From on-shell to off-shell open gauge theories

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We present an alternative quantization for irreducible open gauge theories. The method relies on the possibility of modifying the classical Becchi-Rouet-Stora-Tyutin (BRST) operator and the gauge-fixing action written as in Yang-Mills type theories in order to obtain an on-shell invariant quantum action by using equations characterizing the full gauge algebra. From this then follows the construction of an off-shell version of the theory. We show how it is possible to build off-shell BRST algebra together with an invariant extension of the classical action. This is realized via a systematic prescription for the introduction of auxiliary fields.

I. INTRODUCTION

It is well known that an on-shell quantization of general gauge theories, i.e., gauge theories which are reducible and/or whose classical gauge algebra is closed only on shell (for a review see Ref. [1]), can be successfully realized in the Lagrangian approach by the Batalin-Vilkovisky (BV) formalism [2].

In this framework, the field content of the theory is doubled by the introduction of the so-called antifields. The procedure consists, through the elimination of the antifields via a gauge-fixing fermion of ghost number (-1), in the construction of the quantum theory in which the effective Becchi-Rouet-Stora-Tyutin (BRST) transformations are nilpotent on shell.

Let us note that the BV approach is not the only alternative to quantize reducible and/or open gauge theories. Indeed, the introduction of a set of auxiliary fields, as in supersymmetric theories [3] or in BF theories [4], may close the gauge algebra, and then gives the possibility of using the standard BRST formalism in the context of the Faddeev-Popov procedure [5].

However, no systematic prescription exists in order to introduce these auxiliary fields so that an approach that will be able to realize the on-shell as well as the off-shell quantization of general gauge theories in a systematic way will appear to be superior to all other available schemes.

Recently [6] we have shown for the case of simple supergravity, how an on-shell quantization approach of the theory can lead us, via a convenient procedure, to find out the structure of auxiliary fields as well as the full BRST operator that realizes off-shell quantization of the theory. The aim of the present paper is to extend the analysis developed in Ref. [6] in order to discuss general irreducible open gauge theories, irrespective of the underlying classical action.

The paper is organized as follows: In Sec. II we perform on-shell quantization for a general irreducible open gauge theory by using the structure of the gauge algebra. This is a new more natural quantization procedure, in the sense that we will not be relying on any set of extra fields. Section III is divided into two sections. In the first one we show how it is possible to introduce a set of auxiliary fields to build the full off-shell quantum action and the associated off-shell BRST symmetry for the case of irreducible open gauge theories of type (2,2). The invariant extension of the classical action is also given. In the next section, a complete generalization is given. In Sec. IV the specific problem of the construction of the minimal set of auxiliary fields for any given irreducible theory is analyzed. Section V is devoted to concluding remarks.

II. ON-SHELL QUANTIZATION

Let us consider an arbitrary gauge theory whose classical action $S(\Phi^i)$ possesses local gauge symmetries,

$$\Delta S = 0, \tag{1}$$

with

$$\Delta \Phi^i = (-)^{i\alpha} R^i_\alpha \varepsilon^\alpha, \tag{2}$$

where $\{\Phi^i, i=1,\ldots,N\}$ describes the set of classical fields of the theory and the operators R^i_α are acting on the parameters $\{\varepsilon^\alpha, \alpha=1,\ldots,d\}$ of the d symmetries of S and $i(\alpha)$ is the parity of $\Phi^i(\varepsilon^\alpha)$. The invariance condition (1) leads to Noether's identity

$$R^i_\alpha \frac{\delta S}{\delta \Phi^i} = 0. \tag{3}$$

Dealing with irreducible symmetries [1], we also have

$$\forall X^\alpha_A: \quad R^i_\alpha X^\alpha_A = 0 \Rightarrow X^\alpha_A = 0, \tag{4}$$

where A represents an arbitrary set of indices.

The condition (3) allows one to define the d operators Δ_α:

$$\Delta_\alpha \Phi^i = R^i_\alpha, \tag{5}$$

which satisfy

$$\Delta_\alpha S = 0. \tag{6}$$

The graded commutator of two transformations is then given by

$$[\Delta_\alpha,\Delta_\beta]\Phi^i=R^j_\alpha\frac{\delta R^i_\beta}{\delta\Phi^j}-(-)^{\alpha\beta}R^j_\beta\frac{\delta R^i_\alpha}{\delta\Phi^j}. \tag{7}$$

Considering that the set of the R^i_α is complete, i.e., all the symmetries of S are known, one can easily find that the most general form of the gauge algebra reads [1]

$$[\Delta_\alpha,\Delta_\beta]\Phi^i=T^\lambda_{\alpha\beta}R^i_\lambda+V^{ij}_{\alpha\beta}\frac{\delta S}{\delta\Phi^j}. \tag{8}$$

Therefore the properties of the gauge algebra will depend on the nature of the structure functions $T^\lambda_{\alpha\beta}$ and the nonclosure functions $V^{ij}_{\alpha\beta}$, which depend in general on the classical fields and are graded antisymmetric with respect to $(\alpha\beta)$ and (ij).

In view of Eq. (8), the generalized graded Jacobi identity can be put in the form

$$\sum_{(\alpha\beta\gamma)}\{R^k_\alpha T^\lambda_{\beta\gamma,k}R^i_\lambda-(-)^{\alpha(\beta+\gamma)}T^\sigma_{\beta\gamma}T^\lambda_{\sigma\alpha}R^i_\lambda+\{R^k_\alpha V^{ij}_{\beta\gamma,k}$$

$$-(-)^{\alpha(\beta+\gamma)}[(-)^{\alpha i}V^{ik}_{\beta\gamma}R^j_{\alpha,k}+(-)^{ij+1}(-)^{\alpha j}V^{jk}_{\beta\gamma}R^i_{\alpha,k}$$

$$+T^\sigma_{\beta\gamma}V^{ij}_{\sigma\alpha}]\}S_{,i}\}=0, \tag{9}$$

where $\Sigma_{(\alpha\beta\gamma)}$ means a cyclic sum over α, β, γ and ",k" means a variation with respect to Φ^k.

However, the standard BRST approach consists of the replacement of the local gauge invariance by a global one. This symmetry is encoded in an operator δ defined via the replacement of the gauge parameters ε^α by the ghost fields c^α with parity $(\alpha+1)$ and ghost number $(+1)$, we have

$$\delta\Phi^i=(-1)^{i(\alpha+1)}R^i_\alpha c^\alpha, \tag{10}$$

which maintains the classical action invariant.

It is easy to show that the action of δ on Φ^i is nilpotent on shell, so that

$$\delta^2\Phi^i=V^{ij}S_{,j}, \tag{11}$$

where

$$V^{ij}=\frac{1}{2}(-)^{\beta(\alpha+1)}(-)^{(i+j)(\alpha+\beta)}V^{ij}_{\alpha\beta}c^\alpha c^\beta,$$

provided that the transformation of the ghost is given by

$$\delta c^\lambda=-\frac{1}{2}(-)^{\beta(\alpha+1)}(-)^{\lambda(\alpha+\beta)}T^\lambda_{\alpha\beta}c^\alpha c^\beta, \tag{12}$$

which is also nilpotent on shell. Indeed, by using the graded Jacobi identity, we obtain

$$R^i_\lambda\delta^2 c^\lambda=(-)^{i(\lambda+1)}\{\delta V^{ij}-[(-)^{i+j(\lambda+1)}V^{ik}R^j_{\lambda,k}c^\lambda$$

$$+(-)^{ij+1}(i\rightleftharpoons j)]\}S_{,j}. \tag{13}$$

This means that $R^i_\lambda\delta^2 c^\lambda$ vanishes on shell and because R^i_λ describes irreducible transformations, then $\delta^2 c^\lambda$ also vanishes on shell and can be cast in the form

$$\delta^2 c^\alpha=Z^{\alpha j}S_{,j}, \tag{14}$$

where the new nonclosure functions $Z^{\alpha j}$ satisfy Eq. (13). This characteristic equation represents the fact that $Z^{\alpha j}$ are not completely independent from V^{ij}. It can also be derived by acting δ on Eq. (11) and written as

$$\{\delta V^{ij}-[(-)^{j(i+k+1)}V^{kj}(\delta\Phi^i)_{,k}+(-)^{j(\alpha+1)}Z^{\alpha j}(\delta\Phi^i)_{,\alpha}$$

$$+(-)^{ij+1}(i\rightleftharpoons j)]\}S_{,j}=0, \tag{15}$$

where ",α" means a variation with respect to c^α. One can remark that the above equation is of the third order in ghost, and indicates the possibility of existence of a new characteristic function V^{ijk} defined by

$$\delta V^{ij}-[(-)^{j(i+k+1)}V^{kj}(\delta\Phi^i)_{,k}+(-)^{j(\alpha+1)}Z^{\alpha j}(\delta\Phi^i)_{,\alpha}$$

$$+(-)^{ij+1}(i\rightleftharpoons j)]=V^{ijk}S_{,k}, \tag{16}$$

where V^{ijk} are restricted by the total graded antisymmetry, $V^{ijk}=(-)^{ij+1}V^{jik}=(-)^{kj+1}V^{ikj}$.

