High-energy head-on collisions of particles and the hoop conjecture

Hirotaka Yoshino* and Yasusada Nambu†
Department of Physics, Graduate School of Science, Nagoya University, Chikusa, Nagoya 464-8602, Japan
(Received 18 April 2002; published 19 September 2002)

We investigate the apparent horizon formation for high-energy head-on collisions of particles in D-dimensional spacetime. The apparent horizons formed before the instant of collision are obtained analytically. We apply the D-dimensional hoop conjecture to these solutions and show that the horizon formation becomes more difficult for larger D.

DOI: 10.1103/PhysRevD.66.065004 PACS number(s): 11.10.Kk, 04.20.Cv, 04.50.+h, 04.70.Bw

I. INTRODUCTION

The brane world scenario has been discussed by many authors recently. The scenario suggests that the Planck energy can be as low as the $O(\text{TeV})$ scale [1]. If the Planck energy is at the TeV scale, it is possible to create black holes using accelerators, such as the CERN Large Hadron Collider (LHC) [2]. Further, the collisions of cosmic rays with our atmosphere produce energy beyond that of the LHC and their observation will show the existence of large extra dimensions or will place improved bounds on the fundamental Planck scale. Hence we would like to better understand the process of black hole formation via particle collisions. For this purpose, we investigate the formation of the apparent horizon for a system of head-on collisions of high-energy particles.

To simplify the analysis, we follow the method adopted by Eardley and Giddings [3]. First, the tension of the brane which is expected to be on the Planck scale can be neglected if the center of mass energy is substantially larger than the Planck scale. Second, the geometry of the extra dimensions plays no essential role if the geometrical scales of the extra dimensions are large compared to the horizon radius for the center of mass energy. Thus we consider head-on collisions in D-dimensional Einstein gravity. The metric with a high-energy point particle is obtained by infinitely boosting the Schwarzschild black hole metric with fixed total energy μ. The resulting system becomes a massless point particle accompanied by a plane-fronted gravitational shock wave which is the Lorentz-contracted longitudinal gravitational field of the particle. Combining the two shock waves, we can set up a high-energy collision. This system was originally developed by D'Eath and Payne [4]. Black hole formation with the impact parameter for $D=4$ was investigated by Eardley and Giddings [3], and they showed that the apparent horizon which encloses two particles exists at the instant of collision for a sufficiently small impact parameter.

We examine head-on collisions using a different slicing of the spacetime: we expect that the apparent horizon forms before the collision of the two particles. We construct the solutions for the apparent horizons analytically and discuss how the dimension D affects the formation of the horizon from the viewpoint of the hoop conjecture [5].

*Electronic address: hyoshino@allegro.phys.nagoya-u.ac.jp
†Electronic address: nambu@allegro.phys.nagoya-u.ac.jp

II. HIGH-ENERGY PARTICLE COLLISIONS AT THE SPEED OF LIGHT

The gravitational solution for the each incoming particles can be found by boosting the rest-frame D-dimensional Schwarzschild solution,

$$ds^2=-\left(1-\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}\right)dt^2+\left(1-\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\frac{1}{r^{D-3}}\right)^{-1}dr^2+r^2 d\Omega_{D-2}^2, \tag{1}$$

where $d\Omega_{D-2}^2$ and Ω_{D-2} are the line element and volume of the unit $(D-2)$-sphere and G_D is the D-dimensional gravitational constant. The Aichelburg-Sexl solution [6] is found by taking the limit of large boost and small mass with fixed total energy μ. The resulting metric represents a massless particle moving in the $+z$ direction with the speed of light:

$$ds^2=-d\bar{u}d\bar{v}+\sum_{i=1}^{D-2}d\bar{x}_i^2+\Phi(\bar{x}_i)\delta(\bar{u})d\bar{u}^2, \tag{2}$$

where $\bar{u}=\bar{t}-\bar{z}$ and $\bar{v}=\bar{t}+\bar{z}$. Φ depends only on the transverse radius $\bar{\rho}=\sqrt{\bar{x}_i\bar{x}^i}$ and takes the form

$$\Phi=\begin{cases}-8G_4\mu\log\bar{\rho} & \text{for}\quad D=4, \quad (3)\\ \dfrac{16\pi\mu G_D}{\Omega_{D-3}(D-4)}\dfrac{1}{\bar{\rho}^{D-4}} & \text{for}\quad D>4. \quad (4)\end{cases}$$

