

Gauge-fixing dependence of Φ -derivable approximations

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We examine the problem of gauge dependence of the two-particle irreducible (2PI) effective action and its Φ -derivable approximations in gauge theories. The dependence on the gauge-fixing condition is obtained. The result shows that Φ -derivable approximations, defined as truncations of the 2PI effective action at a certain order, have a controlled gauge dependence, i.e. the gauge dependent terms appear at higher order than the truncation order. Furthermore, using the stationary point obtained for the approximation to evaluate the complete 2PI effective action boosts the order at which the gauge dependent terms appear to twice the order of truncation. We also comment on the significance of this controlled gauge dependence.

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I. INTRODUCTION

Perturbative approaches to the study of equilibrium and nonequilibrium properties of hot and dense media may lead to inconsistencies and are often plagued with infrared divergences. These problems are linked to the fact that calculations in terms of the bare quantities of the underlying quantum field theory (and perturbative approximations thereof) fail to describe the collective phenomena in the medium. A strategy to tackle this handicap of the theory is to work with dressed quantities, in which the most relevant effects of the interacting ensemble are accounted for. These dressed quantities are obtained by means of nonperturbative resummation schemes, which usually involve solving a set of self-consistent equations.

An arbitrary resummation scheme will however not guarantee that the conservation laws of the original theory are preserved by the dressed quantities. A way to solve this problem is by formulating the scheme in terms of an action functional that respects the symmetries of the original theory. A particular kind of such action functionals was first introduced in the study of nonrelativistic Fermi systems by Luttinger and Ward [1], De Dominicis and Martin [2], and Baym [3] and later generalized to relativistic field theories by Cornwall, Jackiw, and Toumboulis [4]. These functionals, which are derived from the so-called two-particle irreducible (2PI) effective action, involve a diagrammatic expansion in terms of 2PI skeleton graphs. A particular choice for an action functional is obtained by truncating this diagrammatic series, in this way defining what is called a Φ -derivable approximation. A variational principle applied to the resulting action leads to a set of self-consistent equations from which the dressed quantities are obtained.

A manifest advantage of such a functional formulation is that global symmetries of the original theory are preserved. Additionally, the variational principle used to determine the dressed quantities guarantees thermodynamic consistency [3]. All these useful properties make Φ -derivable approximations a very attractive mathematical framework for the study of properties of high-energy plasmas. In particular, they may prove useful for QCD plasmas, whose interest has grown in recent years due to the possibility of creating quark-gluon plasma in heavy-ion collision experiments at Brookhaven

and CERN. Calculations of thermodynamical quantities such as the entropy [5] and free energy [6] have been achieved using these methods. In those calculations a resummation of the physics encoded in the hard thermal loops (HTL) was performed. It is important to mention the fact that, due to the remarkable symmetry properties of the HTL, the results in Refs. [5,6] are manifestly gauge invariant. Nonequilibrium properties can also be formally studied within these approximation schemes [7]. They could be used to shed some light on important issues such as thermalization and loss of initial correlations. Very interesting results in this direction have been obtained [8] with scalar models.

However, an extension of these approximation schemes beyond the HTL regime in the study of QCD plasmas is still lacking. There are two main problems involved. One is that renormalization seems to be a nontrivial issue, as shown in explicit calculations for scalar theories [9]. To deal with this obstacle, a recent approach based on Bogolubov-Parasiuk-Hepp-Zimmermann (BPHZ) renormalization has been proposed by van Hees and Knoll [10]. The other main problem is the fact that gauge invariance may be lost in the approximations. This is because, in general, the solutions for the dressed propagators and/or vertices do not satisfy Ward identities. In particular this implies that thermodynamical quantities computed within these approximations will suffer from gauge dependence. This pathology shows up as an explicit dependence on the choice of gauge-condition.

In this paper we study the problem of gauge dependence of the 2PI effective action and its Φ -derivable approximations. In Sec. II we review the general formalism of Φ -derivable approximations and introduce the notation to be used. In Sec. III we apply the formalism to gauge theories and determine the dependence of the 2PI effective action under a change of the gauge-fixing condition. From the result one sees that the 2PI effective action is gauge independent at its stationary point. This was already shown for the 1PI effective action and expected from general arguments [11]. In Sec. IV we apply the result of Sec. III to the Φ -derivable approximations that result from truncating the 2PI effective action at a certain order. We show that these approximations have a controlled gauge-fixing dependence, i.e. the gauge-dependent terms appear at higher order. We discuss in Sec. V that the use of Φ -derivable approximations restricts the

choices of gauge fixing available, if they are indeed to be good approximations to the exact theory. This prevents the high-order gauge-dependent terms to take arbitrarily large values, such that the gauge dependence will be indeed controlled.

II. 2PI EFFECTIVE ACTION AND Φ -DERIVABLE APPROXIMATIONS

The generating functional for correlation functions can be written as

$$Z[J, K] = \int \mathcal{D}\varphi e^{i\{S[\varphi] + J_i\varphi^i + 1/2\varphi^i K_{ij}\varphi^j\}}, \quad (1)$$

where $S[\varphi]$ is the action, φ represents the fields and the J and K are auxiliary external sources. We use a shorthand notation where Latin indices stand for all field and current attributes [i.e. $\varphi(x) \rightarrow \varphi_i$] and summation and/or integration over repeated indices is understood, i.e. $J_i\varphi^i = \int d^4x J(x)\varphi(x)$.¹ The generating functional of connected diagrams W is defined from Z as

$$W[J, K] = -i \log(Z[J, K]). \quad (2)$$

The expectation value of a functional $O[\varphi]$ is given by

$$\langle O[\varphi] \rangle \equiv \frac{\int \mathcal{D}\varphi O[\varphi] e^{i\{S[\varphi] + J_i\varphi^i + 1/2\varphi^i K_{ij}\varphi^j\}}}{\int \mathcal{D}\varphi e^{i\{S[\varphi] + J_i\varphi^i + 1/2\varphi^i K_{ij}\varphi^j\}}} = iO\left[\frac{\delta}{\delta(iJ)}\right]W. \quad (3)$$

