Isoperimetric inequality for higher-dimensional black holes

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The initial data sets for the five-dimensional Einstein equation have been examined. The system is designed such that the black hole ($\simeq S^3$) or the black ring ($\simeq S^2\times S^1$) can be found. We have found that the typical length of the horizon can become arbitrarily large but the area of characteristic closed two-dimensional submanifold of the horizon is bounded above by the typical mass scale. We conjecture that the isoperimetric inequality for black holes in n-dimensional space is given by $V_{n-2}\lesssim GM$, where V_{n-2} denotes the volume of a typical closed $(n-2)$-section of the horizon and M is typical mass scale, rather than $C\lesssim(GM)^{1/(n-2)}$ in terms of the hoop length C, which holds only when $n=3$.

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I. INTRODUCTION

There is much interest in higher dimensional space-times in the context of the unified theory of elementary particles. It is exciting if the existence of extra dimensions is confirmed in high energy experiments. In this aspect, the notion of the brane world [1] is an attractive idea. This phenomelogical model provides us with a new way of thinking about our universe, in which a size of the extra dimensions can be large because the standard model particles and gauge interactions are confined to the boundary of the higher-dimensional space-time. According to this scenario, the gravitational interaction at the short distance determined by the size of the extra dimensions is modified effectively on the brane, so that we might be able to see the extra dimensions by the gravitational experiments below 1 mm. If the extra dimensions are large, the higher dimensional Planck scale may be given by rather low energy. The possibility of TeV gravity, in which the fundamental Planck scale is set around TeV, has been much disscussed.

It is suggested that small black holes might be produced at the CERN Large Hadron Collider (LHC) [2–4]. This argument follows from the hoop conjecture [5]; a black hole with horizon forms if and only if the typical length (hoop length) C and the mass M satisfies $C\lesssim 4\pi GM$. Note that this statement might be valid only for four space-time dimensions. The property of the higher-dimensional black holes has not so far been fully explored, though there is much attention to this issue [6–10]. We need reliable knowledge about such black holes to predict phenomelogical results. We here consider black holes with small size compared with the extra dimensions, such that they are well described by the asymptotically flat black hole solutions (treatment of Planck size black holes is beyond the scope of this paper). The purpose of this paper is to consider the higher-dimensional generalization of the hoop conjecture.

In four dimensions, the hoop conjecture is believed to be valid. Though it is loosely formulated, it seems to have at least the following three meanings: (i) If the massive object is compactified into a small region, there must be a black hole; (ii) a black hole is small; and (iii) a highly deformed black hole does not form. The first one (i) has been proved by Schoen and Yau [11] (see also Ref. [12]), which can be regarded as the if part of the hoop conjecture. A precise statement concerning the second proposition (ii) is, for example, given by the Penrose inequality [13–15], which states that the square-root of the area of the apparent horizon A is bounded above by the [Arnowitt-Deser-Misner (ADM)] mass: $\sqrt{A}\leqslant 4\pi GM_{ADM}$. Thus the Penrose inequality may serve as a part of the only if part of the hoop conjecture. For the last statement (iii), which is also the only if part of the conjecture, we rely on the numerical works (e.g., Refs. [16–18]). There is also a problem concerning the precise formulation of the conjecture [19], such as the definition of the hoop length C.

At first glance, the hoop conjecture is not valid for higher-dimensional space-times, since there is black string solutions. In four dimensions, the length scale of the horizon cannot be so much larger than the Schwarzschild radius, while this is not the case in higher dimensions. The simplest example is the four-dimensional Schwarzschild space-time times the real line, which is the five-dimensional vacuum solution representing the gravitational field of the infinitely long $S^2\times\mathbf{R}$ black hole.

Nevertheless, we expect that higher dimensional black holes are also governed by some isoperimetric inequality. In what follows, we investigate initial data set for the five-dimensional Einstein equation and estimate the size of the black holes. Then we show the existence of such an isoperimetric inequality and give its physical reasoning.