We can also introduce a function $Z^{\alpha ij}$ from the ghost nonclosure function $Z^{\alpha i}$ by acting δ on Eq. (14) and then find the following characteristic equation:

$$\delta Z^{\alpha i}-(-)^{i(\alpha+\beta+1)}Z^{\beta i}(\delta c^\alpha)_{,\beta}-(-)^{i(\alpha+k)}V^{ki}(\delta c^\alpha)_{,k}$$

$$+(-)^{\alpha+1}Z^{\alpha k}(\delta\Phi^i)_{,k}$$

$$=Z^{\alpha ij}S_{,j}, \tag{17}$$

where $Z^{\alpha ij}=(-)^{ij+1}Z^{\alpha ji}$.

It is worth noting that another application of δ on Eq. (16) [Eq. (17)] leads to an equation which allows us to introduce another function of type V^{ijkl} ($Z^{\alpha ji}$), and so on for all orders of application of δ. The general characteristic functions produced in this way are all related by equations derived in the same way as Eqs. (16) and (17). We denote the characteristic functions defined from an equation of order n in application of δ by $V^{i_1\cdots i_n}_n$ and $Z^{\alpha i_1\cdots i_{n-1}}_n$. They are graded antisymmetric with respect to the indices i_l ($l=1,\ldots,n-1,n$). At an order $(n+1)$ we find the following characteristic equations:

$$\delta V^{i_1\cdots i_n}_n-\sum_{m=2}^n(-)^m\{V^{ki_{n-m+2}\cdots i_n}_m(V^{i_1\cdots i_{n-m+1}}_{n-m+1})_{,k}$$

$$-Z^{\alpha i_{n-m+2}\cdots i_n}_m(V^{i_1\cdots i_{n-m+1}}_{n-m+1})_{,\alpha}\}=V^{i_1\cdots i_{n+1}}_{n+1}S_{,i_{n+1}}, \tag{18}$$

$$\delta Z_n^{\alpha i_1 \cdots i_{n-1}} - \sum_{m=2}^{n} (-)^m \{ Z_m^{\beta i_{n-m+1} \cdots i_{n-1}} (Z_{n-m+1}^{\alpha i_1 \cdots i_{n-m}})_{,\beta}$$

$$- Z_m^{\alpha i_1 \cdots i_{m-2} k}(V_{n-m+1}^{i_{m-1} \cdots i_{n-1}})_{,k}$$

$$+ V_m^{k i_{n-m+1} \cdots i_{n-1}} (Z_{n-m+1}^{\alpha i_1 \cdots i_{n-m}})_{,k} \}$$

$$= Z_{n+1}^{\alpha i_1 \cdots i_n} S_{,i_n}, \tag{19}$$

where graded antisymmetrization over all independent combinations related to the indices $(i_1, \ldots, i_n)$ must be carried out. Note that the functions V_n and Z_n have parity $(i_1 + \cdots + i_n + n \bmod 2)$ and $(\alpha + i_1 + \cdots + i_{n-1} + n \bmod 2)$ and ghost numbers (n) and $(n+1)$, respectively.

The existence of these characteristic functions V_n and Z_n permits a classification for irreducible open gauge theories. We will say that a theory is of type (p,q) in the case where $V_n = 0 (Z_n = 0)$ for $n > p (n > q)$. For example, global supersymmetric theories as well as super-Yang-Mills theories are of type $(2,1)$ while simple supergravity is of type $(2,2)$.

In what follows we turn to discuss how to construct the quantum theory of a classical open gauge theory of type (p,q). It is obvious that a δ-exact form of the gauge fixing action cannot be suitable to build the full invariant quantum action because of the on-shell nilpotency of the BRST operator δ. To this end, we generalize the prescription discussed in Ref. [6] for the case of simple supergravity by simply modifying the classical BRST operator δ. As a consequence the gauge-fixing action written as in Yang-Mills theories must be also modified so that the complete quantum action becomes invariant. We first introduce the gauge fermion Ψ of ghost number (-1) to implement the gauge constraints $F_\alpha = 0$ associated to all the invariances of the classical action S, we have

$$\Psi = \bar{c}^\alpha F_\alpha, \tag{20}$$

where $\bar{c}^\alpha (\alpha = 1, \ldots, d)$ represent the antighosts with parity $(\alpha + 1)$ and ghost number (-1), which allow us as usual to define the Stueckelberg auxiliary fields b^α through the action of the transformation δ, so that

$$\delta \bar{c}^\alpha = b^\alpha, \quad \delta b^\alpha = 0. \tag{21}$$

Let us note that the gauge-fixing functions F_α depend only on the classical fields Φ^i, since the gauge symmetries are considered as irreducible.

At the quantum level we have to define a modified BRST operator Q. This will be done by introducing a set of operators δ_n given by

$$\delta_0 \Phi^i = \delta \Phi^i, \tag{22a}$$

$$\delta_n \Phi^i = \frac{1}{n!}(-)^{in + a_n} V_{n+1}^{i i_1 \cdots i_n} \Psi_{,i_1} \cdots \Psi_{,i_n}, \quad n = 1, \ldots, p-1, \tag{22b}$$

for the classical fields, and

$$\delta_0 c^\alpha = \delta c^\alpha, \tag{23a}$$

$$\delta_n c^\alpha = \frac{1}{n!}(-)^{(\alpha+1)n + a_n} Z_{n+1}^{\alpha i_1 \cdots i_n} \Psi_{,i_1} \cdots \Psi_{,i_n},$$

$$n = 1, \ldots, q-1, \tag{23b}$$

where $a_n = \sum_{r=2}^{n} i_r \sum_{s=1}^{r-1} (i_s + 1)$ gives to $(-)^{a_n} \Psi_{,i_1} \cdots \Psi_{,i_n}$ the same graded symmetry properties as $V_{n+1}^{i i_1 \cdots i_n}$ and $Z_{n+1}^{\alpha i_1 \cdots i_n}$. For the other fields $\bar{c}^\alpha$ and b^α the action of the δ_n operators is taken to be trivial, i.e., $\delta_0 \bar{c}^\alpha = \delta \bar{c}^\alpha$, $\delta_0 b^\alpha = \delta b^\alpha$, and $\delta_n \bar{c}^\alpha = \delta_n b^\alpha = 0$ for $n > 0$. We are now able to define the effective BRST operator Q,

$$Q\Phi^i = \sum_{n=0}^{p-1} \delta_n \Phi^i, \quad Qc^\alpha = \sum_{n=0}^{q-1} \delta_n c^\alpha, \tag{24a}$$

$$Q\bar{c}^\alpha = \delta_0 \bar{c}^\alpha, \quad Qb^\alpha = \delta_0 b^\alpha, \tag{24b}$$

which leaves invariant the following full quantum action S_q;

$$S_q = S + \sum_{n=0}^{p-1} \frac{1}{n+1} \delta_n \Psi. \tag{25}$$

The first term $(n=0)$ of the gauge-fixing action, $S_{gf} = \sum_{n=0}^{p-1} 1/n+1 \, \delta_n \Psi$, leads to the standard result of the Yang-Mills type theories while the other terms describe higher ghost couplings which characterize open gauge theories. To prove the invariance of the quantum action (25) under the effective BRST symmetry defined by Eqs. (24a) and (24b) we take advantage of the characteristic equations (18) and (19) together with the on-shell nilpotency (11) and (14) of the classical BRST operator δ.

