Counting domain walls in $\mathcal{N}=1$ super Yang-Mills theory

Adam Ritz

Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, United Kingdom

Mikhail Shifman and Arkady Vainshtein

Theoretical Physics Institute, University of Minnesota, 116 Church St. SE, Minneapolis, Minnesota 55455
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We study the multiplicity of BPS domain walls in $\mathcal{N}=1$ super Yang-Mills theory, by passing to a weakly coupled Higgs phase through the addition of fundamental matter. The number of domain walls connecting two specified vacuum states is then determined via the Witten index of the induced world volume theory, which is invariant under the deformation to the Higgs phase. The world volume theory is a sigma model with a Grassmanian target space which arises as the coset associated with the global symmetries broken by the wall solution. Imposing a suitable infrared regulator, the result is found to agree with recent work of Acharya and Vafa in which the walls were realized as wrapped $D4$-branes in type IIA string theory.

I. INTRODUCTION

It is well known that $\mathcal{N}=1$ super Yang-Mills (SYM) theory with a gauge group G exhibits h distinct vacua where h, the dual Coxeter number of the group, is equal to the adjoint Casimir invariant T_G. This vacuum structure results from the spontaneous breakdown $\mathbb{Z}_{2T_G} \to \mathbb{Z}_2$ of the discrete $\mathbb{Z}_{2T_G}$ remnant of the anomalous $U(1)_R$ symmetry. As initially suggested by Witten [1], the relevant order parameter is the gluino condensate $\langle \lambda^2 \rangle \equiv \langle 0 | \mathrm{Tr} \lambda^2 | 0 \rangle$, first demonstrated to be nonzero within the framework of the Veneziano-Yankielowicz effective Lagrangian [2]. Subsequently, the exact calculation of this condensate [3–5] was performed via a controlled deformation to weak coupling. The result takes the form

$$\langle \lambda^2 \rangle_k = \frac{16\pi^2}{g^2} M_{\mathrm{PV}}^3 \exp\left(\frac{2\pi i(\tau + k)}{T_G} \right) = 3 T_G \Lambda^3 \exp\left(\frac{2\pi i k}{T_G} \right), \tag{1}$$

where $\tau = 4\pi i / g^2 + \theta/2\pi$ is the complex gauge coupling, M_{PV} is the Pauli-Villars regulator scale, and the subscript $k = 0, \ldots, T_G - 1$ marks the k-th vacuum. The second equality defines the renormalization group invariant dynamical scale Λ.

The presence of discrete vacuum states implies the existence of topologically stable domain walls interpolating between them. Moreover, one expects Bogomol'nyi-Prasad-Sommerfield (BPS) saturated domain walls to exist since the $\mathcal{N}=1$ supersymmetry algebra contains a central charge $\mathcal{Z}$ whose operator form is [6–8]

$$\mathcal{Z} = \frac{T_G}{8\pi^2} \int dz \frac{\partial}{\partial z} \lambda^2, \tag{2}$$

where z is the spatial coordinate perpendicular to the wall plane. The expectation value of the central charge is nonva-

nishing for states in the sector $|kn\rangle$, corresponding to domain wall configurations interpolating between the nth and $(k+n)$th vacua:

$$\mathcal{Z}_{kn} = \langle kn | \mathcal{Z} | kn \rangle = \frac{T_G}{8\pi^2} (\langle \lambda^2 \rangle_{k+n} - \langle \lambda^2 \rangle_n). \tag{3}$$

The absolute value of $\mathcal{Z}_{kn}$ provides a lower bound for the mass per unit area (i.e. the wall tension) of $|kn\rangle$-sector states. This bound is saturated by BPS domain walls whose tension T [making use of Eqs. (1), (3)] is given by

$$T_k = |\mathcal{Z}_{kn}| = \frac{3}{4\pi^2} T_G^2 \Lambda^3 \sin \frac{\pi k}{T_G}, \tag{4}$$

which depends only on k. We will refer to these BPS walls as k-walls. They preserve two of the four supercharges, and thus form short 1/2-BPS multiplets containing one bosonic and one fermionic state.

In this paper, we address the problem of counting the number of such BPS supermultiplets for domain walls interpolating between two given vacua in $\mathcal{N}=1$ SYM theory, and we limit ourselves henceforth to gauge group $SU(N)$ where $T_{SU(N)} = N$. The number of BPS multiplets ν_k is counted by the Cecotti-Fendley-Intriligator-Vafa (CFIV) index [9–11],

$$\nu_k = \mathrm{Tr}_{kn}[F(-1)^F], \tag{5}$$

where F is the fermion number, and the trace (suitably defined) runs over all states in the $|kn\rangle$ sector. The index counts walls as solitonic objects in $3+1$D, and thus there is an implicit infrared regulator required to make the total wall mass finite.

An alternative way to view the CFIV index is that it counts the number of supersymmetric vacua in the $(2+1)$D theory induced on the world volume of the wall. We can then equivalently rewrite ν_k as given by the Witten index [1] of the world volume theory:

$$\nu_k = \mathrm{Tr}_{\mathrm{WV}}[(-1)^F]. \tag{6}$$

It is this alternative point of view which will be particularly useful for us.

The status of ν_k as an index refers to its invariance under smooth D-term deformations, as shown in [9], and it is this feature which makes it a valuable quantity in theories with four supercharges, where D-terms are generally unconstrained. The index does, however, depend on the F-terms but these are quantities over which $\mathcal{N}=1$ supersymmetry exercises some control.

For a strongly coupled system like $\mathcal{N}=1$ SYM theory, the invariance of the index under various deformations is crucial for the counting problem to be tractable. It is clear that to make progress in evaluating Eq. (5), we should first deform the theory to a weak coupling regime where the F-terms upon which the index depends—in this case the holomorphic superpotential—are calculable. Our strategy will then be to determine ν_k in this regime, and then rely on holomorphy of the superpotential, and the independence of ν_k on the D-terms to return to pure SYM theory and deduce the corresponding wall spectrum. To this end, there are many possible deformations one could consider. Before outlining the particular route we will follow, it is appropriate to consider some other recent approaches to this problem, which in part motivated our work.[1]

One particular deformation, in which there has been considerable recent interest [20–24], involves a geometric realization of $\mathcal{N}=1$ gauge theories as the low energy decoupling limit of M theory on a 7-manifold of G_2 holonomy. Making use of a smooth geometric transition in the moduli of the G_2 manifold [22,24] (see also [25,26]) leads to a tractable large volume regime which exhibits many of the features of the confining phase of SYM theory [20,21]. For SU(N), the dual 7-manifold is topologically $\mathbb{R}^4 \times S^3/\mathbb{Z}_N$, and Acharya and Vafa [23] proposed that in this context BPS walls correspond to $M5$-branes wrapping the Lens space $S^3/\mathbb{Z}_N$. Reducing to type IIA string theory via the fiber of the Hopf map leads to $D4$-branes wrapping the S^2 base, pierced by N units of Ramond-Ramond flux. When this S^2 is large, the low energy on the unwrapped $(2+1)$-dimensional part of the $D4$-brane preserves two supercharges, with a field content consisting of a photon, a photino, a massless neutral scalar and another spinor, where the gauge half-multiplet is given a topological mass by the presence of a Chern-Simons term of level N [27,28].

In calculating the index ν_k in this system one can identify (and factor out) the massless fields with the expected translational modes of the wall. The remaining massive gauge modes can be dualized to massive scalars which decouple, with the caveat that the Chern-Simons term induces a remnant set of quantum mechanical zero modes described by a Landau system,

$$S_L = \int dt \left[\frac{1}{2e^2}\dot{A}_i^2 - \frac{N}{8\pi}\epsilon_{ij}A_i\dot{A}_j \right]. \qquad (7)$$

Here $A_i = A_i(t)$ $(i=1,2)$ are the spatially homogeneous modes of the gauge field. The number of vacuum states,[2] given by the degeneracy of the lowest Landau level [27,28], is N provided the theory has an infrared regulator on the world volume. Consequently, one concludes that the degeneracy of 1-wall multiplets is N [23].

This discussion was extended [23] to k-walls by wrapping k $D4$-branes around the S^2. The simplest twisted theory arising at low energies is then $\mathcal{N}=1$ Chern-Simons Yang-Mills (CSYM) theory in $(2+1)$D with gauge group U(k). The massless sector has now expanded to include a real scalar transforming in the adjoint of $U(k)$, whose eigenvalues we naturally associate with the positions of the constituent 1-walls. All but one of these fields should gain a mass if the configuration is to form a bound state, but the required mechanism was not apparent in this construction. Nonetheless, it was argued that provided this lifting took place, vacua would arise as in the corresponding $\mathcal{N}=2$ CSYM theory [30,31] at the origin of the moduli space, with the index taking the value

$$\nu_k = \binom{N}{k} \equiv \frac{N!}{k!(N-k)!}, \qquad (8)$$

which reduces to N for $k=1$ as above. Once again an infrared regulator on the world volume was a necessary condition.

