N}\geqslant 2$ it is first of all necessary to develop a gauge formulation of the higher spin fields carrying mixed symmetry representations of the AdS algebras $o(d-1,2)$. This problem is now under investigation.[6]

It is tempting to speculate that, once the two-row mixed symmetry higher spin AdS_5 fields are included, the condition that the elements of the higher spin algebra have to commute with $N_{\mathcal{N}}$ can be relaxed and (symplectic) AdS_5 dual versions of the $osp(2\mathcal{N},8)$ conformal boundary models might be constructed. These models are expected to contain all types of gauge (massless) fields in AdS_5 having one of the algebras $hu(n,m|8)$, $ho(n,m|8)$, or $husp(n,m|8)$ as the gauge algebra. In that case we arrive at the remarkable possibility that the generalized $sp(8)$ AdS_5/CFT_4 correspondence will relate the bulk model that describes AdS_5 massless fields of all spins (types) to the boundary conformal model describing 4D conformal massless fields of all spins. This is the AdS_5/CFT_4 analogue of the Flato-Fronsdal theorem relating the AdS_5 massless fields to the tensor product of the $sp(8)$ (super)singletons. Once such a generalization is really possible, it will lead to surprising conclusions on the higher spin AdS/CFT correspondence which, in fact, would imply space-time dimension democracy.

Indeed, the following extension of the Flato-Fronsdal theorem is likely to take place

$$S_{osp(L,2M)}\otimes S_{osp(L,2M)}=\sum_s m^{0\,s}_{osp(L,2M)}=S_{osp(2L,4M)}\,, \tag{9.3}$$

where $S_{osp(L,2M)}$ denotes the (super)singleton representation of $osp(L,2M)$ while $m^{0\,s}_{osp(L,2M)}$ denotes all massless unitary representations of $osp(L,2M)$ characterized by the spin parameters s. The chain of identities can be continued to the left provided that L and M are even. For $L=2^q$ and $M=2^p$ the chain continues down to the case of $sp(2)$ or $sp(4)$ with the appropriate truncations in the Clifford sector associated with L if necessary (say, by singling out the bosonic or fermionic constituents of some of the supersingletons). Since the tensor product of the representations is associated with the bilinear currents built from the boundary fields, the conclusion is that the generalized (symplectic) higher dimensional models are expected to be dual to the nonlinear effective theories built from the lowest dimensional (higher spin) models.

The equality $S_{osp(L,2M)}\otimes S_{osp(L,2M)}=S_{osp(2L,4M)}$ is obvious because the supersingleton S of $S_{osp(L,2M)}$ is the Fock module generated by L fermionic and $2M$ bosonic oscillators. By definition, its tensor square is the Fock module generated by two sorts of the same oscillators which is equivalent to the supersingleton module of $S_{osp(2L,4M)}$. The fact that $S_{osp(2L,4M)}$ is equivalent to the sum of all massless representations of $osp(L,2M)$ is less trivial. It is in agreement with the definition of masslessness given by Günaydin in [90,35]. However, to make this definition consistent with the property that massless fields (except for the scalar and spinor) are gauge fields, it is necessary [98,92,99] to prove that the unitary representations corresponding to the gauge massless higher spin fields are at the boundary of the unitarity region of the modules of $osp(L,2M)$, thus being associated with certain singular vectors, decoupling of which manifests the gauge symmetry.[7]

As conjectured in [11,12], the higher spin AdS/CFT correspondence is expected to correspond to the limit $g^2n\rightarrow 0$, where n is the number of boundary conformal supermultiplets and g is the boundary coupling constant. An interesting related question is whether the free 4D boundary theories discussed in this paper admit nonlinear deformations preserving the infinite-dimensional higher spin symmetries $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ (or some of their deformations). Let us argue that, most probably, these symmetries are broken by interactions to lower symmetries.[8] One argument is based on knowledge [7,8] of the full nonlinear higher spin dynamics in $d=4$.