Furthermore, using again the characteristic Eqs. (18) and (19), we find that the effective BRST operator Q is nilpotent on shell at the quantum level, i.e. with respect to the quantum equations of motion derived from the quantum action (25). Indeed, we have

$$Q^2 \Phi^i = A^{ik} S_{q,k} + B^{\alpha i} S_{q,\alpha}, \tag{26a}$$

$$Q^2 c^\alpha = B'^{\alpha} \dot{S}_{q,i}, \tag{26b}$$

$$Q^2 \bar{c}^\alpha = Q^2 b^\alpha = 0, \tag{26c}$$

where

$$A^{ik} = \sum_{n=1}^{p-1} \frac{(-)^{n-1}}{(n-1)!}(-)^{(i+k)(n+1) + a_{n-1}} V_{n+1}^{iki_1 \cdots i_{n-1}}$$

$$\times \Psi_{,i_1} \cdots \Psi_{,i_{n-1}}, \tag{27a}$$

$$B^{\alpha i} = -\sum_{n=1}^{q-1} \frac{(-)^{\alpha(n+1)}}{(n-1)!}(-)^{i(n+\alpha) + a_{n-1}} Z_{n+1}^{\alpha i i_1 \cdots i_{n-1}}$$

$$\times \Psi_{,i_1} \cdots \Psi_{,i_{n-1}}, \tag{27b}$$

$$B'^{\alpha i} = \sum_{n=1}^{q-1} \frac{1}{(n-1)!}(-)^{(\alpha+i)(n+1)+a_{n-1}}Z_{n+1}^{\alpha i i_1\cdots i_{n-1}}$$

$$\times \Psi_{,i_1}\cdots\Psi_{,i_{n-1}}. \tag{27c}$$

It is remarkable that the used prescription, which simply consists of the modification of the classical BRST operator and of the gauge-fixing action written as in Yang-Mills theories, provides a natural on-shell quantization scheme for open irreducible gauge theories in the sense that it does not need to rely on any set of extra fields (such as antifields).

III. OFF-SHELL QUANTIZATION

We are now going to discuss how we can introduce auxiliary fields, as a generalization of the approach developed in Ref. [6], so that we end up with an off-shell structure for open gauge theories. To this end, and for the sake of the procedure, we perform first the generalization for classical open gauge theories of type (2,2), then a complete generalization will be straightforwardly given.

A. Open gauge theories of type (2,2)

In this case the theory is only characterized by the functions V^{ij} and $Z^{\alpha i}$ and all the remaining characteristic functions V_n and Z_n for $n>2$ vanish. Also for simplicity and to present computations leading to insight in the generalization of the analysis in Ref. [6] to open gauge theories, we consider an open gauge algebra of type (2,2) in which the classical degrees of freedom (Φ^i) as well as the different parameters of the classical symmetry (ε^α) are taken to have odd parity.

For this considered theory the characteristic equations associated to the gauge algebra (18),(19) become

$$\delta V^{ij} - S(ij)\left[-V^{kj}\frac{\delta(\delta\Phi^i)}{\delta\Phi^k} + Z^{\alpha j}\frac{\delta(\delta\Phi^i)}{\delta c^\alpha}\right] = 0, \tag{28a}$$

$$S(ijk)\left[-V^{lk}\frac{\delta V^{ij}}{\delta\Phi^l} + Z^{\alpha k}\frac{\delta V^{ij}}{\delta c^\alpha} + i\leftrightarrow j\right] = 0, \tag{28b}$$

$$\delta Z^{\alpha i} - \left[-\frac{\delta(\delta c^\alpha)}{\delta c^\beta} + V^{ki}\frac{\delta(\delta c^\alpha)}{\delta\Phi^k} - Z^{\alpha k}\frac{\delta(\delta\Phi^i)}{\delta\Phi^k}\right] = 0, \tag{29a}$$

$$S(jk)\left[-Z^{\beta j}\frac{\delta Z^{\alpha k}}{\delta c^\beta} + V^{ij}\frac{\delta Z^{\alpha k}}{\delta\Phi^i}\right] = 0, \tag{29b}$$

where $S(\cdots)$ means that a symmetrization over the indices in brackets is carried out.

Let us now introduce the space C of the $(d\times d)$ invertible matrices. One can define on C (of dimension d^2) a basis of d^2 matrices,

$$\{\Gamma^A\}_{A=1,\ldots,d^2}, \tag{30}$$

which satisfies the orthonormality condition

$$\text{tr}(\Gamma^A\Gamma^B) = d\,\delta^{AB}, \tag{31}$$

where the trace operation is considered as the scalar product on the matrix space. One may also define the inverse basis of Eq. (30) $\{\bar{\Gamma}^A\}_{A=1,\ldots,d^2}$ satisfying

$$\bar{\Gamma}^A_{\alpha\lambda}\Gamma^B_{\lambda\beta} = \Gamma^A_{\alpha\lambda}\bar{\Gamma}^B_{\lambda\beta} = \delta^{AB}\delta_{\alpha\beta}. \tag{32}$$

Furthermore, each matrix M belonging to C may be also decomposed into a symmetric matrix and an antisymmetric one, i.e., $M_{\alpha\beta} = M_{(\alpha\beta)} + M_{[\alpha\beta]}$ where $M_{(\alpha\beta)} = \frac{1}{2}[M_{\alpha\beta} + M_{\beta\alpha}]$ and $M_{[\alpha\beta]} = \frac{1}{2}[M_{\alpha\beta} - M_{\beta\alpha}]$. In other terms this means that C can be decomposed into two subspaces, i.e., $C = C_0 \oplus C_1$ where C_0 is the subspace of the symmetric matrices of dimension $d(d+1)/2$ and C_1 is the subspace of the antisymmetric matrices of dimension $d(d-1)/2$. From all the possible basis on C, we will choose the one which is built from the basis of C_0 and C_1, in order to have

$$(\Gamma^A)^T = (-)^A\Gamma^A, \tag{33}$$

where $A=0(=1)$ for the Γ^A belonging to $C_0(C_1)$. Let us now show that the introduction of such a basis for C is of great help in the introduction of auxiliary fields and then in performing the off-shell quantization of the theory. To this end, one can put the full quantum action of the theory (25) in the form

$$S_q = S + \frac{1}{4}V^{ij}_{\alpha\beta}F_{\rho,i}F_{\sigma,j}c^\alpha c^\beta\bar{c}^\rho\bar{c}^\sigma + Q\Psi, \tag{34}$$

where $F_{\rho,i} = \delta F_\rho/\delta\Phi^i$. We will focus on the second part of the right-hand side of Eq. (34),

$$\tilde{S}_\Lambda = \frac{1}{4}V^{ij}_{\alpha\beta}F_{\rho,i}F_{\sigma,j}c^\alpha c^\beta\bar{c}^\rho\bar{c}^\sigma. \tag{35}$$

By noting $F_{\rho,i}F_{\sigma,j} = F_{\rho\sigma,ij}$, we can perform a kind of Fierzing [3] on Eq. (35). This is based on the observation that the term $V^{ij}_{\alpha\beta}F_{\rho\sigma,ij}$ can be viewed for fixed α and σ as a $d\times d$ matrix which can be expanded into the complete set of Γ^A, we have

$$V^{ij}_{\alpha\beta}F_{\rho\sigma,ij} = C^A_{\alpha\sigma}\Gamma^A_{\beta\rho}, \tag{36}$$

where all the $C^A_{\alpha\sigma}$ are completely determined by Eq. (31),

$$C^A_{\alpha\sigma} = \frac{(-)^A}{d}V^{ij}_{\alpha\lambda}\Gamma^A_{\lambda\delta}F_{\delta\sigma,ij}. \tag{37}$$

Doing the same operation once again on $V^{ij}_{\alpha\lambda}F_{\delta\sigma,ij}$ in Eq. (37), the action (35) can be cast in the form

$$\tilde{S}_\Lambda = \frac{(-)^B}{4d^2}F_{\delta,i}\Gamma^B_{\delta\lambda}V^{ij}_{\lambda\tau}\Gamma^A_{\tau\gamma}F_{\gamma,j}(\bar{c}^\alpha\Gamma^A_{\alpha\beta}c^\beta)(\bar{c}^\rho\Gamma^B_{\rho\sigma}c^\sigma). \tag{38}$$

We are now able to make the following identifications for the auxiliary fields:

$$P^A \equiv (\bar{c}^\alpha \Gamma^A_{\alpha\beta} c^\beta). \tag{39}$$

These fields have even parity and ghosts number zero. The action (38) will then take the form

$$\widetilde{S}_\Lambda = W^{BA} P^A P^B, \tag{40}$$

where

$$W^{BA} = \frac{(-)^B}{4d^2} F_{,i} \Gamma^B V^{ij} \Gamma^A F_{,j}. \tag{41}$$

By a direct calculation one finds: $F_{,i} \Gamma^B V^{ij} \Gamma^A F_{,j} = (-)^{A+B} F_{,i} \Gamma^A V^{ij} \Gamma^B F_{,j}$, so that $W^{BA} = W^{AB}$, and then no symmetrization is required in Eq. (40).