One may view the picture arising from this construction not just as a quantitative prediction for the spectrum of BPS walls (8), but also as an interesting conjecture about the structure of the world volume theory, and it is interesting to contrast it with expectations from field theory. One unresolved issue is the existence of additional moduli associated with the constituent 1-wall positions in a k-wall system, as noted above. Given the suggestions that the effective wall cutoff may scale with N in SYM theory, one possible means of reconciling these descriptions is to take the large N limit, where the binding energies are suppressed. This is indeed the expected regime where the geometric transition is induced [20]. Having this regime in mind, we will find a world volume description within supersymmetric QCD (SQCD) which matches surprisingly well with this construction.

A second tractable deformation of $\mathcal{N}=1$ SYM theory, which will also provide a useful reference point, involves compactification on $\mathbb{R}^3 \times S^1$ [32,33]. In this process, the SU(N) gauge symmetry can be broken to its maximal Abelian subgroup by a Wilson line $\phi^a = \int_{S^1}dx^1 A_1^a$, $a = 1,\ldots,N-1$, associated with the Cartan components of the gauge field A_μ^a along the compact direction. If the radius of the S^1 is much smaller than Λ^{-1}, and $\phi^a \sim 1$, then we arrive at a weakly coupled effective U(1)$^{N-1}$ gauge theory in $(2+1)$D. Furthermore, on this Coulomb branch the photons can be dualized to periodic scalars σ^a [34], and the

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[1]For additional related work on domain walls see [12–19].

[2]As an aside, this domain wall system provides an interesting viewpoint on the subtle vacuum structure of Maxwell-Chern-Simons theory [27,29].

system is then described as an $\mathcal{N}=2$ Kähler sigma model with target space $T^{2(N-1)}/S_{N-1}$ parametrized by the complex fields

$$V^a = \phi^a + \tau \sigma^a. \tag{9}$$

This moduli space is lifted by a nonperturbative superpotential generated by 3D instantons (BPS monopole configurations) [32,33]. Its general form for gauge group SU(N) has been determined in various ways [32,33,35–37] and has the structure of a complexified affine-Toda potential

$$\mathcal{W} \propto \left[\sum_{a=1}^{N-1} e^{-V^a} + e^{2\pi i \tau} e^{\Sigma_a V^a} \right]. \tag{10}$$

This superpotential therefore leads as expected to N chirally asymmetric vacua, and the corresponding condensates may be continued back to $\mathbb{R}^3 \times \mathbb{R}^1$ as the complex structure manifest in Eq. (10) was shown to be independent of the radius of the circle [33]. In this system, the wall configurations allowed by Eq. (10) may be counted individually as there are no additional moduli. Each wall forms a single multiplet, and in the $|kn\rangle$ sector one finds the same overall multiplicity ν_k [23,38] as in Eq. (8), which is consistent given that we again have an explicit infrared regulator.

With these approaches in mind, we will follow an alternate strategy, deforming $\mathcal{N}=1$ SYM theory to weak coupling. To this end we replace SYM theory by SQCD with $N_f \geq N-1$ massive fundamental flavors, a route also followed for the exact calculation of the gluino condensate [5]. When the tree-level mass terms are large compared to the dynamical scale, we return to pure SYM theory in the infrared. However, taking the masses as small perturbations, we pass to a Higgs phase where the gauge fields are heavy and the theory is well-defined in the infrared [3,39]. The low energy dynamics of this system is in terms of meson quasi-moduli M, and in the case of $N_f=N-1$ flavors one has the standard 1-instanton induced Affleck-Dine-Seiberg (ADS) superpotential [3] in addition to the mass terms,

$$\mathcal{W} = \text{Tr}(mM) + \frac{(\Lambda_{N-1})^{2N+1}}{\det M}. \tag{11}$$

Note that with a diagonal ansatz for the meson moduli, this superpotential formally coincides with Eq. (10), up to issues related to the compactification of the target space. Indeed, the structure of this superpotential already allows us to infer that the wall degeneracy will be nontrivial. As recalled in more detail below, BPS domain walls describe straight line trajectories in the $\mathcal{W}$ plane. Thus, for $N=2$ with a single chiral meson field, we see from Eq. (11) that the wall trajectories will be given by the roots of a quadratic equation. This suggests a 2-fold wall degeneracy, consistent with Eq. (8), which was indeed observed in earlier investigations of this system [7].

While the similarity with Eq. (10) is apparent, the Higgs phase approach can now be seen as a way of bridging both the above regimes, and thus testing some aspects of the M-theoretic construction. For gauge group SU(N), the num-ber of meson moduli entering Eq. (11) increases as $\mathcal{O}(N^2)$, and we will find that the BPS wall solutions exhibit a moduli space coordinatized by Goldstone modes which arise from broken flavor symmetries. It will be convenient to focus on the case $N_f=N$, where the full quantum flavor symmetry is manifest, and the corresponding description of the Higgs phase is discussed in more detail in the next section. We find that wall solutions break the flavor symmetries in such a way that the world volume theory, after factoring out the translational mode, is an $\mathcal{N}=1$ Grassmannian sigma model in (2+1)D. The vacuum states of this theory therefore count the number of BPS wall supermultiplets, Eq. (6).

We make significant use of the fact that the index is independent of variations in the Kähler potential. This and holomorphy of the superpotential are sufficient to ensure that the index is preserved under the flow back to pure SYM theory. We will explicitly show in this way that ν_k is given precisely by Eq. (8) *provided* a suitable infrared regulator is in place in full accord with [23], e.g. compactifying one spatial world volume dimension. However, when the external infrared regulator is removed, the status of the wall multiplicity count is less clear.

The layout of the paper is as follows. In Sec. II we discuss the structure of the Higgs phase in SQCD when $N_f=N$. With this background in hand, we turn to the calculation of the BPS wall moduli space in Sec. III, while the index calculation, and subtleties related to the need for an infrared regulator, are discussed in Sec. IV. Some aspects of the Higgs phase system with fewer flavors are discussed in Sec. V, and we conclude in Sec. VI with some additional comments on the world volume dynamics.

II. FROM SYM THEORY TO SQCD

A convenient tractable deformation of $\mathcal{N}=1$ SYM theory is obtained by adding $N_f=N$ chiral superfields, Q_f and $\bar{Q}^g$ ($f,g=1,\ldots,N_f$), transforming respectively in the fundamental and anti-fundamental representations of the gauge group (the gauge indices are suppressed). This matter content is sufficient to fully break the gauge symmetry of the theory through the Higgs mechanism in any vacuum in which the matter fields have a nonzero vacuum expectation value. One may then integrate out the gauge fields obtaining an effective description of the meson moduli matrix M,

$$M_f^g = Q_f \bar{Q}^g, \tag{12}$$

which involves a quantum constraint, first discussed by Seiberg [39]. On the non-baryonic Higgs branch, which we can restrict our attention to here, this constraint takes the form

$$\det M = (\Lambda_N)^{2N}, \tag{13}$$

which defines a manifold of complex dimension N^2-1 in $\mathbb{C}^{N^2}$. Here Λ_N is the dynamical scale of SQCD with $N_f=N$.

If we introduce a tree-level mass term

$$\mathcal{W}_{\text{tree}} = \text{Tr}(mM), \tag{14}$$

where m_f^g is the mass matrix in flavor space, then the effective superpotential can be written as

$$\mathcal{W} = \text{Tr}(mM) + \lambda[\det M - (\Lambda_N)^{2N}], \tag{15}$$

where λ is a Lagrange multiplier. There are N chirally asymmetric vacua of the theory, which lie at

$$\langle M \rangle_k = m^{-1}\mu\Lambda_N^2\omega_N^k, \quad k = 0, \dots N-1, \tag{16}$$

where $\omega_N^k = \exp(2\pi i k/N)$ is an N-th root of unity and we have defined

$$\mu = (\det m)^{1/N}. \tag{17}$$

For the weak coupling Higgs regime to set in we require that, for some of the meson fields, $\langle M \rangle \gg \Lambda_N^2$. From Eq. (16) we see that this is only possible when the mass matrix m is hierarchical. For example, we can choose the mass matrix in block diagonal form, with the mass of the N-th flavor m_N^N much larger than all the others,

$$m = \begin{pmatrix} m_N^N & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & m' & \\ 0 & & & \end{pmatrix},$$

$$m_N^N \gg (m')_f^g, \quad f,g = 1, \dots, N-1. \tag{18}$$

We can then integrate out the heaviest flavor, which resolves the constraint in Eq. (15) with $M_N^N = \Lambda_N^{2N}/\det M'$ (the fields M_f^N and M_N^g vanish), leading to the ADS superpotential [3] for the theory with $N_f = N-1$ flavors, as introduced earlier in Eq. (11),

$$\mathcal{W} = \text{Tr}(m'M') + \frac{(\Lambda_{N-1})^{2N+1}}{\det M'},$$

$$(\Lambda_{N-1})^{2N+1} = m_N^N(\Lambda_N)^{2N}, \tag{19}$$

where m' and M' refer to the reduced theory. If all remaining entries in the mass matrix m' are much less than Λ_{N-1} then the N vacua of the theory lie at weak coupling and one can reliably calculate the index ν_k by counting solutions of the classical BPS equations.

