New nonlocal effective action

A. O. Barvinsky*
Theory Department, Lebedev Physics Institute, Leninsky Prospect 53, Moscow 117924, Russia

V. F. Mukhanov†
Sektion Physik, LMU, Theresienstrasse 37, Munich, Germany
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We suggest a new method for the calculation of the nonlocal part of the effective action. It is based on the resummation of the perturbation series for the heat kernel and its functional trace at large values of the proper time parameter. We derive a new, essentially nonperturbative, nonlocal contribution to the effective action in spacetimes with dimensions $d>2$.

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I. INTRODUCTION

The effective action is among the fundamental ideas of modern quantum field theory. Calculated analytically for a given background field, it gives information about the induced energy-momentum tensor of quantum fields and quantum corrections to the classical equations of motion. The nonlocal part of the effective action should contain, for instance, particle creation effects. For black holes it should be able to account simultaneously for vacuum polarization and asymptotic Hawking radiation. Various important applications of the effective action can also be found in fundamental string theory. The Lorentzian effective action, which we actually need, can be obtained from the Euclidean one $\Gamma[\phi]$ via analytic continuation. In turn, $\Gamma[\phi]$ can be defined by the following path integral:

$$\exp(-\Gamma[\phi(x)])=\int D\varphi\,\exp\left(-S[\varphi]+(\varphi-\phi)\frac{\delta\Gamma[\phi]}{\delta\phi}\right), \tag{1.1}$$

where $\phi(x)$ is a given mean field, and the functional integration over quantum fields $\varphi(x)$ is assumed. The general semiclassical expansion of $\Gamma[\phi]$ begins with the one-loop contribution, which is given by the Gaussian path integral

$$\exp\{-\Gamma[\phi(x)]\}=\int D\varphi\,\exp\left(-\frac{1}{2}\int dx\sqrt{g}\varphi(x)\times\hat{F}(\nabla,\phi(x))\varphi(x)\right). \tag{1.2}$$

The operator $\hat{F}(\nabla,\phi(x))$ here determines the propagation of small field disturbances $\varphi(x)$ on the background of $\phi(x)$ and in the bosonic case can generically be written down as

$$\hat{F}=-\Box+V(x), \tag{1.3}$$

where $\Box=\nabla^2\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the Laplacian in the Euclidean field theory, which becomes the d'Alembertian when analytically continued to the Lorentzian sector, and $V(x)$ is the potential term. Note that, for some fields, the one-loop contribution is exact, for instance, for the scalar field without self-coupling. For a properly defined measure the Gaussian integral (1.2) can be formally calculated as

$$\Gamma=\frac{1}{2}\ln\left(\prod_\lambda\ \lambda\right)=\frac{1}{2}\sum_\lambda\ \ln\lambda=\frac{1}{2}\mathrm{Tr}\ln\hat{F}, \tag{1.4}$$

where λ are the eigenvalues of the operator $\hat{F}$ corresponding to appropriately normalized eigenfunctions $\phi_\lambda(x)$, $\int dx\sqrt{g}\phi_\lambda(x)\phi_{\lambda'}(x)=\delta_{\lambda\lambda'}$. Here the functional trace Tr does not depend on a particular basis in the functional space of disturbances φ and, therefore, in an appropriate representation it reduces to the integral over spatial coordinates x of the diagonal element of the operator kernel.

The effective action (1.4) is, of course, ultraviolet divergent and should be regularized, with the subsequent interpretation of explicitly isolated divergences in terms of infinite renormalizations of the coupling constants of the theory. These divergences are well understood and it is unlikely that anything new can be added here. Therefore, we concentrate on more interesting finite, and generally nonlocal, contributions to the one-loop effective action. These contributions depend on infrared properties of the theory and contain nontrivial information about real physical effects. Analytical calculational schemes for Γ are usually based on the following integral representation of the functional trace of $\hat{F}$:

$$\mathrm{Tr}\ln\hat{F}=-\int_0^\infty\frac{ds}{s}\mathrm{Tr}\,e^{-s\hat{F}}, \tag{1.5}$$

where all local divergences can be easily isolated with the aid of dimensional regularization. The kernel

$$K(s|x,y)\equiv\exp(-s\hat{F})\delta^{(d)}(x,y), \tag{1.6}$$

where d is spacetime dimensionality, obviously satisfies the heat kernel equation

*Email address: barvin@td.lpi.ac.ru
†Email address: mukhanov@theorie.physik.uni-muenchen.de

$$\frac{\partial}{\partial s}K(s|x,y) = -\hat{F}K(s|x,y) \equiv [\Box - V(x)]K(s|x,y), \tag{1.7}$$

with the initial condition

$$K(0|x,y) = \delta^{(d)}(x,y) \tag{1.8}$$

at $s=0$. The auxiliary parameter s is usually called the proper time. Thus, calculating the effective action can be reduced to the solution of the Cauchy problem for $K(s|x,y)$. In fact, what we actually need is the coincidence limit of this function, since, in the representation we used, the functional trace of the operator $e^{-s\hat{F}}$ corresponds to the integration of the diagonal elements of K over the spacetime coordinates x, so that

$$\Gamma = -\frac{1}{2}\int dx \left(\int_0^\infty \frac{ds}{s} K(s|x,x) \right). \tag{1.9}$$

It is clear that the success of the calculation mainly depends on our ability to find an analytical solution of the heat kernel equation and carry out the integration over the proper time in Eq. (1.9). The integral is obviously divergent as $s \to 0$. As we have already mentioned above, this divergence can be easily isolated and interpreted in terms of the local ultraviolet properties of the theory. On the other hand, the behavior of the integral at infinity, $s \to \infty$, determines infrared properties of the theory and carries the physical information, e.g., particle creation. If the field has a big positive mass the proper time integral is convergent at $s \to \infty$. However, in the case of massless fields, the situation is much less trivial. The infrared convergence here depends on the approximation scheme used to calculate $K(s|x,y)$. Then, it is often even unclear to what extent the obtained effective action reflects the physical properties of the theory rather than the features of the approximation scheme used.

Below, we discuss the known calculational techniques, namely, the local Schwinger-DeWitt expansion [1,2], nonlocal covariant perturbation theory [3–5], and the modified gradient expansion [6], and point out why all of them fail when applied to interesting physical problems. Instead of them, we suggest a new method based on resummation of perturbation series and calculate new, essentially nonperturbative terms in the effective action. This method becomes indispensable in low-dimensional models ($d \leqslant 2$) where all previously known techniques are inapplicable. In this paper, we demonstrate how our method works in flat space of dimension $d>2$, while the generalization to the curved space and low-dimensional case will be considered in [7].

One of the main results of this paper is an exact (nonperturbative in V) late-time asymptotics for the heat kernel which in a spacetime of dimension $d>2$ for the coincidence limit takes the following form:

$$K(s|x,x) = \frac{1}{(4\pi s)^{d/2}} \left(1 + \frac{1}{\Box - V} V(x) \right)^2, \quad s \to \infty. \tag{1.10}$$

This asymptotics can also be found when the arguments of $K(s|x,y)$ are different—see Sec. IV for details. Another important result is the (nonperturbative in the potential) expression

$$\mathrm{Tr}\, K(s) = \frac{1}{(4\pi s)^{d/2}} \int dx \left\{ 1 - s \left(V(x) + V \frac{1}{\Box - V} V(x) \right) \right\}, \quad s \to \infty, \tag{1.11}$$

for the functional trace of $K(s)$.[1]

To avoid excessive use of integration signs we employ here and throughout the paper the following shorthand notation:

$$\frac{1}{\Box - V} J(x) \equiv \int dy\, G(x,y) J(y), \tag{1.12}$$

where $G(x,y)$ is the Green's function of the operator $-\hat{F} = \Box - V$ with zero boundary conditions at spacetime infinity, that is,

$$(\Box - V)G(x,y) = \delta^{(d)}(x,y), \quad G(x,y) \to 0, \quad |x| \to \infty, \tag{1.13}$$

and $J(x)$ can be any function of various field quantities like powers of the potential, its derivatives, etc. We always presume that the spacetime has a positive definite (Euclidean) signature, so that the Laplacian $\Box$ is negative definite assuming zero boundary conditions at infinity. Moreover, we consider only non-negative potentials $V(x) \geqslant 0$, so that the whole operator $\hat{F} = -\Box + V$ is positive definite. Therefore, the Green's function (1.13) is uniquely defined and guarantees that the nonlocal expression (1.12) makes sense for $d>2$.

As we shall see, the asymptotics (1.10) and (1.11) are the cornerstone of the technique we develop for the calculation of nonlocal contributions to the effective action. In particular, they lead to essentially nonperturbative terms which can be explicitly calculated for two broad classes of potentials with compact support, namely, for those that are, respectively, very small or very big in units of the inverse size of their support. For small potentials we get the terms which *replace* the conventional Coleman-Weinberg contribution to the effective action. In four dimensions, for instance, these terms read

[1]Note that this expression for $\mathrm{Tr}\, K(s)$ cannot be obtained directly by integrating the asymptotics (1.10) over the whole spacetime because for a given s this asymptotics fails at $|x|^2 > s$. Its derivation is given in Sec. IV and Appendix B, where we show that the expressions (1.10) and (1.11) are in complete agreement with each other.

$$\Delta\Gamma=\frac{1}{64\pi^2}\int d^4xV^2(x)\ln\left(\int d^4yV^2(y)\right)$$
$$-\frac{1}{64\pi^2}\int d^4xV^2(x)\ln\left(\int d^4yV\frac{\mu^2}{V-\Box}V(y)\right), \tag{1.14}$$

where the mass parameter μ^2 reflects the usual ultraviolet renormalization ambiguity. On the contrary, in the case of big potentials the Coleman-Weinberg action is *supplemented* by a nonlocal term of the form

$$\Delta\Gamma=\frac{1}{64\pi^2}\int_{|x|\leqslant R}d^4x\left\langle V+V\frac{1}{\Box-V}V\right\rangle^2. \tag{1.15}$$

Here R is the size of the compact support of $V(x)$, that is, $V(x)=0$ at $|x|>R$, and $\langle\cdots\rangle$ denotes the spacetime averaging of the corresponding quantity over this compact domain. The expressions obtained are both nonlocal and nonanalytic in the potential $V(x)$.