II. MOMENTARILY STATIC INITIAL DATA SET FOR THE FIVE-DIMENSIONAL EINSTEIN EQUATION

Let us consider the initial data set $(g_{\mu\nu},K_{\mu\nu})$ on a four-dimensional Cauchy surface Σ^4, where $g_{\mu\nu}$ is the induced metric on Σ^4 and $K_{\mu\nu}=g_\mu{}^{\lambda 5}\nabla_\lambda n_\nu$ (n_ν denotes the unit normal to Σ^4) is the extrinsic curvature of Σ^4. The Hamiltonian and the momentum constraints are given by

$$R-K_{\mu\nu}K^{\mu\nu}+K^2=16\pi G\varrho \tag{1}$$

and

$$\nabla_\nu(K^{\mu\nu}-Kg^{\mu\nu})=8\pi G J^\mu, \tag{2}$$

respectively, where $\varrho := {}^5G(n,n)$ denotes the energy density and $J^\mu := g^{\mu\nu 5}G_\nu(n)$ is the energy flux. Let us consider the momentarily static initial data set

$$K_{\mu\nu}=0 \tag{3}$$

and assume the conformally flat metric

$$g=f^2\delta_{\mu\nu}dx^\mu dx^\nu. \tag{4}$$

Then the momentum constraint (2) is solved with $J^\mu=0$ and the Hamiltonian constraint (1) becomes

$$\nabla_0^2 f=-\frac{8\pi G}{3}f^3\varrho, \tag{5}$$

where ∇_0 denotes the flat connection. We consider the vacuum case $\varrho=0$ so that an initial data is described by a harmonic function f on E^4.

We are interested in the possibility of the formation of highly nonspherical black holes in higher dimensions. As typical cases in five dimensions, we shall consider the initial data sets for spindle, disk and ring shaped black holes.

A. Spherical black holes

A spherically symmetric black hole gives reference values of the volume, area and circumference of the horizon to the other cases discussed below. We consider the metric with spherical symmetry of the form

$$g=f^2(r)[dr^2+r^2d\chi^2+r^2\sin^2\chi(d\vartheta^2+\sin^2\vartheta^2d\varphi^2)], \tag{6}$$

and consider a point source at the origin,

$$f^3\varrho=\frac{M_{ADM}}{2\pi^2r^2}\delta(r), \tag{7}$$

where M_{ADM} is the ADM mass, that is, total gravitational mass of the system. Then the solution of Eq. (5) becomes

$$f=1+\frac{2GM_{ADM}}{3\pi r^2}. \tag{8}$$

This gives just an initial data for the Schwarzschild spacetime.

The location of the black hole in the sense of the apparent horizon is given by the minimal surface for the momentarily static initial data set ($K_{\mu\nu}=0$). A spherical minimal surface centered at the origin satisfies

$$(rf)_{,r}=0. \tag{9}$$

The solution of the above equation is given by

$$r=r_s:=\left(\frac{2GM_{ADM}}{3\pi}\right)^{1/2}. \tag{10}$$

The volume of the minimal surface H_s, area of its S^2-section S_s and the length of the circumference C_s of S are given by

$$\mathrm{Vol}(H_s)=2\pi^2[r_sf(r_s)]^3=2\pi^2\left(\frac{8GM_{ADM}}{3\pi}\right)^{3/2}, \tag{11}$$

$$\mathrm{Area}(S_s)=4\pi[r_sf(r_s)]^2=\frac{32GM_{ADM}}{3}, \tag{12}$$

$$\mathrm{Length}(C_s)=2\pi r_sf(r_s)=2\pi\left(\frac{8GM_{ADM}}{3\pi}\right)^{1/2}, \tag{13}$$

respectively.

B. Spindle black holes

Let us consider the metric with axial and spherical symmetry of the form

$$g=f^2(\rho,z)[dz^2+d\rho^2+\rho^2(d\vartheta^2+\sin^2\vartheta^2d\varphi^2)], \tag{14}$$

and consider the uniform line source of the length L located at z-axis,

$$f^3\varrho=\frac{M_{ADM}}{4\pi L\rho^2}\delta(\rho)\,\theta(L/2-|z|), \tag{15}$$

where θ is the Heaviside's step function.