Mean fields ϕ^i and connected correlation functions $G^{ijk\dots}$ can then be obtained by functional differentiations of $W[J]$ as

$$i\frac{\delta W}{\delta(iJ_i)} = \phi^i, \quad i\frac{\delta^2 W}{\delta(iJ_i)\delta(iJ_j)} = G^{ij}, \quad (4)$$

$$i\frac{\delta^N W}{\delta(iJ_i)\delta(iJ_j)\delta(iJ_k)\dots} = G^{ijk\dots}.$$

Functional differentiations of $W[J, K]$ with respect to the bilocal currents K may generate also disconnected diagrams. For example, differentiating once with respect to K leads to

$$i\frac{\delta W[J, K]}{\delta(iK_{ij})} = \frac{1}{2}(\phi^i\phi^j + G^{ij}). \quad (5)$$

A functional Legendre transform in the mean field ϕ^i and the two-point function G^{ij} leads to the so-called 2PI effective action

$$\Gamma[\phi, G] = W[J, K] - J_i\phi^i - \frac{1}{2}K_{ij}(\phi^i\phi^j + G^{ij}). \quad (6)$$

From its definition one can derive the relations

$$\frac{\delta\Gamma[\phi, G]}{\delta\phi^i} = -J_i - K_{ij}\phi^j,$$

$$\frac{\delta\Gamma[\phi, G]}{\delta G^{ij}} = -\frac{1}{2}K_{ij}. \quad (7)$$

With the help of Eq. (7) one can write the expression for the expectation value of a functional $O[\varphi]$ in terms of the 2PI effective action as

$$\langle O[\varphi] \rangle = e^{-i\Gamma[\phi, G]} \int \mathcal{D}\varphi O[\varphi] e^{i\{S[\varphi] - \{\delta\Gamma[\phi, G]/\delta\phi_i\}(\varphi - \phi)_i - \{\delta\Gamma[\phi, G]/\delta G_{ij}\}[(\varphi - \phi)_i(\varphi - \phi)_j - G_{ij}]\}}. \quad (8)$$

The 2PI effective action can be cast into the convenient form [1–4, 7]

$$\Gamma[\phi, G] = S_0[\phi] + ic \text{Tr}\{\log(G^{-1}) + G(G_0^{-1} - G^{-1})\} - i\Phi[\phi, G], \quad (9)$$

where S_0 is the free part of action, G_0 is the bare two-point function $(-i\delta^2 S_0[\varphi]/\delta\varphi\delta\varphi)^{-1}$ and c is a constant equal to

1/2 for bosons and -1 for fermions. The functional $\Phi[\phi, G]$ consists on the sum of all two-particle-irreducible (2PI) *skeleton diagrams* with bare vertices and dressed propagators. In this context skeleton diagrams are those without self-energy insertions. Non-2PI diagrams with mean field insertions are also included in the definition of Φ .² For example, in the case of a theory with quartic interactions (such as $\lambda\phi^4$), the above expression can be written graphically as

¹The time integration involved in this functional product can also run along a contour \mathcal{C} in the complex plane such as the ones used in the real and imaginary time formalisms of thermal field theory. This detail will however not be important in our calculations, so we will omit the subscript \mathcal{C} in the integrations.

²In the literature [4] Φ is usually defined in such a way that it only contains strict 2PI diagrams. This involves a redefinition of the action to include mean fields and tadpoles. We prefer to the above notation where all interaction parts are placed in Φ . Of course, both definitions agree when $\phi = 0$.

$$\Gamma[\phi, G] = S_0[\phi] + ic \left\{ \text{Diagram 1} + \text{Diagram 2} \right\} - i \left\{ \frac{1}{24} \text{Diagram 3} + \frac{1}{4} \text{Diagram 4} + \frac{1}{8} \text{Diagram 5} + \frac{1}{12} \text{Diagram 6} + \frac{1}{48} \text{Diagram 7} + \dots \right\} \quad (10)$$

where the thick lines are *dressed* propagators, the small lollipops are the mean fields and the cross-hatched blob is the one-particle irreducible self-energy Π , defined from the dressed propagator via the usual Dyson equation $G^{-1} = G_0^{-1} + i\Pi$.

In practice one has to restrict to an approximated version of the 2PI effective action that results from considering only a certain subset of diagrams in the functional Φ . This defines a Φ -derivable approximation. One typically considers the loop expansion of skeleton diagrams as pictured in Eq. (10) (and referred to as the *skeleton-loop expansion* in the following), and truncates it at a given order. In this way the 2PI effective action Γ is split into two pieces: the truncated part Γ_0 , and the higher order part Γ_1 . Then one takes Γ_0 as the approximated effective action. This action defines *approximate* mean fields and two-point functions ϕ_{ap} and G_{ap} , which result from the stationarity condition, i.e. from the implicit functional equation (7) for vanishing sources J and K , as

$$\left. \frac{\delta \Gamma_0[\phi, G]}{\delta \phi} \right|_{\phi_{\text{ap}}, G_{\text{ap}}} = 0 \quad \text{and} \quad \left. \frac{\delta \Gamma_0[\phi, G]}{\delta G} \right|_{\phi_{\text{ap}}, G_{\text{ap}}} = 0. \quad (11)$$