The 4D conformal system analyzed in Sec. IV describes a set of 4D massless fields of all spins which decomposes into irreducible representations of $sp(8)$. From [6,7] it is known that such sets of massless fields admit consistent interactions in AdS_4 but not in flat space. The interactions are introduced in terms of higher spin potentials rather than in terms of the (higher spin) Weyl tensors discussed in this paper. This breaks down the usual 4D conformal symmetry. The breaking of the conformal symmetry is expected to be of the spontaneous type via the vacuum expectation values of certain auxiliary fields needed to provide consistent higher spin dynamics. This results in $CFT_d\rightarrow AdS_d$ deformation with respect to the d-dimensional coupling constant $g^2\sim\Lambda\kappa^{d-2}$, where Λ and κ are the cosmological constant and the gravitational constant, respectively. Let us note that by AdS_d we assume a universal covering of anti–de Sitter space-time (or an appropriate symplectic generalization discussed below), which, although being curved, is topologically R^d. Note that since the AdS_d geometry is conformally flat it should be possible to have AdS/CFT correspondence with the boundary CFT theory formulated in the AdS space-time rather than in

[6]Note that after the original version of this paper was sent to hep-th some progress in this direction was achieved in [95–97].

[7]Let us note that beyond the AdS_3 and AdS_4 cases in which the symplectic and orthogonal tracks are equivalent, the concept of masslessness may be different for, say, symplectic AdS^M (i.e., symplectic bulk) and orthogonal AdS_d (i.e., the usual bulk) theories. For the symplectic algebras $osp(L,2^p)$, which contain the (maximally embedded) AdS subalgebras $o(2p,2)$ or $o(2p+1,2)$, the values of the lowest energies compatible with unitarity are expected to be higher than the lowest energies of the lowest weight unitary representations of their AdS_d subalgebras. (I am grateful to R. Metsaev for a useful discussion of this point.) In fact, there is nothing special in this phenomenon, which would just signal that the extra symplectic dimensions play a real role. Very much the same story happens for the usual AdS_d algebras $o(d-1,2)$: the lowest energies of $o(d-1,2)$ are higher than those of its lower-dimensional subalgebra $o(d-2,2)$ [98,99]. Let us note that, from this perspective, Günaydin's identification [90] of the massless representations of AdS algebras with those that belong to the tensor product of the singleton and doubleton representations is likely to be true for the symplectic track rather than for the usual AdS_d one.

[8]I am grateful to E. Witten for a stimulating discussion of this issue.

the Minkowski one. (To the best of our knowledge this technically more involved possibility has so far not been investigated.) As a result, in the framework of higher spin gauge theories, AdS^{2M}/AdS^{M} correspondence is likely to replace the usual AdS/CFT correspondence. [Abusing notation, we use the notation AdS^{M} for the generalized space-time identified below with $Sp(M)$.] Perhaps the breakdown of the conformal higher spin symmetries down to the AdS higher spin symmetries can be understood as a result of the conformal anomaly arising in the process of approaching conformal infinity [100]. Also, let us note that since the AdS/CFT correspondence refers to the conformal boundary of the bulk space a possible argument against the infinite chain of AdS/CFT dualities (1.19) based on the fact that the boundary of a boundary is zero is avoided just because the full conformal symmetry is expected to be broken.

The formulation of the full nonlinear 4D higher spin dynamics of [7] provides us with some hints on the character of the breaking of the "conformal" $sp(2M)$ by interactions. The full nonlinear formulation of the 4D higher spin dynamics was given in terms of the star product algebra with eight spinor generating elements. In other words, the construction of [7] has explicit local $hu(1,1|8)$ symmetry [extension to $hu(n,m|8)$ is trivial by considering matrix versions of the model along the lines of [25]] and, in particular, $sp(8)$ as its finite-dimensional subalgebra. These local symmetries are broken by the vacuum expectation values of the auxiliary fields called S to $hu(1,1|4)\oplus hu(1,1|4)$ containing $sp(4)\oplus sp(4)$. (The doubling is due to the Klein operators.) The lesson is that the higher spin interactions break the conformal $hu(n,m|2M)$ symmetry to $hu(n',m'|M)$ (for M even).

