Calibrations, monopoles, and fuzzy funnels

Neil R. Constable*
Center for Theoretical Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Neil D. Lambert†
Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855
(Received 9 July 2002; published 30 September 2002)

We present new non-Abelian solitonic configurations in the low energy effective theory describing a collection of N parallel D1-branes. These configurations preserve 1/4, 1/8, 1/16 and 1/32 of the spacetime supersymmetry. They are solutions to a set of generalized Nahm equations which are related to self-duality equations in eight dimensions. Our solutions represent D1-branes which expand into fuzzy funnel configurations ending on collections of intersecting D3-branes. Supersymmetry dictates that such intersecting D3-branes must lie on a calibrated three-surface of spacetime and we argue that the generalized Nahm equations encode the data for the construction of magnetic monopoles on the relevant three-surfaces.

DOI: 10.1103/PhysRevD.66.065016 PACS number(s): 11.10.Kk

I. INTRODUCTION

In the absence of gauge fields branes are embedded into spacetime so as to minimize their worldvolume. In addition, requiring that the branes are supersymmetric leads to the condition that their worldvolumes are in a preferred class of sub-manifolds of spacetime known as calibrated surfaces [1]. These surfaces were first applied to supersymmetric brane configurations within string theory compactifications in [2,3]. The theory of calibrations is also important for understanding intersecting D-branes which preserve some fraction of supersymmetry in flat space [4–6]. In this context the combined worldvolume of a set of intersecting Dp-branes can be regarded as a single $(p+1)$-dimensional hypersurface embedded into spacetime. The condition that the embedding is supersymmetric is equivalent to the condition that the $(p+1)$-dimensional hypersurface is calibrated but now these surfaces are generically non-compact.

In particular it was shown in [5] that the calibration equations of [1] are realized as a Bogomol'nyi-Prasad-Sommerfield (BPS) condition in the low energy effective field theory of a *single* Dp-brane. For this reason we refer to these as *Abelian* embeddings. As is well known, multiple coincident Dp-branes give rise to a non-Abelian gauge symmetry in the low energy effective theory [7]. This has lead to many interesting implications for the spacetime interpretation of D-branes and has given invaluable new geometrical insights into non-Abelian gauge theory phenomena such as monopoles and instantons. It is therefore of interest to understand *non-Abelian* embeddings of D-branes into spacetime. Ideally one would like to derive the non-Abelian generalization of the BPS conditions obtained for a single Dp-brane in [5]. However, this issue is at present complicated by the fact that the non-Abelian generalization of the Born-Infeld (BI) action is poorly understood and in particular there is virtually no understanding of non-Abelian κ-symmetry—see [8–10] for some recent discussion and progress on this issue.

In this paper we will initiate a study of non-Abelian embeddings by studying the simplest example, namely D1-branes. Because of the absence of a suitable notion of non-Abelian κ supersymmetry we will restrict our analysis to the Yang-Mills approximation. We expect that any supersymmetric solution to the Yang-Mills equations of motion can be lifted to a the full non-linear equations of motion. In the study of Abelian embeddings this corresponds to considering free scalar Maxwell theories and is trivial, i.e. most of the interesting solutions crucially involve the non-linear terms. However, we will find that non-trivial solutions exist in the non-Abelian Yang-Mills approximation.

More specifically we will see that the resulting solitons are generalizations of the D1$\perp$D3 system in which N D1-branes end on a single D3-brane. From the point of view of the effective theory living on the D3-brane the D1-branes act as point sources of magnetic charge. These monopole configurations are realized as BPS solutions to the equations of motion which represent the worldvolume of the D3-brane protruding into the ambient spacetime in a spike-like configuration known as a BIon [11–13]. As a BPS configuration in a $U(1)$ field theory the BIon is a completely Abelian object. This system can instead be studied as a solitonic object in the non-Abelian theory describing the N D1-branes. The BPS equations for this system are known to be given by the Nahm equations [14] describing BPS monopoles in $(3+1)$-dimensional Yang-Mills-Higgs theory [15]. The Nahm data used in the construction of monopoles in the D3-brane theory are therefore naturally encoded in the non-Abelian dynamics of the D1-branes which end on the D3-branes.

It was shown in [16] that such solitonic solutions in the D1-brane theory in fact describe a non-commutative (or fuzzy) funnel configuration which opens up into a D3-brane orthogonal to the worldvolume of the D1-branes. Further, it was also shown that this funnel actually acts as a source for the Ramond-Ramond four form of type IIB string theory in precisely the correct fashion to be identified with the D3-

*Email address: constabl@lns.mit.edu
†Email address: nlambert@physics.rutgers.edu

brane on which the D1-branes are ending. This construction therefore provides a self-contained description of the D1⊥D3 intersection which is complementary to that of the BIon spike.

The fuzzy funnel configurations described in [16] are solutions to Nahm equations and describe D1-branes ending on a single D3-brane. As such, these constructions involve only three of the eight transverse scalars in the effective theory describing the D1-branes. It is natural to ask whether one may find analogous solitonic configurations which involve more, perhaps all, of the transverse scalars. This question was addressed in [17] where such solutions involving five scalars were found. These were shown to correspond to D1-branes ending on a collection of D5-branes. While stable these configurations are not supersymmetric. It is the purpose of the present investigation to discuss new solitonic non-Abelian D1-brane configurations which involve five, six and seven of the transverse scalars while preserving various fractions of the supersymmetry preserved by the original D1-branes.

In the following we will denote the transverse scalars of the non-Abelian D1-brane theory by Φ^i, $i=1,\ldots,8$ and take the configuration to lie along the x^9 direction in spacetime. We will show that there exist a class of supersymmetric configurations which are solutions to the generalized Nahm equations

$$\frac{\partial\Phi^i}{\partial x^9}=\frac{1}{2}c_{ijk}[\Phi^j,\Phi^k], \tag{1.1}$$

where c_{ijk} is a totally antisymmetric constant tensor which we will determine exactly below. These equations can also be derived as the dimensional reduction of the higher-dimensional self-duality conditions in [18]. In the special case that only three of the transverse scalars are non-trivial c_{ijk} becomes ε_{ijk} and Eq. (1.1) is simply the standard Nahm equations discussed in [15,16] corresponding to the D1-branes ending on a single D3-brane. The more general configurations involving more than three non-Abelian scalars will turn out to correspond to fuzzy funnel configurations in which the D1-branes end on collections of intersecting D3-branes. As discussed above such intersecting D3-branes are supersymmetric if their worldvolumes lie on a calibrated three-manifold [1]. Defining the three form $\omega=(1/3!)c_{ijk}dx^i\wedge dx^j\wedge dx^k$ we will show that the non-trivial components of ω are in a one-to-one correspondence with the three-form which calibrates the D3-brane intersection on which the D1-branes are ending. We are thus led to conjecture that the generalized Nahm equations encode the necessary data to construct magnetic monopole solutions on various calibrated three-manifolds via an analogous method to the standard Nahm construction of magnetic monopoles on $\mathbb{R}^3$. In addition we will argue that these generalized Nahm equations also encode data on the moduli of the associated calibrated manifold.

The remainder of this paper is organized as follows. In Sec. II we derive the generalized Nahm equations as a BPS condition in the non-Abelian worldvolume theory describing N D1-branes. In Sec. III we discuss how solutions to the generalized Nahm equations are related to supersymmetric D3-brane configurations. Specifically, we make precise our claims that the generalized Nahm equations encode the data describing magnetic monopoles on calibrated three-manifolds. We also discuss the connection between our construction and the notion of higher dimensional self-dual gauge fields [18]. In Sec. IV we provide explicit solutions to the generalized Nahm equations and demonstrate that these solutions can be interpreted as fuzzy funnels which open up into the various D3-brane configurations discussed in Secs. II and III. In Sec. V we examine the moduli associated with our solutions and identify the deformations which correspond to the moduli of the calibrated surfaces on which the D1-branes are ending. We conclude with some comments and open problems.