These *approximate* mean fields and two-point functions defined from the truncated action Γ_0 differ from the *exact* ones, which are obtained from the stationary point of the complete 2PI effective action as

$$\left. \frac{\delta \Gamma[\phi, G]}{\delta \phi} = \frac{\delta(\Gamma_0 + \Gamma_1)}{\delta \phi} \right|_{\phi_{\text{ex}}, G_{\text{ex}}} = 0$$

$$\text{and} \quad \left. \frac{\delta \Gamma[\phi, G]}{\delta G} = \frac{\delta(\Gamma_0 + \Gamma_1)}{\delta G} \right|_{\phi_{\text{ex}}, G_{\text{ex}}} = 0. \quad (12)$$

We end this section by noting that one could also construct more general effective actions by including higher-point external sources $J_i, K_{ij}, L_{ijk}, \dots$ into the functional $W = W[J_i, K_{ij}, L_{ijk}, \dots]$ and performing a Legendre transform as follows

$$\Gamma[\phi_i, G_{ij}, G_{ijk}, \dots]$$

$$= W[J_i, K_{ij}, L_{ijk}, \dots] - J_i \phi^i - \frac{1}{2} K_{ij} (\phi^i \phi^j + G^{ij})$$

$$- \frac{1}{6} L_{ijk} (G^{ijk} + 3G^{ij} \phi^k + \phi^i \phi^j \phi^k) - \dots \quad (13)$$

This form of the effective action can be rewritten as a diagrammatic series in terms of skeleton diagrams of the n -point vertex functions [12] and can be used for generalized Φ -derivable approximations.

III. GAUGE-FIXING DEPENDENCE OF THE 2PI EFFECTIVE ACTION

We consider now the case of a pure Yang-Mills theory with gauge group G . Its action is given by

$$S_{YM} = - \int d^4x \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}(x), \quad (14)$$

where $F_{\mu\nu} \equiv F_{\mu\nu}^a T_a = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu, A_\nu]$ is the field-strength tensor of the gauge field $A_\mu = A_\mu^a T_a$, g is the (unrenormalized) coupling constant and T_a are the generators of the Lie algebra of the gauge group G .

The action is invariant under gauge transformations $U(x) \in G$ of the gauge potential A_μ :

$$A_\mu \rightarrow U A_\mu(x) = U(x) A_\mu(x) U^{-1}(x) - \frac{i}{g} [\partial_\mu U(x)] U^{-1}(x). \quad (15)$$

This invariance implies that the functional integrals over gauge field configurations are ill-defined. One gets around this difficulty by the Faddeev-Popov gauge-fixing procedure, which introduces a gauge-breaking term S_{GF} into the action. In the context of Becchi-Rouet-Stora (BRS) quantization this term is realized in a useful manner by introducing some auxiliary fields: the Faddeev-Popov fermionic ghost fields c^a and \bar{c}^a and the bosonic Lautrup-Nakanishi fields B^a . The gauge-fixing is implemented through the condition $C^a[A] = 0$, where a typical choice is the covariant gauge $C^a[A] = \partial^\mu A_\mu^a$. The gauge-fixed action then reads

$$S = S_{YM} + S_{GF}$$

$$= \int d^4x \left\{ - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}(x) - \bar{c}_a(x) \frac{\delta C^a[A]}{\delta A_{b\mu}} (D_\mu c(x))_b \right.$$

$$\left. + B_a(x) C^a[A] - \frac{1}{2} \xi B_a(x) B^a(x) \right\}, \quad (16)$$

where $D_\mu \equiv \partial_\mu - ig T^a A_a$ is the covariant derivative and ξ is the gauge-fixing parameter.

The action obtained by adding this gauge-fixing term is no longer invariant under local gauge transformations (15). However, it is invariant under BRS transformations, which are defined as

$$\delta_{BRS} A_\mu^a = \epsilon (D_\mu c)^a,$$

$$\delta_{BRS} c^a = i \epsilon g c^2,$$

$$\delta_{BRS} \bar{c}^a = -\epsilon B^a,$$

$$\delta_{BRS} B^a = 0, \quad (17)$$

where ϵ is an infinitesimal global anticommuting parameter

and c^2 is a short-hand notation for $(T_a c^a)(T_b c^b) = 1/2[T^a, T^b]c^a c^b$. The Lautrup-Nakanishi field B has been introduced to ensure the nilpotency of the BRS charge Q_{BRS} , defined as $\delta_{BRS} = \epsilon Q_{BRS}$. It allows for a convenient rewriting of the gauge-breaking term as a complete BRS variation

$$S_{GF} = Q_{BRS} \int d^4x \left\{ \frac{1}{2} \xi \bar{c}_a(x) B^a(x) - \bar{c}_a(x) C^a[A] \right\} \\ \equiv Q_{BRS} \Psi. \quad (18)$$

Using the notation of Sec. II the generating functional $Z[J, K]$ for this gauge theory can be compactly written as

$$Z[J, K] = \mathcal{N}_\xi \int \mathcal{D}\varphi e^{i\{S_{YM} + Q_{BRS}\Psi + J_i \varphi^i + 1/2 \varphi^i K_{ij} \varphi^j\}} \\ = e^{i\{\Gamma[\phi, G] + J_i \phi^i + 1/2 K_{ij}(\phi^i \phi^j + G^{ij})\}}, \quad (19)$$

where φ denotes collectively all fields $\{A_\mu^a(x), c^a(x), \bar{c}^a(x), B^a(x)\}$, J and K denote all their associated currents $\{J_A, J_c, J_{\bar{c}}, J_B, K_{AA}, K_{c\bar{c}}, K_{BB}\}$, and Latin indices stand for both space-time and group indices, i.e. $A_\mu^a(x) \rightarrow A^i$. This notation is used to allow one to write formulas in a compact way, though one should bear in mind that the ghost fields c and \bar{c} and their associated local currents J_c and $J_{\bar{c}}$ are anti-commuting variables. \mathcal{N}_ξ is a ξ -dependent infinite constant generated during the Faddeev-Popov gauge-fixing procedure. Its gauge parameter dependence can be seen already in the free theory and can be absorbed into the action by rescaling the ghost fields by $\xi^{-1/4}$. Hence this constant will not play a role in the following.