The paper is organized as follows. In Sec. II we consider the known approximation schemes and discuss their applicability in the infrared region. Section III is devoted to the nonlocal and nonlinear resummation of the Schwinger-DeWitt perturbation series, corresponding to the so-called connected graph expansion of the heat kernel. In Sec. IV, as an extension of this resummation, we derive the asymptotics (1.10) and discuss its relation to the functional trace (1.11) of the heat kernel. The nonperturbative, nonlocal contributions to the effective action are obtained in Sec. V with the aid of the new technique based on a piecewise smooth approximation for the heat kernel. In two appendixes we give details of the resummation technique and derive from the covariant perturbation theory the asymptotics of the heat kernel trace (1.11) up to the first subleading order in $1/s$ inclusive.

II. APPROXIMATION SCHEMES AND INFRARED PROPERTIES OF THE EFFECTIVE ACTION

In flat space, which we consider in this paper, the solution of the heat kernel equation can be easily found if the potential vanishes. For an arbitrary spatially dependent potential the analytical expressions are, of course, available only in certain approximations. In the general case, it is convenient to factorize the ''zero potential'' part of the solution explicitly and use the following ansatz for $K(s|x,y)$:

$$K(s|x,y)=\frac{1}{(4\pi s)^{d/2}}\exp\left[-\frac{|x-y|^2}{4s}\right]\Omega(s|x,y), \tag{2.1}$$

where the factor singular in s guarantees that the initial condition (1.8) is satisfied, provided that Ω is analytic in s at $s=0$ and $\Omega(0|x,y)=1$. If $V=0$, then $\Omega\equiv1$, and hence all nontrivial information about the potential is encoded in the deviation of Ω from unity.

The most well-known approximation used for the calculation of $K(s|x,y)$ is the so-called local Schwinger-DeWitt expansion, where Ω is written down as a series in growing powers of the proper time s. This expansion is a very powerful tool for revealing local ultraviolet properties of the theory. However, when applied in the infrared region, it gives a finite result only for massive fields. If the potential $V(x)$ has a large positive constant part, that is,

$$V(x)=m^2+v(x), \tag{2.2}$$

where m^2 is the squared mass of the field, then the function $\Omega(s|x,y)$ contains an overall exponential factor e^{-sm^2} and the part independent of mass is expanded in powers of s:

$$\Omega(s|x,y)=e^{-sm^2}\sum_{n=0}^{\infty}a_n(x,y)s^n. \tag{2.3}$$

Here $a_n(x,y)$ are the two-point Schwinger-DeWitt coefficients, whose coincidence limits $(x\to y)$ are explicitly calculable in general field theories, including gravity. Substituting Eq. (2.3) into Eq. (2.1) and then the obtained expression at $x=y$ into Eq. (1.9), one gets

$$\Gamma=-\frac{1}{2(4\pi)^{d/2}}\int dx\sum_{n=0}^{\infty}\left(\int_0^{\infty}ds s^{n-d/2-1}e^{-sm^2}\right)a_n(x,x). \tag{2.4}$$

It is important that the exponent e^{-sm^2} is not expanded here in powers of s. Therefore, in the proper time integral it provides a cutoff at the upper limit, so that the powers of s in this expansion get effectively replaced by powers of $1/m^2$. The first $(d/2+1)$ integrals in Eq. (2.4) diverge at $s\to0$ and should be regularized. To do that, we apply the dimensional regularization method; namely, by replacing the dimensionality d by 2ω, we calculate the integrals in the domain of their convergence and then analytically continue the result to $\omega\to d/2$. In spaces with even number of dimensions, which we mainly consider in what follows, this gives rise to the contribution $\Gamma_{\mathrm{div,\,log}}$ containing the pole at $\omega=d/2$ and the term logarithmic in m^2:

$$\Gamma_{\mathrm{div,log}}=\frac{1}{2(4\pi)^{d/2}}\int dx\sum_{n=0}^{d/2}\frac{(-m^2)^{d/2-n}}{(d/2-n)!}$$
$$\times\left[\frac{1}{\omega-d/2}-\Gamma'\left(\frac{d}{2}-n+1\right)+\ln\frac{m^2}{4\pi\mu^2}\right]a_n(x,x), \tag{2.5}$$

where $\omega\to d/2$. The pole corresponds to an infinite ultraviolet renormalization of the terms proportional to $a_0,\ldots,a_{d/2}$ in the original Lagrangian. Other terms in the expansion (2.4) are finite and give the infrared contribution to the total action

$$\Gamma=\Gamma_{\mathrm{div,ln}}-\frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\int dx\sum_{n=d/2+1}^{\infty}\frac{\Gamma(n-d/2)}{(m^2)^n}a_n(x,x). \tag{2.6}$$

The Schwinger-DeWitt coefficients $a_n(x,x)$ are the homogeneous polynomials of dimensionality $2n$ in the units of inverse length, which are built of $v(x)$ and its multiple derivatives. Therefore, on dimensional grounds, they can be symbolically written down as

$$a_n(x,x) \sim v^k(x)(\nabla^i v^j)(x),$$

where i denotes an overall number of derivatives acting in all possible ways on j factors of $v(x)$, and k powers of $v(x)$ stay undifferentiated. The positive integers (k,j,i) are related to n as $2(k+j)+i=2n$. It is clear that the infinite series in Eq. (2.6) represents the expansion in growing powers of the following dimensionless quantities:

$$\frac{v(x)}{m^2} \ll 1, \quad \frac{\nabla^i v(x)}{m^{2+i}} \ll 1, \tag{2.7}$$

which obviously should be much smaller than unity. Only in this case are the first few terms in the asymptotic series (2.6) reliable.

Thus, the Schwinger-DeWitt expansion is applicable only in theories with *small and slowly varying fields* as compared to a big mass parameter. This expansion contains only local terms. This is not surprising because all nonlocal effects, e.g., particle creation, are very small for heavy particles in a weak external field and cannot be handled by this method. The Schwinger-DeWitt technique can be easily extended to curved spacetime and to theories with covariant derivatives built with respect to an arbitrary fiber-bundle connection. In this case, the perturbation potential $v(x)$ will also depend on the spacetime curvature tensor and fiber-bundle curvatures (commutator of covariant derivatives). The smallness of fields and their derivatives includes the requirement of the smallness of these curvatures and their derivatives as well. Despite its universality, the Schwinger-DeWitt expansion becomes inefficient when the ratios in Eq. (2.7) become of the order of unity, and completely fails for massless fields. In the last case all integrals over the proper time integral are infrared divergent. This divergence has, of course, no physical meaning and is an artifact of the approximation technique used.

There are two known ways to proceed with massless fields. One possibility is the resummation of all terms that contain the undifferentiated potential $V(x)$ in the local Schwinger-DeWitt series (2.3). They are summed up to form an exponent similar to e^{-sm^2}:

$$\Omega(s|x,x) = e^{-sV(x)} \sum_{n=0}^{\infty} \tilde{a}_n(x,x) s^n. \tag{2.8}$$

This method was suggested in [6], where a regular technique for the calculation of the modified Schwinger-DeWitt coefficients $\tilde{a}_n(x,y)$ was also presented. The proper time integral in Eq. (1.9) now has an infrared cutoff at $s \sim 1/V(x)$ and in this case the effective action is similar to Eqs. (2.5),(2.6), where m^2 is replaced by $V(x)$ and $a_n(x,x)$ by $\tilde{a}_n(x,x)$. It is convenient to write this action as a sum of three terms

$$\Gamma = \Gamma_{\rm div} + \Gamma_{\rm CW} + \Gamma_{\rm fin}, \tag{2.9}$$

where the divergent part is equal to

$$\Gamma_{\rm div} = \frac{1}{2(4\pi)^{d/2}} \int dx \sum_{n=0}^{d/2} \frac{(-V)^{d/2-n}}{(d/2-n)!} \times \left[\frac{1}{\omega - d/2} - \Gamma'\left(\frac{d}{2}-n+1\right) - \ln 4\pi \right] \tilde{a}_n(x,x). \tag{2.10}$$

The pole part of this action coincides with that of Eq. (2.5) if we take the $m^2 \to 0$ limit of Eq. (2.5). Actually, in this case, only the term proportional to $a_{d/2}$ survives in $\Gamma_{\rm div,\,log}$ and by virtue of the relation between twiddled and untwiddled coefficients, namely,

$$a_{d/2}(x,x) = \sum_{n=0}^{d/2} \frac{(-V)^{d/2-n}}{(d/2-n)!} \tilde{a}_n(x,x), \tag{2.11}$$

the pole parts of Eqs. (2.5) and (2.10) are the same. The terms proportional to $\Gamma'(d/2-n+1)$ perform finite renormalization of the local terms $V^{d/2-n}\tilde{a}_n$. The logarithmic terms of Eq. (2.5) are replaced in the modified action (2.9) by

$$\Gamma_{\rm CW} = \frac{1}{2(4\pi)^{d/2}} \int dx \sum_{n=0}^{d/2} \frac{(-V)^{d/2-n}}{(d/2-n)!} \ln\frac{V}{\mu^2} \tilde{a}_n = \frac{1}{2(4\pi)^{d/2}} \int dx \ln\frac{V}{\mu^2} a_{d/2}. \tag{2.12}$$