II. GENERALIZED NAHM EQUATIONS FROM D1-BRANES

The effective action for N D1-branes obtained by quantization of open strings is a non-linear generalization of the Yang-Mills action with gauge group $U(N)$. We will choose conventions where the fields, which take values in the Lie algebra $u(N)$, are anti-Hermitian. The worldvolume gauge field has explicitly been set to zero. In the following we will work in static gauge so that the worldvolume coordinates are identified with those of spacetime as, $\sigma^1=t$ and $\sigma^2=x^9$. Furthermore, we will be considering only the leading order terms appearing in an expansion of the full non-Abelian Dirac-Born-Infeld effective action[1]

$$S=-T_1\int dtdx^9\Big[N+\lambda^2\mathrm{Tr}\Big(\frac{1}{2}\partial^a\Phi^i\partial_a\Phi^i+\frac{1}{4}[\Phi^i,\Phi^j][\Phi^i,\Phi^j]+\cdots\Big)\Big], \tag{2.1}$$

where $\lambda=2\pi l_s^2$ and $i,j=1,\ldots,D$ labels the number of non-vanishing transverse directions. The terms involving the scalar fields are easily recognized as the bosonic sector of maximally supersymmetric Yang-Mills theory with gauge group $U(N)$ in two spacetime dimensions described by the Lagrangian

$$\mathcal{L}=T_1\lambda^2\mathrm{Tr}\Big(\frac{1}{2}\partial_\mu\Phi^i\partial^\mu\Phi^i+\frac{1}{4}[\Phi^i,\Phi^j]^2\Big). \tag{2.2}$$

After subtracting off the ground state energy of the D1-branes, the energy of a static configuration can be written as

$$\begin{aligned}E&=T_1\lambda^2\int dx^9\mathrm{Tr}\Big(\frac{1}{2}\Phi^{i\prime}\Phi^{i\prime}+\frac{1}{4}[\Phi^i,\Phi^j][\Phi^i,\Phi^j]\Big)\\&=T_1\lambda^2\int dx^9\Big\{\frac{1}{2}\mathrm{Tr}\Big(\Phi^{i\prime}-\frac{1}{2}c_{ijk}[\Phi^j,\Phi^k]\Big)^2+T\Big\},\end{aligned} \tag{2.3}$$

where a prime denotes differentiation with respect to x^9 and

[1]We assume here the conventions of [19].

we have introduced a constant, totally anti-symmetric tensor c_{ijk} which will be specified below. In Eq. (2.3) we have performed the usual Bogomol'nyi construction and written the energy density as a squared term plus a topological piece given by

$$T=\frac{T_1\lambda^2}{3}c_{ijk}\mathrm{Tr}(\Phi^i\Phi^j\Phi^k)'. \tag{2.4}$$

We must also impose that the two quartic terms in Eq. (2.3) agree, that is we must impose that

$$\frac{1}{2}c_{ijk}c_{ilm}\mathrm{Tr}([\Phi^j,\Phi^k][\Phi^l,\Phi^m])=\mathrm{Tr}([\Phi^i,\Phi^j][\Phi^i,\Phi^j]). \tag{2.5}$$

It now follows that the Bogomol'nyi equation is

$$\Phi^{i\prime}=\frac{1}{2}c_{ijk}[\Phi^j,\Phi^k], \tag{2.6}$$

and hence the energy of such a configuration depends only on the boundary conditions of the coordinates.

Equation (2.6) can be thought of as a generalized Nahm equation. Indeed, in the case that $\Phi^i\neq 0$ for $i=1,2,3$ and $c_{ijk}=\varepsilon_{ijk}$ (2.6) is precisely the Nahm equation [14] describing BPS monopoles on $\mathbb{R}^3$. The emergence of this equation in the low energy D1-brane theory was first pointed out in [15]. This was further studied in [16] where it was found that this equation has supersymmetric solutions which represent the expansion of the parallel D1-branes into a non-commutative funnel structure which opens up into an orthogonal D3-brane filling the x^1,x^2,x^3 directions in spacetime. As will be seen below Eq. (2.6) possesses supersymmetric non-commutative funnel solutions, with the appropriate choices of c_{ijk}, involving 5, 6 and 7 transverse scalars. These solutions preserve 1/4, 1/8 and 1/16 of the supersymmetry preserved by the original collection of D1-branes. Further, instead of expanding into a single D3-brane, these configurations are found to be expanding into collections of intersecting D3-branes.

In order to proceed note that since Eq. (2.5) is a constraint on the fields we must also ensure that the equation of motion is satisfied

$$\Phi^{i\prime\prime}=-[[\Phi^i,\Phi^j],\Phi^j], \tag{2.7}$$

or, from Eq. (2.6),

$$\frac{1}{2}c_{ijk}c_{jlm}[[\Phi^l,\Phi^m],\Phi^k]=-[[\Phi^i,\Phi^j],\Phi^j]. \tag{2.8}$$

Multiplying Eq. (2.8) by Φ^i and taking the trace we find that the constraint (2.5) is automatically satisfied. Thus solutions to the generalized Nahm equations, along with the constraint (2.8), are guaranteed to be solutions of the full equations of motion.

We are interested in supersymmetric solutions of Eq. (2.6). The general supersymmetry variation is

$$\delta\lambda=\left(\frac{1}{2}\partial_\mu\Phi_i\Gamma^{\mu i}+\frac{1}{4}[\Phi^i,\Phi^j]\Gamma^{ij}\right)\epsilon. \tag{2.9}$$

Here the Γ matrices form the Spin(1,9) Clifford algebra. In our case $\delta\lambda=0$ becomes

$$0=\sum_{i<j}[\Phi^i,\Phi^j]\Gamma^{ij}(1+c_{ijk}\Gamma^{ijk9})\epsilon. \tag{2.10}$$

Note that ϵ is the preserved supersymmetry on the D1-brane worldvolume. To solve the supersymmetry condition (2.10) we define the projectors

$$P_{ij}=\frac{1}{2}(1+c_{ijk}\Gamma^{ijk9}), \tag{2.11}$$

where there is no sum on i,j. In the cases we will consider, for a given pair i,j, c_{ijk} is only non-zero for at most one value of k. In this case, provided that we normalize $c_{ijk}=\pm 1$, we find that $P_{ij}^2=P_{ij}$. Hence we set $P_{ij}\epsilon=0$ for each pair i,j such that $c_{ijk}\neq 0$ for some k.[2] Of course to find a non-trivial solution for ϵ, and hence preserve some fraction of the D1-brane's 16 supersymmetries, we must impose that the matrices Γ^{ijk9} which appear in the P_{ij} projectors commute with each other. It is not hard to see that $[\Gamma^{ijk9},\Gamma^{i'j'k'9}]=0$ if and only if $i=i'$ and $j\neq k\neq j'\neq k'$, etc., i.e. the sets $\{i,j,k\}$ and $\{i',j',k'\}$ have exactly one element in common. Note that it is possible that some combination of projectors imply that other projectors are automatically satisfied. The number of preserved spacetime supersymmetries is 16×2^{-k} where k is the number of independent projectors P_{ij} that we impose.

Once we have a particular set of mutually commuting projectors P_{ij} we can return to the supersymmetry transformation (2.10). This now becomes

$$\sum_{c_{ijk}=0}[\Phi^i,\Phi^j]\Gamma^{ij}\epsilon=0, \tag{2.12}$$

where the sum is over pairs i,j such that $c_{ijk}=0$ for all k. The projectors can then be used to reduce this equation to a set of conditions on the commutators $[\Phi^i,\Phi^j]$ alone. As a consequence of the specific form for the c_{ijk} elements that we will use, one can show that Eq. (2.12) and the supersymmetry projectors (2.11) imply Eq. (2.8). To summarize then we must solve the Bogomol'nyi equation (2.6) and the commutator conditions that follow from Eq. (2.12).

III. D3-BRANES AND CALIBRATED GEOMETRY

In this section we wish to understand the spacetime origin of the projectors obtained in Eq. (2.11). Recall that in type IIB string theory there are two ten-dimensional supersymmetry generators ϵ_L and ϵ_R with the same chirality. The pres-

[2]Note that there may be more general solutions to the supersymmetry condition but we expect that these are related to those discussed here by a rotation.

ence of the D1-brane breaks half of the supersymmetry of the vacuum by imposing that $\Gamma^{09}\epsilon_L=\epsilon_R$. Using this relation the projection $\Gamma^{ijk9}\epsilon_L=\mp\epsilon_L$ can be written as $\Gamma^{0ijk}\epsilon_L=\pm\epsilon_R$. We immediately recognize this projector as due to the presence of a (anti-)D3-brane in the $0ijk$ plane. Therefore simultaneously imposing multiple projectors is equivalent to the presence of multiple intersecting D3-branes. A list of all possible orthogonal D3-brane intersections which preserve some fraction of supersymmetry can be found from the M-fivebrane intersections given in [4,5]. Combined with our intuition from [16] these observations suggest that solutions to the generalized Nahm equations (2.6) can be interpreted as the D1-branes opening up into a collection of intersecting D3-branes. That this is indeed the case will be shown in Sec. IV. However, before discussing explicit solutions it is instructive to consider the generalized Nahm equations and the D3-brane configurations they describe in more detail.