Having set up the notation we turn now to study the gauge dependence of the 2PI effective action. We study how it transforms both under a change of the gauge-fixing condition $C^a[A] \rightarrow C^a[A] + \Delta C^a[A]$ and gauge parameter $\xi \rightarrow \xi + \Delta \xi$, or more generally, under a change $\Psi \rightarrow \Psi + \Delta \Psi$. Under this shift of gauge condition, the effective action, the mean field and the two-point function, respectively, change as

$$\Gamma \rightarrow \Gamma' = \Gamma + \Delta \Gamma, \\ \phi \rightarrow \phi' = \phi + \Delta \phi, \quad (20) \\ G \rightarrow G' = G + \Delta G.$$

The currents J_i and K_{ij} are taken to be gauge independent since they are external. This fact allows us to calculate immediately from Eq. (7) how much the first functional derivatives of the effective action vary under the gauge-fixing change, obtaining

$$\Delta \left(\frac{\delta \Gamma}{\delta G_{ij}} \right) = 0, \\ \Delta \left(\frac{\delta \Gamma}{\delta \phi_i} \right) = 2 \Delta \phi_j \frac{\delta \Gamma}{\delta G_{ij}}. \quad (21)$$

The first functional derivatives of Γ are used to find the stationary point by setting them to zero as done in Eq. (12). From Eq. (21) one notices that the stationarity condition itself is only gauge invariant if it is realized *simultaneously* for both arguments ϕ and G .

To compute the variation of the effective action Γ itself one can use the relations (7) to cast Eq. (19) into the convenient form

$$e^{i\Gamma_\Psi[\phi, G]} = \int \mathcal{D}\varphi e^{i\{S_{YM} + Q_{BRS}\Psi - (\varphi - \phi)_i [\delta\Gamma/\delta\phi_i] - [(\varphi - \phi)_i (\varphi - \phi)_j - G_{ij}] \delta\Gamma/\delta G_{ij}\}}. \quad (22)$$

For simplicity and later convenience we denote the field combinations $(\varphi - \phi)_i$ and $[(\varphi - \phi)_i (\varphi - \phi)_j - G_{ij}]$ that appear in the exponent of Eq. (22) as $\tilde{\varphi}_i$ and \tilde{G}_{ij} , respectively. These have the property that their expectation values $\langle \tilde{\varphi}_i \rangle$ and $\langle \tilde{G}_{ij} \rangle$ vanish.

After a change of gauge condition $\Psi \rightarrow \Psi' = \Psi + \Delta \Psi$ Eq. (22) becomes

$$e^{i\Gamma'} = e^{i(\Gamma + \Delta \Gamma)} \\ = \int \mathcal{D}\varphi \{ e^{i\{S_{YM} + Q_{BRS}\Psi - \tilde{\varphi}_i [\delta\Gamma/\delta\phi_i] - \tilde{G}_{ij} [\delta\Gamma/\delta G_{ij}]\}} \\ \times e^{i\{Q_{BRS}\Delta\Psi + \Delta\phi_i [\delta\Gamma/\delta\phi_i] + \Delta G_{ij} [\delta\Gamma/\delta G_{ij}] + \Delta\phi_i \Delta\phi_j [\delta\Gamma/\delta G_{ij}]\}} \}, \quad (23)$$

which using the notation of Eq. (3) leads to

$$e^{i\Delta\Gamma} = \langle e^{i\{Q_{BRS}\Delta\Psi + \Delta\phi_i [\delta\Gamma/\delta\phi_i] + \Delta G_{ij} [\delta\Gamma/\delta G_{ij}] + \Delta\phi_i \Delta\phi_j [\delta\Gamma/\delta G_{ij}]\}} \rangle. \quad (24)$$

This result is valid for any finite change in the gauge-fixing conditions. To proceed further we restrict ourselves to infinitesimal variations $\Delta \Psi$. Then one can expand both sides of Eq. (24) to obtain

$$\Delta \Gamma = \langle Q_{BRS} \Delta \Psi \rangle + \Delta \phi_i \frac{\delta \Gamma}{\delta \phi_i} + \Delta G_{ij} \frac{\delta \Gamma}{\delta G_{ij}} + O(\Delta^2) \quad (25)$$

where we used the fact that $\Delta\phi$ and ΔG are of order $O(\Delta\Psi)$. This can be easily checked. Indeed, following the same steps as to obtain Eq. (24) one gets for the mean field

$$\phi' = \phi + \Delta\phi = e^{-i\Delta\Gamma} \langle \varphi e^{i\{Q_{BRS}\Delta\Psi + \Delta\phi_i[\delta\Gamma/\delta\phi_i] + \Delta G_{ij}[\delta\Gamma/\delta G_{ij}] + \Delta\phi_i\Delta\phi_j[\delta\Gamma/\delta G_{ij}]\}} \rangle, \quad (26)$$

which using Eq. (24) reduces to

$$\phi + \Delta\phi = \frac{\langle \varphi e^{iQ_{BRS}\Delta\Psi} \rangle}{\langle e^{iQ_{BRS}\Delta\Psi} \rangle}, \quad (27)$$

and expanding in $\Delta\Psi$ yields

$$\Delta\phi_i = i\langle \tilde{\varphi}_i Q_{BRS}\Delta\Psi \rangle + O(\Delta^2). \quad (28)$$