The solution of the Hamiltonian constraint (5) is given by

$$\begin{aligned}f&=1+\frac{2GM_{ADM}}{3\pi L}\int_{-L/2}^{L/2}\frac{dz'}{\rho^2+(z'-z)^2}\\&=1+\frac{2GM_{ADM}}{3\pi L\rho}\left(\arctan\frac{z+L/2}{\rho}-\arctan\frac{z-L/2}{\rho}\right).\end{aligned} \tag{16}$$

Note that this massive segment corresponds to another asymptotic end rather than the singularity. One may anyway fill up the segment with some spatially extended gravitational source.

Due to the geometric symmetry imposed, we have only to consider the minimal surface equation for the three-surface, $\rho=r(\xi)\sin\xi$, $z=r(\xi)\cos\xi$, given by

$$\begin{aligned}&r_{,\xi\xi}-4\frac{(r_{,\xi})^2}{r}-3r+\frac{(r_{,\xi})^2+r^2}{r}\Bigg[\frac{r_{,\xi}}{r}\cot\xi-3(r_{,\xi}\sin\xi\\&\quad+r\cos\xi)\frac{f_{,z}}{f}+3(r_{,\xi}\cos\xi-r\sin\xi)\frac{f_{,\rho}}{f}\Bigg]=0,\end{aligned} \tag{17}$$

with the boundary condition, $r_{,\xi}=0$ ($\xi=0,\pi/2$), required from the regularity of the surface. The results are shown in Fig. 1.

We shall look at the various geometric quantities characterizing the horizon. The volume of the horizon H is given by

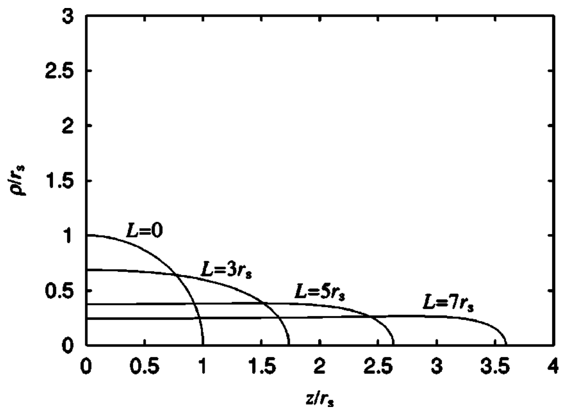

FIG. 1. Horizons for $L=0$, $3r_s$, $5r_s$ and $7r_s$ are depicted in (z,ρ)-plane. The coordinate values are normalized by the radius r_s of the minimal surface of the spherical case $L=0$. We will obtain minimal surfaces for $L>7r_s$ if we wish.

$$\mathrm{Vol}(H)=8\pi\int_0^{\pi/2}f^3\sqrt{(r_\xi)^2+r^2}r^2\sin^2\xi d\xi. \tag{18}$$

Typical area scales are that of the $(\theta=\pi/2)$-section S_1

$$\mathrm{Area}(S_1)=4\pi\int_0^{\pi/2}f^2\sqrt{(r_\xi)^2+r^2}r\sin\xi d\xi, \tag{19}$$

and that of the largest sphere S_2 among ($\xi=$const)-sections

$$\mathrm{Area}(S_2)=\max\{4\pi f^2r^2\sin^2\xi;\xi\in[0,\pi/2]\}. \tag{20}$$

The length scales C_1 and C_2 are defined by the circumferences of S_1 and S_2, respectively;

$$\mathrm{Length}(C_1)=4\int_0^{\pi/2}f\sqrt{(r_\xi)^2+r^2}d\xi, \tag{21}$$

$$\mathrm{Length}(C_2)=\max\{2\pi fr\sin\xi;\xi\in[0,\pi/2]\}. \tag{22}$$

The result is shown in Figs. 2 and 3.