In general intersecting D3-branes which preserve some fraction of the spacetime supersymmetry can be thought of as a single D3-brane which is stretched over a three-manifold in the space in which the branes are embedded. The condition that supersymmetry is preserved is known to be equivalent to the statement that the manifold over which the D3-branes are stretched is a calibrated sub-manifold of the embedding space [2,4–6]. Here the embedding space will simply be $\mathbb{R}^n$ for $n=3,5,6,7$ and the calibrated three-manifolds which we will encounter are known as $\mathbb{R}^3$, Kähler, special Lagrangian and associative sub-manifolds. We recall that a calibration [1] is a closed p-form ω in the bulk space with the property that, for any tangent vector ξ to a p-dimensional sub-manifold, $P[\omega](\xi)\leqslant dvol(\xi)$, where $dvol$ is the induced volume form on the sub-manifold and $P[\omega]$ denotes the pull-back of ω to the worldvolume. A sub-manifold for which this inequality is saturated is said to be calibrated. These distinguished surfaces each represent the minimal volume (i.e. energy) elements of their respective homology classes.

One of the main observations of this paper is that the constants c_{ijk} [for which there are supersymmetric solutions to Eq. (2.6)] are exactly the non-vanishing components of the calibration forms associated with the corresponding three-manifold over which the D3-brane intersection is stretched. Put differently, the generalized Nahm equations describe D1-brane configurations which open up into D3-brane intersections that are stretched over a calibrated three-manifold whose calibration form ω is none other than the three-form $(1/3!)c_{ijk}dx^i\wedge dx^j\wedge dx^k$. Furthermore, the Bogomol'nyi bound (2.4) can be written as

$$E\geqslant T_1\lambda^2\int dx^9\mathrm{STr}P[i_\Phi i_\Phi\omega], \tag{3.1}$$

where i_Φ is the the non-Abelian interior product introduced in [19], STr is the symmeterized trace and P denotes the pull-back to the D1-brane by the non-Abelian scalars Φ^i. Thus the energy of the D1-brane is bounded below by the (non-Abelian) pull-back of the calibrating form to the D1-brane worldvolume. This is in complete analogy with the Abelian Bogomol'nyi bound on the D3-brane worldvolume [2,4–6] and provides a new interpretation of fuzzy funnels as non-Abelian calibrated three-surfaces. This is reminiscent of the notion of a generalized calibration given in [20], which included Abelian worldvolume gauge fields. In particular the S-dual configuration of a fundamental string ending on a Dp-brane is an example of such a (generalized) calibrated p surface.

In order to demonstrate our claims in concrete examples we now summarize the various cases with which we will be concerned. In what follows the c_{ijk} components with $+1$ are chosen to be so, whereas those equal to -1 are then fixed by supersymmetry. This change in sign corresponds to adding an anti-D3-brane. The simplest example is the one already discussed in [16] for which the only non-vanishing scalars are taken to be $\Phi^{1,2,3}$. Here the non-Abelian D1-branes open up into a single D3-brane which spans $\mathbb{R}^3$. The relevant D3-brane configuration, the non-vanishing components of c_{ijk}, the fraction ν of preserved supersymmetries on the D1-brane and the associated Bogomol'nyi equations (2.6) can be summarized as

$$\begin{aligned}
&D3{:}\,1\;\;2\;\;3\\
&D1{:}\qquad\qquad\;\;9\\
&c_{123}=1,\quad \nu=1/2\\
&\Phi^{1\prime}=[\Phi^2,\Phi^3],\quad \Phi^{2\prime}=[\Phi^3,\Phi^1],\quad \Phi^{3\prime}=[\Phi^1,\Phi^2].
\end{aligned} \tag{3.2}$$

Note that c_{ijk} is simply ε_{ijk}, which is indeed the volume form on $\mathbb{R}^3$, and the Bogomol'nyi equations are the standard Nahm equations describing BPS monopoles on $\mathbb{R}^3$ [15,16]. More interesting examples can be found by allowing for more scalars to be turned on. If we take $\Phi^{1,2,3,4,5}$ to be non-trivial then there exist supersymmetric solutions to the generalized Nahm equations describing the following configuration:

$$\begin{aligned}
&D3{:}\,1\;\;2\;\;3\\
&D3{:}\,1\quad 4\;\;5\\
&D1{:}\qquad\qquad\;\;9\\
&c_{123}=c_{145}=1\quad \nu=1/4\\
&\Phi^{1\prime}=[\Phi^2,\Phi^3]+[\Phi^4,\Phi^5],\\
&\Phi^{2\prime}=[\Phi^3,\Phi^1],\quad \Phi^{3\prime}=[\Phi^1,\Phi^2],\\
&\Phi^{4\prime}=[\Phi^5,\Phi^1],\quad \Phi^{5\prime}=[\Phi^1,\Phi^4],\\
&[\Phi^2,\Phi^4]=[\Phi^3,\Phi^5],\quad [\Phi^2,\Phi^5]=[\Phi^4,\Phi^3].
\end{aligned} \tag{3.3}$$

Supersymmetry tells us that such a D3-brane intersection should be stretched over $\mathbb{R}\times\mathcal{M}$ where $\mathbb{R}$ is the common direction and $\mathcal{M}$ is a complex curve. Such complex curves are calibrated by the Kähler form associated with a given complex structure on the four-manifold spanned by $\Phi^{2,3,4,5}$. Forming the complex pairs $Z^1=\Phi^2+i\Phi^3$ and $Z^2=\Phi^4$

$+i\Phi^5$ we see that $c_{ijk}dx^i\wedge dx^j\wedge dx^k$ in this case is nothing but the wedge product of dx^1 with the Kähler form associated with this complex structure.

There are two distinct ways to obtain configurations which preserve $\nu=1/8$ of the supersymmetry. The first is a straightforward generalization of the Kähler case above which is obtained by turning on $\Phi^{1,2,3,4,5,6,7}$. The corresponding D3-brane intersection and generalized Nahm equations are

$$
\begin{aligned}
&D3:\ 1\ 2\ 3\\
&D3:\ 1\quad\ 4\ 5\\
&D3:\ 1\quad\ \ 6\ 7\\
&D1:\quad\ \ 9\\
&c_{123}=c_{145}=c_{167}=1\quad \nu=1/8\\
&\Phi^{1\prime}=[\Phi^2,\Phi^3]+[\Phi^4,\Phi^5]+[\Phi^6,\Phi^7],\\
&\Phi^{2\prime}=[\Phi^3,\Phi^1],\quad \Phi^{3\prime}=[\Phi^1,\Phi^2],\\
&\Phi^{4\prime}=[\Phi^5,\Phi^1],\quad \Phi^{5\prime}=[\Phi^1,\Phi^4],\\
&\Phi^{6\prime}=[\Phi^7,\Phi^1],\quad \Phi^{7\prime}=[\Phi^1,\Phi^6],\\
&[\Phi^2,\Phi^4]=[\Phi^3,\Phi^5],\quad [\Phi^2,\Phi^5]=[\Phi^4,\Phi^3],\\
&[\Phi^2,\Phi^6]=[\Phi^3,\Phi^7],\quad [\Phi^2,\Phi^7]=[\Phi^6,\Phi^3],\\
&[\Phi^4,\Phi^6]=[\Phi^5,\Phi^7],\quad [\Phi^4,\Phi^7]=[\Phi^6,\Phi^5].
\end{aligned}
\tag{3.4}
$$