This conclusion fits the analysis of the embedding of the generalized AdS algebra into the conformal algebra $sp(2M)$. Indeed, to embed the usual AdS_d algebra $o(d-1,2)$ into the d-dimensional conformal algebra $o(d,2)$ one identifies the AdS_d translations with a mixture of the translations and special conformal transformations in the conformal algebra $P^a_{AdS_d}=P^a_{d\,conf}+\lambda^2 K^a_{d\,conf}$. Commutators of such defined AdS_d translations close to d-dimensional Lorentz transformations L^{ab}. P^a_{AdS} and L_{ab} form the AdS_d algebra $o(d-1,2)\subset o(d,2)$ [cf. Eq. (3.15) for the particular case of AdS_4]. This embedding breaks down the explicit $o(1,1)$ dilatational covariance because it mixes the operators P^a and K^a, which have different scaling dimensions.

Let us now analyze the analogous embedding of a generalized AdS subalgebra into the conformal algebra $sp(2M)$ in $\frac{1}{2}M(M+1)$-dimensional generalized space-time. Since we want to keep the dimension of the generalized space-time intact, the generators of AdS translations have to be of the form $P^{AdS}_{\hat\alpha\hat\beta}=P_{\hat\alpha\hat\beta}+\lambda^2\eta_{\hat\alpha\hat\beta\,\hat\gamma\hat\delta}K^{\hat\gamma\hat\delta}$ with some bilinear form $\eta_{\hat\alpha\hat\beta\,\hat\gamma\hat\delta}$. To allow embedding of the generalized AdS superalgebra into the conformal superalgebra with the AdS supercharges being a mixture of the Q and S supercharges of the conformal algebra, i.e., $Q^{AdS}_{\hat\alpha}=Q_{\hat\alpha}+\lambda V_{\hat\beta\hat\alpha}S^{\hat\beta}$, $\eta_{\hat\alpha\hat\beta\hat\gamma\hat\delta}$ has to have a factorized form, i.e.,

$$P^{AdS}_{\hat\alpha\hat\beta}=P_{\hat\alpha\hat\beta}+\lambda^2 V_{\hat\alpha\hat\gamma}V_{\hat\beta\hat\delta}K^{\hat\gamma\hat\delta}, \tag{9.4}$$

with some antisymmetric bilinear form $V_{\hat\alpha\hat\beta}$. We require $V_{\hat\alpha\hat\beta}$ to be nondegenerate, which assumes that M is even (for the case of odd M the resulting generalized AdS algebra is not semisimple). The commutator of such defined generalized AdS translations closes to the subalgebra $sp(M)$ of $sl_M\subset sp(2M)$, which leaves invariant the antisymmetric bilinear form $V_{\hat\alpha\hat\beta}$. The full generalized AdS subalgebra is

$$sp(M)\oplus sp(M)\subset sp(2M). \tag{9.5}$$

Its Lorentz subalgebra $sp^l(M)$ is identified with the diagonal $sp(M)$ while AdS translations belong to the coset space $sp(M)\oplus sp(M)/sp^l(M)$. For $M=2$ one recovers the usual $3d$ embedding $o(2,2)\sim sp(2)\oplus sp(2)\subset sp(4)\sim o(3,2)$. Analogously to the 3D case, the $\frac{1}{2}M(M+1)$-dimensional space-time where the generalized AdS algebra $sp(M)\oplus sp(M)$ acts is the group manifold $Sp(M)$, while the two $sp(M)$ symmetry algebras are induced by its left and right actions on itself. In particular, the ten-dimensional generalized space-time associated with the AdS phase of 4D massless fields of all spins is $Sp(4)$.