This is nothing but the spacetime integral of the Coleman-Weinberg effective potential. For instance, in four dimensions the leading term is the original Coleman-Weinberg effective potential $V^2 \ln(V/\mu^2)/64\pi^2$, while the rest represents corrections due to the derivatives of $V(x)$. Similarly to Eq. (2.6), the finite part $\Gamma_{\rm fin}$ is an infinite series

$$\Gamma_{\rm fin} = -\frac{1}{2} \int dx \left(\frac{V(x)}{4\pi}\right)^{d/2} \sum_{n=d/2+1}^{\infty} \Gamma(n-d/2) \frac{\tilde{a}_n(x,x)}{V^n(x)}. \tag{2.13}$$

The modified Schwinger-DeWitt coefficients do not contain the undifferentiated potential and the typical structure of the terms entering $\tilde{a}_n(x,x)$ is $\nabla^m V^j(x)$, where $m+2j=2n$. Every V here should be differentiated at least once and therefore $m \geqslant j$. Thus the coefficients $\tilde{a}_n$ can be symbolically written down as

$$\tilde{a}_n(x,x) \sim \sum_{j=1}^{[2n/3]} \nabla^{2n-2j} V^j, \tag{2.14}$$

where the upper value of j is the integer part of $2n/3$.

This perturbation theory is efficient as long as the potential is slowly varying or bounded from below by a large positive constant, so that

$$\frac{\nabla^2 V(x)}{V^2(x)} \ll 1, \quad \frac{[\nabla V(x)]^2}{V^3(x)} \ll 1, \ldots . \tag{2.15}$$

The case of bounded potentials reproduces the original Schwinger-DeWitt expansion for nonvanishing mass. Therefore, let us consider the potentials which vanish at spacetime infinity ($|x| \to \infty$); namely, we assume the case of a power-like falloff

$$V(x) \sim \frac{1}{|x|^p}, \quad \nabla^m V(x) \sim \frac{1}{|x|^{p+m}}, \quad |x| \to \infty \tag{2.16}$$

for some positive p. For such a potential terms of the perturbation series (2.13) behave as

$$\frac{\tilde{a}_n}{V^n} \sim \sum_{j=1}^{[2n/3]} |x|^{(p-2)(n-j)} \tag{2.17}$$

and thus decrease with increasing n only if $p<2$. For $p \geqslant 2$, the modified gradient expansion completely breaks down. It makes sense only for slowly decreasing potentials of the form (2.16) with $p<2$. In this case the potential $V(x)$ is not integrable over the whole spacetime [$\int dx V(x) = \infty$] and moreover even the operation $(1/\Box)V(x)$ is not well defined.[2] Therefore the above restriction is too strong to account for many interesting physical problems. In addition, similarly to Eq. (2.6), the asymptotic expansion (2.13) is completely local. It does not allow us to capture nonlocal effects, which are exponentially small for potentials satisfying Eq. (2.15).

The way to overcome this difficulty and to take into account nonlocal effects was suggested in the form of covariant perturbation theory (CPT) [3–5]. In this theory the full potential $V(x)$ is treated as a perturbation and the solution of the heat equation is found as a series in its powers. From the viewpoint of the Schwinger-DeWitt expansion it corresponds to an infinite resummation of all terms with a given power of the potential and arbitrary number of derivatives. The result reads as

$$\mathrm{Tr}\, K(s) \equiv \int dx K(s|x,x) = \sum_{n=0}^{\infty} \mathrm{Tr}\, K_n(s), \tag{2.18}$$

where

$$\mathrm{Tr}\, K_n(s) = \int dx_1 dx_2 \cdots dx_n F_n(s|x_1, x_2, \ldots, x_n) \times V(x_1) V(x_2) \cdots V(x_n), \tag{2.19}$$

and the nonlocal form factors $F_n(s|x_1, x_2, \ldots, x_n)$ were explicitly obtained in [3–5]. It was shown that at $s \to \infty$ the terms in this expansion behave as

$$\mathrm{Tr}\, K_n(s) = O\left(\frac{1}{s^{d/2-1}}\right), \quad n \geqslant 1, \tag{2.20}$$

and therefore in spacetime dimension $d \geqslant 3$ the integrals in Eq. (1.9) are infrared convergent:

$$\int^{\infty} \frac{ds}{s} O\left(\frac{1}{s^{d/2-1}}\right) < \infty. \tag{2.21}$$

In one and two dimensions this expansion for Γ does not exist except for the special case of the massless theory in curved two-dimensional spacetime, when it reproduces the so-called Polyakov action [9,8,4].[3] CPT should always be applicable whenever $d \geqslant 3$ and the potential V is sufficiently small,[4] so that its effective action explicitly features analyticity in the potential at $V=0$. Therefore, its serious disadvantage is that this theory does not allow one to overstep the limits of perturbation scheme and, in particular, discover nonanalytic structures in the action if they exist.

All this implies the necessity of a new approximation technique that would allow us to overcome disadvantages of the above methods. In the rest of this paper we develop such a technique, involving further resummation of the perturbation series. We develop an infrared improved perturbation theory for the heat kernel and reveal new nonlocal and nonanalytical structures in the effective action.

III. RESUMMATION OF PROPER TIME SERIES

We use the exponential ansatz for the function $\Omega(s|x,y)$ defined by Eq. (2.1):

$$\Omega(s|x,y) = \exp[-W(s|x,y)].$$

Our goal is to develop an approximation technique for W similar to CPT, which is an alternative to the expansion in s. By virtue of Eqs. (1.7) and (1.8) the function $W(s|x,y)$ satisfies the equation

$$\frac{\partial W}{\partial s} + \frac{(x-y)^\mu}{s} \nabla_\mu W - \Box W = V - (\nabla W)^2, \tag{3.1}$$

with the initial condition

$$W(s=0|x,y) = 0. \tag{3.2}$$

This equation is nonlinear in W and we solve it by iteration, considering $(\nabla W)^2$ as a perturbation. For this purpose it is convenient to rewrite Eqs. (3.1),(3.2) as an integral equation. In Appendix A, it is shown that this integral equation takes the following form:

[2]For the convergence of the integral in $(1/\Box)V$ the potential $V(x)$ should fall off at least as $1/|x|^3$ in any spacetime dimension [4].

[3]This action can be obtained by integrating the conformal anomaly [9,8].

[4]The conditions of the smallness of the potential are exactly opposite to those of Eq. (2.15), e.g., $V^2/\nabla^2 V \ll 1$. However, this is true only as a rather rough estimate, because CPT is a nonlocal perturbation theory and its actual smallness "parameters" are some nonlocal functionals of the potential.

$$W(s|x,y)=s\int_0^1 d\alpha e^{s\alpha(1-\alpha)\bar{\Box}}\{V(\bar{x})-[\nabla W(s\alpha|\bar{x},y)]^2\}, \tag{3.3}$$

where the operator $\bar{\Box}$ acts on the argument $\bar{x}\equiv\bar{x}(\alpha|x,y)=\alpha x+(1-\alpha)y$. Equation (3.3) can be easily solved if $(\nabla W)^2\ll V$. As we will see later, this condition is satisfied for a broad class of potentials V. The lowest-order approximation for W is obtained by omitting $(\nabla W)^2$ in Eq. (3.3):

$$W_0(s|x,y)=s\int_0^1 d\alpha e^{s\alpha(1-\alpha)\bar{\Box}}V(\bar{x})|_{\bar{x}=\alpha x+(1-\alpha)y}. \tag{3.4}$$

This is a linear but essentially nonlocal functional of the potential. Further terms of the perturbation theory $W_n=O[(\nabla W_0)^n]$ can be graphically represented by connected tree graphs with two derivatives in the vertices, internal lines associated with the nonlocal operator

$$f(-s\Box)=\int_0^1 d\alpha e^{s\alpha(1-\alpha)\Box}, \tag{3.5}$$

and external lines given by Eq. (3.4).[5] Note that this connected graph structure arises in the exponential and when expanded gives rise to the disconnected graphs. In context of the heat kernel expansion this property was observed in [10]. Resummation of the perturbation series in V explicitly features exponentiation of the quantities containing only connected graphs. Here we showed how this exponentiated set of connected graphs directly arises from the solution of the simple nonlinear equation (3.1). The ''propagator'' (3.5) was worked out within the covariant perturbation theory in [3–5] and was also obtained in [6] by direct summation of gradient series.

At this stage the efficiency of the connected graph expansion is not yet obvious. Crudely, it runs in powers of the dimensionless quantity $[sf(-s\Box)\nabla]^2V(x)$ which, at least naively, should be small for slowly varying or/and small potentials. Apart from this, infrared properties of the effective action strongly depend on the lowest-order approximation for W Eq. (3.4). The effective action involves only diagonal elements of the two-point function $W(s|x,y)$, which look much simpler than Eq. (3.4):

$$W_0(s|x,x)=s\int_0^1 d\alpha e^{s\alpha(1-\alpha)\Box}V(x). \tag{3.6}$$

Note that, at small s, the function W_0 can be expanded as $W_0=sV+O(s^2)$. The only term with undifferentiated potential entering W_0 is linear in s, while all other terms contain derivatives of the potential V. The same is also true for the exact W, which differs from W_0 by higher powers of the differentiated potential. This completely agrees with the modified gradient expansion discussed in Sec. II. However, the expression (3.6) directly involves the nonlocal operator and its late-time behavior is very different from that naively expected in the modified gradient expansion and in CPT.