Thus, we conclude that to ensure the validity of weak coupling analysis the mass matrix should be of the hierarchical form (18) which diminishes the flavor symmetry. Recall that the classical Kähler potential $\mathcal{K}$ of the underlying N-flavor theory is $U(N) \times U(N)$ symmetric, although only $SU(N) \times SU(N) \times U(1)$ is realized canonically in terms of the meson moduli, where $\mathcal{K} = \text{Tr}(\bar{M}M)^{1/2}$. This symmetry is broken by the superpotential (15) to at most $SU(N)$, when all masses are equal. The mass matrix (18) breaks this further to $SU(N-1)$. Nonetheless, the reason we have emphasized

the N-flavor perspective is that we can restore the maximal $SU(N)$ flavor symmetry of the superpotential, despite the presence of the hierarchical mass matrix (18), by a holomorphic field rescaling. In practice it is convenient to introduce the dimensionless fields X_f^g,

$$X = mM(\mu\Lambda_N)^{-2}. \tag{20}$$

In terms of X the superpotential (15) is manifestly $SU(N)$ symmetric,

$$\mathcal{W} = \mu\Lambda_N^2[\text{Tr}X + \lambda(\det X - 1)]. \tag{21}$$

Of course, this is at the expense of diminishing the symmetry of the Kähler potential, which in terms of X becomes $\mathcal{K} \propto \text{Tr}[\bar{X}\bar{m}^{-1}m^{-1}X]^{1/2}$.

The crucial point, as emphasized above, is that the CFIV index ν_k does not depend on nonsingular deformations of the Kähler potential [9]. Thus it is convenient to choose a field rescaling that maximizes the symmetry of the superpotential, as we can then deform the metric back to a more symmetric form $\mathcal{K} \propto \text{Tr}(\bar{X}X)^{1/2}$ if so desired, for the purpose of analyzing the BPS equations. Formally, this procedure is equivalent to taking a symmetric mass matrix in Eq. (15). However, we emphasize that our dynamical model is that of Eq. (18) and the deformation described above is appropriate only for calculating an invariant quantity like ν_k.

III. MODULI SPACE OF BPS DOMAIN WALLS

Having deformed $\mathcal{N}=1$ SYM theory to SQCD in the Higgs phase, we can again verify that the vacuum structure and supersymmetry algebra still imply the existence of 1/2-BPS domain walls with tension determined by the central charge. However, the expression for the central charge operator itself is modified [8]. Ignoring total superderivatives, and making use of the Konishi relation [40], this operator can be written in the form [6,7,41]

$$\mathcal{Z} = \int dz \frac{\partial}{\partial z}\{2\widehat{\mathcal{W}}\}_{\theta=0},$$

$$\widehat{\mathcal{W}} = \mathcal{W}_{\text{tree}} - \frac{T_G - \sum_f T(R_f)}{16\pi^2}\text{Tr}\,W^2, \tag{22}$$

where z is again the transverse coordinate to the wall. The first term is due to the tree level superpotential, while the second represents the anomalous contribution, given in pure SYM theory by Eq. (2). For $N_f = N$ flavors, which is our main focus here, the anomalous term vanishes since $T_{SU(N)} = N$, and $T(R_f) = 1$.

In the Higgs regime at weak coupling this expression reduces in the $|kn\rangle$ sector to the simple form

$$\mathcal{Z}_{kn} = 2[\mathcal{W}_{k+n} - \mathcal{W}_n], \tag{23}$$

where $\mathcal{W}$ is now the effective superpotential (21) depending on the moduli X while $\mathcal{W}_k$ is the value of this superpotential in the kth vacuum,

$$X_k = \omega_N^k \cdot \mathbb{1}, \quad \mathcal{W}_k = N\mu\Lambda_N^2 \omega_N^k, \qquad (24)$$

where $\mathbb{1}$ is the $N \times N$ unit matrix. For reference, the explicit expression for the central charge is given by

$$\mathcal{Z}_{kn} = |\mathcal{Z}_{kn}| e^{i\gamma_{kn}}$$

$$= 4iN\mu\Lambda_N^2 \sin\frac{\pi k}{N} \exp\left(\frac{i\pi(2n+k)}{N}\right), \qquad (25)$$

which leads to Eq. (4) in pure SYM theory, via the decoupling relation $16\pi^2\mu\Lambda_N^2 \to 3N\Lambda^3$ as $\mu \to \infty$.

BPS walls in this system satisfy the first order differential equations [11,42,43],

$$g_{\bar{a}b} \partial_z X^b = e^{i\gamma} \partial_{\bar{a}} \bar{\mathcal{W}}, \qquad (26)$$

where we have chosen a convenient basis to expand the meson matrices, in which the Kähler metric is given by $g_{\bar{a}b} = \partial_{\bar{a}}\partial_b\mathcal{K}$, and the derivatives $\partial_{\bar{a}}$ and ∂_b are taken over $\bar{X}$ and X. Finally, $\gamma \equiv \gamma_{kn}$ is the phase of the central charge $\mathcal{Z}_{kn}$ as defined in Eq. (25). An important consequence of the BPS equations is that

$$\partial_z\mathcal{W} = e^{i\gamma} \|\partial_i X\|^2, \qquad (27)$$

and thus the domain wall describes a straight line in the $\mathcal{W}$ plane connecting the two vacua [11,42,43].

In calculating the number of solutions to these equations, with specified boundary conditions in the $|kn\rangle$ sector, we will need a more precise characterization of the dependence of the CFIV index on variations of the superpotential itself. The structure of the BPS mass spectrum implies that changes can occur only if a marginal stability condition is satisfied—where three vacua align in the $\mathcal{W}$ plane [42,43]. This allows considerable freedom (beyond that of deforming the Kähler metric) in perturbing the system in order to verify the existence or otherwise of BPS wall solutions.

We will make use of this freedom as follows, following the construction of Cecotti and Vafa [11]. First, since we consider massive vacua, one can expand the superpotential to quadratic order about each vacuum, and the set of all linearized solutions to Eq. (26) forms a cycle Δ_j in field-space diffeomorphic to a sphere. It then follows [11] that the weighted soliton number, namely the index ν_k, is given by the intersection number of the cycles associated with the two vacua [11]

$$\nu_k = \Delta_n \circ \Delta_{n+k}. \qquad (28)$$

This formulation of the index is manifestly topological, and provides a clear picture of how robust it is under deformations. In particular, it follows from Eq. (28) that ν_k counts all trajectories in the punctured $\mathcal{W}$ plane (with the vacua excised) which are homotopic to the straight line connecting the vacua describing the exact wall solution. Moreover, the change in the soliton spectrum as paths wrap around other

vacua, crossing curves of marginal stability via changes in the mass parameters for example, can also be understood as the intersection numbers change in such a process according to Picard-Lefschetz monodromies [11].

This point of view will prove useful in analyzing the BPS equations below. However, one must bear in mind that this approach refers strictly to $(1+1)$D, and thus to a compactification of SQCD on a torus T^2. The stability of the index under decompactification must also be addressed for any direct application of the results to $(3+1)$D. Taking these questions in turn, for the remainder of this section we analyze the space of wall solutions in SQCD, which we will demonstrate includes continuous flavor moduli, and then move to a discussion of ν_k itself in the following section.

A. Broken symmetries and Goldstone modes

As we will now demonstrate, the BPS wall solutions in this theory possess a nontrivial bosonic moduli space $\mathcal{M}$. In fact, on the general grounds that a k-wall spontaneously breaks translational invariance, we have the isometric decomposition,

$$\mathcal{M} = \mathbb{R} \times \tilde{\mathcal{M}}, \qquad (29)$$

where the factor $\mathbb{R}$ reflects the center of mass position z_0, while $\tilde{\mathcal{M}}$ denotes the reduced moduli space. The consistency of this decomposition with supersymmetry can be made explicit if we lift $\mathcal{M}$ to the corresponding supermanifold which also encodes the fermionic moduli. In particular, the two fermionic moduli associated with z_0 lift $\mathbb{R}$ to the supermanifold $\mathbb{R}^{1,2}$; quantization of these moduli naturally explains the two state multiplet structure, described algebraically in Sec. I, from the semiclassical point of view. A final point to emphasize is that, since only two supercharges are realized on the moduli, $\mathcal{M}$ is a real manifold, not endowed with any Kähler structure.