Since no ghost terms are explicitly occurring in the action (40) obtained for the d^2 fields P^A, it can also be considered at the classical level in the way that classically, we can put

$$\widetilde{S} = S + W^{AB} P^A P^B, \tag{42}$$

which will represent the classical extension of the classical action S of the theory. Before investigating the symmetries of this action, an important remark must be pointed out in order to show that the fields P^A totally play the role of auxiliary fields of the theory. The fact that the classical extension (42) is algebraic in P^A (it contains no derivative terms in P^A) allows us to see that they are nonpropagating (nondynamical) fields. They must also not introduce any new degrees of freedom to the classical theory, i.e., their equations of motion derived from Eq. (42) must be completely solved. This is simply guaranteed by the implicit functions theorem [7]. Indeed, at the dynamical level, the equations of motion of the d^2 fields P^A read

$$\frac{\delta \widetilde{S}(\Phi, P)}{\delta P^A} = 0, \tag{43}$$

and the above-mentioned theorem affirms that the condition

$$\det \frac{\delta^2 \widetilde{S}(\Phi, P)}{\delta P^A \delta P^B} \neq 0 \tag{44}$$

ensures that the system of the d^2 equations defined by Eq. (43) possesses a unique system of d^2 solutions $P^A_0(\Phi^i_0)$, where Φ^i_0 are the solutions of the N equations of motion of the classical fields Φ^i, i.e., $(\delta \widetilde{S}(\Phi, P)/\delta \Phi^i)_{\Phi^i_0} = 0$. The condition (44) must be viewed as crucial to check if any given classical theory can admit a structure of auxiliary fields.

In view of Eq. (42), for any open gauge theory of type (2,2), the condition (44) leads to the fact that W^{AB} must have an inverse $\bar{W}^{AB}$ such that

$$\bar{W}^{AB} W^{BC} = \delta^{AC}, \tag{45a}$$

$$W^{AB} \bar{W}^{BC} = \delta^{AC}. \tag{45b}$$

Let us remark that these two conditions lead for $\bar{W}^{AB}$ as W^{AB} to the same symmetry property.

Now, one can show that the action $\widetilde{S} = S + W^{AB} P^A P^B$ is invariant under the action of the operator Δ defined by

$$\Delta \Phi^i = R^i_\alpha c^\alpha + K^{iA}_\alpha c^\alpha P^A, \tag{46a}$$

$$\Delta P^A = L^{iA}_\alpha c^\alpha \frac{\delta \widetilde{S}}{\delta \Phi^i} + E^{AB}_\alpha c^\alpha P^B, \tag{46b}$$

where

$$K^{iA}_\alpha[\Phi] = -\frac{1}{2d} V^{ij}_{\alpha\beta} \Gamma^A_{\beta\lambda} F_{\lambda,j}, \tag{47a}$$

$$L^{iA}_\alpha[\Phi] = -\frac{1}{2} \bar{W}^{AB} K^{iB}_\alpha, \tag{47b}$$

$$E^{AB}_\alpha[\Phi] = -\frac{1}{2} \bar{W}^{AC} \frac{\delta W^{CB}}{\delta \Phi^l} R^l_\alpha. \tag{47c}$$

One may note that the explicit form of $K^{iA}_\alpha[\Phi]$ (47a) which extends the classical symmetry in Eq. (46a) can be simply derived by performing rearrangement of type (36) in the on-shell BRST transformation $Q\Phi^i$ on the term $V^{ij}_{\alpha\beta} F_{\lambda,j}$ viewed for fixed α.

The rest of our task is basically twofold. On the one hand we have to check the Δ invariance of the full quantum action

$$\widetilde{S}_q = S + W^{AB} P^A P^B + \Delta \Psi, \tag{48}$$

which contains the gauge-fixing terms. On the other hand, one has to show that the defined BRST operator Δ is nilpotent off shell in order to achieve the proof that the above-introduced fields P^A are the desired auxiliary fields. However, one can remark that in view of Eq. (48) together with the Δ invariance of $\widetilde{S}$, the Δ invariance of $\widetilde{S}_q$ simply requires that $\Delta^2 \Psi = 0$ which is equivalent to showing the off-shell nilpotency of Δ on the classical fields Φ^i, and this is because of the exclusive dependence on Φ^i of the gauge-fixing functions (20) for irreducible open gauge theories. To this end, one has to add to the definition of Δ [Eqs. (46a) and (46b) and Eqs. (47a) and (47c)] its action on the ghost fields

$$\Delta c^\lambda = -\frac{1}{2} T^\lambda_{\alpha\beta} c^\alpha c^\beta + H^{\lambda A}_{\alpha\beta} c^\alpha c^\beta P^A, \tag{49a}$$

where

$$H^{\lambda A}_{\alpha\beta}[\Phi] = \frac{1}{3d} Z^{\lambda j}_{\alpha\beta\gamma} \Gamma^A_{\gamma\delta} F_{\delta,j}, \tag{49b}$$

where the functions $Z^{\lambda j}_{\alpha\beta\gamma}$ acting on the ghosts $c^\alpha c^\beta c^\gamma$ realize the nonclosure functions $Z^{\lambda j}$ defined in Eq. (14), i.e., $Z^{\lambda j} = \frac{1}{3} Z^{\lambda j}_{\alpha\beta\gamma} c^\alpha c^\beta c^\gamma$. This leads to the Δ invariance of $\widetilde{S}_q$:

$$\Delta \widetilde{S}_q = 0. \tag{50}$$

We note, in particular, that to prove this we have used besides the characteristic Eqs. (28a) and (28b) the trivial but very helpful identity

$$W^{AB} = -\frac{1}{2d} F_{,i}(\Gamma^A)^T K^{iB}. \tag{51}$$

We turn now to show the off-shell nilpotency of the BRST operator Δ. On the classical fields Φ^i it is simply derived from Eq. (50) which implies $\Delta^2 \Psi = 0$, and then

$$\Delta^2 \Phi^i F_{\lambda,i} = 0. \tag{52}$$

On this ground, a particular observation on the gauge-fixing functions can be done. These functions $F_\lambda[\Phi]$ must not possess any invariance whatever the transformation on Φ^i, i.e., for any set of transformations $\Delta_\omega \Phi^i \equiv X^i_\omega[\Phi]$, we must have

$$\Delta_\omega F_\lambda = 0 \Rightarrow X^i_\omega = 0, \tag{53}$$

where "ω" labels the set of transformations of Φ^i. This clearly leads to

$$\forall X^i_\omega[\Phi]: X^i_\omega F_{\lambda,i} = 0 \Rightarrow X^i_\omega = 0. \tag{54}$$

This condition on the gauge fixing functions allows us from Eq. (52) to prove the off-shell nilpotency of Δ on the classical fields Φ^i. That condition remains essential if we undertake to show this off-shell nilpotency by a direct computation of $\Delta^2 \Phi^i$. Indeed, it permits us to obtain

$$K^{iA}_\alpha \bar{W}^{AB} K^{jB}_\beta = V^{ij}_{\alpha\beta}, \tag{55}$$

which is necessary to the direct proof of

$$\Delta^2 \Phi^i = 0. \tag{56}$$

Let us be precise that in deriving Eq. (55) we have used the other trivial but useful identity $[K^{iA}_\alpha \bar{W}^{AB} K^{jB}_\beta - V^{ij}_{\alpha\beta}]\Gamma^D_{\beta\sigma} F_{\sigma,i} = 0$ together with the condition (53) and the inverse basis of the Γ^A matrices.

We have now to show the off-shell nilpotency of Δ on the ghost fields c^α. To this end, besides the characteristic Eqs. (29a) and (29b) we use

$$Z^{\lambda j}_{\alpha\beta\gamma} = -H^{\lambda A}_{\alpha\beta} L^{jA}_\gamma, \tag{57}$$

which is easily proven from the identity $[Z^{\lambda j}_{\alpha\beta\gamma} + H^{\lambda A}_{\alpha\beta} L^{jA}_\gamma]\Gamma^B_{\rho\sigma} F^j_\sigma = 0$ in the same way that we have done for Eq. (55). Therefore we find

$$\Delta^2 c^\alpha = 0. \tag{58}$$

Finally, the off-shell nilpotency of Δ on the auxiliary fields P^A can be simply deduced from Eqs. (56) and (58). Indeed, the evaluation of $\Delta^3 \Phi^i = \Delta(\Delta^2 \Phi^i) = \Delta^2(\Delta \Phi^i)$ leads to

$$\Delta^2 \Phi^k \frac{\delta(\Delta \Phi^i)}{\delta \Phi^k} + \Delta^2 c^\alpha \frac{\delta(\Delta \Phi^i)}{\delta c^\alpha} + \Delta^2 P^A \frac{\delta(\Delta \Phi^i)}{\delta P^A} = 0, \tag{59}$$

which, in view of Eqs. (56), (58), and (46a) implies

$$\Delta^2 P^A K^{iA}_\alpha = 0, \tag{60}$$

then, by the application of $\Gamma^B_{\alpha\beta} F_{\beta,i}$, which is nondegenerate in view of Eq. (53) and the existence of $\bar{\Gamma}^A$, it follows that

$$W^{AB} \Delta^2 P^A = 0. \tag{61}$$

Using the fact that an inverse for W^{AB} must exist, we get

$$\Delta^2 P^A = 0, \tag{62}$$

which ends up with the proof that the BRST operator Δ given by the above prescription applied for open gauge theories of type (2,2) is nilpotent off shell.