Similarly one computes the variation of the two-point function obtaining

$$\Delta G_{ij} = i\langle \tilde{G}_{ij} Q_{BRS}\Delta\Psi \rangle + O(\Delta^2), \quad (29)$$

which verifies our statement above. Moreover, Eqs. (28) and (29) can be used to write Eq. (25) as

$$\begin{aligned} \Delta\Gamma = & \langle Q_{BRS}\Delta\Psi \rangle + i\langle \tilde{\varphi}_i Q_{BRS}\Delta\Psi \rangle \frac{\delta\Gamma}{\delta\phi_i} \\ & + i\langle \tilde{G}_{ij} Q_{BRS}\Delta\Psi \rangle \frac{\delta\Gamma}{\delta G_{ij}} + O(\Delta^2). \end{aligned} \quad (30)$$

One expects the stationary point of the effective action, i.e. when its functional derivatives are set to zero, to be gauge-independent. That is still not obvious from Eq. (30) since it appears that the first term in the right-hand side (RHS) would not vanish. For that, one can make use of the following trick [13]. Consider the expectation value of the gauge-fixing change $\Delta\Psi$, namely

$$\langle \Delta\Psi \rangle = e^{-i\Gamma} \int \mathcal{D}\varphi \Delta\Psi e^{i\{S_{YM} + Q_{BRS}\Psi - \tilde{\varphi}_i[\delta\Gamma/\delta\phi_i] - \tilde{G}_{ij}[\delta\Gamma/\delta G_{ij}]\}}. \quad (31)$$

One can do a BRS transformation $\varphi \rightarrow \varphi + \epsilon Q_{BRS}\varphi$ on the field variables in the path-integral. This transformation leaves the measure invariant and only amounts to a shift of the integration variable, so the equation remains the same. The left-hand side can be however rewritten so that

$$\langle \Delta\Psi \rangle = e^{-i\Gamma} \int \mathcal{D}\varphi (\Delta\Psi + \epsilon Q_{BRS}\Delta\Psi) e^{i\{S_{YM} + Q_{BRS}\Psi - \tilde{\varphi}_i[\delta\Gamma/\delta\phi_i] - \tilde{G}_{ij}[\delta\Gamma/\delta G_{ij}] - \epsilon Q_{BRS}\tilde{\varphi}_i[\delta\Gamma/\delta\phi_i] - \epsilon Q_{BRS}\tilde{G}_{ij}[\delta\Gamma/\delta G_{ij}]\}}, \quad (32)$$

where we have used the fact that $\Delta\Psi[\varphi + \epsilon Q_{BRS}\varphi] = \Delta\Psi + \epsilon Q_{BRS}\Delta\Psi$, which follows from the definition (18). The BRS charge Q_{BRS} appearing in this expression does not operate on the mean fields ϕ and two-point functions G that are part of $\tilde{\varphi}$ and \tilde{G} , but only on the fields to be path-integrated over. Expanding the rhs of Eq. (32) in the anticommuting parameter ϵ leads to

$$\begin{aligned} \langle Q_{BRS}\Delta\Psi \rangle = & -i\left\langle \Delta\Psi Q_{BRS}\left(\tilde{\varphi}_i \frac{\delta\Gamma}{\delta\phi_i}\right) \right\rangle \\ & -i\left\langle \Delta\Psi Q_{BRS}\left(\tilde{G}_{ij} \frac{\delta\Gamma}{\delta G_{ij}}\right) \right\rangle, \end{aligned} \quad (33)$$

where the quantities have been reorganized so that the equation is valid for all fields, both commuting (A_μ and B) and anticommuting (\bar{c} and c). Combinations like $\tilde{\varphi} \delta\Gamma/\delta\phi$ or $\tilde{G} \delta\Gamma/\delta G$ are always commuting so it is preferable to have them in this form.

The same procedure can be applied also to the expectation values $\langle \tilde{\varphi}_a \Delta\Psi \rangle$ and $\langle \tilde{G}_{ab} \Delta\Psi \rangle$ to obtain

$$\begin{aligned} \langle \tilde{\varphi}_a Q_{BRS}\Delta\Psi \rangle = & \langle \Delta\Psi Q_{BRS}\tilde{\varphi}_a \rangle - i\left\langle \tilde{\varphi}_a \Delta\Psi Q_{BRS}\left(\tilde{\varphi}_j \frac{\delta\Gamma}{\delta\phi_j}\right) \right\rangle \\ & - i\left\langle \tilde{\varphi}_a \Delta\Psi Q_{BRS}\left(\tilde{G}_{jk} \frac{\delta\Gamma}{\delta G_{jk}}\right) \right\rangle, \end{aligned} \quad (34)$$

$$\begin{aligned} \langle \tilde{G}_{ab} Q_{BRS}\Delta\Psi \rangle = & \langle \Delta\Psi Q_{BRS}\tilde{G}_{ab} \rangle \\ & - i\left\langle \tilde{G}_{ab} \Delta\Psi Q_{BRS}\left(\tilde{\varphi}_j \frac{\delta\Gamma}{\delta\phi_j}\right) \right\rangle \\ & - i\left\langle \tilde{G}_{ab} \Delta\Psi Q_{BRS}\left(\tilde{G}_{jk} \frac{\delta\Gamma}{\delta G_{jk}}\right) \right\rangle. \end{aligned} \quad (35)$$

These results enable us to write the change in the effective action $\Delta\Gamma$ only in terms proportional to its functional derivatives simply by substituting Eqs. (33)–(35) into Eq. (30).