To show that, let us first find the coordinate representation of the operator (3.5), that is, the kernel $f(-s\Box)\delta^{(d)}(x,y)$. Using a well-known result for the exponentiated $\Box$ operator

$$e^{s\alpha(1-\alpha)\Box}\delta^{(d)}(x,y)=\frac{1}{[4\pi s\alpha(1-\alpha)]^{d/2}}\times\exp\left(-\frac{|x-y|^2}{4s\alpha(1-\alpha)}\right), \tag{3.7}$$

one can write

$$\int_0^1 d\alpha e^{s\alpha(1-\alpha)\Box}\delta^{(d)}(x,y)=\frac{e^{-|x-y|^2/2s}}{(4\pi s)^{d/2}}\int_0^\infty d\beta\frac{(1+\beta)^{d-2}}{\beta^{d/2}}\times\exp\left[-\frac{|x-y|^2}{4s}\left(\frac{1}{\beta}+\beta\right)\right], \tag{3.8}$$

where we have changed the integration variable, $\alpha=\beta/(1+\beta)$, $0\leqslant\beta<\infty$. For $d\geqslant 3$, the integral can be easily calculated and the result is expressed as a sum of McDonald functions of the argument $|x-y|^2/2s$:

$$f(-s\Box)\delta^{(d)}(x,y)=\frac{2e^{-|x-y|^2/2s}}{(4\pi s)^{d/2}}\sum_{k=1}^{d-1}C_{k-1}^{d-2}K_{k-d/2}\times(|x-y|^2/2s), \tag{3.9}$$

where C_{k-1}^{d-2} are the binomial coefficients. For very large s, the argument of $K_{k-d/2}(z)$ is small. Using the asymptotics $K_\nu(z)\simeq\Gamma(|\nu|)(2/z)^{|\nu|}/2$, $z\to 0$, we find that at large s the form factor is dominated by the following term:

$$f(-s\Box)\delta^{(d)}(x,y)=\frac{1}{2s}\frac{\Gamma(d/2-1)}{\pi^{d/2}|x-y|^{d-2}}+O\left(\frac{1}{s^2}\right). \tag{3.10}$$

Taking into account the fact that

$$\frac{1}{\Box}\delta^{(d)}(x,y)=-\frac{\Gamma(d/2-1)}{4\pi^{d/2}|x-y|^{d-2}}, \tag{3.11}$$

we finally obtain

$$W_0(s|x,x)=sf(-s\Box)V=-\frac{2}{\Box}V(x)+O\left(\frac{1}{s}\right). \tag{3.12}$$

This behavior agrees with the formal asymptotics found by the Laplace method in [4]. Therefore, at $s\to\infty$, the function

[5]This graphical interpretation should not be taken too literally because integration over α parameter(s) also involves the argument $\bar{x}\equiv\bar{x}(\alpha|x,y)$ of the potential.

W_0 approaches a constant. The nonlocal functional (3.12) is well defined for $d \geqslant 3$ only if the potential V vanishes fast enough at $|x| \to \infty$.

For split arguments the asymptotics of $W_0(s|x,y)$ is more intricate. In this case the form factor (3.5) no longer arises as a whole because the integration parameter α appears in Eq. (3.4) also in the argument $\bar{x} = \alpha x + (1-\alpha)y$ of the potential $V(\bar{x})$. Applying the Laplace method, one can show that the integral (3.4) is dominated by the contribution of the end points $\alpha = 0$ and $\alpha = 1$. These contributions are different because $\bar{x}(\alpha = 0) = y$ and $\bar{x}(\alpha = 1) = x$, whence

$$W_0(s|x,y) = -\frac{1}{\Box} V(x) - \frac{1}{\Box} V(y) + O\left(\frac{1}{s}\right). \quad (3.13)$$

Substituting this expression in Eq. (3.3) and solving the integral equation by iterations one can find the late-time asymptotics $W_\infty(x,y)$ for the exact $W(s|x,y)$

$$W(s|x,y) = W_\infty(x,y) + O\left(\frac{1}{s}\right) \quad (3.14)$$

as a nonlocal gradient series

$$W_\infty(x,y) = -\frac{1}{\Box} V(x) + \frac{1}{\Box}\left(\frac{1}{\Box}\nabla V(x)\right)^2 + \cdots + (x \leftrightarrow y). \quad (3.15)$$

It is remarkable, however, that this series can be ''summed up'' and the nonlocal expression for $W_\infty(x,y)$ can be found exactly in terms of the Green's function of the original operator $\hat{F} = -\Box + V$.

IV. LATE-TIME ASYMPTOTICS OF THE HEAT KERNEL

By substituting the late time ansatz (3.14) in Eq. (3.1), it is easy to see that the first two terms in its left-hand side vanish at $s \to \infty$, while the rest reduce to the equation for $W_\infty(x,y)$:

$$(\Box - V) e^{-W_\infty(x,y)} = 0. \quad (4.1)$$

Despite the positivity of the operator $-\Box + V$ with Dirichlet boundary conditions at $|x| \to \infty$, this equation admits nontrivial solutions. In fact, $e^{-W_\infty(x,y)}$ does not have to go to zero at infinity.[6] In view of the iterative solution (3.15) it should tend to some unknown function of y,

$$e^{-W_\infty(x,y)} \to C(y), \quad |x| \to \infty. \quad (4.2)$$

Equation (4.1) with this boundary condition is then solved by

$$e^{-W_\infty(x,y)} = C(y)\Phi(x), \quad (4.3)$$

if the new function $\Phi(x)$ satisfies the equation

$$(\Box - V)\Phi(x) = 0, \quad (4.4)$$

and $\Phi(x) \to 1$ at $|x| \to \infty$. The solution of this problem for $\Phi(x)$ is uniquely determined in terms of the Green's function (1.13):

$$\Phi(x) = 1 + \frac{1}{\Box - V} V(x). \quad (4.5)$$

This function of x is a nonlocal and essentially nonlinear functional of the potential, and it will play a very important role in what follows.

The heat kernel is symmetric in its arguments x,y and, therefore, the unknown function $C(y)$ should coincide with $\Phi(y)$. Thus, finally we obtain the following exact late-time asymptotics:

$$e^{-W_\infty(x,y)} = \Phi(x)\Phi(y). \quad (4.6)$$

Expanding $W_\infty(x,y)$ in powers of the potential V one gets

$$\begin{aligned} W_\infty(x,y) &= -\ln\Phi(x) - \ln\Phi(y) \\ &= -\frac{1}{\Box} V(x) - \frac{1}{\Box} V \frac{1}{\Box} V(x) \\ &\quad + \frac{1}{2}\left(\frac{1}{\Box} V(x)\right)^2 + \cdots + (x \leftrightarrow y). \end{aligned} \quad (4.7)$$

The first term here is in agreement with the perturbative asymptotics (3.15). However, beyond that, the iterative solution (3.15) seems to be in contradiction with Eq. (4.7). The series (3.15) runs in powers of the differentiated potential, while the expansion (4.7) contains only the undifferentiated potential. However, integration by parts in the second term of Eq. (3.15) exactly reproduces the second and third terms of Eq. (4.7). Thus, both expansions are equivalent, but the first one reveals more explicitly the gradient of the potential as a small parameter, while in Eq. (4.7) the smallness is a result of nontrivial cancellations between different terms.

Finally we write down the exact late-time asymptotics for the heat kernel, advocated in the Introduction,

$$K(s|x,y) = \frac{1}{(4\pi s)^{d/2}} \Phi(x)\Phi(y), \quad s \to \infty. \quad (4.8)$$

Its heuristic interpretation is rather transparent. The heat kernel can be decomposed in the series

$$K(s|x,y) = \sum_\lambda e^{-\lambda s} \Phi_\lambda(x)\Phi_\lambda(y), \quad (4.9)$$

where λ and Φ_λ are, respectively, the eigenvalues and eigenfunctions of the operator $\hat{F} = -\Box + V(x)$. Since $\hat{F}$ is a positive semidefinite operator, only the lowest eigenmode with $\lambda = 0$ survives in this expression in the limit $s \to \infty$. The appropriate eigenfunction satisfies the equation $\hat{F}\Phi_{\lambda=0} = 0$, which coincides with Eq. (4.4) and therefore $\Phi_{\lambda=0}(x) = \Phi(x)$. The spectrum of the operator is continuous and the eigenmodes are not square integrable [$\Phi_0(x) \to 1$ at

[6] The boundary condition $K(s|x,y) \to 0$ at $|x| \to \infty$ is enforced by the Gaussian factor in Eq. (2.1), even for nonvanishing finite $\Omega(s|x,y) = \exp(-W(s|x,y))$.

$|x|\to\infty$]. This is why the integral over the spectrum, denoted above by Σ_λ, yields within the steepest decent approximation the powerlike asymptotics $1/s^{d/2}$, rather than the exponential one. Of course, these arguments are not very rigorous. The zero mode $\Phi(x)$ with unit boundary condition at infinity does not even belong to the continuous spectrum of modes normalized to the delta function. Nevertheless, as we have seen, this particular mode gives the leading contribution to the late-time asymptotics of the heat kernel.