B. Open gauge theories of type (p,q)

Although the general case of open gauge theories of type (p,q) contains more characteristic gauge functions as well as more associated characteristic equations (18), (19), almost all of the general features leading to build up the off-shell version of an on-shell open gauge theory are expressed in the case of theories of type (2,2). Indeed, the typical rearrangement introduced in Eq. (36) together with the field redefinition (39) which allow us to identify the auxiliary fields of the theory and the crucial condition (44) remains unchanged and sufficient to formally find out the off-shell BRST operator and the classical extension for any given open gauge theory of gauge fields Φ^i enriched with the set of auxiliary fields. We then only concentrate on particular remarks that stand out in the general case, all other results will be directly given. These remarks are basically twofold. The first one affects the general form of the action obtained for the on-shell quantum theory (25). This action clearly contains higher order ghost-antighost couplings and could be recast in the form

$$S_q = S + Q\Psi - \sum_{n=1}^{p-1} \frac{n}{n+1} \delta_n \Psi, \tag{63}$$

where Q is the on-shell BRST operator defined by Eqs. (24a) and (24b). Expressing each term of $\sum_{n=1}^{p-1} n/n+1 \, \delta_n \Psi$ occurring in the above expression by using Eq. (22b) one obtains

$$\frac{n}{n+1} \delta_n \Psi = \frac{(-)^n n}{(n+1)!} (-)^{a_{n+1}} V^{i_1 \cdots i_{n+1}}_{n+1} \Psi_{,i_1} \cdots \Psi_{,i_{n+1}}, \tag{64}$$

developing then the $V^{i_1 \cdots i_n}_n$ functions in terms of the ghost fields in the same way we have done in Eq. (11),

$$V^{i_1 \cdots i_n}_n = \frac{1}{n}(-)^{\sum_{s=1}^{n-1}(\alpha_s+1)\sum_{r=1}^n \alpha_r}(-)^{\sum_{r,s=1}^n(i_r\alpha_s)}$$

$$\times V^{i_1 \cdots i_n}_{\alpha_1 \cdots \alpha_n}[\Phi] c^{\alpha_1} \cdots c^{\alpha_n}, \tag{65}$$

and also expressing the gauge fermion Ψ in the function of the antighost fields, using $\Psi = F_\beta[\Phi]\bar{c}^\beta$, we find that for any order "n" in Eq. (63) a term of type

$$V^{i_1 \cdots i_{n+1}}_{\alpha_1 \cdots \alpha_{n+1}} F_{\beta_1,i_1} \cdots F_{\beta_{n+1},i_{n+1}} c^{\alpha_1} \cdots c^{\alpha_{n+1}} \bar{c}^{\beta_1} \cdots \bar{c}^{\beta_{n+1}}$$

contributes to the quantum action. They are of even order in ghost-antighost pairs whatever the integer "n" is. By performing the Fierz-like rearrangement [see Eq. (36)] $n+1$ times on each coefficient of these terms using the orthonormality property of the basis $\{\Gamma^A\}_{A=1,\ldots,d}$ (31) we show that they can be put in the form

$$V^{i_1 \cdots i_{n+1}}_{\alpha_1 \cdots \alpha_{n+1}} F_{\beta_1,i_1} \cdots F_{\beta_{n+1},i_{n+1}}$$

$$= \frac{1}{d^{n+1}} V^{i_1 \cdots i_{n+1}}_{\rho_1 \cdots \rho_{n+1}} F_{\sigma_1,i_1} \cdots F_{\sigma_{n+1},i_{n+1}} \Gamma^{A_1}_{\rho_1 \sigma_1} \cdots \Gamma^{A_{n+1}}_{\rho_{n+1} \sigma_{n+1}}$$

$$\times \Gamma^{A_1}_{\beta_1 \alpha_1} \cdots \Gamma^{A_{n+1}}_{\beta_{n+1} \alpha_{n+1}}, \qquad (66)$$

where a sum over $(A_1, \ldots, A_{n+1})$ is underlaid. Thus the higher order terms in the quantum action (63) acquire the form

$$\widetilde{S}_\Lambda = \sum_{n=1}^{p-1} W^{A_1 \cdots A_{n+1}}_{n+1}[\Phi](\bar{c}\Gamma^{A_1}c) \cdots (\bar{c}\Gamma^{A_{n+1}}c), \quad (67)$$

where all the coefficients $W^{A_1 \cdots A_{n+1}}_{n+1}[\Phi]$ are completely defined by Eqs. (64), (65), and (66). We are now able to perform the same identifications as in the previous section for the auxiliary fields (39), i.e., $P^A \equiv (\bar{c}\Gamma^A c)$. To step forward we have to make another remark which can be crucial for the practical application of our prescription. In the general case the d^2 fields P^A constructed in this way have no defined Grassmannian parity. Indeed, since the ghost and antighost fields $(\bar{c}^\alpha, c^\beta)$ associated with the classical symmetry parameters (ε^α) have various Grassmannian parities, any bilinear combination of them will not have any defined parity. For that reason this redefinition is taken to be purely formal. For practical application we have to split the formal set of fields P^A into sets having well defined parities. This can be easily done in the following way. The general set of the "d" symmetries can be divided into the set of the "d_b" bosonic symmetries and the set of the "d_f" fermionic ones, such that $d = d_b + d_f$. Then, each Γ^A of the d^2 elements of the basis of the $d \times d$ matrix space C can take the following block matrix form:

$$\Gamma^A_{d \times d} \equiv \begin{bmatrix} \Gamma^{A^1}_{d_b \times d_b} & \Gamma^{A^2}_{d_f \times d_b} \\ \Gamma^{A^3}_{d_b \times d_f} & \Gamma^{A^4}_{d_f \times d_f} \end{bmatrix}, \qquad (68)$$

which can be condensed in the notation $\Gamma^A \equiv (\Gamma^{A^a})_{a=1,\ldots,4}$, where each value of "$a$" denotes one of the four sectors of Γ^A. Then the set of the fields P^A can be viewed as a supermultiplet containing the bosonic as well as the fermionic auxiliary fields, i.e., $P^A \equiv (P^{A^a})_{a=1,\ldots,4}$, where every auxiliary field P^{A^a} is introduced by the field redefinition $P^{A^a} \equiv (\bar{c}\Gamma^{A^a}c)$. The d_b^2 fields P^{A^1} and the d_f^2 fields P^{A^4} are bosonic while the $d_f \times d_b$ fields P^{A^2} and the $d_b \times d_f$ fields P^{A^3} are fermionic. All of them are of ghost number zero. Then the action (67) which formally reads $\widetilde{S}_\Lambda = \sum_{n=1}^{p-1} W^{A_1 \cdots A_{n+1}}_{n+1}[\Phi]P^{A_1} \cdots P^{A_{n+1}}$, will be practically written as

$$\widetilde{S}_\Lambda = \sum_{n=1}^{p-1} \sum_{a_1 \cdots a_{n+1}=1}^{4} W^{A_1^{a_1} \cdots A_{n+1}^{a_{n+1}}}_{n+1}[\Phi]P^{A_1^{a_1}} \cdots P^{A_{n+1}^{a_{n+1}}}, \tag{69}$$

where the functions $W^{A_1^{a_1} \cdots A_{n+1}^{a_{n+1}}}_{n+1}[\Phi]$ are completely derived upon the Γ^A dependence of $W^{A_1 \cdots A_{n+1}}_{n+1}[\Phi]$, see Eq. (66) and the definition of the P^{A^a}.

In what follows we pursue only with the formal notation P^A for the auxiliary fields, but keeping in mind that for practical applications we have to go back to the fields P^{A^a} in order to obtain the correct representation of the auxiliary fields.