One notices that the first terms coming from the rhs of Eqs. (34) and (35) cancel exactly those that come from Eq. (33) when both are substituted into Eq. (30). In this way terms with a single functional derivative do not appear in $\Delta\Gamma$. After some rearrangements one is then left with

$$\begin{aligned}\Delta\Gamma &= \frac{1}{2} \left\langle \Delta\Psi Q_{BRS} \left(\tilde{\varphi}_i \frac{\delta\Gamma}{\delta\phi_i} \tilde{\varphi}_j \frac{\delta\Gamma}{\delta\phi_j} \right) \right\rangle \\ &+ \left\langle \Delta\Psi Q_{BRS} \left(\tilde{\varphi}_i \frac{\delta\Gamma}{\delta\phi_i} \tilde{G}_{jk} \frac{\delta\Gamma}{\delta G_{jk}} \right) \right\rangle \\ &+ \frac{1}{2} \left\langle \Delta\Psi Q_{BRS} \left(\tilde{G}_{ij} \frac{\delta\Gamma}{\delta G_{ij}} \tilde{G}_{kl} \frac{\delta\Gamma}{\delta G_{kl}} \right) \right\rangle + O(\Delta^2),\end{aligned}\quad (36)$$

which can be cast into the compact result

$$\Delta\Gamma[\phi, G] = \frac{1}{2} \left\langle \Delta\Psi Q_{BRS} \left(\tilde{\varphi}_i \frac{\delta\Gamma}{\delta\phi_i} + \tilde{G}_{jk} \frac{\delta\Gamma}{\delta G_{jk}} \right)^2 \right\rangle + O(\Delta^2),\quad (37)$$

where the average and Q_{BRS} only apply to the fields φ contained in $\tilde{\varphi}$ and \tilde{G} .

Equation (37) gives the variation of the 2PI effective action caused by a change in the gauge condition and is the main result of this paper. One sees that when the functional derivatives of Γ are set to zero this variation vanishes, and then the effective action is gauge-fixing independent. This situation occurs precisely at the stationary point, i.e. at the *exact* mean fields ϕ_{ex} and two-point functions G_{ex} .

However, the quantities ϕ_{ex} and G_{ex} are not gauge-fixing independent themselves. Indeed, one can explicitly compute their gauge dependence by applying the condition (12) to Eqs. (28) and (29), obtaining in this manner

$$\Delta\phi_{\text{ex}}^i = i \langle \Delta\Psi Q_{BRS}(\tilde{\varphi}_{\text{ex}})^i \rangle, \quad (38)$$

$$\Delta G_{\text{ex}}^{ij} = i \langle \Delta\Psi Q_{BRS}(\tilde{G}_{\text{ex}})^{ij} \rangle. \quad (39)$$

For the case of the effective action $\Gamma[\phi_i, G_{ij}, G_{ijk}, \dots]$ including higher-point correlation functions, the same procedure leads to the generalized result

$$\begin{aligned}\Delta\Gamma &= -\frac{1}{2} \left\langle \Delta\Psi Q_{BRS} \left(\tilde{\varphi}_i \frac{\delta\Gamma}{\delta\phi_i} + \tilde{G}_{ij} \frac{\delta\Gamma}{\delta G_{ij}} \right. \right. \\ &\left. \left. + \tilde{G}_{ijk} \dots \frac{\delta\Gamma}{\delta G_{ijk} \dots} + \dots \right)^2 \right\rangle + O(\Delta^2),\end{aligned}\quad (40)$$

where the quantities $\tilde{G}_{ijk} \dots$ are given by

$$\tilde{G}_i = \tilde{\phi}_i = \varphi_i - \phi_i, \quad (41a)$$

$$\tilde{G}_{ij} = (\varphi - \phi)_i (\varphi - \phi)_j - G_{ij}, \quad (41b)$$

$$\begin{aligned}\tilde{G}_{ijk} &= (\varphi - \phi)_i (\varphi - \phi)_j (\varphi - \phi)_k - G_{ijk} - G_{ij} (\varphi - \phi)_k \\ &- G_{jk} (\varphi - \phi)_i - G_{ki} (\varphi - \phi)_j,\end{aligned}\quad (41c)$$

$$\begin{aligned}\tilde{G}_{ijkl} &= (\varphi - \phi)_i (\varphi - \phi)_j (\varphi - \phi)_k (\varphi - \phi)_l - G_{ijk} (\varphi - \phi)_l \\ &- G_{jkl} (\varphi - \phi)_i - G_{kli} (\varphi - \phi)_j - G_{lij} (\varphi - \phi)_k \\ &- G_{ij} G_{kl} - G_{ik} G_{jl} - G_{il} G_{kj} \\ &- \{G_{ij} [(\varphi - \phi)_k (\varphi - \phi)_l - G_{kl}] + 5 \text{ permutations}\} \\ &- G_{ijkl},\end{aligned}\quad (41d)$$

etc.