Once again this intersection should be stretched over $\mathbb{R}\times\mathcal{M}$ where now $\mathcal{M}$ is to be regarded as a complex curve embedded into six-dimensional Euclidean space. Forming complex pairs as $Z^1=\Phi^2+i\Phi^3$, $Z^2=\Phi^4+i\Phi^5$ and $Z^3=\Phi^6+i\Phi^7$ we see that $c_{ijk}dx^i\wedge dx^j\wedge dx^k$ in this case is again the wedge product of dx^1 with the Kähler form. A more interesting example which preserves the same amount of supersymmetry is provided by turning on $\Phi^{1,2,3,4,5,6}$ as follows:

$$
\begin{aligned}
&D3:\ 1\ 2\ 3\\
&D3:\quad 2\quad 4\quad 6\\
&D3:\quad\ \ 3\quad 5\ 6\\
&\bar{D}3:\ 1\quad\ 4\ 5\\
&D1:\quad\ \ 9\\
&c_{123}=c_{145}=c_{246}=-c_{356}=1,\quad \nu=1/8\\
&\Phi^{1\prime}=[\Phi^2,\Phi^3]+[\Phi^4,\Phi^5],\\
&\Phi^{2\prime}=[\Phi^3,\Phi^1]+[\Phi^4,\Phi^6],\\
&\Phi^{3\prime}=[\Phi^1,\Phi^2]+[\Phi^6,\Phi^5],\\
&\Phi^{4\prime}=[\Phi^5,\Phi^1]+[\Phi^6,\Phi^2],\\
&\Phi^{5\prime}=[\Phi^1,\Phi^4]+[\Phi^6,\Phi^3],\\
&\Phi^{6\prime}=[\Phi^2,\Phi^4]+[\Phi^5,\Phi^3],\\
&[\Phi^2,\Phi^5]+[\Phi^3,\Phi^4]+[\Phi^6,\Phi^1]=0.
\end{aligned}
\tag{3.5}
$$

Note that there is no direction shared by all of the D3-branes. Supersymmetry implies that this intersection should be stretched over a special Lagrangian three-manifold embedded into six-dimensional Euclidean space. Defining our complex coordinates to be $Z^1=\Phi^1+i\Phi^6$, $Z^2=\Phi^2+i\Phi^5$ and $Z^3=\Phi^3+i\Phi^4$ we may introduce the holomorphic three-form $\psi=dZ^1\wedge dZ^2\wedge dZ^3$. It is then straightforward to see that c_{ijk} represents the non-vanishing components of $\omega=Re(\psi)$ which is the calibration form for a special Lagrangian surface embedded into the Euclidean six-dimensional space endowed with the above complex structure [1].

The final example again involves the seven scalars $\Phi^{1,2,3,4,5,6,7}$ and leads to the following configuration:

$$
\begin{aligned}
&D3:\ 1\ 2\ 3\\
&\bar{D}3:\quad\ 3\ 4\quad\ \ 7\\
&\bar{D}3:\quad\ 3\quad 5\ 6\\
&D3:\ 1\quad\ \ \ 6\ 7\\
&D3:\ 1\quad\ 4\ 5\\
&D3:\quad 2\quad 4\quad 6\\
&\bar{D}3:\quad 2\quad\ 5\quad 7\\
&D1:\quad\ \ 9\\
&c_{123}=c_{145}=c_{167}=c_{246}=-c_{257}=-c_{347}=-c_{356}=1,\\
&\nu=1/16\\
&\Phi^{1\prime}=[\Phi^2,\Phi^3]+[\Phi^4,\Phi^5]+[\Phi^6,\Phi^7],\\
&\Phi^{2\prime}=[\Phi^3,\Phi^1]+[\Phi^4,\Phi^6]+[\Phi^7,\Phi^5],\\
&\Phi^{3\prime}=[\Phi^1,\Phi^2]+[\Phi^7,\Phi^4]+[\Phi^6,\Phi^5],\\
&\Phi^{4\prime}=[\Phi^5,\Phi^1]+[\Phi^6,\Phi^2]+[\Phi^3,\Phi^7],\\
&\Phi^{5\prime}=[\Phi^1,\Phi^4]+[\Phi^3,\Phi^6]+[\Phi^2,\Phi^7],\\
&\Phi^{6\prime}=[\Phi^2,\Phi^4]+[\Phi^5,\Phi^3]+[\Phi^7,\Phi^1],\\
&\Phi^{7\prime}=[\Phi^1,\Phi^6]+[\Phi^5,\Phi^2]+[\Phi^4,\Phi^3].
\end{aligned}
\tag{3.6}
$$

Here the components of the three-form c_{ijk} are precisely the non-zero components of the unique three-form in $\mathbb{R}^7$ which is invariant under the exceptional group G_2. These particular c_{ijk} can be identified with the octonionic structure

constants and the calibration form ω calibrates so-called associative three-surfaces in $\mathbb{R}^7$ [1].

In what follows we will refer to these various cases as $\mathbb{R}^3$, Kähler, special Lagrangian and associative, respectively. Before proceeding let us make several comments on the generalized Nahm equations.

First we note that these equations are not new but can be recognized as the dimensional reduction of a higher-dimensional self-duality condition

$$F_{IJ}=\frac{1}{2}t_{IJKL}F^{KL}, \tag{3.7}$$

where $\{x^I\}=\{x^i,x^9\}$ and $t_{9ijk}=c_{ijk}$, $t_{ijkl}=0$ [18]. In particular the generalized Nahm equations arise by assuming that the gauge field A_I depends only on x^9 and choosing the gauge $A_9=0$. These equations were previously analyzed in [21–26].

Next we note that in the $\mathbb{R}^3$ and associative cases there is no constraint equation since there are no pairs i,j such that $c_{ijk}=0$ for all k. Further, in these cases the c_{ijk} satisfy

$$c_{ijk}c_{lmk}=\delta_{il}\delta_{jm}-\delta_{jl}\delta_{im}+\gamma_{ijlm} \tag{3.8}$$

where γ_{ijlm} is antisymmetric in i,j,l,m. Hence Eq. (2.8) follows from the Jacobi identity.

As a final comment we note that in all of the above cases the D1-brane should appear as a monopole on the calibrated surface. Applying S-duality to our configurations merely has the effect of changing the D1-branes into fundamental strings. By definition a D3-brane is a suitable end point for a fundamental string, and the condition that it is calibrated implies that some supersymmetry is preserved. Therefore we expect that smooth solutions representing D1-branes, i.e., monopoles, on calibrated three-surfaces exist. Further, T-dualizing along the x^1,x^2,x^3 directions produces a configuration of intersecting D4-branes with a single D0-brane (corresponding to the first D3-brane). This system should therefore correspond to an instanton on a calibrated four-surface and are related to non-trivial solutions to Eq. (3.7). Indeed it was shown in [30] that supersymemtric wrapped D-branes in manifolds of special holonomy give rise to cohomological field theories whose equations of motion localize to solutions of Eq. (3.7). Starting with the above equations such a class of solutions is obtained by taking $\partial_{x^9}\Phi^i=0$. Hence the above equations all become constraints on the commutators. These solution may therefore be interpreted as supersymmetric D0-brane states and in particular we expect that the examples of Kähler calibrations are related to the construction of complex curves in (M)atrix theory [31].

IV. SOLUTIONS

In this section we will provide some explicit solutions to the generalized Nahm equations which can be interpreted as fuzzy funnels that open up into the D3-brane intersections discussed in the preceding section. Our focus here is to demonstrate that solutions to the generalized Nahm equations do in fact exist and describe D1-branes ending on supersymmetric D3-brane intersections. We therefore present only a set of very simple configurations which correspond to the D3-brane intersections at the origins of both the Higgs and Coulomb branches. We will argue that the generalized Nahm equations capture all of the physics of these intersections in the following section when we analyze deformations of the solutions presented here.[3]

For the purposes of orienting the reader we will begin by briefly reviewing the fuzzy funnel solutions to the basic Nahm equations appearing in Eq. (3.2) which were originally presented in [16]. Taking only $\Phi^{1,2,3}$ to be non-vanishing we make the ansatz

$$\Phi^i=f(x^9)\alpha^i, \tag{4.1}$$

where $[\alpha^i,\alpha^j]=2\varepsilon^{ijk}\alpha^k$ is an n-dimensional representation of $su(2)$. This is easily seen to solve the Nahm equations so long as $f'=2f^2$ which gives

$$\Phi^i=-\frac{1}{2}\frac{1}{x^9-a}\alpha^i, \tag{4.2}$$

where a is an arbitrary constant of integration. The profile of this solution is clearly that of a fuzzy, or $su(2)$ valued, funnel which opens up into a three-dimensional surface i.e., $\mathbb{R}^3$, as $x^9\to a$. At each finite value of x^9 the cross section of the funnel is a fuzzy two-sphere. The x^9-dependent radius of the funnel is given by

$$R(x^9)^2\equiv-\frac{\lambda^2}{n}\sum_{i=1}^{3}\mathrm{Tr}[\Phi^i(x^9)^2]=\frac{c_2\lambda^2}{4}\frac{1}{(x^9-a)^2} \tag{4.3}$$

where c_2 is the quadratic Casimir of the $su(2)$ representation. We will work with the n-dimensional irreducible representation for which we have $c_2=n^2-1$.