Let us now introduce the classical extension $\widetilde{S}(\Phi,P)$ of the classical action of the theory $S(\Phi)$,

$$\widetilde{S}(\Phi,P) = S(\Phi) + \sum_{n=1}^{p-1} W^{A_1 \cdots A_{n+1}}_{n+1}[\Phi]P^{A_1} \cdots P^{A_{n+1}}, \tag{70}$$

and by applying the same procedure as for the (2,2)-type open gauge theories, one expands in ghost-antighost pairs the on-shell BRST operator Q acting on the gauge fields Φ^i [Eq. (24a)] in order to obtain the off-shell BRST symmetry of the classical action (70). Each term of Q, i.e., $\delta_n\Phi^i = 1/n!$ $(-1)^{i_n+a_n}V^{ii_1 \cdots i_n}_{n+1}\Psi_{,i_1} \cdots \Psi_{,i_n}$ clearly contains "n" pairs $(\bar{c}^\alpha, c^\beta)$, then by performing "$n$" times the Fierz-like rearrangement and also making the suitable identification for the auxiliary fields P^A we obtain the following BRST transformation on Φ^i:

$$\Delta\Phi^i = \delta\Phi^i + \sum_{n=1}^{p-1} K^{iA_1 \cdots A_n}_{n\alpha}[\Phi]c^\alpha P^{A_1} \cdots P^{A_n}, \tag{71}$$

where δ is the standard BRST operator.

In order to consider the fields P^A as auxiliary fields we still impose the general condition $\det \delta^2\widetilde{S}(\Phi,P)/\delta P^A \delta P^B \neq 0$. To this purpose it is convenient to put the action (70) in the form

$$\widetilde{S}(\Phi,P) = S + \hat{W}^{AB}[\Phi,P]P^A P^B, \tag{72}$$

where

$$\hat{W}^{AB}[\Phi,P] = W^{AB}_2[\Phi]$$

$$+ \sum_{n=3}^{p-1} W^{ABC_1 \cdots C_{n-2}}_n[\Phi]P^{C_1} \cdots P^{C_{n-2}},$$

then the condition (44) will just imply that $\hat{W}^{AB}[\Phi, P]$ must have an inverse $\hat{W}^{AB}_{\text{inv}}[\Phi, P]$ such that $\hat{W}^{AB}_{\text{inv}}\hat{W}^{BC} = \delta^{AC}$ and $\hat{W}^{AB}\hat{W}^{BC}_{\text{inv}} = \delta^{AC}$. In the same way the BRST transformation (71) could be cast in the form

$$\Delta\Phi^i = \delta\Phi^i + \hat{K}^{iA}_\alpha[\Phi, P]c^\alpha P^A, \qquad (73)$$

where

$$\hat{K}^{iA}_\alpha[\Phi, P] = K^{iA}_{1\alpha}[\Phi] + \Sigma^{p-1}_{n=2}K^{iAA_1\cdots A_{n-1}}_{n\alpha}[\Phi]P^{A_1}\cdots P^{A_{n-1}}.$$

Then by defining the action of Δ on the auxiliary fields

$$\Delta P^A = -\frac{1}{2}\hat{W}^{AB}_{\text{inv}}\hat{K}^{iB}_\alpha c^\alpha \tilde{S}_{,i} - \hat{W}^{AC}_{\text{inv}}\delta\hat{W}^{AB}, \qquad (74)$$

a tedious but straightforward calculation leads to the Δ invariance of the classical extension $\tilde{S}(\Phi, P)$.

The last step will consist of showing the off-shell nilpotency of the BRST operator Δ. To this purpose we supplement the definition of Δ with its application on the ghost fields c^α in the same spirit as in the case of the gauge fields Φ^i. First we begin to expand in ghost-antighost pairs the on-shell BRST operator Q acting on c^α [Eq. (24a)], this involves functions of the type $Z^{\alpha i_1\cdots i_n}_{\lambda_1\cdots\lambda_{n+2}}$ which realize the characteristic functions of type $Z^{\alpha i_1\cdots i_n}_{n+1}$ [see Eq. (23b)] by acting on the $(n+2)$th order term $c^{\lambda_1}\cdots c^{\lambda_{n+2}}$ as well as the gauge fixing terms $F_{\beta_1, i_1}\cdots F_{\beta_{n+1}, i_{n+1}}$ related with the nth order term in antighost fields $\bar{c}^{\beta_1}\cdots\bar{c}^{\beta_{n+1}}$. Thus each term in the definition of the on-shell BRST operator Q contributes with a term of order "n" in ghost-antighost pairs $(\bar{c}^\alpha, c^\beta)$, then performing "$n$" times the Fierz-like rearrangement (36) and also applying the prescribed identification for the auxiliary fields P^A we obtain the following form for the BRST transformation on c^α [for best insight, one may return to Eqs. (49a) and (49b)]

$$\Delta c^\alpha = \delta c^\alpha + \sum^{q-1}_{n=1}H^{\alpha A_1\cdots A_n}_{n\rho\sigma}[\Phi]c^\rho c^\sigma P^{A_1}\cdots P^{A_n}, \qquad (75)$$

which can be easily put in the more convenient expression

$$\Delta c^\alpha = \delta c^\alpha + \hat{H}^{\alpha A}_{\rho\sigma}[\Phi, P]c^\rho c^\sigma P^A, \qquad (76)$$

where $\hat{H}^{\alpha A}_{\rho\sigma}[\Phi, P] = H^{\alpha A}_{1\rho\sigma}[\Phi] + \Sigma^{q-1}_{n=2}H^{\alpha A_1\cdots A_{n-1}}_{n\rho\sigma}[\Phi] \times c^\rho c^\sigma P^{A_1}\cdots P^{A_{n-1}}$. Then we can show by a last tedious calculation that the obtained BRST operator Δ defined by Eqs. (73), (74), and (75) is nilpotent off shell, i.e.,

$$\Delta^2 X = 0, \qquad (77)$$

where X describes all the fields of the theory. However, let us note that in addition to the characteristic equations [(18) and (19)] the proof of the off-shell nilpotency of Δ requires the condition (53) be imposed on the gauge fixing functions.

Once we get the off-shell nilpotency of Δ, the gauge fixing action occurring in the full quantum action of the theory can be put in the usual Δ-exact form, i.e., $S_q = \tilde{S} + \Delta\Psi$.

IV. MINIMAL AND NONMINIMAL SET OF AUXILIARY FIELDS

We are now going to investigate one of the most typical features of theories that contain auxiliary fields. For those theories we remark that the number of auxiliary fields is not unique, but in all cases we may find a minimal set of these fields (for a review see Refs. [3] and [8]). In this section we will see how this statement can be analyzed and reproduced in the general framework of the ideas suggested in this paper. We first deal with theories of type (2,2) then we briefly discuss the general case (p, q) which does not bring anything new to the spirit of the approach.

In the above sections we showed how we can start with an on-shell open gauge theory to end up with the corresponding off-shell version. The procedure is essentially based on the identification (39) for the auxiliary fields, i.e.,

$$P^A \equiv (\bar{c}^\alpha \Gamma^A_{\alpha\beta}c^\beta), \qquad (78)$$

which are clearly of number "d^2." The set of the "d^2" matrices $\{\Gamma^A\}_{A=1,\ldots,d^2}$ can be always split into the two sets of the symmetric matrices $\{\Gamma^A_0\}$ of number $d(d+1)/2$ and the antisymmetric matrices $\{\Gamma^A_1\}$ of number $d(d-1)/2$. This fact together with the identification (78) permit us to split the set of auxiliary fields noted Λ_p into two parts. The first one Λ^0_p contains $d(d+1)/2$ auxiliary fields P^A_0 defined by

$$P^A_0 \equiv \bar{c}^\alpha \Gamma^A_{0\alpha\beta}c^\beta, \qquad (79a)$$

and the second part Λ^1_p contains $d(d-1)/2$ auxiliary fields P^A_1 defined by

$$P^A_1 \equiv \bar{c}^\alpha \Gamma^A_{1\alpha\beta}c^\beta. \qquad (79b)$$

Our task consists now in showing that we could eliminate one of the two above representations of auxiliary fields without affecting the other one. To this aim one can remark that the auxiliary fields P^A appear in the off-shell version of the theory at two levels: in the classical extension of the classical action (42) and in the off-shell BRST operator Δ. In both of them they are associated with coefficients that involve the characteristic functions of the theory and the different gauge fixing functions. It is the last dependence that will be investigated. We first introduce from the gauge fixing functions $F_\alpha[\Phi]$ a set of functions $F^A_\alpha[\Phi]$ defined by

$$F^A_\alpha[\Phi] = \Gamma^A_{\alpha\beta}F_\beta[\Phi]. \qquad (80)$$

Such a definition is guaranteed by the existence of the inverse basis $\bar{\Gamma}^A$. Thus we have

$$F_\alpha[\Phi] = \bar{\Gamma}^A_{\alpha\beta}F^A_\beta[\Phi]. \qquad (81)$$

We can observe that the inverse basis $\bar{\Gamma}^A$ can also be decomposed into symmetric and antisymmetric parts Γ_0^A and Γ_1^A in the way that using Eq. (32) we obtain

$$F_\alpha[\Phi] = \bar{\Gamma}_{0\alpha\beta}^A F_{0\beta}^A[\Phi] + \bar{\Gamma}_{1\alpha\beta}^A F_{1\beta}^A[\Phi]. \tag{82}$$