IV. GAUGE-FIXING DEPENDENCE OF Φ -DERIVABLE APPROXIMATIONS

Our main interest is to study the gauge dependence of Φ -derivable approximations. As previously mentioned, they are obtained after one truncates the *skeleton-loop expansion* of the 2PI effective action at a certain order. For definiteness let us consider a truncation at L loops, which translates into a truncation at $O(g^{2L-2})$ for the coupling constant g . Then Γ is split into two pieces

$$\Gamma[\phi, G] = \Gamma_0[\phi, G] (\text{of } O(g^{2L-2})) + \Gamma_1[\phi, G] (\text{of } O(g^{2L})), \quad (42)$$

where the truncated part Γ_0 is used to generate *approximate* mean fields ϕ_{ap} and two-point functions G_{ap} from the stationarity condition (11).

The separation of the effective action Γ into Γ_0 and Γ_1 can be performed directly on the result (37) for the variation $\Delta\Gamma$. Evaluated at the approximate mean fields ϕ_{ap} and two-point functions G_{ap} yields

$$\begin{aligned}\Delta(\Gamma_0 + \Gamma_1)[\phi_{\text{ap}}, G_{\text{ap}}] &= -\frac{1}{2} \left\langle \Delta\Psi Q_{BRS} \left(\frac{\delta\Gamma_1}{\delta\phi_i} \Big|_{\phi_{\text{ap}}, G_{\text{ap}}} \tilde{\varphi}_{i,\text{ap}} \right. \right. \\ &\left. \left. + \frac{\delta\Gamma_1}{\delta G_{jk}} \Big|_{\phi_{\text{ap}}, G_{\text{ap}}} \tilde{G}_{jk,\text{ap}} \right)^2 \right\rangle + O(\Delta^2),\end{aligned}\quad (43)$$

where we used the fact that ϕ_{ap} and G_{ap} correspond to the stationary point of Γ_0 .³

On one hand, Eq. (43) implies that the *truncated* effective action Γ_0 evaluated at its corresponding physical mean fields ϕ_{ap} and propagators G_{ap} is gauge independent up to the order of truncation, i.e. $O(g^{2L-2})$. This is so since Γ_1 is of order $O(g^{2L})$ and the rhs of Eq. (43) is of order $O(\Gamma_1^2)$, so to first order $\Delta\Gamma_0 \approx -\Delta\Gamma_1 \approx O(g^{2L})$.

On the other hand, Eq. (43) tells us that the *complete* action Γ evaluated at the *approximate* mean fields and propa-

³Since one works at the stationary point of Γ_0 instead of the exact one obtained from Γ this implies immediately from Eq. (7) that expectation values $\langle \dots \rangle$ are here evaluated at the values of the currents given by $J' = -\delta\Gamma_1/\delta\phi + 2\phi\delta\Gamma_1/\delta G$ and $K' = -2\delta\Gamma_1/\delta G$. However, since $\Gamma_1 \sim O(g^{2L})$, the expectation values are in first approximation equal to those obtained with vanishing currents.

gators obtained from Γ_0 is gauge-fixing independent up to order $O(g^{4L})$, i.e. twice the order of Γ_1 . This is a consequence of having the square of the functional derivatives of Γ_1 in the rhs of Eq. (43) and can be understood with a diagrammatic argument. To see that, first note⁴ that the diagrams in the loop expansion of the 2PI effective action Γ are *skeleton* diagrams, so without any self-energy insertions. The approximate propagator G_{ap} obtained from truncating the skeleton series of Γ to Γ_0 at L loops [or at $O(g^{2L-2})$] is the solution of the variational condition (12), which can be interpreted as a dressing of the bare propagator with all the self-energy contributions that come from cutting one line in the 2PI diagrams of Γ_0 . Evaluating the effective action Γ at G_{ap} entails substituting this propagator in the diagrams of the *skeleton-loop expansion*. The outcome can be expanded perturbatively to compare directly with the usual *perturbative loop expansion* of the 1PI effective action. One can check that both expansions match perfectly up to $2L$ loops, or $O(g^{4L-2})$. They differ at $2L+1$ loops because, by construction, diagrams that would result from dressing skeleton diagrams of Γ_1 with self-energy contributions to G_{ap} coming also of Γ_1 , do not appear in the expansion of the skeleton series considered. However, they are present in the perturbative loop expansion. The importance of the fact that both expansions match up to $2L$ loops is that, since the perturbative loop expansion is gauge-invariant at every loop order, one can immediately conclude that so must be the skeleton-loop expansion of $\Gamma[G_{\text{ap}}]$ up to $2L$ loops, or, in other words, up to $O(g^{4L})$.

In this manner, Eq. (43) shows that Φ -derivable approximations, as truncations to the 2PI effective action, have a controlled gauge-fixing dependence, in the sense that gauge dependent terms appear at higher orders.

V. CHOICE OF GAUGE CONDITION

A large body of experience with gauge theory has led to the common view that one should not tamper with gauge invariance. Yet, we explore here the possibility of accepting a controlled amount of gauge dependence in the computation of physical quantities. The question is then, what is a good choice of gauge fixing? To be specific, consider the class of covariant gauges described by $C^a = \partial^\mu A_\mu^a$. Then we have to decide on a reasonable choice for the gauge parameter ξ . Evidently, ξ should be such that it does not upset the assumption that Γ_1 may be neglected compared to Γ_0 .

That such upset can happen is easier to see in the more familiar perturbative case. There we have the loop expansion in terms of bare propagators. Consider for simplicity diagrams without ghosts. The gauge propagators have a longitudinal part proportional to ξ , so in a given diagram with I internal lines we would have the factor ξ^I . In terms of the number of three- and four-point vertices V_3 and V_4 and the number of loops L it can be written as $\xi^{2L-2+V_3/2}$. Together with the powers g^{2L-2} in the bare coupling constant g , the diagram has an overall factor $(g\xi)^{2L-2}\xi^{V_3/2}$. Taking ξ big

enough, say $|\xi| > 1/g$, various terms belonging to different orders of g in the perturbation expansion would be shuffled. This will evidently upset our ordering principle.