The decomposition (29) implies that each k-wall supermultiplet corresponds to a unique vacuum on $\tilde{\mathcal{M}}$. Therefore, provided we decouple moduli associated with the translational zero mode, the problem of calculating ν_k is reduced to that of finding the Witten index of the world volume theory on the wall. Consequently, we now turn to the problem of deducing the structure of $\tilde{\mathcal{M}}$. Given that the low energy description of SQCD outlined in Sec. II is of Landau-Ginzburg form, it is natural to expect that this space is determined purely by the flavor symmetries broken by the wall. In the remainder of this subsection, we will present the basic symmetry argument which determines $\tilde{\mathcal{M}}$. In the following subsection, we show how this arises from a more direct analysis of the BPS equations.

As reviewed in the previous section, the superpotential (21) exhibits an SU(N) global symmetry which is preserved by the vacua and is also supported by the deformed Kähler metric. This superpotential depends only on the eigenvalues $\{\eta_i\}$, $i=1,\ldots,N$ of the dimensionless meson matrix X. In terms of these eigenvalues it takes the form
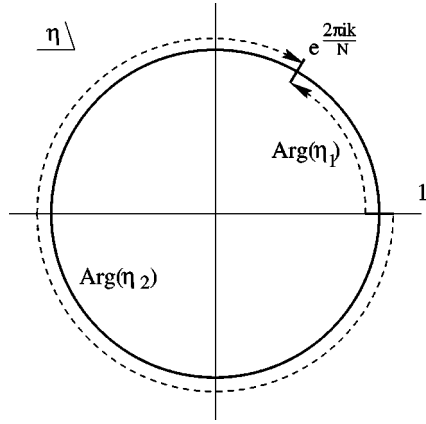
FIG. 1. Possible winding trajectories η_1 and η_2 for the eigenvalues.

$$\mathcal{W}=\mu\Lambda_N^2\left[\sum_{i=1}^{N}\eta_i+\lambda\left(\prod_{i=1}^{N}\eta_i-1\right)\right].\qquad(30)$$

Now consider a k-wall trajectory, choosing the $|k0\rangle$ sector for simplicity. The N eigenvalues are restricted to be the same in each vacuum, i.e. $\eta_i=1$ at $z\to-\infty$ and $\eta_i=e^{2\pi ik/N}$ at $z\to+\infty$ for all $i=1,\ldots,N$. The modulus of the field is unity in each vacuum and thus the only "pseudo topological" means of characterizing the eigenvalue trajectory in the wall is via its winding number,

$$w_i=\frac{1}{2\pi i}\int_\Gamma\frac{d\eta_i}{\eta_i},\qquad(31)$$

where Γ denotes the wall trajectory. Up to integer multiples this is clearly k/N for the k-wall, but we can also keep track of the additional windings. In principle there are an infinite number of possibilities, but its clear that energetically only two will occur in stable walls, namely,

$$w_1=k/N,\quad w_2=(k/N)-1\qquad(32)$$

as exhibited in Fig. 1.

The index is independent of the Kähler potential and thus one can use the freedom to perform diffeomorphisms of the Kähler metric to set all fields characterized by the same winding numbers equal to each other. To see that this is possible note that two trajectories η_a and η_b in a given wall with the same winding, say w_1, can be mapped into each other by an analytic mapping on a domain given by the complex plane with the vacuum points deleted. In other words from this point of view there are only two inequivalent trajectories in the wall, up to permutation.

Now let us prove that for k-wall trajectories there are precisely k eigenvalues that have winding w_1 and $N-k$ having winding w_2. Thus, if $N(w_i)$ is the number of fields having winding w_i, then

$$N(w_1)=k,\quad N(w_2)=N-k.\qquad(33)$$

To verify this, denote the trajectories with the windings w_1 and w_2 by η_1 and η_2, respectively. If $N(w_1)=l$, the superpotential (30) takes the form

$$\mathcal{W}=\mu\Lambda_N^2\{l\,\eta_1+(N-l)\,\eta_2+\lambda[\,\eta_1^l\,\eta_2^{N-l}-1]\}.\qquad(34)$$

The constraint imposed by the Lagrange multiplier ensures that $\eta_1=\eta_2^{-(N-l)/l}$. The vacuum values of the fields then require $k=nl$ with $n\in\mathbb{Z}$, and under the assumption that the fields do not undergo multiple windings in the wall, as argued above, then we have $k=l$.

Consequently, since we have two sets of field configurations in the wall with respectively k and $N-k$ coinciding eigenvalues, we find that k-wall trajectories preserve the following (continuous) subgroup of the SU(N) flavor symmetry:

$$\mathrm{SU}(k)\times\mathrm{SU}(N-k)\times\mathrm{U}(1).\qquad(35)$$

Note that the rank of symmetry group is preserved, following from the fact that the constraints on $\{\eta_i\}$ do not break the Cartan torus of SU(N). Thus far, we have not kept track of the global structure of the symmetry groups. However, to determine the precise coset associated with the broken symmetry generators, and thus the Goldstone modes induced on the wall world volume, it is sufficient to realize that, since our description involves only adjoint-valued meson fields, the center of each SU(p) symmetry group, for $p=N,k$ or $N-k$, acts trivially. Thus, the nonabelian symmetry groups are strictly of the form SU(p)/$\mathbb{Z}_p$.

Therefore, taking this global structure into account, we deduce that the reduced moduli space for k-walls is the complex Grassmannian[3]

$$\widetilde{\mathcal{M}}_k=G(k,N)\equiv\frac{\mathrm{U}(N)}{\mathrm{U}(k)\times\mathrm{U}(N-k)}\qquad(36)$$

reflecting the Goldstone modes induced by the broken flavor symmetries. For 1-walls, $G(1,N)=\mathbb{CP}^{N-1}$. This result apparently depends on the number of flavors, but can nonetheless be used to determine the world volume Witten index as we shall discuss in more detail below.

One may note that the information on eigenvalue trajectories is actually sufficient to deduce the "classical" wall degeneracy. It is given by the number of possible permutations of the eigenvalues (as discussed in [38]), leading to the result for ν_k given in Eq. (8). However, this argument neglects infrared effects on the wall, an understanding of which requires a more detailed analysis of the world volume dynamics. Before turning to this, we will present in the next subsection a more explicit derivation of $\widetilde{\mathcal{M}}$ following from the BPS equations.

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[3]We thank A. Smilga for helpful discussions on the geometry of soliton moduli.

B. Analysis of the BPS equation

The BPS equations (26) depend not just on the form of the superpotential, but also on the Kähler metric, and it is the latter dependence which, while irrelevant for obtaining ν_k, determines the precise wall profile. Therefore, it is useful to disentangle the crucial dependence on the moduli, arising from the superpotential, from the inessential details relating to the actual profile of the wall.

To this end, we first introduce a basis of Hermitian matrices $\{T_0, T_A\}$, $A = 1, \ldots, N^2 - 1$, where $T_0 = \mathbb{1}/\sqrt{N}$ and $\{T_A\}$ are orthonormal generators for the Lie algebra of SU(N), satisfying $\mathrm{Tr}(T_A T_B) = \delta_{AB}$. Then we can expand the dimensionless meson matrix in this basis,

$$X = \sqrt{N}(X^0 T_0 + i X^A T_A), \qquad (37)$$

where $\{X^0, X^A\} \in \mathbb{C}$. In this basis, the vacua lie at

$$\langle X^0 \rangle = \omega_N^k, \quad \langle X^A \rangle = 0, \qquad (38)$$

and on imposing the constraint obtained by integrating out the Lagrange multiplier λ, the superpotential takes the form

$$\mathcal{W} = \mu \Lambda_N^2 X^0. \qquad (39)$$

BPS wall profiles are then given formally by parametrizing the constraint that $\mathcal{W}$, and thus X^0, must follow a straight line connecting the two vacua, as follows from Eq. (27). In other words, *any* k-wall trajectory, parametrized by a fiducial scale $t \in [0,1]$, is given by the relation

$$X^0(t) = f(t) \equiv (1-t) + t\omega_N^k, \qquad (40)$$

with X^A subject to the constraint

$$\det[X^0(t)\mathbb{1} + i\sqrt{N}X^A T_A] = 1 \qquad (41)$$

with the appropriate boundary conditions at the vacua. The existence, and the moduli space, of wall solutions then devolves on the analysis of this constraint.

In essence, we have simply shifted the nontrivial field dependence from the superpotential to the metric. With a suitable coordinate choice the BPS equations will then provide a nontrivial profile for a single coordinate, consistent with Eq. (27), with the other coordinates either fixed or remaining constant and thus leading to moduli. We turn first to the simplest case with gauge group SU(2).