If we want to calculate the contribution to the effective action due to the late-time behavior of the heat kernel we need its functional trace—the spacetime integral of the coincidence limit $K(s|x,x)$. Unfortunately, the expression (4.8) cannot be used directly to calculate $\mathrm{Tr}\,K(s)$ for a given s. The point is that this asymptotic expression taken at a fixed large s is applicable only for $|x|^2<s$ and fails at $|x|^2\gg s$.[7] At the same time, when calculating the trace we have to integrate over the whole spacetime up to $|x|\to\infty$ and therefore need the heat kernel behavior for $|x|^2\gg s$. The attempt to disregard this subtlety and integrate the coincidence limit of Eq. (4.8) over x results in a poorly defined quantity—the spacetime integral is strongly divergent at infinity. Nevertheless, one can use the expression (4.8) to find $\mathrm{Tr}\,K(s)$ with the aid of the following somewhat subtler procedure.

First let us write the variational relation

$$\delta\,\mathrm{Tr}\,K(s)=-s\,\mathrm{Tr}\,[\delta V K(s)]=-s\int dx\,\delta V(x)K(s|x,x), \tag{4.10}$$

where, of course, $K(s)=\exp[s(\Box-V)]$. Then it follows that

$$\frac{\delta\,\mathrm{Tr}\,K(s)}{\delta V(x)}=-sK(s|x,x). \tag{4.11}$$

Substituting $K(s|x,x)$ from Eq. (4.8) in the right-hand side of this relation we obtain the following functional differential equation:

$$\frac{\delta\,\mathrm{Tr}\,K(s)}{\delta V(x)}=-\frac{s}{(4\pi s)^{d/2}}\Phi^2(x). \tag{4.12}$$

This equation satisfies the integrability condition, because the variational derivative

$$\frac{\delta\Phi^2(x)}{\delta V(y)}=2\Phi(x)\Phi(y)\frac{1}{\Box-V}\delta(x,y) \tag{4.13}$$

is symmetric in x and y. Therefore Eq. (4.12) can be integrated to determine $\mathrm{Tr}K(s)$. The solution subject to the obvious boundary conditions at $V=0$ reads

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{d/2}}\int dx(1-sV\Phi),\quad s\to\infty. \tag{4.14}$$

[7]This follows from the derivation of Eq. (4.8) above, which is based on discarding the second term of Eq. (3.1) linearly growing in $(x-y)/s$.

One can easily check that this expression satisfies Eq. (4.12).

It is remarkable that in the covariant perturbation theory the leading and next subleading (in s) terms of the heat kernel trace can be explicitly calculated for $s\to\infty$ in every order of expansion in powers of V. The corresponding infinite series can be explicitly summed up to yield essentially nonlocal and nonlinear in $V(x)$ expression for $\mathrm{Tr}\,K(s)$. This is done in Appendix B to the first subleading order inclusive. The following very simple and concise result reads as:

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{d/2}}\int dx\left\{1-sV\Phi-2\nabla_\mu\Phi\frac{1}{\Box-V}\nabla^\mu\Phi+O\left(\frac{1}{s}\right)\right\} \tag{4.15}$$

in terms of the function $\Phi(x)$ and its derivatives. As we see, it exactly reproduces the leading order term of Eq. (4.14), $O(s/s^{d/2})$, and also gives a nontrivial $O(1/s^{d/2})$ correction. Below we use this asymptotics for obtaining new types of nonlocal effective action.

V. EFFECTIVE ACTION

The functional trace of the heat kernel is everything we need for the calculation of the effective action. Unfortunately, only its asymptotics are known; namely, at small s one can use the modified gradient expansion (2.1), (2.8) and at large s the nonlocal and nonlinear expression (4.15). The goal of this section is to unify both of these approximations to get the expression for the effective action which would incorporate both the ultraviolet and infrared properties of the theory. The calculation will be explicitly done in the four-dimensional case. The generalization to other dimensions $d>2$ is straightforward.

The key idea is to replace $\mathrm{Tr}\,K(s)$ in Eq. (1.9) by some approximate function $\mathrm{Tr}\,\bar{K}(s)$ such that the integral over s,

$$\bar{\Gamma}=-\frac{1}{2}\int\frac{ds}{s}\mathrm{Tr}\,\bar{K}(s), \tag{5.1}$$

becomes explicitly calculable. The difference $\mathrm{Tr}\,K(s)-\bar{K}(s)$ can then be treated as a perturbation. Certainly, the efficiency of this procedure very much depends on the successful choice of $\bar{K}(s)$. Here we exploit the simplest possibility—namely, let us take two simple functions $\mathrm{Tr}\,\bar{K}_<(s)$ and $\mathrm{Tr}\,\bar{K}_>(s)$, which coincide with the leading asymptotics of $\mathrm{Tr}\,K(s)$ at $s\to0$ and $s\to\infty$ and use them to approximate $\mathrm{Tr}\,K(s)$ at, respectively, $0\leqslant s\leqslant s_*$ and $s_*\leqslant s<\infty$. In turn, the value of s_* will be determined from the requirement that these two functions match at s_*. This will guarantee the stationarity of $\bar{\Gamma}$ with respect to the choice of s_*, $\partial\bar{\Gamma}/\partial s_*=0$. We will discuss the justification of this procedure a little later, while now let us proceed with the calculation of $\bar{\Gamma}$.

At small s we use the lowest-order term of the modified gradient expansion

$$\mathrm{Tr}\,K_<(s)=\frac{1}{(4\pi s)^2}\int dx\,\exp(-Vs),\quad s<s_*,\qquad (5.2)$$

and disregard all terms containing derivatives of the potential V. The rest of the asymptotic series (2.8), containing $\tilde{a}_n$, will be treated by perturbations. Correspondingly, at large $s>s_*$ we use the late-time asymptotics (4.14):

$$\mathrm{Tr}\,K_>(s)=\frac{1}{(4\pi s)^2}\int dx(1-sV\Phi),\quad s>s_*.\qquad (5.3)$$

The requirement of stationarity of $\bar{\Gamma}$ with respect to s_* leads to the equation

$$\int dx\,\exp(-Vs_*)=\int dx(1-s_*V\Phi),\qquad (5.4)$$

which determines the value of s_* as some nontrivial functional of the potential, $s_*=s_*[V(x)]$. Unfortunately this functional is not calculable explicitly in general, but nevertheless, as we will see below, it can be obtained for two rather broad classes of potentials. The action (5.1) can be written down as a sum of two contributions:

$$\bar{\Gamma}=\Gamma_<+\Gamma_>$$
$$=-\frac{1}{2}\int_0^\infty\frac{ds}{s}\mathrm{Tr}\,K_<(s)-\frac{1}{2}\int_{s_*}^\infty\frac{ds}{s}[\mathrm{Tr}\,K_>(s)-\mathrm{Tr}\,K_<(s)].\qquad (5.5)$$

The first integral here has already been calculated and is given by the sum of the expressions (2.10) and (2.12) with $\tilde{a}_0=1$ and $\tilde{a}_n=0$, $n\geqslant 1$, which in our particular case gives rise to

$$\Gamma_<=\Gamma_{\mathrm{div}}+\Gamma_{\mathrm{CW}}$$
$$\equiv\frac{1}{64\pi^2}\int dx\left[\left(-\frac{1}{2-\omega}+2\mathbf{C}-3-\ln 4\pi\right)V^2+V^2\ln\frac{V}{\mu^2}\right],\quad \omega\to 2,\qquad (5.6)$$

where $\mathbf{C}=0.577\ldots$ is Euler's constant. The first term here is responsible for the renormalization of the original action and the second one is just the Coleman-Weinberg potential. The second integral in Eq. (5.5) can also be calculated exactly. Integrating by parts and taking into account Eq. (5.4), we obtain

$$\Gamma_>\equiv-\frac{1}{2}\int_{s_*}^\infty\frac{ds}{s}[\mathrm{Tr}\,K_>(s)-\mathrm{Tr}\,K_<(s)]$$
$$=\frac{1}{64\pi^2}\int dx\left[\frac{V\Phi}{s_*}-\frac{Ve^{-s_*V}}{s_*}+V^2\Gamma(0,s_*V)\right],\qquad (5.7)$$

where $\Gamma(0,x)$ is an incomplete gamma function, $\Gamma(0,x)=\int_x^\infty dt\,t^{-1}e^{-t}$, with the following asymptotics:

$$\Gamma(0,x)\sim\begin{cases}\ln\dfrac{1}{x}-\mathbf{C}, & x\ll 1,\\ \dfrac{1}{x}e^{-x}, & x\gg 1.\end{cases}\qquad (5.8)$$

Further steps strongly depend on the class of potential, for which the consistency of the piecewise approximation (5.2),(5.3) should be carefully analyzed.

A. Small potential

The approximation (5.2),(5.3) is efficient only if the ranges of the validity of the two asymptotic expansions (respectively, for small and big s) overlap with each other and the point s_* belongs to this overlap. In this case the corrections due to the deviation of $\mathrm{Tr}\,\bar{K}(s)$ from the exact $\mathrm{Tr}\,K(s)$ are uniformly bounded everywhere and one can expect that Eq. (5.1) will give a good zeroth order approximation to an exact result. Below we show that this necessary requirement can be satisfied at least for two rather wide classes of potential $V(x)$.