The energy of this configuration may be obtained by evaluating Eq. (2.3) on the solution presented in Eq. (4.2). We find

$$\begin{aligned}E&=\frac{T_1\lambda^2}{3}\int dx^9 c_{ijk}\mathrm{Tr}(\Phi^i\Phi^j\Phi^k)'\\&=T_3(1-1/n^2)^{-1/2}\int 4\pi R^2 dR,\end{aligned} \tag{4.4}$$

where we have used $T_1=T_3(2\pi l_s)^2$ and we have identified the physical radius of the funnel R with the radial coordinate in the space spanned by x^1,x^2,x^3. It is now clear that for large n the energy of the funnel configuration can be identified with the energy of a single, flat D3-brane sitting at $x^9=a$ filling the x^1,x^2,x^3 directions. To further support the claim that the funnel is indeed opening up into a D3-brane

[3]We note that a general family of solutions to the associative example were obtained in [21,22,24,25]. However, the fields were not in $u(N)$ and hence they cannot be readily embedded into the D1-brane effective action.

we note that the non-Abelian Wess-Zumino couplings identified in [19,28] can be evaluated as

$$\begin{aligned} i\lambda\mu_1 \int \mathrm{STr}P[i_\Phi i_\Phi C^{(4)}] &= \frac{i\lambda}{2}\mu_1 \int dt dx^9 C^{(4)}_{tkji}\mathrm{Tr}(\Phi^{k\prime}[\Phi^i,\Phi^j]) \\ &= i\mu_3(1-1/n^2)^{-1/2}\int 4\pi R^2 dR C_{0123}, \end{aligned} \tag{4.5}$$

indicating that the funnel is acting as a source for precisely the correct Ramond-Ramond four-form field to be identified as a D3-brane.

We will now discuss solutions to the generalized Nahm equations. It will be shown below that the Kähler and special Lagrangian cases can all be obtained as special cases of the associative example. Thus we will begin our analysis with the configuration in Eq. (3.6). The generalized Nahm equations in Eq. (3.6) can be solved by taking

$$\Phi^i = f(x^9)A^i, \tag{4.6}$$

where A^i are a set of $N\times N$ constant matrices which satisfy $\frac{1}{4}c_{ijk}[A^j,A^k]=A^i$. A solution is given by

$$\begin{aligned} A^1 &= \mathrm{diag}(\alpha^1,0,0,\alpha^1,\alpha^1,0,0) \\ A^2 &= \mathrm{diag}(\alpha^2,0,0,0,0,\alpha^1,\alpha^1) \\ A^3 &= \mathrm{diag}(\alpha^3,\alpha^1,\alpha^1,0,0,0,0) \\ A^4 &= \mathrm{diag}(0,\alpha^3,0,0,\alpha^2,\alpha^2,0) \\ A^5 &= \mathrm{diag}(0,0,\alpha^3,0,\alpha^3,0,\alpha^3) \\ A^6 &= \mathrm{diag}(0,0,\alpha^2,\alpha^2,0,\alpha^3,0) \\ A^7 &= \mathrm{diag}(0,\alpha^2,0,\alpha^3,0,0,\alpha^2), \end{aligned} \tag{4.7}$$

where α^a satisfy $[\alpha^a,\alpha^b]=2\varepsilon^{abc}\alpha^c$ are now regarded as an n-dimensional representation of $su(2)$ so that $N=7n$. Notice that each diagonal block contains exactly one copy of the $su(2)$ generators. Substituting this ansatz into the generalized Nahm equation leads to $f'=2f^2$. Hence we again find funnel-like solutions of the form

$$\Phi^i = -\frac{1}{2}\frac{1}{x^9-a}A^i. \tag{4.8}$$

Analogous block diagonal solutions to the higher-dimensional self-duality equation (3.7) have previously been constructed [27]. These have the interpretation of four-dimensional instantons embedded onto calibrated four-surfaces in $\mathbb{R}^8$.

It is easy to see that

$$\begin{aligned} (\Phi^1)^2+\ldots+(\Phi^7)^2 &= -\frac{c_2}{4}\frac{1}{(x^9-a)^2}\mathrm{diag}(1,1,1,1,1,1,1), \end{aligned} \tag{4.9}$$

where $c_2=-(1/n)\mathrm{Tr}[(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2]$ is the Casimir invariant of $su(2)$ in our n-dimensional representation. In this case the D1-branes should not be thought of as expanding into a fuzzy six sphere.[4] To see this recall that each block in Eq. (4.7) contains exactly one complete set of the $su(2)$ generators. As a result each block gives rise to a single copy of the fuzzy funnel described above. This configuration should therefore be viewed as expanding into seven intersecting fuzzy two spheres each of which, as $x^9\to a$, become one of the D3-branes making up the intersection of Eq. (3.6).

In fact the ansatz made in Eq. (4.6) can easily be generalized to include a different profile for each diagonal block. This leads to seven independent integration constants and therefore corresponds to orthogonal D3-branes which are separated in the x^9 direction. One may also trivially add an identity matrix to each diagonal block. This corresponds to separating the fuzzy funnels in directions orthogonal to the D1-branes. In this section we will restrict ourselves to discussing the case where all of the D3-branes are located at the same position in x^9 as well as in the transverse directions—see the following section for more details.

Under these assumptions we may take

$$R^2(x^9) = \frac{c_2\lambda^2}{4}\frac{1}{(x^9-a)^2} \tag{4.10}$$

to be the x^9-dependent radius of each of the fuzzy funnels. The energy of this configuration is then easily evaluated to be

$$\begin{aligned} E &= \frac{T_1\lambda^2}{3}\int dx^9 c_{ijk}\mathrm{Tr}(\Phi^i\Phi^j\Phi^k)' \\ &= 7T_3(1-1/n^2)^{-1/2}\int 4\pi R^2 dR, \end{aligned} \tag{4.11}$$

which is precisely the energy one expects from the D3-brane intersection in Eq. (3.6). Finally we may verify our interpretation of the funnel by examining the induced RR couplings

$$\begin{aligned} i\lambda\mu_1\int \mathrm{STr}P[i_\Phi i_\Phi C^{(4)}] &= \frac{i\lambda}{2}\mu_1\int dt dx^9 C^{(4)}_{tkji}\mathrm{Tr}(\Phi^{k\prime}[\Phi^i,\Phi^j]) \\ &= i\mu_3(1-1/n^2)^{-1/2}\int 4\pi R^2 dR(C_{0123}+C_{0145} \\ &\quad +C_{0167}+C_{0246}-C_{0257}-C_{0347}-C_{0356}) \end{aligned} \tag{4.12}$$

[4]We would like to thank R. Myers for a discussion on this point.

which indicates that this non-Abelian embedding of D1-branes is a source for precisely the correct Ramond-Ramond fields to be identified with the configuration (3.6).

In order to find similar solutions to the other generalized Nahm equations we observe that all of the equations (and constraints) presented in Eqs. (3.3), (3.4) and (3.5) can be viewed as special cases of the final, associative case, Eq. (3.6). In particular the special Lagrangian case (3.5) follows by setting $\Phi^7=0$ and the constraint arises from the Φ^7 equation in Eq. (3.6). A solution to these equations is then found by simply deleting the second, fourth and seventh blocks from each of the matrices in Eq. (4.7). We find $\Phi^i = f(x^9)A^i$ where

$$\begin{aligned} A^1&=\mathrm{diag}(\alpha^1,0,\alpha^1,0)\\ A^2&=\mathrm{diag}(\alpha^2,0,0,\alpha^1)\\ A^3&=\mathrm{diag}(\alpha^3,\alpha^1,0,0)\\ A^4&=\mathrm{diag}(0,0,\alpha^2,\alpha^2)\\ A^5&=\mathrm{diag}(0,\alpha^3,\alpha^3,0)\\ A^6&=\mathrm{diag}(0,\alpha^2,0,\alpha^3). \end{aligned} \tag{4.13}$$

Assuming the same profile for each complete set of $su(2)$ generators this solution corresponds to a collection of six intersecting fuzzy funnels which open into the D3-brane intersection described in Eq. (3.5). On the other hand, by imposing the constraint relations in Eq. (3.4) directly on the equations in Eqs. (3.6) we obtain the generalized Nahm equations corresponding to the Kähler case in Eq. (3.4) which have the simple solutions $\Phi^i=f(x^9)A^i$ with

$$\begin{aligned} A^1&=\mathrm{diag}(\alpha^1,\alpha^1,\alpha^1)\\ A^2&=\mathrm{diag}(\alpha^2,0,0)\\ A^3&=\mathrm{diag}(\alpha^3,0,0)\\ A^4&=\mathrm{diag}(0,\alpha^2,0)\\ A^5&=\mathrm{diag}(0,\alpha^3,0)\\ A^6&=\mathrm{diag}(0,0,\alpha^2)\\ A^7&=\mathrm{diag}(0,0,\alpha^3), \end{aligned} \tag{4.14}$$

representing three intersecting fuzzy funnels which this time open into the D3-brane intersection of Eq. (3.4).