1. Gauge group SU(2)

We will study first the theory with gauge group SU(2), and two flavors. In this case, the constraint (41) takes the form

$$\sum_{a=0}^{3} (X^a)^2 = 1, \qquad (42)$$

where $a = (0,A) = 0, \ldots, 3$. This constraint yields a smooth complex submanifold of $\mathbb{C}^4$, known as the deformed conifold.[4]

The real section of the deformed conifold (42) defines a 3-sphere, and we see that the two vacua of the theory given by Eq. (38) lie at antipodal points on this S^3. The geometry of the space becomes more transparent by defining a radial coordinate r,

$$r^2 = \mathrm{Tr}(\bar{X}X), \qquad (43)$$

where constant values of r define a foliation with sections having the generic form SU(2)$\times$SU(2)/U(1), where the U(1) acts in such a way that this space is topologically $S^2 \times S^3$; see [44]. However, at the minimal radius $r = 1$ the coset becomes SU(2)$\times$SU(2)/SU(2) and these sections collapse to an S^3 identifiable as the real section above. Thus, the manifold is conical for large r, while the apex of the cone is rounded off to an S^3.

Supersymmetry demands that we use a Kähler metric on the deformed conifold. Such a metric preserving the SU(2)$\times$SU(2) action apparent from the $r \neq 1$ sections will be a function only of r^2. In terms of the Kähler potential $K = K(r^2)$, and the meson matrix X, the metric will take the form

$$ds_C^2 = K'(r^2)\mathrm{Tr}(d\bar{X}dX) + K''(r^2)|\mathrm{Tr}(\bar{X}dX)|^2. \qquad (44)$$

Kähler metrics on the deformed conifold, which are also Ricci flat, were first obtained in [44]. Supersymmetry does not impose the latter constraint here,[5] but in fact we will not need to choose a precise form for $K(r^2)$ away from the apex. In this region, the symmetry we have imposed is sufficient to ensure that the metric (44) takes the asymptotic form [44,45]

$$ds_{r\to 1}^2 = \Lambda_2^2\left[d\rho^2 + d\Omega_3^2 + \frac{1}{2}\rho^2 d\Omega_2^2\right], \qquad (45)$$

where $\rho = \sqrt{2(r-1)}$ and $d\Omega_2^2$ represents a 2-surface (topologically S^2) which shrinks as $\rho \to 0$, while $d\Omega_3^2$ is a round S^3—equivalent to the real section—which remains with finite volume at the apex. Thus, the metric reduces locally near $r = 1$ to $\mathbb{R}^3 \times S^3$.

The crucial simplifying feature is that it is not just the vacua, but the entire wall trajectory which lies on the real section. To see this we adopt a different viewpoint on the deformed conifold, first discussed by Stenzel [46,47]. The manifold is symplectic, given by the co-tangent bundle $T^*(S^3)$ over S^3, and this is made manifest by introducing real coordinates x^a and momenta p_a via

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[4]The singular conifold is recovered in the classical limit, $\Lambda \to 0$, where Eq. (42) reduces to $\Sigma_{a=0}^3 (X_{\mathrm{class}}^a)^2 = 0$ with $X_{\mathrm{class}} = \lim_{\Lambda \to 0}(\Lambda^2 X)$.

[5]In contrast, with $\mathcal{N}=2$ SUSY the Higgs branch metric is required to be hyper-Kähler, which would imply Ricci flatness.

$$X^a = \cosh(\sqrt{p_b p_b})x^a + i\frac{\sinh(\sqrt{p_b p_b})}{\sqrt{p_c p_c}}p_a,$$

$$a,b,c = 0,1,2,3. \tag{46}$$

The defining constraint then takes the form

$$\sum_{a=0}^{3}(x^a)^2 = 1, \quad \sum_{a=0}^{3}x^a p_a = 0, \tag{47}$$

which describes the canonical phase space of a dynamical system with configuration space S^3, with momenta lying in the cotangent space.

Now, from the condition that the superpotential lie along a real line connecting the two vacua, see Eqs. (39),(40), we deduce that $p_0(t)=0$, and the superpotential reduces to $\mathcal{W} = \mu\Lambda_N^2\cosh(\sqrt{p_A p_A})x^0$, where $A=1,2,3$. Since $\cosh(\sqrt{p_A p_A})$ is strictly positive, and the vacua lie on the real section so that $p_A(0)=p_A(1)=0$, we can smoothly deform the Kähler potential if necessary so that $p_A(t)=0$, $A = 1,2,3$, for all t. One can readily verify that this solves the constraints (47). The solution then corresponds to a zero "energy" configuration of the analog system, where $E = \sum_{a=0}^{3}(p_a)^2/2$, and remains entirely within the real section S^3 at $r=1$.

By deformation, we have chosen the Kähler metric on the field theory moduli space to respect the maximal $SU(2)\times SU(2)$ symmetry, and so the corresponding metric on S^3 will be the round one. Re-expressing the real coordinates $\{x^a\}$, $a=0\ldots3$, on S^3 in terms of Euler angles $\{\theta,\xi,\phi\}$, the metric takes the form

$$ds_{r=1}^2 = \Lambda_2^2 d\Omega_3^2,$$

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta(d\xi^2 + \sin^2\xi d\phi^2), \tag{48}$$

while the superpotential, restricted to the real section, is given by

$$\mathcal{W}|_{\text{trajectory}} = 2\mu\Lambda_2^2\cos\theta. \tag{49}$$

The vacua lie at the poles $\theta=0,\pi$, and the BPS equations reduce to

$$\partial_z\theta = 2\mu\sin\theta, \quad \partial_z\xi = \partial_z\phi = 0, \tag{50}$$

which are solved by the sine-Gordon soliton,

$$\theta(z) = 2\arctan e^{2\mu(z-z_0)}. \tag{51}$$

This solution, schematically represented in Fig. 2, is characterized by three moduli $\{z_0,\xi_0,\phi_0\}$. The angular modes are Euler angles on the 2-sphere, and so we recover the moduli space deduced earlier from symmetry considerations,

$$\mathcal{M}^{N=2} = \mathbb{R}\times\mathbb{CP}^1, \tag{52}$$

where since $N=2$, we have necessarily been considering minimal walls with $k=1$.
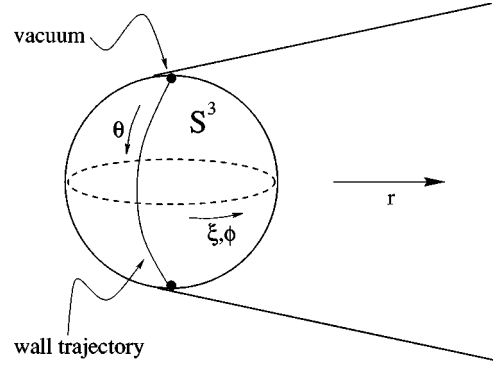

FIG. 2. Wall trajectory on the resolved S^3 at the tip of the deformed conifold.

In preparation for later analysis, we can compactify the spatial dimensions on a 2-torus of area L^2. Then, at excitation energies $E\ll L^{-1}$, we are effectively reducing the system to one of the kinks in $(1+1)$D, where this reduction to quantum mechanical moduli makes sense. Integrating over the wall profile leads to the corresponding bosonic moduli space Lagrangian which takes the form

$$L_{\text{bose}} = -M + \frac{1}{2}M\dot{z}_0^2 + R_{\tilde{\mathcal{M}}}(\dot{\xi}^2 + \sin^2\xi\dot{\phi}^2), \tag{53}$$

where $M=T_1 L^2$ is the kink mass, determined by the wall tension $T_1 = 8\mu\Lambda_2^2$, while integrating over the wall profile leads to a scale

$$R_{\tilde{\mathcal{M}}} \sim L^2\frac{\Lambda_2^2}{\mu} \tag{54}$$

for the flavor moduli space $\tilde{\mathcal{M}}$.

2. Gauge group SU(N)

The general case can be understood by following a slightly less direct approach. It will be convenient to describe generic k-walls as embedded sine-Gordon solitons, although we emphasize that this simply reflects a particular choice of the Kähler metric. In more detail, along the wall trajectory we make the change of variables $t=\sin^2\theta/2$ where $\theta \in [0,\pi]$. The superpotential (39),(40) can then be written as

$$\mathcal{W}|_{\text{trajectory}} = N\mu\Lambda_N^2 e^{i\pi k/N}\left[\cos\frac{k\pi}{N} - i\sin\frac{k\pi}{N}\cos\theta\right]. \tag{55}$$

If we now choose a Kähler potential of the form

$$\mathcal{K} = \Lambda_N^2[\theta^2 + \text{Tr}f(\bar{\tilde{X}}\tilde{X})], \tag{56}$$

where f is any smooth function along the wall trajectory and $\tilde{X}=X-(1/N)(\text{Tr}X)\mathbb{1}$ is the trace-free part of X, then the BPS equation for k-walls takes the form

$$\partial_z \theta = N \mu \sin \frac{k\pi}{N} \sin \theta, \qquad (57)$$

which is again solved by the sine-Gordon soliton,

$$\theta(z) = 2 \arctan e^{2\tilde{\mu}(z-z_0)}, \quad \tilde{\mu} = \frac{N}{2} \mu \sin \frac{k\pi}{N}. \qquad (58)$$

This profile is dependent entirely on the choice of the metric (56), which naturally generalizes Eq. (48). However, since we are not concerned with its precise form the only point of concern is whether it introduces additional singularities. To check this, we need to compare it with the canonical metric at weak coupling, which is the only regime in which the metric is known. One can verify that the mapping to Eq. (56) is nonsingular in the vicinity of the wall trajectory.