The delta function appearing in Eq. (2) shows that the two coordinate systems are discontinuously connected at $\bar{u}=0$. A continuous coordinate system can be introduced via

$$\begin{aligned}\bar{u}&=u,\\ \bar{v}&=v+\Phi\,\theta(u)+\frac{u}{4}\theta(u)(\nabla_i\Phi\nabla^i\Phi),\\ \bar{x}_i&=x_i+\frac{u}{2}\nabla_i\Phi(x_i)\,\theta(u),\end{aligned} \tag{5}$$

where θ is the Heaviside step function and ∇_i is the $(D-2)$-dimensional flat-space derivative. We can superpose the two solutions to obtain the exact geometry outside the future light cone of the collision of the shocks:

$$ds^2=-dudv+(H_{ik}^{(1)}H_{jk}^{(1)}+H_{ik}^{(2)}H_{jk}^{(2)}-\delta_{ij})dx^i dx^j, \quad (6)$$

where

$$H_{ij}^{(1)}=\delta_{ij}+\frac{u}{2}\theta(u)\nabla_i\nabla_j\Phi^{(1)}(\boldsymbol{x}),$$
$$H_{ij}^{(2)}=\delta_{ij}+\frac{v}{2}\theta(v)\nabla_i\nabla_j\Phi^{(2)}(\boldsymbol{x}). \quad (7)$$

Here $\boldsymbol{x}\equiv(x^i)$ is the point in flat $(D-2)$-space that is transverse to the direction of particle motion. The apparent horizon is defined as a closed spacelike $(D-2)$-surface on which the outer null geodesic congruence has zero convergence. It was shown that the apparent horizon exists in the union of the two shock waves $u=0>v$ and $v=0>u$ [3]. This apparent horizon consists of two flat disks with radii

$$\rho_0\equiv\left(\frac{8\pi\mu G_D}{\Omega_{D-3}}\right)^{1/(D-3)}, \quad (8)$$

and ρ_0 gives the characteristic scale for each dimension D.

III. TIME SLICING AND APPARENT HORIZONS

To treat the collision of particles as a time evolutional process, we consider the following slice of spacetime:

$$\text{region I:} \quad t=z, t\leqslant T,$$
$$\text{region II:} \quad z=-t, t\leqslant T, \quad (9)$$
$$\text{region III:} \quad t=T, -T\leqslant z\leqslant T,$$

where $T\leqslant 0$ and the particles collide at $T=0$. In order to find the apparent horizon on the above slice, we first prepare surfaces with zero expansion in regions I and III, and then connect them smoothly by requiring that the null normal coincides at the junction of regions I and III, $t=z=T$. In region I, the surface that has zero expansion is given by

$$v=-\Phi+\text{const}, \quad (10)$$

and its null normal k_1^a is

$$k_1^u=(\rho_0/\rho)^{-(D-3)},$$
$$k_1^v=(\rho_0/\rho)^{D-3}, \quad (11)$$
$$k_1^\rho=1.$$

In region III, the surface that has zero expansion is given by

$$az=\pm f(a\rho), \quad (12)$$

where a is a constant of integration determined by the matching condition at the junction. For $D=4$, the function $f(x)$ is given by

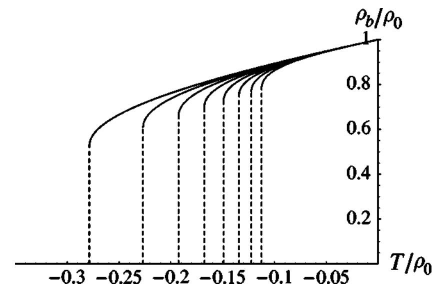

FIG. 1. The relation between T and the horizon radius ρ_b for $D=4,\ldots,11$. The intersection of the dotted line and T/ρ_0 axis is the time T_c when the apparent horizon appears. The value of $|T_c/\rho_0|$ decreases as D increases.