The modified gradient expansion is well applicable in the overlap range of the parameter s if

$$s\nabla\nabla V\ll V\qquad (5.9)$$

[cf. Eq. (2.15) with s replaced by effective cutoff $s=1/V$] and the applicability of the large s expansion in the same domain reads as

$$s\int dx V\Phi\gg\int dx\nabla_\mu\Phi\frac{1}{V-\Box}\nabla^\mu\Phi,\qquad (5.10)$$

which means that the subleading term (quadratic in $\nabla_\mu\Phi$) of the late time expansion (4.15) is much smaller than the leading second term.

To implement these requirements, let us make some simplifying assumptions. Instead of the power law falloff, assume that $V(x)$ has a compact support of finite size R,

$$V(x)=0,\quad |x|\geqslant R.\qquad (5.11)$$

Let us also assume that the potential is sufficiently smooth inside its support and the characteristic magnitude of $V(x)$ is given by V_0. Then the derivatives of the potential are bounded and satisfy the following obvious estimate:

$$\nabla\nabla V\sim\frac{V_0}{R^2},\qquad (5.12)$$

so that Eq. (5.9) reads as $sV_0/R^2\ll V_0$, or

$$s\ll R^2.\qquad (5.13)$$

To find out what the criterion (5.10) means let us make a further assumption, namely, that the potential V is small. In

this case it can be disregarded in the Green's functions and $1/(\Box - V)$ can be replaced by $1/\Box$. Therefore the following estimates hold:

$$\frac{1}{\Box - V} V(x) \sim \int_{|y| \leqslant R} dy \frac{1}{|x-y|^{d-2}} V(y)$$

$$\sim \frac{1}{R^{d-2}} R^d V_0 \sim V_0 R^2,$$

$$\int dx V \Phi \sim V_0 R^d,$$

$$\int dx \nabla_\mu \Phi \frac{1}{V - \Box} \nabla^\mu \Phi \simeq V_0^2 R^{d+4}. \tag{5.14}$$

Roughly, every Green's function gives the factor R^2, every derivative $1/R$, integration gives the volume of compact support R^d, etc. Applying these estimates to Eq. (5.10) we get $sV_0R^d \gg V_0^2 R^{d+4}$, whence

$$s \gg V_0 R^4. \tag{5.15}$$

Combining this with Eq. (5.13) one gets the following range of overlap of our asymptotic expansions:

$$R^2 \gg s \gg V_0 R^4. \tag{5.16}$$

It immediately follows from here that this overlap domain is not empty only if

$$V_0 R^2 \ll 1. \tag{5.17}$$

Moreover, the assumption of disregarding the potential in the Green's function is also justified in this case since $V \sim V_0 \ll 1/R^2 \sim \Box$. In other words this bound means that the potential is small in units of the inverse size of its compact support.

Now let us check whether s_* introduced above belongs to the overlap domain (5.16). Note that if it is really so then $s_* V$ in Eq. (5.4) is much smaller than unity because in the overlap range one has $sV \sim sV_0 \ll R^2 V_0 \ll 1$. Hence the exponent in the left-hand side of Eq. (5.4) can be expanded in powers of $s_* V$, and the resulting equation for s_* becomes[8]

$$\int dx \left(1 - s_* V + \frac{s_*^2}{2} V^2 + O[(s_* V)^3] \right) = \int dx (1 - s_* V \Phi). \tag{5.18}$$

Its solution has the following form:

$$s_* \simeq 2 \frac{\int dx V (1 - \Phi)}{\int dx V^2} = 2 \frac{\int dx V [1/(V - \Box)] V}{\int dx V^2}. \tag{5.19}$$

Taking into account the estimates (5.14) we see that the point $s_* \sim R^2$ belongs to the upper edge of the interval (5.16). The late-time expansion is fairly well satisfied here, but the small s expansion is on the verge of breakdown. At this level of generality it is hard to overstep the uncertainty of this estimate. There is a hope that numerical coefficients in more precise estimates (with concrete potentials) can be large enough to shift s_* to the interior of the interval (5.16) and thus make our approximation completely reliable.

Bearing in mind all these reservations let us proceed with the calculation of the effective action. Using the small x asymptotics (5.8) in the expression (5.7) we get

$$\Gamma_> \simeq \frac{1}{64\pi^2} \int d^4x \left[-V \frac{1 - \Phi}{s_*} + V^2 \left(\ln \frac{1}{s_* V} - \mathbf{C} + 1 \right) \right]. \tag{5.20}$$

It is interesting to note that in the whole action $\bar{\Gamma} = \Gamma_< + \Gamma_>$ the Coleman-Weinberg term disappears and the final answer reads

$$\bar{\Gamma} \simeq \frac{1}{64\pi^2} \left(-\frac{1}{2-\omega} + \mathbf{C} - 2 - \ln 4\pi \right) \int d^4x V^2$$

$$+ \frac{1}{64\pi^2} \int d^4x V^2 \ln \left(\int dx V^2 \right)$$

$$- \frac{1}{64\pi^2} \int d^4x V^2 \ln \left(\int dx V \frac{\mu^2}{V - \Box} V \right). \tag{5.21}$$

The first term here differs from $\Gamma_{\rm div}$ in Eq. (5.6) by a finite renormalization of the local V^2 term, while the two other terms have the entirely new nonlinear and nonlocal structure (1.14) advocated in the Introduction. The ultraviolet renormalization mass parameter μ^2 makes the argument of the second logarithm dimensionless—it plays the same role as for the Coleman-Weinberg potential, but now it enters the new essentially nonlocal structure.

It is natural that the original Coleman-Weinberg term for the case of small potentials (5.17) gets replaced by the other qualitatively new nonlocal structure. Potentials that are small in units of the inverse size of their support are qualitatively very different from nearly constant potentials for which the Coleman-Weinberg action was originally derived. In the case of small potentials the spacetime gradients dominate over their magnitude and, therefore, one should not expect that the Coleman-Weinberg term would survive the inclusion of nonlocal structures.

[8]Note that the quadratic term should be retained in the expansion of $e^{-s_* V}$ if we want to get a nontrivial solution for s_*.

B. Big potential

Remarkably, the case of the small potentials (5.17) is not the only one when one can find a nonempty domain of overlap where both asymptotics for $\mathrm{Tr}K(s)$ are applicable; namely, the opposite case of big potentials (in units of the inverse size of their support)

$$V_0R^2 \gg 1 \tag{5.22}$$

is equally good. The key observation here is that in this case the kernel of the Green's function $1/(\Box - V)$ can be replaced within the compact support of V by $-1/V(\Box \sim 1/R^2 \ll V_0 \sim V)$ and correspondingly

$$\frac{1}{\Box - V}V(x) \sim -\frac{1}{V}V = -1, \tag{5.23}$$

$$\int dx \nabla_\mu \Phi \frac{1}{V-\Box}\nabla^\mu \Phi \simeq \frac{R^4}{V_0R^2}. \tag{5.24}$$

Therefore, the criterion of applicability of the late-time expansion (5.10) becomes $s \gg 1/V_0^2R^2$. Together with Eq. (5.13) it yields the new overlap range

$$R^2 \gg s \gg \frac{1}{V_0^2R^2} \tag{5.25}$$

which is obviously not empty if the potential satisfies Eq. (5.22).

To find s_* in this case we have to solve Eq. (5.4) for the case when s_*V is no longer a small quantity. Since V is big, the exponent in Eq. (5.4) can be replaced by zero inside the compact support, $\exp[-s_*V(x)] \sim 0$, $|x| \leqslant R$, and by 1 outside it where the potential vanishes, $\exp[-s_*V(x)] \sim 1$, $|x| > R$. Rewriting the integrals in both sides of Eq. (5.4) as a sum of contributions of $|x| \leqslant R$ and $|x| > R$, we see that the contribution of the noncompact domain gets canceled and the equation becomes

$$s_* \int_{|x|\leqslant R} dx V\Phi \simeq \int_{|x|\leqslant R} dx. \tag{5.26}$$

Then it follows that s_* is approximately given by the inverse of the function $V\Phi(x)$ *averaged* over the compact support of the potential:

$$s_* \simeq \frac{1}{\langle V\Phi \rangle}, \tag{5.27}$$

$$\langle V\Phi \rangle \equiv \frac{\int_{|x|\leqslant R} dx V\Phi}{\int_{|x|\leqslant R} dx}. \tag{5.28}$$

A qualitative estimate of $<V\Phi> \sim V_0$ implies that $s_* \sim 1/V_0$ and it belongs to the middle of the interval (5.25). This makes the case of big potentials fairly consistent. On the other hand, the value of $\Phi(x)$ is close to zero inside the potential support [see Eq. (5.23)], so most likely the estimate for $<V\Phi>$ is smaller by at least one power of the quantity $1/V_0R^2$, which is the basic dimensionless small parameter in this case. Therefore the magnitude of s_* becomes bigger by one power of V_0R^2, $s_* \simeq R^2$, which is again near the upper boundary of the overlap interval (5.25). Similarly to the small potential case, a more rigorous analysis is needed (maybe for more concretely specified potentials) to account for subtle edge effects at the boundary of the compact support, which might shift the value of s_* to a safe region inside Eq. (5.25).