With this profile in hand we can now investigate the moduli of this solution. The corresponding reduced moduli space $\tilde{\mathcal{M}}$ is determined by the space of smooth solutions to the constraint (41) with fixed boundary conditions at the appropriate vacua.

Geometrically, the manifold (41) is a determinental variety of complex dimension $N^2 - 1$, which is rather difficult to analyze directly, except in the SU(2) case discussed above, and so we will adopt a different approach motivated by the discussion in Sec. III A. First of all, note that although the flavor symmetry of the theory is SU(N), the symmetry of the constraint (41) is its complexification SL(N,$\mathbb{C}$). This allows us to diagonalize the meson matrix $X = \sqrt{N}(X^0 T_0 + i X^A T_A)$ by the adjoint action of SL(N,$\mathbb{C}$), with $X \rightarrow \text{diag}\{\eta_i\}$ as before, where now the eigenvalues are functions of a single complex variable x. Following the arguments of Sec. III A, to solve the constraint

$$\prod_{i=1}^{N} \eta_i(x(t)) = 1, \qquad (59)$$

we must impose $\eta_i = \eta_j$ for $i,j = 1, \ldots, k$, and $\eta_k = \eta_l$ for $k,l = k+1, \ldots, N$. We can then represent the diagonalization of X in the form

$$X \rightarrow f(t)\mathbb{1} + x\sqrt{N}\Omega, \qquad (60)$$

where we have used Eq. (40) and Ω is the following generator of the Cartan subalgebra

$$\Omega = \text{diag}\left\{ -\sqrt{\frac{N-k}{Nk}}\mathbb{1}_k, \sqrt{\frac{k}{N(N-k)}}\mathbb{1}_{N-k} \right\}, \qquad (61)$$

with $\mathbb{1}_k$ the $k \times k$ unit matrix.

The constraint (59) then reduces to

$$\left(f(t) + \sqrt{\frac{N-k}{k}}x \right)^k \left(f(t) - \sqrt{\frac{k}{N-k}}x \right)^{N-k} = 1. \qquad (62)$$

The number of solutions $x(t)$ to this equation asymptoting to the vacua $x(0) = x(1) = 0$ can be obtained as follows [38]. Note first that the constraint is resolved by defining a new variable y,

$$y^k = f(t) - \sqrt{\frac{k}{N-k}}x,$$

$$y^{-(N-k)} = f(t) + \sqrt{\frac{N-k}{k}}x. \qquad (63)$$

An apparent phase ambiguity which is dropped in this transition is actually fake once the vacua are fixed. Then, eliminating the x dependence we obtain

$$\frac{\mathcal{W}_{\text{ansatz}}}{\mu \Lambda_N^2} = N f(t) = k y^{k-N} + (N-k)y^k, \qquad (64)$$

which we recognize as the constraint that the ADS superpotential, evaluated within the ansatz (60), follows the straight line wall trajectory. Note that another phase ambiguity has been dropped to ensure the correct asymptotic vacua.

Since the vacua are massive, and thus the second derivative of the superpotential is finite, there are at most two possible trajectories emanating from each vacuum point. However, a perturbative analysis shows that only one of these can interpolate between both. Existence of this unique solution can be demonstrated [38] by taking the trial solution $y(t) = e^{2\pi it/N}$ and showing that its image in the punctured $\mathcal{W}_{\text{ansatz}}$ plane (with the vacua excised) is homotopic to a straight line, the latter describing the exact wall trajectory.

Thus, we have found precisely one solution for all k associated with the ansatz (60) and the generator Ω. Consequently, following standard arguments, the moduli space of these solutions is given by the coadjoint orbit of Ω under the symmetry group, which in this case is SU(N). This is the manifold swept out by the adjoint action of SU(N) mod the stability group of Ω, which we see immediately is SU(k)$\times$SU($N-k$)$\times$U(1) up to discrete factors. Thus, taking into account the fact that the center of each nonabelian group acts trivially, we recover the result obtained earlier that the reduced moduli space,

$$\tilde{\mathcal{M}}_k = G(k,N), \qquad (65)$$

is the complex Grassmannian of k-planes in $\mathbb{C}^N$.

Having factored out the transverse position modulus, which is decoupled and not visible in the construction above, we find that the moduli associated with the broken flavor symmetries induce a nontrivial (2+1)D sigma model on the wall world volume, the supersymmetric vacuum states of which—to be identified with inequivalent k-walls—we will count in the next section.

IV. THE WORLD VOLUME WITTEN INDEX

Having determined the structure of the bosonic moduli space as $\mathbb{R} \times \tilde{\mathcal{M}}_k$, the calculation of ν_k simplifies in that the massless field associated with the translational zero mode is

factorized, and its associated multiplet decoupled. We then identify each inequivalent k-wall with a unique vacuum state of the $\mathcal{N}=1$ sigma model on $\widetilde{\mathcal{M}}_k$, and count then with the Witten index. The index is conveniently defined by imposing an infrared regulator on the spatial coordinates of the world volume. In general, the result is insensitive to removal of the regulator if the theory has a mass gap, an issue which requires some dynamical knowledge of the system in question. In this section, we discuss the status of the index for the world volume sigma model on $\widetilde{\mathcal{M}}_k$ by systematically removing the infrared regulators.

A. Compactification on T^2

We first consider a fully regularized system putting the spatial part of the world volume on a torus. Then in the low-energy limit, the world volume theory reduces to a quantum mechanical problem with the moduli dependent only on time. In effect, we are now analyzing the quantum mechanical moduli of kinks in $(1+1)$D, as discussed in detail in [11]. Given that the flavor moduli parametrize the Grassmannian $G(k,N)$, which is compact, the techniques for calculating the index and thus the number of quantum ground states were described in [1], and the result is given by [11]

$$\nu_k = \chi(G(k,N)) = \binom{N}{k} \equiv \frac{N!}{k!(N-k)!}, \qquad (66)$$

where $\chi(G(k,N))$ is the Euler characteristic of the Grassmannian. The resulting spectrum of k-walls is consistent with the results of [23]. Note that the result is independent of the original SQCD Kähler metric, and depends only on the topology of the bosonic moduli space. This is a necessary consistency check as we have relied on the independence of the result under smooth deformations of the metric [9]. Moreover, the invariance of the index under small perturbations of the superpotential, for which no vacua become aligned [11], is now transparent. Specifically, were we to perturb the meson mass terms slightly, thus reducing the residual symmetry of the wall, the spectrum would not change as this would deform the metric on $G(k,N)$ but clearly not its topology.

Let us note that in the context of kinks, the degeneracy (66) has an interesting interpretation. The moduli space for 1-walls is $\mathbb{CP}^{N-1}$, and the corresponding degeneracy from Eq. (66) is N. It is natural to interpret this in terms of the walls forming an N-plet of SU(N), which is the isometry group of $\mathbb{CP}^{N-1}$ (see also [38]). The degeneracy (66) then implies that composite k-walls fall into antisymmetric tensor multiplets of SU(N), namely the k-th fundamental representation. This implies that 1-walls, when reduced to kinks in $(1+1)$D, are "fermionic" in flavor, consistent with expectations for solitons in similar Landau-Ginzburg systems.

B. Compactification on S^1

We now decompactify one cycle of the torus. The index obtained above will remain valid provided vacuum states cannot disappear to infinity in the process of decompactification. In this case, we are left with the moduli dynamics

described by an $\mathcal{N}=1$ Grassmannian sigma model in $(1+1)$D. Such systems are well understood, the nontrivial infrared behavior of the $(1+1)$D sigma model restores the original SU(N) symmetry, allowing a dynamically generated mass gap for the flavor moduli. Since the vacua are massive, we are guaranteed that on considering scales below the gap the problem reduces to one of quantum mechanics as above. Thus we can conclude that the number of discrete supersymmetric vacua, and therefore the spectrum of k-walls in SYM theory compactified on $\mathbb{R}^3 \times S^1$, is still given by Eq. (66). This conclusion is clearly consistent with the results obtained by direct compactification of SYM theory [23,38], as reviewed in Sec. I.