Thus, for even M we obtain that the AdS subalgebra of the conformal algebra acting in the $\frac{1}{2}M(M+1)$-dimensional space-time is isomorphic to the direct sum of the two conformal algebras of the generalized $[M(M+2)/8]$-dimensional space-time. The process can be continued to lower dimensions provided that $M=2^q$. Let us note that the fact that the AdS algebra is semisimple may indicate that the corresponding reduced higher spin algebra acquires more supersymmetry. A particularly nice scenario would be that the AdS reduction of the $\mathcal{N}$-extended conformal higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|2M)$ in the generalized space-time $Sp(M)$ is $hu(2^{\mathcal{N}},2^{\mathcal{N}}|M)$. In that case, the extension $\mathcal{N}-1\rightarrow\mathcal{N}$ would imply the doubling of the even sector because of the new unimodular bosonic element $\phi_{\mathcal{N}+1}\overline{\phi}^{\mathcal{N}+1}$ built from the additional Clifford elements.[9] Then, the breaking of the free field conformal symmetry $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|2M)$ to the AdS^M one by interactions would imply

$$hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|2M)\rightarrow hu(2^{\mathcal{N}},2^{\mathcal{N}}|M), \tag{9.6}$$

which would lead along with Eq. (9.3) to the chain of correspondences

$$\begin{aligned}\cdots AdS^{2M,\mathcal{N}}&AdS^{M,\mathcal{N}+1}\\ &\rightarrow AdS^{M,\mathcal{N}+1}/AdS^{M/2,\mathcal{N}+2}\\ &\rightarrow AdS^{M/2,\mathcal{N}+2}/AdS^{M/4,\mathcal{N}+3}\rightarrow\cdots\end{aligned} \tag{9.7}$$

with $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|M)$ realized either as AdS^M higher spin algebra in the generalized space-time $Sp(M)$ or as the conformal higher spin algebra in the generalized space-time $Sp(\frac{1}{2}M)$. [We assume that the proposed scenario is going to

[9] Let us note that this scenario does not sound too unrealistic taking into account that the reduction of the star product sector algebra allows for introducing unimodular Klein-type operators built from the bosonic oscillators.

work when all relevant algebras $sp(m)$ have even m. The chain of correspondences continues down to the lowest dimensions for $M=2^q$.]

Let us stress that this scenario is mainly justified by the observation that the full 4D $sp(8)$ conformal massless higher spin multiplets expected to provide a boundary theory for the AdS_5 bulk higher spin theory have spectra identical to those of AdS_4 higher spin theories thus requiring deformation of the flat boundary geometry to the anti–de Sitter one in a phase with higher spin interactions respecting higher spin gauge symmetries. [Note that an analogous observation was made in [14], where it was found that the 3D free conformal higher spin theories describe the same sets of massless fields (scalar and spinor) as the nonlinear AdS_3 higher spin theories constructed in [101].] Since the standard AdS/CFT duality is a nonlinear mapping of the bulk fields to the boundary currents bilinear in the elementary boundary fields [2,4], the resulting generalized space-time dimension democracy suggests a chain of nonlinear mappings with the higher dimensional models equivalent to the theories of composite fields of the lower dimensional ones.

The suggested chain of AdS/CFT correspondences can be true for full higher spin theories based on the algebras $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ [say, as conjectured in Eq. (9.7)] but makes no sense for reduced theories based on the algebras $hu_\alpha(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ and their further reductions. Once a theory is truncated to the subsector singled out by the condition (4.40), say, to the $\mathcal{N}=4$ SYM theory, no full $CFT_d \to AdS_d$ deformation correspondence can be expected. In other words, a reduction to the usual space-times and symmetries is expected to break the correspondence chain (1.19) at some point. Note that such a reduction is likely to result from some sort of spontaneous breaking mechanism with a Higgs-type field φ acquiring a vacuum expectation value proportional to $N_{\mathcal{N}}$, thus reducing the full higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ to its subalgebra which is the centralizer of $N_{\mathcal{N}}$.