$$f(x)=\cosh^{-1}x, \quad (13)$$

and for $D>4$,

$$f(x)=-\frac{x^{-D+4}}{D-4}\times{}_2F_1\left(\frac{1}{2},\frac{D-4}{2(D-3)},\frac{3D-10}{2(D-3)},x^{2(3-D)}\right)-\sqrt{\pi}\,\frac{\Gamma((D-4)/2(D-3))}{\Gamma(1/2(3-D))}, \quad (14)$$

where ${}_2F_1$ is the Gauss hypergeometric function. The null normal k_3^a of the surface is given by

$$k_3^u=(a\rho)^{D-3}-\sqrt{(a\rho)^{2(D-3)}-1},$$
$$k_3^v=(a\rho)^{D-3}+\sqrt{(a\rho)^{2(D-3)}-1}, \quad (15)$$
$$k_3^\rho=1.$$

Matching these surfaces and null normals at the junction $t=z=T$, we have

$$f(a\rho_b)=-aT, \quad (16)$$

$$(\rho_0/\rho_b)^{D-3}=(a\rho_b)^{D-3}+\sqrt{(a\rho_b)^{2(D-3)}-1}, \quad (17)$$

where ρ_b is the radius of the surface at the junction. From this, the relation between T and ρ_b can be given parametrically as

$$\frac{T}{\rho_0}=-\xi f\left(\frac{1}{\xi}(2\xi^{3-D}-1)^{1/2(3-D)}\right), \quad (18)$$

$$\frac{\rho_b}{\rho_0}=(2\xi^{3-D}-1)^{1/2(3-D)}, \quad (19)$$

where $0\leqslant\xi\leqslant 1$. Figure 1 shows the relation between T and ρ_b for each D. We denote the time when the apparent horizon appears as $T=T_c$. The value of $|T_c/\rho_0|$ becomes small as D increases. For large D, we have

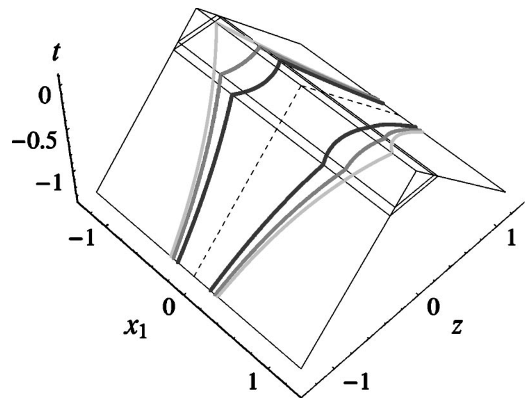

FIG. 2. The apparent horizon for $D=4$ at $T/\rho_0=-0.278$, $-0.225,0$. The dark line is the horizon at $T=T_c=-0.278\rho_0$, and the light line is the horizon at $T=0$. The unit of the axis is ρ_0.

$$\rho_b/\rho_0 \approx D^{-1/2D}, \quad T_c/\rho_0 \approx -1/D \tag{20}$$

at $T=T_c$. The intersection of the $z=$const plane and the surface in region III is a $(D-3)$-dimensional sphere, of which the expansion is positive and proportional to $D-3$. Thus the surface has negative expansion on the $(\rho/\rho_0, z/\rho_0)$ plane and its curvature on this plane increases with increase of the spacetime dimension D. This leads to a decrease in the distance of the two particles at horizon formation. The shapes of the apparent horizons for $D=4$ and $D=5$ are shown in Fig. 2 and Fig. 3.

IV. HOOP CONJECTURE

Now we examine the difference in horizon formation for various spacetime dimensions using the hoop conjecture. The hoop conjecture gives the criterion for black hole formation in four-dimensional general relativity [5]. It states that an apparent horizon forms when and only when the mass M of the system gets compacted into a region of which the circumference C satisfies