Upon this decomposition, the classical extension (42) reads

$$\tilde{S} = S + W_{00}^{AB} P_0^A P_0^B + W_{11}^{AB} P_1^A P_1^B + W_{10}^{AB} P_1^A P_0^B$$
$$+ W_{01}^{AB} P_0^A P_1^B, \tag{83}$$

with

$$W_{00}^{AB} = \frac{1}{4d^2} F_0^{Ai} V^{ij} F_0^{Bj}, \tag{84a}$$

$$W_{11}^{AB} = \frac{1}{4d^2} F_1^{Ai} V^{ij} F_1^{Bj}, \tag{84b}$$

$$W_{10}^{AB} = \frac{1}{4d^2} F_1^{Ai} V^{ij} F_0^{Bj}, \tag{84c}$$

$$W_{01}^{AB} = \frac{1}{4d^2} F_0^{Ai} V^{ij} F_1^{Bj}. \tag{84d}$$

Note that W_{00} and W_{11} are symmetric in A and B, and $W_{10}^{AB} = W_{01}^{BA}$. We are now able to choose between the elimination of the fields P_0^A or P_1^A. This will be simply done by taking advantage of the freedom in the manner that we choose the gauge fixing functions. If we want, for example, to eliminate the fields P_0^A it is sufficient to choose the gauge fixing functions such that in Eq. (82) we have

$$F_{0\beta}^A = 0. \tag{85}$$

From this and from Eqs. (84a) and (84d), the only coefficient that remains in Eq. (83) is W_{11} and only the auxiliary fields P_1^A take part in the classical extension of the action. In order to completely eliminate the P_0^A it is necessary to show that they do not appear in the BRST operator Δ. Indeed, from Eqs. (46a) and (46b) and (49) we find

$$\tilde{Q}\Phi^i = R_\alpha^i c^\alpha + K_{1\alpha}^{iA} c^\alpha P_1^A, \tag{86a}$$

$$\tilde{Q}C^\lambda = -\frac{1}{2} T_{\alpha\beta}^\lambda c^\alpha c^\beta + H_{1\alpha\beta}^{\lambda A} c^\alpha c^\beta P_1^A, \tag{86b}$$

$$\tilde{Q}P_1^A = L_{1\alpha}^{iA} c^\alpha \frac{\delta \tilde{S}_0}{\delta \Phi^i} + E_{1\alpha}^{AB} c^\alpha P_1^B, \tag{86c}$$

$$\tilde{Q}P_0^A = 0, \tag{86d}$$

with

$$K_{1\alpha}^{iA}[\Phi] = -\frac{1}{2d} V_{\alpha\beta}^{ij} F_{1\beta}^{Aj}, \tag{87a}$$

$$H_{1\alpha\beta}^{\lambda A}[\Phi] = \frac{1}{3!d} Z_{\alpha\beta\gamma}^{\lambda j} F_{1\gamma}^{Aj}, \tag{87b}$$

$$L_{1\alpha}^{iA}[\Phi] = -\frac{1}{2} \bar{W}_{11}^{AB} K_{1\alpha}^{iB}, \tag{87c}$$

$$E_{1\alpha}^{AB}[\Phi] = -\frac{1}{2} \bar{W}_{11}^{AC} \frac{\delta W_{11}^{CB}}{\delta \Phi^l} R_\alpha^l, \tag{87d}$$

where $\bar{W}_{11}$ is the inverse of W_{11}.

Thus the condition (85) is sufficient to the elimination of the auxiliary fields P_0^A. Moreover, one can note that if instead of Eq. (85) we have chosen the gauge fixing functions such that $F_{1\beta}^A = 0$, then the fields P_1^A will be eliminated. So we have defined two possible configurations for the auxiliary fields. For a given open gauge theory, the choice of the gauge fixing functions such that $F_{0\beta}^A = 0$ leads to the set Λ_p^1 of the $d(d-1)/2$ auxiliary fields P_1^A. This will be named *the minimal set of auxiliary fields*. The other choice of the gauge fixing functions such that $F_{1\beta}^A = 0$ which leads to the set Λ_p^0 of the $d(d+1)/2$ auxiliary fields P_0^A will be named *the nonminimal set of auxiliary fields*.

Since the keystone for the determination of the minimal (or nonminimal) set of auxiliary fields is the choice of the gauge fixing functions via the decomposition (82), no particular generalization is needed in the case of theory of type (p,q). The condition $F_{0\beta}^A = 0 (F_{1\beta}^A = 0)$ remains sufficient to obtain the minimal (nonminimal) set of auxiliary fields for general open gauge theories. Nevertheless, one can recall that for a practical application (where both bosonic and fermionic symmetries are responsible for the opening of the classical algebra), we have to deal with the set $(P^{A^a})_{a=1,\dots,4}$ of the genuine auxiliary fields with well defined parities obtained from the formal set (P^A) as it is shown in Sec. III B. In order to understand what will occur to the minimal and nonminimal configurations of auxiliary fields, we must notice that the "$d^2 = (d_b + d_f)^2$" matrices Γ^A expressed such as in Eq. (68) lead to the $[d_b(d_b-1)/2 + d_f(d_f-1)/2 + d_b d_f]$ antisymmetric matrices of the base of C_1 and the $[d_b(d_b+1)/2 + d_f(d_f+1)/2 + d_b d_f]$ symmetric matrices of the base of C_0. Therefore the minimal set Λ_p^1 will contain $[d_b(d_b-1)/2 + d_f(d_f-1)/2]$ bosonic and $(d_b d_f)$ fermionic auxiliary fields, while the nonminimal set Λ_p^0 will contain $[d_b(d_b+1)/2 + d_f(d_f+1)/2]$ bosonic and $(d_b d_f)$ fermionic auxiliary fields.

To end this section, we will briefly discuss the particular case of simple supergravity (D=4 and N=1) to show how the procedure developed in this paper can be practically applied. In this theory [3] the classical dynamical gauge fields are the vierbein e_μ^a and the gravitino ψ_μ^A with $a = 1, \dots, 4$ labeling the flat Minkowski space, $\mu = 1, \dots, 4$ labeling the curved Riemannian space, and $A = 1, \dots, 4$ is related to the $N=1$ supersymmetry. One recalls that the theory admits a

vanishing torsion leading to a nonpropagating spin connection ω_μ^{ab}. The symmetries of the theory are the diffeomorphism, the Lorenz and the supersymmetry transformations. Their associated ghost fields are c^μ, c^{ab}, and c^A, respectively. The classical BRST operator associated to the classical symmetries of the theory have the following on-shell property [3]

$$\delta^2 \psi_\mu = V_{\mu\nu} \frac{\delta S}{\delta \bar{\psi}_\nu}, \tag{88a}$$

$$\delta^2 c^{ab} = Z_\nu^{ab} \frac{\delta S}{\delta \bar{\psi}_\nu}, \tag{88b}$$

$$\delta^2 X = 0 \quad \text{for all others fields} \tag{88c}$$

with $\bar{\psi}_\nu = \psi_\nu^T C$, where C is the charge conjugation matrix, and the supersymmetric index is omitted for simplicity. This on-shell structure follows easily from the open structure of the superalgebra of the simple supergravity. The characteristic functions $V_{\mu\nu}$ and Z_ν^{ab} are given by

$$V_{\mu\nu} = \frac{1}{8} \bar{c} \gamma^a c \left(\frac{1}{4} g_{\mu\nu} \gamma_a - \frac{1}{2} e \varepsilon_{\mu\nu\rho\tau} e_b^\tau \gamma_5 \gamma^b \right)$$
$$+ \frac{1}{8} \bar{c} \sigma_{ab} c \left(e_\mu^a e_\nu^b + \frac{1}{2} g_{\mu\nu} \sigma^{ab} - \frac{1}{2} e \varepsilon_{\mu\nu\rho\tau} e_b^\tau \gamma_5 \right.$$
$$\left. - \frac{1}{2} e \varepsilon_{\mu\nu\rho\tau} e^{\rho a} e^{\tau b} \gamma_5 \right), \tag{89}$$

$$Z_\mu^{ab} = \frac{1}{8} \bar{c} \gamma_a e_\mu^a \sigma^{ab} \gamma_5 c \bar{c} \gamma_5, \tag{90}$$