The argument against a nontrivial deformation of the full higher spin conformal symmetries to a nonlinear theory, based on the peculiarities of the higher spin dynamics requiring AdS geometry, fails to be directly applicable to models based on the algebras $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ with $\mathcal{N}\leqslant 4$ because the corresponding supermultiplets do not contain higher spins. Although the problem is formulated in flat space-time, this possibility is not strictly speaking, ruled out by the Coleman-Mandula-type theorems because conformal theories do not admit a well-defined S matrix. Indeed, some of the models of interest were argued to admit a conformal quantum phase compatible with higher spin symmetries [102]. In the framework of classical field theory, the problem is to find a nonlinear deformation of Eqs. (3.9), (4.35) with the matter field $|\Phi\rangle$ contributing to the right hand side of Eq. (3.9). Provided that the deformed equations are formally consistent, the appropriately deformed conformal higher spin symmetries will also be guaranteed. It is *a priori* not excluded that a nonlinear deformation of the free field dynamics compatible with conformal higher spin symmetries, e.g., in the $\mathcal{N}=4$ SYM theory, may exist. On the other hand, a potential difficulty is due to the possible anomaly resulting from the divergency of the star product of the Fock vacua (4.22) and (4.64) in the $(|\Phi\rangle * \langle\Psi|)$-like bilinear terms.

X. CONCLUSIONS AND OUTLOOK

In this paper, infinite-dimensional 4D conformal higher spin symmetries have been realized on free massless supermultiplets. The explicit form of the higher spin transformations is given by virtue of the unfolded formulation of the equations of motion for massless fields in the form of the covariant constancy condition for the appropriate Fock fiber bundle. Such conformal field theories were conjectured to be boundary dual to nonlinear higher spin theories in the bulk AdS space [13]. In [11,12] it was conjectured that the AdS/CFT duality for higher spin theories should correspond to the weak coupling regime $g^2 n \to 0$ in the superstring picture. To verify these conjectures it is now necessary to build the AdS_5 higher spin theory. Progress in this direction for the simplest case of $\mathcal{N}=0$ higher spin theory was achieved in [23] where some cubic higher spin interactions were found. To extend these results to $\mathcal{N}\neq 0$ and, in particular, to $\mathcal{N}=4$ it is necessary to extend the results of [23] to higher spin gauge fields carrying mixed symmetry massless representations of the AdS_5 algebra associated with the two-row Young diagrams.

As a by-product of our formulation it is shown how the $osp(L,8)$ symmetry is realized on the infinite set of free boundary conformal fields of all spins. This result is interesting from various points of view. First of all, it was argued by many authors [33–37] that the algebras $osp(m,2^n)$ and, in particular, $osp(1,32)$ and $osp(1,64)$ play a fundamental role for the M-theory interpretation of superstring theory. It is usually believed that the related symmetries are broken by the brane charges. From the results of this paper it follows that the algebras of this type can be unbroken if an infinite number of massless fields of all spins is allowed. A natural mechanism of spontaneous breaking of the symplectic symmetries to the usual (AdS or conformal) symmetry algebras might result from a scalar field φ in the (bulk or boundary) theory, which acquires a nonzero vacuum expectation value $\varphi = N_{\mathcal{N}} + \cdots$, where $N_{\mathcal{N}}$ is the operator (1.9) that breaks $osp(\mathcal{N},8)$ to $su(2,2|2\mathcal{N})$ and the higher spin algebra $hu(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$ to $hu_0(2^{\mathcal{N}-1},2^{\mathcal{N}-1}|8)$. In that case the breaking of the symmetries associated with the so-called central charge coordinates results from a condensate of the higher spin fields.

The new equations (7.45) and (7.46) of the scalar and svector (symplectic vector) fields in the manifestly $sp(2M)$ conformally invariant $\frac{1}{2}M(M+1)$-dimensional extended space-time are formulated. These equations encode in a concise form the dynamical equations for all types of massless fields in the 3D and 4D cases for $M=2$ and $M=4$, respectively. Remarkably, the proposed $sp(2M)$ invariant equations are compatible with unitarity, as follows from the Bogolyubov transform duality of their unfolded formulation to the unitary singleton representation of $sp(2M)$. The superextension of these equations is also given in the form of an infinite chain of equations in the extended superspace associated with $osp(L,2M)$.