$$H_4 \equiv C/4\pi G_4 M \lesssim 1. \tag{21}$$

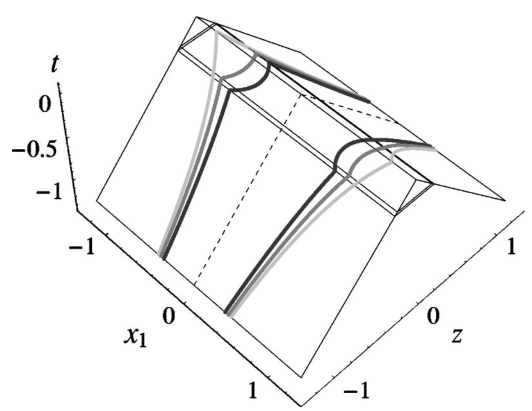

FIG. 3. The apparent horizon for $D=5$ at $T/\rho_0=-0.227$, $-0.2,0$. The dark line is the horizon at $T=T_c=-0.227\rho_0$, and the light line is the horizon at $T=0$. The unit of the axis is ρ_0.

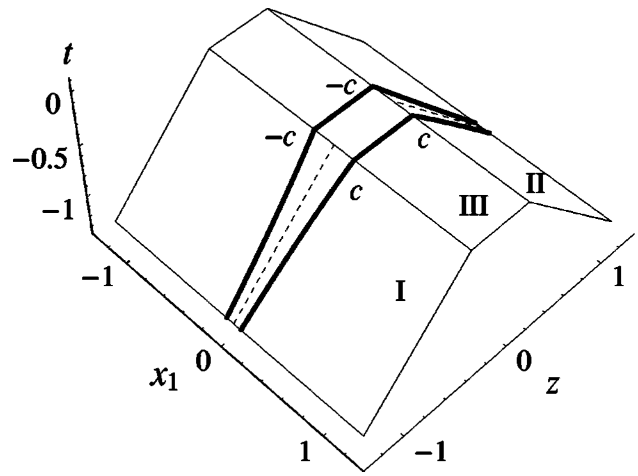

FIG. 4. The closed loop used to calculate the circumference. We calculate C by taking $c\to 0$.

As $4\pi G_4 M$ is the circumference of the four-dimensional Schwarzschild horizon, we can expect that the criterion for black hole formation in D-dimensional Einstein gravity is given by

$$H_D \equiv C/2\pi r_h(M) \lesssim 1, \tag{22}$$

where $r_h(M)$ is the Schwarzschild radius of D-dimensional spacetime. This criterion was implicitly used to estimate the total cross section for black hole production via non-head-on collisions [2].

To calculate the values of H_D and H_4, we must specify the definition of the mass of the system. In this paper, we use the total energy $E=2\mu$ as the mass of the system. The circumference C is defined as the minimum length which encloses two particles. We take the loop as shown in Fig. 4 and calculate C by taking the limit $c\to 0$. C reduces to $4|T|$, which is twice the distance of the two particles. The value of H_D at $T=T_c$ is shown in Table I. As D increases, the value of H_D decreases and the mass M must be compacted into a region with smaller circumference C than $2\pi r_h$ to produce a black hole. This reflects the decrease in $|T_c|/\rho_0$ with increase in D. The value of H_4 at $T=T_c$ is also shown in Table I. This result can be written as

$$H_4 = F(D)\frac{(G_D E)^{1/(D-3)}}{G_4 E}, \tag{23}$$

TABLE I. The values of H_4 and H_D at $T=T_c$ for $D=4$–11.

D	H_D	H_4
4	0.1773	0.1773
5	0.1567	$0.0722[(G_5E)^{1/2}/(G_4E)]$
6	0.1348	$0.0527[(G_6E)^{1/3}/(G_4E)]$
7	0.1176	$0.0444[(G_7E)^{1/4}/(G_4E)]$
8	0.1042	$0.0396[(G_8E)^{1/5}/(G_4E)]$
9	0.0936	$0.0364[(G_9E)^{1/6}/(G_4E)]$
10	0.0849	$0.0340[(G_{10}E)^{1/7}/(G_4E)]$
11	0.0777	$0.0321[(G_{11}E)^{1/8}/(G_4E)]$

where $F(D)=0.03$–0.2. The D-dimensional gravitational constant is related to the Planck energy as

$$M_p^{D-2}=\frac{(2\pi)^{D-4}}{4\pi G_D}. \tag{24}$$

Using this formula, Eq. (23) becomes

$$H_4=F(D)\left(\frac{M_4}{M_p}\right)^2\left(\frac{8\pi^2 M_p}{E}\right)^{(D-4)/(D-3)}. \tag{25}$$

If the Planck energy is at the TeV scale, M_4/M_p is $\sim 10^{16}$ and H_4 becomes $\sim 10^{32}$. Thus the mass does not need to be compacted into a small region of which the circumference is $C \lesssim 4\pi G_4 M$ to produce a black hole.