In concluding this subsection, we note that the $(1+1)$D sigma model provides another interesting point of view on the degeneracy (66). In particular, the N-plet wall multiplet structure seems in this case rather closely tied to restoration of the SU(N) symmetry in the infrared.

C. Decompactification and an alternative regulator

On decompactifying the second cycle of the torus, we have removed all infrared regulators and the status of the index (66), in as far as it correctly describes the wall multiplicity, devolves on the infrared dynamics of the $\mathcal{N}=1$ Grassmannian sigma model in $(2+1)$D. This system, specifically the $\mathbb{CP}^{N-1}$ model, has received less attention than the corresponding models in $(1+1)$D. In perturbation theory there is no evidence for infrared divergences, and this conclusion extends to leading order in the large N expansion [48,49]. However, due to the fact that the flavor modes can be combined into complex chiral multiplets, there is no obvious symmetry [50] or anomaly constraint [32] which forbids a mass term. Moreover, one must also bear in mind that the UV divergences of the model are cut off physically at a scale given by the inverse width of the wall, which is $\mathcal{O}(m)$ in SQCD, and this introduces an additional scale. Thus, without a more detailed understanding of the dynamics of this UV regularized $\mathcal{N}=1$ $G(k,N)$ sigma model, the status of the wall multiplicity count remains unclear after decompactification.

In contrast, the index itself can be regulated in an alternative manner via a perturbation of the theory which lifts the additional flavor moduli.[6] In practice, we require a perturbation which lifts the off-diagonal elements of the meson matrix, so that the system reduces to a theory of the eigenvalues $M \rightarrow \text{diag}\{\eta_i\}$ with

$$\mathcal{W} = \mu \Lambda_N^2 \left[\sum_{i=1}^{N} \eta_i + \lambda \left(\prod_{i=1}^{N} \eta_i - 1 \right) \right] \qquad (67)$$

as in Eq. (30). One may then construct wall solutions which possess no flavor moduli, and determine the multiplicity directly as in [23,38], finding the result (8) once again.

There are several possible mechanisms for lifting the off-

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[6]We thank A. Losev for suggesting this approach and for related discussions.

diagonal modes and all have certain side effects. One can simply perturb the superpotential $\mathcal{W}\to\mathcal{W}+\delta\mathcal{W}$ with nonlinear terms $\delta\mathcal{W}$ which break the SU(N) flavor symmetry, but this generically introduces new vacua, and care is needed in discarding any spurious wall solutions which decouple as the perturbation is removed. Alternatively, one may weakly gauge the flavor symmetry, under which the meson matrix M transforms in the adjoint. The decoupling limit for the gauge modes then enforces a D-term constraint, $\mathrm{Tr}[M,\bar{M}]^2=0$, ensuring that the off-diagonal modes of M are lifted, leading to Eq. (67). This procedure does introduce an additional decoupled set of light U(1) fields, but has the merit of retaining the symmetry structure we expect to be important in the pure SYM regime, as is apparent on comparison with the approach of compactifying on $\mathbb{R}^3\times S^1$; see Eq. (10).

Perturbing the original theory in this way indicates that the index is, as it should be, stable under different choices for the regulator. However, its connection to the physical wall multiplicity still rests on the question of stability under removal of the regulator. As we noted above, this can be rephrased as a dynamical question about the vacuum structure of the world volume sigma model. On this issue, we will limit ourselves here to a few comments describing the two possible scenarios.

The first is that a nonperturbative mechanism generates a mass for the flavor modes, which implies that the index remains unchanged. In this regard, recall that supersymmetric nonlinear sigma models are most conveniently studied by embedding them in a corresponding gauged linear sigma model [51,52]. For the $\mathbb{CP}^{N-1}$ model, this gauge theory is Abelian and at first sight there are no obvious nonperturbative effects which could generate a mass gap. However, the presence of the UV cutoff complicates this issue, as the UV completion of the theory may allow nonperturbative mass generation. A classic example, although not directly relevant here, is the Polyakov mass for the photon [32,34] in U(1) theories where, from the low energy perspective the nonperturbative mechanism involves "singular" instantons, which are resolved above the cutoff scale.

With this in mind it is intriguing to note that, if we assume for a moment that a mass gap for the flavor modes were to arise via some mechanism, one could integrate them out within the linear sigma model, which in effect corresponds to flowing back to pure SYM theory. This process is known to induce a standard kinetic term for the gauge fields, and in (2+1)D will also lead to a Chern-Simons term [48,49]. In the case of 1-walls, the resulting system would be $\mathcal{N}=1$ Maxwell-Chern-Simons theory at level N (up to higher derivative corrections), remarkably consistent with the world volume theory obtained by Acharya and Vafa [23]. This connection can also be made, at a formal level, within the compactified system, where the light flavor fields have masses scaling inversely with the volume.

The second scenario is that the flavor modes remain massless at the quantum level, and *a priori* there is no obvious inconsistency with this. More precisely, as we flow back to $\mathcal{N}=1$ SYM theory, these modes must disappear, but this can occur without direct mass generation. In particular, the flavor

modes may "freeze" in this limit. Recall for the SU(2) case, that the Kähler class of the (2+1)D $\mathbb{CP}^1$ sigma model scales as

$$M_{\tilde{\mathcal{M}}}\propto\frac{\Lambda_2^2}{\mu}=\frac{3\Lambda^3}{8\pi^2\mu^2}, \qquad (68)$$

where Λ is the $\mathcal{N}=1$ SYM scale defined in Eq. (1) and $\mu=(\det m)^{1/N}$. Thus, as we begin to decouple the matter fields, the moduli space shrinks. In the absence of any mass generation, it is plausible that in the limit that $\mu\to\infty$, the manifold $\tilde{\mathcal{M}}$ shrinks to zero size, and the corresponding flavor modes are frozen, and consequently decouple. However, this conclusion requires a significant assumption about the behavior of the SQCD Kähler metric in this regime. It would be natural for the scale (68) to have corrections of $\mathcal{O}(\mu/\Lambda)$ which may significantly change its form when $\mu\gtrsim\Lambda$.

V. BREAKING THE FLAVOR SYMMETRY AND $N_f=N-1$

Thus far, we have purposefully chosen to work with N flavors, to make the multiplet structure of the walls under global symmetries quite explicit. This required us to make use of the invariance of the index under D-term deformations, so as to restore the maximal SU(N) symmetry of the N-flavor theory. Recall that in Sec. II, for consistency, we required that the mass matrix was hierarchical to ensure a weak coupling regime, which implies that the weak coupling flavor symmetry of the underlying theory is at most SU($N-1$).

While this approach was convenient for obtaining the index, it is also instructive to see how the explicit breaking of flavor symmetries is manifest at the level of the flavor moduli space $\tilde{M}$, in a regime where the direct relation to Goldstone modes is lost. For ease of illustration, we focus on the simplest case with gauge group SU(2). In the analysis of Sec. III, as just described, we deformed the Kähler metric so as to restore the maximal SU(2) symmetry of the theory, despite the hierarchical mass matrix of Eq. (18),

$$m=\mathrm{diag}\{m_1,m_2\}, \quad m_1\ll m_2\ll\Lambda_2. \qquad (69)$$

At the level of the Kähler potential, setting $m_1\neq m_2$ breaks the symmetry from SU(2)$\times$SU(2)$\times$U(1) $\cong$SO(4)$\times$SO(2) to SO(2)$\times$SO(2).

This reduction in symmetry can be traced through to the metric structure of the wall moduli space as follows. Recall that in the analysis of Sec. III, the wall profile described a path between the poles of an S^3—the real section of the deformed conifold—with the S^3 having the round SO(4)-invariant metric. It is convenient to visualize the S^3 via stereograhic projection as $\mathbb{R}^3\cup\{\infty\}$. The poles of the S^3 and thus the two vacua of the theory are projected to 0 and ∞, and hence the wall trajectory is described by a line from the origin to infinity in $\mathbb{R}^3$. This line is parametrized by Euler angles on the sphere at fixed radius, and hence this construction realizes the $\mathbb{CP}^1\cong S^2$ flavor moduli space as a submanifold of $\mathbb{R}^3$. In Sec. III, the SO(4) symmetry of the original Kähler metric ensures an SO(3)-invariant round metric on
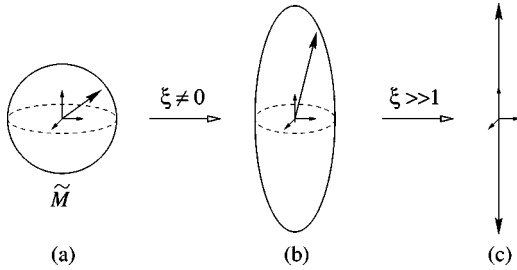
FIG. 3. A schematic representation of the flavor moduli space realized topologically as $S^2 \subset \mathbb{R}^3$, showing the dependence of the induced metric on the hierarchical structure of the masses: (a) equal masses implying SO(3) isometry; (b) $\xi \neq 0$ implying SO(2) isometry; and (c) the decoupling limit where $\xi \gg 1$. The overall scale has been chosen for convenience. The bold arrows indicate an example of the dominant flavor mode profile in the wall.

the S^2. However, with unequal masses, the metric induced on the S^2 has only an SO(2) isometry, which we can arrange to generate rotations in the horizontal plane. Metrically, the S^2 moduli space is then an ellipsoid, with the ellipticity characterized by the dimensionless parameter,

$$\xi \equiv \frac{m_2 - m_1}{\sqrt{m_1 m_2}}. \qquad (70)$$

This metric structure for the moduli space is exhibited in Fig. 3.