This result can dramatically affect our understanding of the nature of extra dimensions. In fact, we argue that, from the perspective of higher spin gauge theory, the proposed symplectic higher dimensional space-times have a better chance of describing appropriately higher dimensional extensions of the space-time geometry than the traditional Minkowski extension. Among other things, this improves the situation with supersymmetry. Indeed, the main reason why supersymmetry singles out some particular dimensions in the Minkowski track is that the dimension of the spinor representations of the Lorentz algebra increases exponentially with the space-time dimension (as $2^{[d/2]}$) while dimensions of its tensor representations increase polynomially. This implies mismatch between the numbers of bosonic and fermionic coordinates, thus singling out some particular dimensions $d \leqslant 11$ where the number of spinor coordinates is not too high due to imposition of appropriate Majorana and/or Weyl conditions. If our conjecture is true, the higher dimensional models considered so far would correspond to some specific truncations of the hypothetical symplectic theories. The crucial ingredient underlying the ''symplectic track'' conjecture is that the generalized symplectic conformal equations (7.45) and (7.46) admit consistent quantization.

We argued that the generalized symplectic space-time is the group manifold $Sp(M)$ that has the conformal (boundary) symmetry $Sp(2M)$ and AdS (bulk) symmetry $Sp(M) \times Sp(M)$ (M is even). The generalized superspace is $OSp(L,M)$. The usual 3D case corresponds to the case of $M=2$, while the usual 4D geometry is embedded into the ten-dimensional generalized space-time $Sp(4)$. The fact that the generalized space-time is the group manifold is interesting from various points of view and, in particular, because the generalized superstring theories may admit a natural formulation in terms of the appropriate Wess-Zumino-Witten-Novikov models.

The algebras $sp(2^p)$ and the related generalized space-times play a distinguished role in many respects. The odd elements of $osp(L,2^p)$ can be interpreted as forming the spinor representations of the usual Lorentz algebras in $d=2p$ or $d=2p+1$ dimensional space-times, so that the theories of this class admit an interpretation in terms of the usual Minkowski track space-time symmetries and supersymmetries. In particular, the generalized space-time coordinates $X^{\hat{\alpha}\hat{\beta}}$ are equivalent to a set of antisymmetric tensor coordinates $x^{a_1\cdots a_n}$

$$X^{\hat{\alpha}\hat{\beta}}=\sum_{n=0}^{d}(\Gamma^{\hat{\alpha}\hat{\beta}}_{a_1\cdots a_n}+\Gamma^{\hat{\beta}\hat{\alpha}}_{a_1\cdots a_n})x^{a_1\cdots a_n} \qquad (10.1)$$

associated with all those antisymmetrized combinations of the Γ-matrices $\Gamma^{\hat{\alpha}\hat{\beta}}_{a_1\cdots a_n}$ which are symmetric in the indices $\hat{\alpha}$ and $\hat{\beta}$. The dynamical equations (7.45) and (7.46) amount to some sets of differential equations with respect to the generalized coordinates $x^{a_1\cdots a_n}$. An interesting possibility consists of the interpretation of the dynamics of branes in the Minkowski track picture as point particles in the generalized spaces of the symplectic track.

Another exciting possibility is that in the framework of the full (i.e., symplectic) higher spin theories the chain of AdS/CFT correspondences can be continued (1.19) to link together higher spin theories in symplectic space-times of various dimensions $\frac{1}{2}M(M+1)$ via a nonlinear field-current correspondence [2,4]. The dramatic effect of this would be ''space-time dimension democracy'' establishing duality between higher spin gauge theories in different dimensions. Since higher spin gauge theory is expected to describe a symmetric phase of the theory of fundamental interactions, like superstring theory and M theory, this would imply that the analogous dualities are to be expected in the superstring theory, although in a hidden form as a result of spontaneous breakdown of the higher spin symmetries and, in particular, the $osp(L,2M)$ supersymmetry. From this perspective the dimensions $M=2^p$ again play a distinguished role because the analogue of the Flato-Fronsdal theorem (9.3) is expected to be true for the generalized space-times $Sp(2^p)$ with all p. In other words, the conjectured chain of dualities links all theories that admit an interpretation in terms of the usual space-time spinors and tensors to each other via the nonlinear generalized AdS/CFT correspondence (1.19).

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