V. SUMMARY AND DISCUSSION

We have investigated the temporal evolution of the apparent horizon for high energy particle collisions. The apparent horizon which encloses the two particles appears at $T=T_c$. Its radius increases in time and reaches ρ_0 at $T=0$. We calculated H_D and found that H_D decreases as D increases. This means that if we increase the spacetime dimension the size of the hoop that encloses the system should be much smaller than $2\pi r_{\rm h}$. Therefore, the formation of an apparent horizon becomes more difficult for larger D. On the other hand, $H_4 = H_D\cdot r_{\rm h}/2G_4M$ gives a large value $\sim 10^{32}$ regardless of the decrease in H_D. This is because the horizon radius $r_{\rm h}$ becomes far larger than $2G_4M$. As the horizon radius corresponds to the length scale which encloses the system, this leads to the conclusion that a black hole is easily formed in the TeV scale scenario.

Finally we discuss the validity of the hoop conjecture. Obviously, H_4 does not give a picture of the hoop conjecture because its value at horizon formation is far larger than unity. The ratio H_D also does not give a picture of the hoop conjecture because its value at horizon formation is much smaller than unity. However, we used rough estimated values of the circumference C and the mass M to evaluate H_D and H_4. The energy of a shock wave is distributed in the transverse direction of the particle motion, and our estimation of the circumference C is too small because the region surrounded by this circumference does not enclose much of the gravitational energy. In our previous paper [7], we stated that H_4 with Hawking's quasilocal mass $M_{\rm H}(S)$ [8] becomes a better parameter to judge horizon formation for a system with motion. We must calculate $H_4^{(\rm H)}(S) = C(S)/4\pi G_4 M_{\rm H}(S)$ for all surfaces S and then take their minimum value. Even though the Hawking mass in multidimensional spacetime has not been calculated in this paper, we expect that $H_D^{(\rm H)} \lesssim 1$ will become a condition for the horizon formation. The value H_D decrease as D increases even if we use the quasilocal mass because H_D should reflect the decrease in $|T_c|/\rho_0$.

Although we can regard $C/2\pi r_{\rm h}(M) \lesssim 1$ as the condition for horizon formation in D-dimensional gravity, it does not give a unique condition. The topology of the apparent horizon is not restricted to be an S^{D-2} surface in multidimensional spacetime. Emparan and Reall derived a solution for the rotating black ring in $D=5$ [9]. For an apparent horizon which does not have S^{D-2} topology, the criterion for its formation may take another form. Our criterion $C/2\pi r_h(M) \lesssim 1$ is applicable only to a horizon with S^{D-2} topology.

ACKNOWLEDGMENTS

The authors would like to thank Akira Tomimatsu and Masaru Shibata for helpful discussions.

[1] N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, Phys. Lett. B **429**, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G.R. Dvali, *ibid.* **436**, 257 (1998).
[2] T. Banks and W. Fischler, hep-th/9906038; S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. **87**, 161602 (2001); S.B. Giddings and S. Thomas, Phys. Rev. D **65**, 056010 (2001).
[3] D.M. Eardley and S.B. Giddings, gr-qc/0201034.
[4] P.D. D'Eath and P.N. Payne, Phys. Rev. D **46**, 658 (1992); **46**, 675 (1992); **46**, 694 (1992).
[5] K.S. Thorne, in *Magic without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972).
[6] P.C. Aichelburg and R.U. Sexl, Gen. Relativ. Gravit. **2**, 303 (1971).
[7] H. Yoshino, Y. Nambu, and A. Tomimatsu, Phys. Rev. D **65**, 064034 (2002).
[8] S. Hawking, J. Math. Phys. **9**, 598 (1968).
[9] R. Emparan and H.S. Reall, hep-th/0110260.