where $e = \det(e_\mu^a)$ and $g_{\mu\nu} = e_\mu^a e_{a\mu}$. These characteristic functions are related upon characteristic equations [6] of type (28a) and (28b) and (29a) and (29b) and show that simple supergravity is of type (2,2). Since the only symmetry that is responsible for the opening of the classical algebra is supersymmetry, and following the procedure presented in this paper, the complete set of auxiliary fields will contain $d^2 = 4^2 = 16$ bosonic fields. To step forward and find out the complete representation of the auxiliary fields we need to define a convenient basis for the 4×4 matrix. Such a basis is given by the 16 matrix $\{\Gamma^A\}_{A=1,\dots,16} \equiv (C, C\gamma^a, 2C\sigma^{ab}, C\gamma_5 \gamma^a, C\gamma_5)$, where γ^a are the Dirac matrix, $\sigma^{\mu\nu} = \frac{1}{4}[\gamma^a, \gamma^b]$, and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. By taking advantage of the properties of the Dirac matrices one can show that this set of matrices split into the set of the six antisymmetric matrices $(C, C\gamma_5, C\gamma_5 \gamma^a)$ and the ten symmetric ones $(C\gamma^a, 2C\sigma^{ab})$. According to this basis the 16 bosonic degrees of freedom expected for the auxiliary fields will be distributed with respect to the following multiplet representation $[S(scalar), P(pseudoscalar), A_5^a(pseudovector)]$ for the minimal set and $[A^a(vector), E^{ab}(2nd\text{-}rank\ antisymmetric\ tensor)]$ for the nonminimal one. These are the standard results occurring in simple supergravity. Let us note that once we choose the standard gauge fixing function for supergravity, i.e., $F = e \gamma^\mu \psi_\mu$, we can see that the only coefficient W_{11}^{AB} [Eq. (84b)] that remains in the minimal representation of auxiliary fields acquires the following simple form:

$$W_{11} = -\frac{e}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & g_{\mu\nu} \end{bmatrix}, \tag{91}$$

which arises from the particular property of the characteristic function $V_{\mu\nu}$, that for any arbitrary spinor φ we have $\bar{c} \gamma_\nu V_{\mu\rho} \gamma^\rho \varphi = 0$. This directly leads, from Eq. (83), to the usual classical extension

$$\tilde{S} = S_{cl} - \frac{e}{3} (S^2 - P^2 + A_a A^a). \tag{92}$$

One can also easily derive the associated BRST symmetry which is nilpotent off shell from the general equations (86a) and (86d) and (87a) and (87d) and find the standard results (see Refs. [3] and [6]).

V. CONCLUDING REMARKS

In this paper we have presented a prescription leading to the construction of an off-shell BRST quantization scheme for irreducible open gauge theories. We first obtained the on-shell BRST full quantum action together with its associated on-shell BRST symmetries. This is realized upon taking advantage of the characteristic functions related to corresponding equations that characterize general open gauge algebras. From this follows the construction of the off-shell version of the theory. To this aim, we used a suitable field redefinition which permits us to find out the necessary set of auxiliary fields which leads to the classical extension of the classical action of the theory as well as to the off-shell BRST operator so that the quantization can be done in the standard way, i.e., as in Yang-Mills type theories. Let us note that we first apply our prescription to theories described by a gauge algebra with vanishing higher-order gauge functions, i.e., theories of type (2,2) which contain all the subtleties required to the insight of the procedure. Then a direct generalization is given for any open gauge theory of type (p,q), with, however, particular technical remarks that stand out in the general case. In the last chapter we study the particular problem of the minimal set of auxiliary fields for any given open gauge theory. Then we end up with a quick formulation of the procedure for simple supergravity and reproduce the standard results.

To quantize gauge systems, the exposed prescription should be compared to the BV approach. The latter is not the unique way to quantize closed and irreducible gauge theories but became impossible to circumvent for open and/or reducible theories for the reason that no systematic procedure for the introduction of auxiliary fields was to date available. At first sight, the comparison clearly stops at the on-shell level for the reason that in the BV procedure, the nilpotency of the BRST operator is guaranteed only on shell after the elimination of the antifields. It is worth noting that at this level both procedures lead to the same higher-order ghost coupling terms in the on-shell full quantum action. However, to step forward and really quantize the theory, one may remark that a systematic procedure for the introduction of auxiliary fields

closes the classical algebra and makes the quantum theory much simpler, since in this case the transformation laws are linear[1] and lead to an off-shell BRST operator together with a complete off-shell invariant action containing all the gauge fixing conditions. One can then easily derive the so-called Ward identities which are necessary in many aspects of the quantized theory, for instance, gauge independence of the partition function as well as perturbational proofs of unitarity and renormalizability are heavily based on these identities [9]. Then, to determine the quantum theory completely one has to add an extra symmetry, i.e., the so-called shift symmetry upon introducing the set of collective fields in order to obtain the quantum equations of motion, i.e., the Schwinger-Dyson equations [10] (see also Ref. [11]) as Ward identities of the complete theory and end up with a physical quantum theory, in the sense that all the physical degrees of freedom are fixed, together with an off-shell structure of the symmetries. This cannot be realized in the BV quantization scheme. In this approach, in order to obtain a theory with all the fixed degrees of freedom, one has to require the elimination of the antifields for the benefit of the gauge fixing functions trough the gauge fixing fermion and this leads inevitably to an on-shell structure of the symmetries. But if we want to quantize the theory effectively and derive the Ward identities one has to reintroduce the antifields and take advantage of the off-shell structure provided by this reintroduction. One can then clearly see that in the BV formalism, a physical quantum theory cannot be obtained together with an off-shell structure contrary to what can be done via the introduction of auxiliary fields that realizes the off-shell nilpotency and allows at the same time the introduction of all the gauge fixing functions without introducing any new physical degrees of freedom in the sense that they are nonpropagating fields. Let us also remark that besides the fact that auxiliary fields simplify greatly the quantization of open gauge theories, they are of particular interest in many specific cases. For example, one can cite globally supersymmetric models such as the Wess-Zumino model, for which it is only with auxiliary fields that one can obtain a tensor calculus [3]. One can also mention the case of BF theories which represent models of reducible theories but have, however, an on-shell structure, and for which the introduction of auxiliary fields realizes the metric independence of the BRST operator and allows one to simplify the proof of the metric independence of the partition function of such theories [4].

However, one should mention that an interesting idea exists in order to extend the BV method to investigate a possible realization of a complete off-shell quantization procedure. In their approach (see Refs. [12–14]) the authors are led to identify the auxiliary fields through the variation of the gauge fixing fermion with respect to the gauge fields of the classical theories. This method leads at first sight to three binding remarks. The first one concerns the nonvanishing ghost number of the auxiliary fields obtained in this way.

This clearly compromises the possibility of considering these fields at the classical level and thus jeopardizes the construction of a classical extension of the theory. The second remark is related to the particular constraints taken by the authors on the gauge functions of the gauge algebra. These constraints imposed for internal consistency reduce considerably the logical simplicity of the theory and potential generalizations (see in particular Ref. [12]). The last remark affects the representation (and then the number) of the auxiliary fields. Indeed, in their approach we see that these fields are inevitably in the same number that of the gauge fields with nonvanishing ghost number and opposite statistic. This can raise the problem of the definition of the minimal set of auxiliary fields. Nevertheless, in Ref. [14] the authors present a clever way to bypass this difficulty for the specific case of simple supergravity, but they take too much advantage of the particularities of the theory to envisage a smooth generalization to general open gauge theories. As a quick comparison, our prescription gives rise to auxiliary fields with vanishing ghost-numbers and their representation is only related, upon the field-redefinition (39), to the symmetries of the classical theory. This permits us in Sec. IV to analyze the question of the minimal representation in a general framework. The constraints used in this section are twofold. The first and more important one [Eq. (44)] is a very general condition related to the nature of any set of auxiliary fields that impose to them that they must not introduce any new degrees of freedom to the classical theory and this condition finds its theoretical meaning in the very general explicit function theorem [7]. The second condition (53) is related to the gauge fixing functions that are taken to not have any kind of invariance, which is not a strong restriction in virtue of the freedom in fixing the gauge.

Finally, one should mention that in order to study all further possible advantages of the auxiliary fields structure, it would be interesting to reinvestigate the prescription presented in this paper in a more formal way[2] and also to find out how to make a generalization to reducible gauge theories. Furthermore, to develop and consolidate our approach, it would be also interesting to apply it for several specific theories. In particular, we plan to use it to give a complete off-shell formulation of the 11-dimensional (11D) supergravity for which the complete structure of the auxiliary fields is unknown. Let us note that 11D supergravity recently became interesting because of its return in the so-called M theory (for a review see Ref. [15]) and only a partial off-shell formulation has been already proposed in Ref. [16].

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[1] One can check from Eqs. (73) and (74) that upon replacing the ghost fields by gauge parameters one can easily see that the obtained transformations are linear.

[2] The essential motivation of the present paper was to show how the introduction of auxiliary fields can be practically realized for irreducible open gauge theories.

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