In a Φ -derivable approximation, however, we consider the loop expansion in terms of dressed propagators where their ξ -dependence is not clear *a priori*. For that one needs to find the stationary point of Γ_0 , and this is done after assuming that Γ_1 may be neglected compared to Γ_0 . Provided we had the explicit form of the dressed propagator, an argument similar to the one above would give the range of ξ that is allowed without upsetting this assumption. Unfortunately, finding the dressed propagators is in general a formidable task. We nevertheless venture the following argument that a good choice for ξ is in the interval $(0,2)$.

Assume that the Φ -derivable approximation gives indeed an approximation to the path integral

$$Z = \int \mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c} \exp \left[\frac{-i}{g^2} \int d^4x \left\{ \frac{1}{4} F_{g=1}^2 + \bar{c} \partial^\mu (\partial_\mu - iA_\mu) c + \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right\} \right], \quad (44)$$

where we have integrated out the B field and rescaled $A \rightarrow A/g$, $(c, \bar{c}) \rightarrow (c/\sqrt{g}, \bar{c}/\sqrt{g})$. Then g^2 in the above action is also the ordering parameter in the skeleton-loop expansion of Φ . If we do not want to upset this power counting, ξ should be treated of order one as $g^2 \rightarrow 0$. For finite g^2 it seems best to choose $1/2\xi$ of the same magnitude as the other numerical coefficients in the action, which are $1/4$ and $1/2$ for F^2 , and 1 for the ghost terms. So this suggests the choice ξ in the range $1-2$. Saddle point arguments for $g^2 \rightarrow 0$ are not upset by letting also $\xi \rightarrow 0$, so it is reasonable to allow also values of $\xi \rightarrow 0$. On the other hand, $\xi \gg 1$ would upset the longitudinal parts of the saddle point regions in functional space (as for the perturbative case above). To allow a continuation of the path integral to imaginary time, ξ has to be positive. All these arguments lead us to conclude that ξ is best taken in the range $0-2$.

VI. CONCLUSIONS AND COMMENTS

In this paper, the gauge dependence of the 2PI effective action that defines a Φ -derivable approximation of a gauge theory has been determined. To obtain it we used its definition as a Legendre transform of a generating functional with bilocal sources and the BRS symmetry of the underlying Yang-Mills action. As expected on general grounds, the result shows that the 2PI effective action is gauge independent at its stationary point. Furthermore, the result has been applied to study the gauge dependence of Φ -derivable approximations, defined as truncations of the 2PI effective action at a certain loop order. Even though correlation functions derived within these approximation schemes are known not to fulfill the Ward identities required by the gauge symmetry, it has been shown that the truncated effective action defined at its stationary point has a controlled gauge-fixing dependence, i.e. the explicit gauge dependent terms appear at higher or-

⁴We take $\phi=0$ for simplicity.

der. Furthermore, if one uses the stationary quantities of the truncated action to evaluate the *complete* 2PI effective action, the gauge dependence appears then at twice the order of truncation.

These features might be interesting for the computation of thermodynamical quantities derived from the 2PI effective action in gauge theories, such as the pressure and entropy. The authors of [5] have calculated the entropy of the quark-gluon plasma with an approximate Φ -derivable approximation, but in order to achieve gauge independence, they had to sacrifice the self-consistency guaranteed by working at the stationary point, hence the word approximate. Their approach was nevertheless strongly motivated from a quasiparticle picture of the quark-gluon plasma, which can be used [14] to describe the lattice results [15]. In any case, our considerations suggest that Φ -derivable approximations may allow a systematic method for computing thermodynamic functions without having to sacrifice its remarkable properties. Gauge-fixing dependent artifacts would appear at high orders, thus making the approximation controllable.

It might seem unsatisfactory that the gauge dependence is not completely removed in Φ -derivable approximations. Yet, we propose here to accept a controlled amount of gauge dependence in physical quantities. We argued that Φ -derivable approximations, provided they are indeed an approximation to the exact gauge theory, implicitly restrict the choices of gauges available. This prevents the high-order gauge artifacts from taking arbitrary values that could render any computed quantity physically meaningless. Both the fact that gauge dependent terms appear at higher orders and that they are constrained by this restriction makes the error introduced by breaking gauge invariance controllable and indicate that Φ -derivable approximations may indeed give reasonable answers to physical quantities. A detailed examination to quan-

tify those gauge dependent terms would involve solving a Φ -derivable approximation for gauge theories. So far, the complete solutions for QED and pure glue QCD even at lowest order (2-loop) have not appeared in the literature.

We would like to note that in the derivation presented in this paper we did not discuss aspects related to regularization and renormalization. This makes the calculations heuristic in some respects. We compared the path integral with a skeleton expansion of the effective action, neither of which has been clearly defined. To make the path integral well defined a regularization is needed, preferably nonperturbative, and this should be compatible with the BRS invariance used. With a lattice regularization this is nontrivial [16]. Another point is that the regularization dependence needs to be removed, or at least shown to be negligible (in the case of QED “triviality” is expected to occur). Renormalization is a nontrivial issue in Φ -derivable approximations [9]. A general renormalization procedure, such as the one recently proposed by van Hees and Knoll [10], would be needed in order to have a well defined path-integral. A detailed study of these issues in Φ -derivable approximations of gauge theories constitutes the subject of further investigations.

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