With the above estimate for $s_* \sim R^2$ the magnitude of s_*V in the expression for the infrared part of the effective action (5.7) becomes big, $s_*V \sim s_*V_0 \sim V_0R^2 \gg 1$, and we use the big x asymptotics in Eq. (5.8) to get the contribution

$$\Gamma_> \simeq \frac{1}{64\pi^2 s_*}\int d^4x V\Phi = \frac{1}{64\pi^2}\int_{|x|\leqslant R} dx \langle V\Phi \rangle^2. \tag{5.29}$$

In this case the Coleman-Weinberg term is not canceled, in complete agreement with what we would expect for big potentials, and the final result reads

$$\overline{\Gamma} = \Gamma_{\mathrm{div}} + \Gamma_{\mathrm{CW}} + \frac{1}{64\pi^2}\int_{|x|\leqslant R} d^4x \langle V\Phi \rangle^2. \tag{5.30}$$

VI. COMMENTS

We developed a new technique for the calculation of late-time asymptotics of the heat kernel and its functional trace. Using these asymptotics we found previously unknown essentially nonlocal and nonperturbative contributions to the effective action for two large classes of potentials with compact supports. Therefore, the generalization of these results to potentials with power-law falloff, which would imply subtler analysis, deserves further studies.

Our results in their present form are applicable only in higher dimensions $d \geqslant 3$. One can easily check that the expression $(1/\Box)V$ is not well defined in low dimensions ($d \leqslant 2$). Note that the logarithmic kernel of the Green's function of the massless field is defined in $d=2$ only up to an additive constant. Moreover, the convolution of the Green's function with the potential makes sense in $d=2$ only if the latter is the total derivative of some other function [4], such as, for instance, the two-dimensional curvature scalar in the Polyakov action [8]. Thus, the extension of our results to low-dimensional models, where other calculational schemes completely fail, is especially important. This will be done in a forthcoming paper [7].

Another possibility is the generalization of our technique to potentials with isolated zeros in the interior of spacetime. Even more interesting is the situation when the potential becomes negative $V(x) < 0$ in some spacetime domains. In this case there is a tachyon instability, and it is worth deriving quantitative criteria describing this instability in terms of the properties of $V(x)$.

Finally, it is important to generalize our results to curved

spacetimes and in addition to consider fields with nontrivial spin-tensor structure. All these issues are addressed in [7]. The nonlocal effective action can then be applied to study interesting physical problems, like quantum black hole evaporation [6], quantum cosmology, etc.

It is worth mentioning that the developed technique for late-time asymptotics of the heat kernel could also be useful in statistical physics for calculating low-temperature partition functions.

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APPENDIX A: INTEGRAL EQUATION FOR *W*

In this appendix we derive the integral form of the equation

$$\frac{\partial W(s|x,y)}{\partial s}+\frac{(x-y)^\mu}{s}\nabla_\mu W(s|x,y)-\Box W(s|x,y)=f(s|x,y), \tag{A1}$$

where

$$f(s|x,y)\equiv V(x)-[\nabla W(s|x,y)]^2, \tag{A2}$$

and $W(s=0|x,y)=0$. With this purpose we first introduce the new function $\widetilde{W}$,

$$W(s|x,y)=e^{-s\Box}\widetilde{W}(s|x,y). \tag{A3}$$

Using the relation

$$e^{s\Box}(x-y)^\mu e^{-s\Box}=(x-y)^\mu+2s\nabla^\mu \tag{A4}$$

one finds that this function satisfies the following equation:

$$\frac{\partial\widetilde{W}}{\partial s}+\frac{(x-y)^\mu}{s}\nabla_\mu\widetilde{W}=e^{s\Box}f, \tag{A5}$$

which no longer contains the $\Box$ term on the left-hand side. To write down the formal solution of this equation in terms of the ''source'' term $f=f(s|x,y)$, let us introduce the characteristic curve $\overline{x}^\mu(t)$ of Eq. (A5), which satisfies the equation

$$\frac{d\overline{x}^\mu(t)}{dt}=\frac{(\overline{x}(t)-y)^\mu}{t}, \tag{A6}$$

with the boundary conditions

$$\overline{x}^\mu(t=0)=y^\mu,\quad \overline{x}^\mu(t=s)=x^\mu. \tag{A7}$$

The solution of Eq. (A6) is

$$\overline{x}^\mu(t)=y^\mu+\frac{(x-y)^\mu}{s}t. \tag{A8}$$

The total derivative of $\widetilde{W}(t|\overline{x}(t),y)$ with respect to t along this characteristic curve is then equal to

$$\frac{d}{dt}\widetilde{W}(t|\overline{x}(t),y)=\left[\frac{\partial}{\partial t}+\frac{(x-y)^\mu}{t}\frac{\partial}{\partial\overline{x}^\mu}\right]\widetilde{W}(t|\overline{x},y)$$

$$=e^{t\overline{\Box}}f(t|\overline{x}(t),y), \tag{A9}$$

where $\overline{\Box}\equiv\partial^2/\partial\overline{x}^\mu\partial\overline{x}_\mu$. Integrating this equation from 0 to s with the initial condition $\widetilde{W}=0$ at $t=0$ and taking into account the boundary conditions (A7) for $\overline{x}(t)$, one gets

$$\widetilde{W}(s|x,y)=\int_0^s dt e^{t\overline{\Box}}f(t|\overline{x}(t),y). \tag{A10}$$

Returning to the original W which is related to $\widetilde{W}$ via Eq. (A3) and taking into account that $\Box=(t/s)^2\overline{\Box}$ we finally obtain

$$W(s|x,y)=s\int_0^1 d\alpha e^{s\alpha(1-\alpha)\overline{\Box}}f(s\alpha|\overline{x},y)|_{\overline{x}=\alpha x+(1-\alpha)y}, \tag{A11}$$

where instead of t the new integration variable $\alpha=t/s$ was introduced. This is exactly the integral form (3.3) of Eq. (A1) that we used in Sec. III.

APPENDIX B: COVARIANT PERTURBATION THEORY AND LATE-TIME BEHAVIOR OF THE FUNCTIONAL TRACE OF THE HEAT KERNEL

Here we consider the nonlocal covariant perturbation theory of [3–5]. In CPT the functional trace of the heat kernel for the covariant second order differential operator is expanded as a nonlocal series in powers of the potential V with explicitly calculable coefficients—nonlocal form factors $F_n(s|x_1,x_2,\ldots,x_n)$. Their leading asymptotic behavior at large s was obtained in [4]. Here we calculate them up to the first subleading order in $1/s$ inclusive for a simple operator $\hat{F}=\Box-V$. Then we explicitly perform an infinite summation of the power series in the potential to obtain the nonlocal and nonlinear expression (4.15) for the late-time behavior of $\mathrm{Tr}\,K(s)$.

According to [4] the heat kernel trace is local in the first two orders of the perturbation theory in the potential (2.18):

$$\mathrm{Tr}\,K_0(s)=\frac{1}{(4\pi s)^{d/2}}\int dx, \tag{B1}$$

$$\mathrm{Tr}\,K_1(s)=-\frac{s}{(4\pi s)^{d/2}}\int dx V(x), \tag{B2}$$

and in higher orders it reads

$$\mathrm{Tr}\,K_n(s)=\frac{(-s)^n}{(4\pi s)^{d/2}n}\int dx\langle e^{s\Omega_n}\rangle V(x_1)\times V(x_2)\cdots V(x_n)|_{x_1=\ldots=x_n=x},\quad n\geqslant 2. \tag{B3}$$

Here Ω_n is a differential operator acting on the product of n potentials

$$\Omega_n=\sum_{i=1}^{n-1}\nabla_{i+1}^2+2\sum_{i=2}^{n-1}\sum_{k=1}^{i-1}\beta_i(1-\beta_k)\nabla_{i+1}\nabla_{k+1}, \tag{B4}$$

expressed in terms of the partial derivatives labeled by the indices i implying that ∇_i acts on $V(x_i)$. It is assumed in Eq. (B3) that after the action of all derivatives on the respective terms all x_i are set equal to x. It is also assumed that the spacetime indices of all derivatives $\nabla=\nabla^\mu$ are contracted in their bilinear combinations $\nabla_i\nabla_k\equiv\nabla_i^\mu\nabla_{\mu k}$. The differential operator (B4) depends on the parameters β_i, $i=1,\ldots,n-1$, which are defined in terms of the parameters α_i, $i=1,\ldots,n$, as

$$\beta_i=\alpha_{i+1}+\alpha_{i+2}+\cdots+\alpha_n,$$

and the angular brackets in $\langle e^{s\Omega_n}\rangle$ imply that this operator exponent is integrated over compact domain in the space of α parameters:

$$\langle e^{s\Omega_n}\rangle\equiv\int_{\alpha_i\geqslant 0}d^n\alpha\,\delta\left(\sum_{i=1}^{n}\alpha_i-1\right)\exp(s\Omega_n).$$

The late-time behavior of $\mathrm{Tr}\,K_n(s)$ is thus determined by the asymptotic behavior of this integral at $s\to\infty$, which can be calculated using the Laplace method. To apply this method, let us note that Ω_n is a *negative* semidefinite operator (this is shown in Appendix B of [4]) which degenerates to zero at n points of the integration domain $(0,\ldots,0,\alpha_i=1,0,\ldots,0)$, $i=1,\ldots,n$. Therefore the asymptotic expansion of this integral is given by the contribution of the corresponding n maxima of the integrand at these points. The integration by parts in Eq. (B3) is justified by the formal identity $\nabla_1+\nabla_2+\cdots+\nabla_n=0$. Using it one can show that the contributions of all these maxima are equal, so that it is sufficient to calculate only the contribution of the point $\alpha_1=1$, $\alpha_i=0$, $i=2,\ldots,n$. In the vicinity of this point it is convenient to rewrite the expression for Ω_n in terms of the independent $(n-1)$ variables $\alpha_2,\alpha_2,\ldots,\alpha_n$, the remaining $\alpha_1=1-\Sigma_{i=2}^n\alpha_i$,

$$\Omega_n=\sum_{i=2}^{n}\alpha_iD_i^2-\sum_{m,k=2}^{n}\alpha_m\alpha_kD_mD_k, \tag{B5}$$

where the operator D_m is defined as

$$D_m=\nabla_2+\nabla_3+\cdots+\nabla_m,\quad m=2,\ldots,n. \tag{B6}$$

Substituting this expression for Ω_n in Eq. (B3) and expanding in powers of the term bilinear in α parameters one gets

$$\mathrm{Tr}\,K_n(s)=\frac{(-s)^n}{(4\pi s)^{d/2}}\int dx\int_0^\infty d^{n-1}\alpha\exp\left(s\sum_{i=2}^{n}\alpha_iD_i^2\right)\times\left(1-s\sum_{m,k=2}^{n}\alpha_m\alpha_kD_mD_k+\cdots\right)V_1V_2\cdots V_n. \tag{B7}$$