In the process of decoupling the heavy flavor, $m_2 \gg m_1$, the ellipticity parameter $\xi \sim \sqrt{m_2/m_1}$ diverges, and the ellipsoidal metric on S^2 becomes singular. As shown schematically in Fig. 3, the corresponding moduli are then "frozen" with two possible orientations, thus reproducing the expected result that there are two inequivalent 1-walls in this theory, and explaining why a direct analysis of the SU(2) theory with 1 flavor would uncover two inequivalent solutions, with no additional moduli. This latter result for the 1-flavor theory has been known for some time [7].

At this point it is worth contrasting the decoupling scenario with our discussion in the previous section. First, note that the limit $m_2 \gg m_1$ serves as a partial alternative to explicitly "lifting" the additional flavor moduli by perturbing the theory with $N_f = N$ flavors, and thus allows a direct calculation of the wall multiplicity. However, this freezing of moduli within weak coupling SQCD is distinct from what may happen on integrating out all the matter fields, which we cannot do here, and returning to the strong coupling regime in pure SYM theory. We see that despite the increasing ellipticity in this case, all mass scales are small relative to Λ_2, and the overall Kähler class of $\tilde{\mathcal{M}}$ which scales as $\mathcal{O}(\Lambda_2^2/\sqrt{m_1 m_2})$ remains large.

This picture of the hierarchical freezing of flavor modes, as some subset of the matter fields are decoupled, allows us to make contact more generally with the picture of the wall spectrum that emerges in SQCD with $N_f = N - 1$ flavors. Recall that in the hierarchical regime (18), we could simply integrate out the N-th flavor, leading to the ADS superpotential (19) which exhibits an SU($N-1$) symmetry if we set the

remaining mass matrix proportional to the identity. In this system, the wall multiplet structure is less explicit, but ν_k must necessarily be the same. As explained above, for SU(2) the agreement follows straightforwardly from the fact that although the additional flavor modes are frozen, this can happen in two possible ways, reproducing the two-wall spectrum obtained some time ago [7].

The case of SU(3) gauge group broken via the Higgs mechanism through the introduction of two flavors can be understood in a similar manner. This system been treated in some detail in the literature. There is now a 2×2 meson matrix M', and the flavor symmetry of the canonical Kähler potential is SU(2)$\times$SU(2) provided the mass matrix m' is chosen proportional to the identity. There are three vacuum states at $\langle M' \rangle \propto \omega_3^k \mathbb{1}_2$ ($k = 0,1,2$), and this theory again possesses only minimal 1-wall solutions.

In searching for classical BPS configurations, it is natural to first introduce a diagonal *ansatz*, namely $(M')_f^g = M \delta_f^g$, $(g,f = 1,2)$. Such field configurations will not break the flavor SU(2), and are flavor-symmetric domain walls. Consequently, there are no massless excitations on the wall world volume, other than the translational modes. Numerical analysis in [15–17,19] demonstrated the existence of a unique flavor-symmetric solution. However, this is not the end of the story as the symmetric ansatz should be relaxed to find all the possible 1-wall solutions. If one demands simply that $(M')_f^g$ is diagonal, with $(M')_1^1 \neq (M')_2^2$, then additional solutions will arise in pairs by permutation of the fields. Perturbative analysis indicates that there are at most four trajectories emanating from each vacuum, and an analysis along the lines of [38] demonstrates that only two of these interpolate to the second vacuum providing true wall configurations. This conclusion is backed up by explicit numerical solutions found[7] in [17] which were confirmed in [19]. More generally, if we retain the dependence on the full 2×2 meson matrix, the flavor asymmetric ansatz will break the SU(2) symmetry down to U(1), inducing flavor moduli parametrizing a $\mathbb{CP}^1$ sigma model on the world volume. When regulated in the infrared, the Witten index is equal to two, which is consistent with the findings above. Thus, in total there are three inequivalent solutions in agreement with our earlier results. In accord with our discussion above regarding freezing of the moduli associated with decoupled fields, we see here that the full moduli space $\mathbb{CP}^2$ of the 3-flavor theory is reduced to $\mathbb{CP}^1$ in the hierarchical mass regime due to the reduction in flavor symmetry. Accounting correctly for the frozen modes ensures that the result for the index in each case is, of course, the same.

This counting of minimal walls, using an unconstrained parametrization of the moduli in the $N-1$ flavor theory, is easily extended. For gauge group SU(N), there is an $(N-1) \times (N-1)$ meson matrix and an explicit SU($N-1$) flavor symmetry. Once again there is a unique flavor symmetric wall, while flavor asymmetric walls induce Goldstone modes

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[7]These authors work instead with a Taylor-Veneziano-Yankielowicz superpotential, but one can compare the results for small m'.

parametrizing a $\mathbb{CP}^{N-2}$ sigma model via the broken flavor symmetry. The corresponding Witten index is $N-1$, leading again to $1+(N-1)=N$ possible 1-wall configurations. Thus the N-plet observed in Sec. IV decomposes in this case as $\mathbf{N}\rightarrow 1+(\mathbf{N}-1)$.

VI. DISCUSSION

Deforming SYM theory to SQCD in the Higgs phase has allowed us to tune the symmetry structure so that classically there was a moduli space of BPS domain walls, enabling a robust calculation of the wall multiplicity given a suitable infrared regulator. Note that this genuine weak coupling approach is in contrast to others for which relevant fields fail to remain weakly coupled throughout the wall trajectory. The results we obtained are consistent with those deduced by Acharya and Vafa [23] using a string dual construction, and the world volume description has intriguing parallels with this work on which we will elaborate further below.

In particular, we will finish with a few comments on the dynamics of the translational moduli, an issue that we have suppressed thus far. Specifically, while the center of mass modulus certainly decouples, one can also study the formation of composite 2-walls from primary 1-walls with adjacent phase boundary conditions. We have emphasized that in this system the positions of the constituents are not moduli (in contrast to certain $\mathcal{N}=2$ domain wall systems [53]), but one can still set up an unstable configuration [54] where the two constituents are well separated and observe the interactions which will be sensitive to the SQCD spectrum. Moreover, one can arbitrarily suppress the binding energy in the large N limit. The binding energy per unit volume follows from the BPS formula,

$$T_2-2T_1=-\frac{3\pi\Lambda^3}{4N}+\mathcal{O}(N^{-3}),\qquad(71)$$

and thus the walls decouple at large N. If we set up such an unstable configuration and allow the walls to evolve to a bound state, there is a moduli space transition,

$$2(\mathbb{R}\times\mathbb{CP}^{N-1})\rightarrow\mathbb{R}\times G(2,N),\qquad(72)$$

which leads to a reduction in bosonic moduli. A remarkable feature of the Grassmannian sigma model is that this reduction in moduli has a consistent interpretation in terms of an enhanced gauge symmetry for the composite. More precisely, if we formulate the corresponding gauged linear sigma models, the above transition corresponds directly to the gauge symmetry enhancement $U(1)\times U(1)\rightarrow U(2)$, recalling that the nonlinear model is realized in the limit in which the gauge kinetic term decouples. In any regime where these gauge modes become dynamical, say at 1-loop, this picture becomes quite consistent with the construction of [23], and indeed more generally with any realization of k-walls in terms of D-branes [55].

The leading behavior of the potential on the asymptotic moduli space of two 1-walls is calculable within the SQCD regime, and aspects of the binding process in Eq. (72) can be studied within the context of the corresponding $\mathcal{N}=1$ gauged linear sigma model [51,52], but we will defer discussion [56] of these features, and other details of the world volume dynamics. It is important to note that, while this analysis is tractable for small m in the SQCD regime, the lack of supersymmetric constraints on the SQCD Kähler metric makes extrapolation to SYM theory at any more than a speculative level fraught with difficulty. This is why the world volume index has a privileged position as essentially the only protected quantity that we are guaranteed can survive this transition.

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