Lastly by setting $\Phi^6=\Phi^7=0$ in Eq. (3.4) we obtain the equations and constraints of the Kähler intersection in Eq. (3.3). These are solved by truncating the matrices in Eq. (4.14) in the obvious way to give $\Phi^i=f(x^9)A^i$ where

$$\begin{aligned} A^1&=\mathrm{diag}(\alpha^1,\alpha^1)\\ A^2&=\mathrm{diag}(\alpha^2,0)\\ A^3&=\mathrm{diag}(\alpha^3,0)\\ A^4&=\mathrm{diag}(0,\alpha^2)\\ A^5&=\mathrm{diag}(0,\alpha^3). \end{aligned} \tag{4.15}$$

As a final comment on all of the solutions presented in this section we note that as the scalars Φ^i are becoming large and quickly varying near $x^9=a$ the Yang-Mills approximation in which we are working is strictly no longer valid. On the other hand, for $x^9 \gg a$ we should be able to trust the solution. As discussed in detail in [16,17] this is the opposite range of approximation that is valid on the worldvolume theory of the D3-branes that the D1-branes are intersecting. Indeed the picture presented here accurately describes the region near the intersection.

V. MODULI

In the previous section we constructed explicit supersymmetric fuzzy funnel solitons. However, these solutions are somewhat trivial since the various components of the scalar fields which describe the different D3-branes commute with each other. Indeed these solutions simply represent several distinct fuzzy funnels, each interpreted as a group of n D1-branes ending on a single D3-brane, embedded into a suitably large matrix. Therefore we would like to obtain more complicated solutions. Unfortunately, finding explicit new solutions is, in general, a difficult task. Here we will perform an analysis of the linearized modes about solutions of the form (4.8). In what follows we will establish the existence of a large number of moduli which lead to non-trivial solutions, at least at the linearized level.

It is convenient to rewrite the solution in the form

$$\Phi^i(x^9)=-\frac{1}{2x^9}A^i_a T^a. \tag{5.1}$$

Here we have placed the D3-branes at $x^9=0$ and T^a, $a=0,1,\ldots,N^2-1$ are the generators of $u(N)$; $[T^a,T^b]=f^{abc}T^c$ with T^0 taken to be the generator of the overall Abelian $u(1)$ in $u(N)=u(1)\oplus su(n)$. In other words we have used the fact that we must embed our matrices into $u(N)$ to express A^i as a linear combination of the generators T^a. Note that since Φ^i solves the Bogomol'nyi equation we can deduce that

$$A^i_a=\frac{1}{4}c_{ijk}f^{abc}A^j_b A^k_c. \tag{5.2}$$

In addition the f^{abc} satisfy the Jacobi identity $f^{abe}f^{cde}+f^{cbe}f^{dae}+f^{dbe}f^{ace}=0$.

A linearized perturbation can also be written as $\delta\Phi^i=\varphi^i_a T^a$ and therefore satisfies the equation

$$x^9 \frac{d\varphi_a^i}{dx^9} = -\frac{1}{2} c_{ijk} f^{abc} A_b^j \varphi_c^k . \tag{5.3}$$

For most examples we will also have to worry about the constraint equation (2.12) but here we will only consider the $\mathbb{R}^3$ fuzzy funnel (3.2) and the associative case (3.6) for which the constraint is absent. Since c_{ijk}, f^{abc} and A_b^j are known constant tensors we have a linear equation for the perturbations φ_a^i. Furthermore, since $(T^a)^\dagger = -T^a$ and $c_{ijk} f^{abc} A_b^j = c_{kji} f^{cba} A_b^j$, the matrix on the right-hand side is real and symmetric. It follows that there is a basis of solutions with eigenvalues λ_a^i and hence there are DN^2 zero modes

$$\varphi_a^i = \epsilon_a^i (x^9)^{\lambda_a^i}, \tag{5.4}$$

where ϵ_a^i is a small parameter. Note that the perturbation is only valid if the second order term in the variation (there are no higher order terms) is small compared to the linear one. This corresponds to $\epsilon_a^i \ll (x^9)^{-\lambda_a^i - 1}$. So any given perturbation is not valid over the entire funnel but is always valid over some region. Strictly speaking we may only trust the Yang-Mills approximation when the derivatives of the gauge fields are small, corresponding to the region $x^9 \to \infty$. Therefore we should only trust zero modes with $\lambda_a^i \leqslant -1$, although we will see exceptions to this. We would also like to see if some of the moduli are associated with the D1-brane itself and if some in fact reflect moduli of the intersecting D3-branes. We expect that zero modes which do not vanish as $x^9 \to \infty$ correspond to moduli of the D1-branes whereas those that do vanish correspond to the geometry of the D3-branes.

Let us start by identifying some obvious zero modes. Firstly we can separate out D center of mass coordinates along each of the transverse directions x^i. This corresponds simply to taking φ^i to be constant and proportional to T^0 (i.e. $\lambda = 0$). Next we can identify the translation mode $\varphi_a^i = \epsilon \Phi^i{}'_a = \frac{1}{2} \epsilon A_a^i / (x^9)^2$ (i.e. $\lambda = -2$). That this solves Eq. (5.3) follows from the identity (5.2). This mode therefore represents the location of the D3-brane along the x^9 direction. Finally it is clear that we can act by gauge transformations on the original solution; $\Phi^i \to g \Phi^i g^{-1}$, $g \in SU(N)$. Expanding $g = e^{h_a T^a}$ with $h_a \ll 1$ leads to $N^2 - 1$ zero modes $\varphi_a^i = -\frac{1}{2} f^{abc} h_b A_c^i / x^9$ (i.e. $\lambda = -1$). Using the Jacobi identity and Eq. (5.2) one sees that this indeed solves Eq. (5.3).

A. $\mathbb{R}^3$

Before analyzing the associative case it is instructive to analyze the simplest case of the original fuzzy funnel with $D=3$, $N=2$, $c_{ijk} = \varepsilon_{ijk}$ and $f^{abc} = 2\varepsilon^{abc}$ if $a,b,c \neq 0$, $f^{0bc} = 0$. Thus there are 12 zero modes. The solution is given in Eq. (4.1) and has $A^i = \alpha^i$ so that $A_a^i = \delta_a^i$. As mentioned above we can identify 3 zero modes $\varphi^i = c^i T^0$ corresponding to the constant center of mass coordinates. The rest of the the zero modes, namely those that are in $su(2)$, are represented by square 3×3 matrices φ_j^i which satisfy

$$x^9 \frac{d\varphi_j^i}{dx^9} = -(\delta_i^j \delta_k^l - \delta_k^j \delta_i^l) \varphi_l^k . \tag{5.5}$$

We can split up φ_j^i into its trace, antisymmetric and symmetric-traceless parts. The trace corresponds to $\varphi_j^i \sim \delta_j^i$ and is just the translational zero mode given above. The antisymmetric part can be written as $\varphi_j^i \sim \epsilon^{ijk} h_k$ and can be seen to correspond to the gauge transformations.