Here the $1/n$ factor disappeared due to the contribution of n equal terms and the range of integration over $\alpha_2,\ldots,\alpha_n$, $\Sigma_{i=2}^n\alpha_i\leqslant 1$, was extended to all positive values of α_i. This is justified since the error we make by extending the integration range is smaller by the factor $O(1/s^{d/2})$ than the leading term of Eq. (B7).[9] Since $d\geqslant 3$ everywhere throughout the paper, this does not affect the first subleading in the $1/s$ term of Eq. (B7).

The second term in parentheses of Eq. (B7) can be rewritten in terms of the derivatives with respect to D_m^2 acting on the exponential, so that

$$\mathrm{Tr}\,K_n(s)=\frac{(-s)^n}{(4\pi s)^{d/2}}\int dx\left(1-\frac{1}{s}\sum_{m,k=2}^{n}D_mD_k\frac{\partial}{\partial D_m^2}\frac{\partial}{\partial D_k^2}+\cdots\right)\int_0^\infty d^{n-1}\alpha\exp\left(s\sum_{i=2}^{n}\alpha_iD_i^2\right)V_1V_2\cdots V_n. \tag{B8}$$

In this form it is obvious that further terms of the expansion in powers of the part of Ω_n quadratic in α bring higher-order corrections of the $1/s$ series. Doing the integral over α here and performing differentiations one obtains

$$\mathrm{Tr}\,K_n(s)=\frac{1}{(4\pi s)^{d/2}}\int dx\left[-s\frac{1}{D_2^2\cdots D_n^2}+2\sum_{m=2}^{n}\frac{1}{D_2^2\cdots D_{m-1}^2}\frac{1}{(D_m^2)^2}\frac{1}{D_{m+1}^2\cdots D_n^2}+2\sum_{m=2}^{n-1}\sum_{k=m+1}^{n}\frac{1}{D_2^2\cdots D_{m-1}^2}\frac{D_m^\mu}{(D_m^2)^2}\frac{1}{D_{m+1}^2\cdots D_{k-1}^2}\frac{D_{k\mu}}{(D_k^2)^2}\frac{1}{D_{k+1}^2\cdots D_n^2}+O\left(\frac{1}{s}\right)\right]V_1V_2\cdots V_n. \tag{B9}$$

[9]Naively, the error induced by such an extension of the integration range is exponentially small in $s\to\infty$, $e^{s\Box}$, $\Box<0$, with the $\Box$ acting on some group of factors in $V_1\cdots V_n$. But in view of the operator nature of $\Box$ the error bound decreases at large s by a powerlike law $e^{s\Box}(V\cdots V)\sim 1/s^{d/2}$. We are grateful to B. L. Voronov for this observation.

The first term in the square brackets gives the leading order term of the late-time expansion. It can be further transformed by taking into account that any operator D_m defined by Eq. (B6) acts as a partial derivative only on the group of factors $V_2V_3\cdots V_{m-1}V_m$ in the full product $V_1\cdots V_n$, $D_mV_1\cdots V_n=V_nV_{n-1}\cdots V_{m+1}\nabla\times(V_mV_{m-1}\cdots V_2)V_1$. Therefore all the operators understood as *acting to the right* can be ordered in such a way

$$\mathrm{Tr}\,K_n(s)=-\frac{s}{(4\pi s)^{d/2}}\int dx\,V_1\frac{1}{D_n^2}V_n\frac{1}{D_{n-1}^2}\times V_{n-1}\cdots\frac{1}{D_2^2}V_2+O\left(\frac{1}{s^{d/2}}\right)$$

that the labels of the D_m^2's can be omitted and all D_m^2 can be identified with boxes also acting to the right:

$$\mathrm{Tr}\,K_n(s)=-\frac{s}{(4\pi s)^{d/2}}\int dx\,\underbrace{V\frac{1}{\Box}V\frac{1}{\Box}\cdots V\frac{1}{\Box}}_{n-1}V(x)+O\left(\frac{1}{s^{d/2}}\right). \tag{B10}$$

Infinite summation of this series is not difficult to perform because this is the geometric progression in powers of the nonlocal operator $V(1/\Box)$ and

$$\mathrm{Tr}\,K(s)=\mathrm{Tr}\,K_0(s)-\frac{s}{(4\pi s)^{d/2}}\times\int dx\sum_{n=0}^{\infty}\left(V\frac{1}{\Box}\right)^nV(x)+O\left(\frac{1}{s^{d/2}}\right),$$

or

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{d/2}}\int dx\left(1-s\Box\frac{1}{\Box-V}V(x)+O(s^0)\right). \tag{B11}$$

The second term here looks like a total derivative. However, it does not vanish because this is a derivative of the nonlocal expression and the corresponding surface term does not vanish at infinity in view of the Green's function asymptotics. This term can be rewritten as

$$\Box\frac{1}{\Box-V}V(x)=V(x)+V\frac{1}{\Box-V}V(x)=V\Phi(x), \tag{B12}$$

and, therefore,

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{d/2}}\int dx[1-sV\Phi(x)+O(s^0)], \tag{B13}$$

where $O(s^0)$ denotes the terms subleading in s which depend on the potential in a nontrivial way. They are given by infinite resummation over n of the second and third terms in square brackets of Eq. (B9). Remarkably, this summation can again be done explicitly. In this case one has to sum multiple geometric progressions.

Indeed, the second term of Eq. (B9) gives rise to the series

$$\frac{2}{(4\pi s)^{d/2}}\int dx\sum_{n=2}^{\infty}\sum_{m=2}^{n}\underbrace{V\frac{1}{\Box}V\cdots\frac{1}{\Box}V}_{n-m}\frac{1}{\Box^2}\times\underbrace{V\frac{1}{\Box}\cdots V\frac{1}{\Box}}_{m-2}V(x). \tag{B14}$$

By summing the two geometric progressions with respect to independent summation indices $0\leqslant n-m<\infty$ and $0\leqslant m-2<\infty$, one finds that this series reduces to

$$\frac{2}{(4\pi s)^{d/2}}\int dxV\frac{1}{(\Box-V)^2}V(x), \tag{B15}$$

which after the integration by parts amounts to

$$\frac{2}{(4\pi s)^{d/2}}\int dx\left(\frac{1}{\Box-V}V(x)\right)^2=\frac{2}{(4\pi s)^{d/2}}\int dx[1-\Phi(x)]^2. \tag{B16}$$

Similarly, the third term of Eq. (B9) gives rise to a triplicate geometric progression which after summation and integration by parts reduces to

$$\frac{2}{(4\pi s)^{d/2}}\int dxV\sum_{i=0}^{\infty}\left(\frac{1}{\Box}V\right)^i\frac{1}{\Box}\nabla^\mu\frac{1}{\Box}\times\sum_{j=0}^{\infty}\left(V\frac{1}{\Box}\right)^j\frac{1}{\Box}\nabla^\mu\frac{1}{\Box}\sum_{l=0}^{\infty}\left(V\frac{1}{\Box}\right)^lV(x)=-\frac{2}{(4\pi s)^{d/2}}\int dx[\nabla_\mu\Phi(x)]\frac{1}{\Box-V}V\frac{1}{\Box}\nabla^\mu\Phi(x). \tag{B17}$$

Taking into account here that

$$\frac{1}{\Box-V}V\frac{1}{\Box}=\frac{1}{\Box-V}-\frac{1}{\Box} \tag{B18}$$

one finds that the sum of Eqs. (B16) and (B17) is equal to

$$\frac{2}{(4\pi s)^{d/2}}\int dx\left((1-\Phi)^2-\nabla_\mu\Phi\frac{1}{\Box-V}\nabla^\mu\Phi+\nabla_\mu\Phi\frac{1}{\Box}\nabla^\mu\Phi\right)$$

$$=-\frac{2}{(4\pi s)^{d/2}}\int dx\,\nabla_\mu\Phi\frac{1}{\Box-V}\nabla^\mu\Phi, \tag{B19}$$

where the cancellation of the first and third terms takes place after rewriting $\nabla_\mu\Phi$ in the third term as $\nabla_\mu(\Phi-1)$ and integrating it by parts.[10] Together with Eq. (B13) the contribution (B19) forms the nonlinear and nonlocal late time expression for the heat kernel trace (4.15) up to the first subleading order in $1/s$ inclusive.

[10]Straightforward integration by parts in $\nabla_\mu\Phi(1/\Box)\nabla^\mu\Phi$ is impossible because $\Phi(x)$ does not vanish at $|x|\to\infty$, while $\Phi(x)-1$ does.

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