The symmetric part is a little more interesting. It gives five zero modes. There are diagonal choices of the form

$$\varphi_j^i \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{5.6}$$

Note that although there are three such modes only two are linearly independent. The corresponding eigenvalue is $\lambda = 1$ so that $\varphi \sim x^9$. In fact we can find the exact solution corresponding to this deformation. In particular consider solutions of the form $\Phi^i = f_i(x^9) \alpha^i$. The Bogomol'nyi equation gives

$$f_1{}' = 2 f_2 f_3, \quad f_2{}' = 2 f_3 f_1, \quad f_3{}' = 2 f_1 f_2 . \tag{5.7}$$

A one parameter solution to these equations is [29]

$$f_1 = -\frac{d}{2} \frac{\mathrm{cn}(dx^9)}{\mathrm{sn}(dx^9)}, \quad f_2 = -\frac{d}{2} \frac{\mathrm{dn}(dx^9)}{\mathrm{sn}(dx^9)}, \quad f_3 = -\frac{d}{2} \frac{1}{\mathrm{sn}(dx^9)}, \tag{5.8}$$

where sn, cn and dn are Jacobi's elliptic functions with parameter k. Expanding in powers of d one finds that Eq. (5.8) corresponds to a linear combination of the zero modes in Eq. (5.6). Clearly one other linearly independent solution can be found in an analogous manner. These functions are generically periodic in x^9 and hence have other poles, indeed they arise in the Nahm construction of charge-two monopoles [29]. Since we are primarily interested in solutions which are well-behaved as $x^9 \to \infty$ we must restrict to the case $k=1$ where the period diverges and the solution becomes

$$f_1 = f_2 = -\frac{d}{2} \frac{1}{\sinh(dx^9)}, \quad f_3 = -\frac{d}{2} \frac{\cosh(dx^9)}{\sinh(dx^9)}. \tag{5.9}$$

Thus as $x^9 \to \infty$, Φ^3 tends to the constant matrix $-(d/2)\alpha^3$, while $\Phi^1, \Phi^2 \to 0$. Clearly the three modes in Eq. (5.6) are interpreted as separating the D1-branes along the x^i axis at infinity.

This leaves three zero modes of the form

$$\varphi^i_j \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.10}$$

These also correspond to eigenvalue $\lambda=1$. Let us again look for the corresponding exact solutions of the Nahm equation. In particular we consider the first zero mode in Eq. (5.10) and let

$$\Phi^1 = g_1(x^9)\alpha^1 + g_2(x^9)\alpha^2, \quad \Phi^2 = g_1(x^9)\alpha^2 + g_2(x^9)\alpha^1, \quad \Phi^3 = g_3(x^9)\alpha^3. \tag{5.11}$$

This leads to the equations $g_1' = 2g_1g_3$, $g_2' = -2g_2g_3$ and $g_3' = 2(g_1^2 - g_2^2)$. The exact solution is provided by the one parameter family

$$g_1 = -\frac{b}{4}\frac{\mathrm{dn}(bx^9)+\mathrm{cn}(bx^9)}{\mathrm{sn}(bx^9)}, \quad g_2 = -\frac{b(1-k^2)}{4}\frac{\mathrm{sn}(bx^9)}{\mathrm{dn}(bx^9)+\mathrm{cn}(bx^9)}, \quad g_3 = -\frac{b}{2}\frac{1}{\mathrm{sn}(bx^9)}. \tag{5.12}$$

If we expand Eq. (5.12) around $b=0$ with $k\neq 1$ then we reproduce the first zero mode in Eq. (5.10) but mixed with the first zero mode in Eq. (5.6). Nevertheless these solutions, along with similar solutions where the ansatz for Φ^1, Φ^2 and Φ^3 are permuted, provide three more linearly independent zero modes. Again these solutions are generically periodic in x^9. When $k=1$ the period is infinite and we in fact find the solution (5.9) again.

In summary we have found 12 zero modes. Three of these correspond to gauge transformations and can be discarded. One of these corresponds to translations of the D3-brane in the x^9 direction. This leaves 8 linearly independent zero modes. To understand these we can consider the case of two D1-branes suspended between two parallel D3-branes separated by a distance v. In this case we must take Φ^i to have poles at $x^9=0$ and $x^9=v$. These boundary conditions are an important part of the Nahm constuction and were derived from string theory in [32,33]. This coincides with Nahm's construction of charge two $SU(2)$ monopoles on $\mathbb{R}^3$ [14] (for a helpful and more recent discussion see [34]). Our counting of 8 physical zero modes then agrees with the dimension of the moduli space of charge two $SU(2)$ monopoles. In particular one can see following the discussion in [34] that Eqs. (5.8) and (5.12) lead to monopoles which are separated along the x^1 and $x^1=x^2$, $x^3=0$ axis', respectively.

B. Associative

Next we consider the associative example with $D=7$, $N=14$. Thus there are some 1372 zero modes. The large number of zero modes is related to the large N that we have used to embed our solution and most zero modes do not reflect any physics but rather the possible embeddings. To proceed it is helpful to split the moduli space into the form

$$\mathcal{M} = \mathcal{M}_G \times \mathcal{M}_{\mathbb{R}^3} \times \tilde{\mathcal{M}}, \tag{5.13}$$

where $\mathcal{M}_G$ consists of the $14^2-1=195$ zero modes corresponding to gauge transformations and do not represent any physical degrees of freedom.

The next part of the moduli space, $\mathcal{M}_{\mathbb{R}^3}$, is obtained by embedding the zero modes of a single $\mathbb{R}^3$ fuzzy funnel into $u(14)$. The 9 $su(2)$ zero modes φ^i_j we found in Sec. V A can be embedded separately into each of the seven block diagonal entries of Eq. (4.7). However, 3 of these are gauge transformations and have already been included $\mathcal{M}_G$. Thus the $su(2)$ zero modes give an additional 42 moduli. In addition we may also embed the $u(1)$ center of mass zero modes that we discussed above into $u(14)$. In particular if we let

$$\varphi^i \sim i\,\mathrm{diag}(c^i_1, c^i_2, \ldots, c^i_7), \tag{5.14}$$

where c^i_j are real, then clearly $[\varphi^i, A^j]=0$ and so Eq. (5.3) is solved if φ^i is a constant, i.e. $\lambda=0$. This gives another 49 zero modes which represent the center of mass of the D1-branes along each of the directions and each of the D3-branes. In total there are 91 linearly independent moduli in $\mathcal{M}_{\mathbb{R}^3}$. Clearly none of these zero modes lead to interesting new solutions. Instead they affect one of the component fuzzy funnels but leave the others invariant. In particular they do not lead to a mixing or interaction between the various off diagonal blocks. These moduli simply represent the relative positions of the D1-branes and D3-branes.

We are, however, still left with 1086 zero modes and these form the rest of the moduli space $\tilde{\mathcal{M}}$. To count the number of physically distinct moduli we note that if g has the form $g=\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_7})$, $\theta_1+\cdots+\theta_7=0$ then $gA^ig^{-1}=A^i$. Thus if φ is a zero mode then so is $g\varphi g^{-1}$ (with the same value of λ). Hence $\tilde{\mathcal{M}}$ has a residual gauge symmetry $U(1)^6$ and the physical part of the moduli space in fact has a quotient form $\mathcal{N}=\tilde{\mathcal{M}}/U(1)^6$. Note that $\mathcal{M}_G$ is mapped to itself under $U(1)^6$ and $\mathcal{M}_{\mathbb{R}^3}$ is invariant. However, we do not know what the fixed points of $U(1)^6$ in $\tilde{\mathcal{M}}$ are; for example, acting with $U(1)^6$ need not always produce as many as 6 new linearly independent zero modes. Therefore we are unable to determine the dimension of $\mathcal{N}$, although it must be *at least* as big as $1086/7 \approx 156$. These zero modes cannot be interpreted as moduli that affect only one of the component fuzzy funnels, i.e. the corresponding solutions are not simply 7 independent fuzzy funnels but rather involve off-diagonal blocks and represent deformations which smooth out the D3-brane intersection.

We should now compare this with the number of zero modes one expects from such a D1/D3-brane intersection. From the intersecting D3-branes there are $4\times 21=84$ scalar modes from the hypermultiplets arising from D3-D3 strings with 4 DN directions. There are also $7\times 6=42$ zero-modes from transverse scalar fields of the D3-branes. However, 7 of these moduli are simply the translational zero modes of each D3-brane along x^9 and have already been included in $\mathcal{M}_{\mathbb{R}^3}$.

Another 7 represent the locations of the branes along x^8 and these have been frozen out from the whole system. In addition any pair of D3-branes that preserve 1/8 of the spacetime supersymmetry may be rotated by a two parameter family of angles in $SU(3) \subset SO(6)$. These contribute $2 \times 21 = 42$ zero modes and hence the total number of D3-brane intersection moduli is 154. These should be viewed as the moduli of the calibrated surface on which the D3-branes are wrapped. Moreover, in addition to the surface moduli we expect that there will be relative $U(1)$ phases of the 7 distinct component monopoles, as well as possible Wilson line moduli.

Unfortunately a more complete analysis of the $\mathcal{N}$ moduli space is beyond the scope of this paper. For example, we have not shown that they all lift to full solutions. It is also important to understand the correct boundary conditions which are crucial in the standard Nahm construction. However, we hope to have convinced the reader that the generalized Nahm equations do admit non-trivial solutions whose zero modes are in a one-to-one correspondence with the moduli space of D1-branes ending on intersecting D3-branes. Recall that from the spacetime point of view these configurations appear as monopoles on calibrated three-surfaces. Hence the moduli space of solutions to the generalized Nahm equations is naturally identified with the moduli space of monopoles on calibrated three-surfaces, including the moduli of the surface itself.

VI. SUMMARY AND COMMENTS

In this paper we have analyzed the general condition for static supersymmetric and non-Abelian embeddings of D1-branes into spacetime. The Bogomol'nyi equations have the form of generalized Nahm equations and the resulting embedding has the interpretation as a non-Abelian calibrated surface. On the worldvolume theory the solutions correspond to D1-branes which open up into non-commutative spheres (i.e. fuzzy funnels). From the spacetime point of view the D1-branes end on configurations of intersecting D3-branes. Although the exact fuzzy funnel solutions we presented in Sec. IV were block diagonal, and so merely represent several distinct fuzzy funnels, in Sec. V we established the existence of a large moduli space of non-trivial solutions at the linearized level.

In the case of the original fuzzy funnel the appearance of the Nahm equation is not surprising. Soon after the discovery of D-branes it was realized that a D1-brane suspended between two parallel D3-branes appears as a monopole on the D3-brane worldvolume. Furthermore, introducing a probe brane provides an explicit realization of the Nahm construction of monopoles [15]. Here we have primarily examined solutions which live on a half-line. In the Nahm construction these correspond to unphysical $U(1)$ monopoles with infinite mass. From the D-brane perspective this infinite mass is simply due to the infinite length of the D1-brane. However, we also discussed the case of parallel D3-branes, or parallel sets of intersecting D3-branes, and obtained solutions which correspond to finite BPS monopoles.

Therefore we expect that there is an associated role for these generalized Nahm equations here. The natural interpretation of such a system is that it encodes the data for BPS monopoles on the calibrated three-surface. In addition we have argued that the geometrical data of the calibrated surface should also be so encoded. Unfortunately it is not clear to us what the recipe is for constructing the monopole fields, or D3-brane geometry, from solutions to the generalized Nahm equations. In [15] the configuration (3.2) was T-dualized into D5-branes ending on a D7-brane. A D1-brane probe analysis then shows how the monopole fields can be reconstructed from the solution and one recovers the Nahm construction [14]. However, in our case we are limited by the fact that the equivalent analysis requires more than ten dimensions where there is no appropriate supersymmetric Yang-Mills theory.

Finally, the configurations that we have obtained in this paper are solutions to the super-Yang-Mills approximation to the full non-Abelian Born-Infeld action. As our solutions are supersymmetric we expect that they will lift to solutions of the full theory although they may receive corrections from higher orders in the non-Abelian field strength. Indeed, the self-duality equations (3.7), from which the generalized Nahm equations can be obtained by dimensional reduction, are known [35] to be corrected at higher orders in $\lambda = 2\pi l_s^2$. It would be very interesting to determine whether the fuzzy funnels presented here also receive such corrections.

ACKNOWLEDGMENTS

We would like to thank B. Acharya, S. Cerkis, D.-E. Diaconescu, D. Fairlie, C. Hofman, C. Houghton, P. Sutcliffe, W. Taylor, D. Tong and J. Troost for useful discussions. We would especially like to thank R. Myers for his initial collaboration on this project. The research of N.R.C. is supported by the NSF under grant PHY 00-96515, the DOE under grant DF-FC02-94ER40818 and NSERC of Canada. NDL would like to thank the Aspen Institute for Physics, McGill University and the Perimeter Institute for their hospitality during the course of this work and was partially supported by the PPARC at King's College London.

[1] R. Harvey and H.B. Lawson, Jr., Acta Math. **148**, 47 (1982).
[2] K. Becker, M. Becker, and A. Strominger, Nucl. Phys. **B456**, 130 (1995).
[3] K. Becker, M. Becker, D.R. Morrison, H. Ooguri, Y. Oz, and Z. Yin, Nucl. Phys. **B480**, 225 (1996).
[4] G.W. Gibbons and G. Papadopoulos, Commun. Math. Phys. **202**, 593 (1999).
[5] J.P. Gauntlett, N.D. Lambert, and P.C. West, Commun. Math. Phys. **202**, 571 (1999).
[6] B.S. Acharya, J.M. Figueroa-O'Farrill, and B. Spence, J. High Energy Phys. **04**, 012 (1998).
[7] E. Witten, Nucl. Phys. **B460**, 335 (1996).

[8] E.A. Bergshoeff, M. de Roo, and A. Sevrin, J. Math. Phys. **42**, 2872 (2001).
[9] E.A. Bergshoeff, A. Bilal, M. de Roo, and A. Sevrin, J. High Energy Phys. **07**, 029 (2001).
[10] J.M. Drummond, P.S. Howe, and U. Lindstrom, ''Kappa-symmetric non-Abelian Born-Infeld actions in three dimensions,'' hep-th/0206148.
[11] C.G. Callan and J.M. Maldacena, Nucl. Phys. **B513**, 198 (1998).
[12] P.S. Howe, N.D. Lambert, and P.C. West, Nucl. Phys. **B515**, 203 (1998).
[13] G.W. Gibbons, Nucl. Phys. **B514**, 603 (1998).
[14] W. Nahm, Phys. Lett. **90B**, 413 (1980); in *Monopoles in Quantum Field Theory*, edited by N.S. Craigie *et al.* (World Scientific, Singapore, 1982).
[15] D. Diaconescu, Nucl. Phys. **B503**, 220 (1997).
[16] N.R. Constable, R.C. Myers, and Ø. Tafjord, Phys. Rev. D **61**, 106009 (2000).
[17] N.R. Constable, R.C. Myers, and Ø. Tafjord, J. High Energy Phys. **06**, 023 (2001); ''Fuzzy funnels: Non-abelian brane intersections,'' hep-th/0105035.
[18] E. Corrigan, C. Devchand, D.B. Fairlie, and J. Nuyts, Nucl. Phys. **B214**, 452 (1983).
[19] R.C. Myers, J. High Energy Phys. **12**, 022 (1999).
[20] O. Barwald, N.D. Lambert, and P.C. West, Phys. Lett. B **463**, 33 (1999).
[21] D.B. Fairlie and T. Ueno, ''Higher-dimensional generalisations of the Euler top equations,'' hep-th/9710079; J. Phys. A **31**, 7785 (1998).
[22] T. Ueno, Phys. Lett. A **245**, 373 (1998).
[23] T.A. Ivanova, in *Tensor and Vector Analysis*, edited by A.T. Fomenko *et al.* (Taylor & Francis, London, 1998), pp. 57–70.
[24] L. Baker and D. Fairlie, J. Math. Phys. **40**, 2539 (1999).
[25] K. Sfetsos, Nucl. Phys. **B629**, 417 (2002).
[26] D. Bak, K. Lee, and J.H. Park, Phys. Rev. D **66**, 025021 (2002).
[27] B.S. Acharya (private communication).
[28] W. Taylor and M. Van Raamsdonk, Nucl. Phys. **B573**, 703 (2000).
[29] S.A. Brown, H. Panagopoulos, and M.K. Prasad, Phys. Rev. D **26**, 854 (1982).
[30] B.S. Acharya, J.M. Figueroa-O'Farrill, B. Spence, and M. O'Loughlin, Nucl. Phys. **B514**, 583 (1998).
[31] L. Cornalba and W. Taylor, Nucl. Phys. **B536**, 513 (1998).
[32] A. Kapustin and S. Sethi, Adv. Theor. Math. Phys. **2**, 571 (1998).
[33] D. Tsimpis, Phys. Lett. B **433**, 287 (1998).
[34] P.M. Sutcliffe, Int. J. Mod. Phys. A **12**, 4663 (1997).
[35] P. Koerber and A. Sevrin, J. High Energy Phys. **10**, 003 (2001).