It seems that there always forms a black hole, however long the massive segment is. When L is sufficiently large, the conformal factor near the origin behaves as that of the infinite line source,

$$f\sim1+\frac{2GM_{ADM}}{3L\rho}. \tag{23}$$

In the region where the conformal factor behaves as the above, the horizon is almost cylindrically symmetric and then is determined by

$$\left[\rho^2\left(1+\frac{2GM_{ADM}}{3L\rho}\right)^3\right]_{,\rho}=0. \tag{24}$$

The root of the above equation is given by

$$\rho=\rho_c:=\frac{GM_{ADM}}{3L}. \tag{25}$$

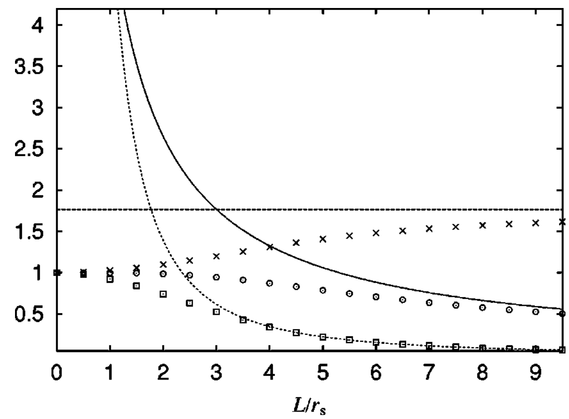

FIG. 2. The volume and typical area of a horizon are plotted as a function of the length L/r_s. These are normalized by those in the spherical case $L=0$: $\mathrm{Vol}(H_s)$ and $\mathrm{Area}(S_s)$, respectively. The volume of the horizon H is depicted by circles, and each area of the S^2-section of H is depicted by crosses $[\mathrm{Area}(S_1)]$ and by squares $[\mathrm{Area}(S_1)]$. The corresponding quantities for the black string with the coordinate length L, $\mathrm{Vol}(H_c)$ (solid line), $\mathrm{Area}(S_{c1})$ (dashed line) and $\mathrm{Area}(S_{c2})$ are also plotted as a function of L/r_s. This figure shows that the area is bounded above.

We shall pay attention to the portion of the black string within the finite interval $-L/2<z<L/2$. The volume of this part is given by

$$\mathrm{Vol}(H_c)=\int_{-L/2}^{L/2}dz\int_0^{\pi}d\vartheta\int_0^{2\pi}d\varphi f^3\rho^2\sin\vartheta|_{\rho=\rho_c}$$
$$=\frac{27\pi r_s}{16L}\mathrm{Vol}(H_s). \tag{26}$$

We shall consider the spheres S_{c1} at $\vartheta=\pi/2$ and S_{c2} at $z=$const. Each area becomes

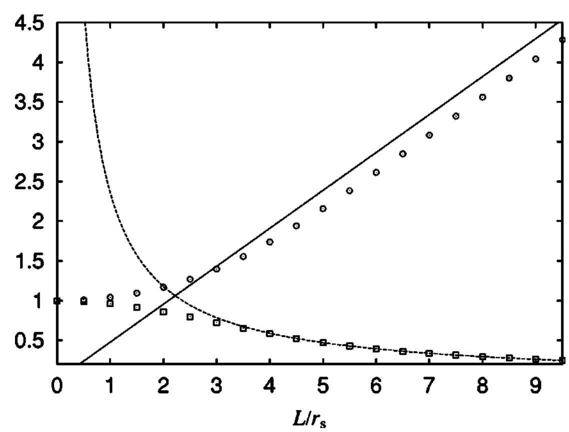

FIG. 3. The typical length scales $\mathrm{Length}(C_1)$ (circles) and $\mathrm{Length}(C_2)$ (squares) of a horizon are plotted as a function of the length of the source L/r_s. All quantities are normalized by those in the spherical case $L=0$. The corresponding length scales $\mathrm{Length}(C_{c1})$ (solid line) and $\mathrm{Length}(C_{c2})$ (dashed line) of a black string with the coordinate length L are also plotted as a function of L/r_s. This figure shows that the length scale of the black hole is unbounded.

$$\mathrm{Area}(S_{c1}) = \int_{-L/2}^{L/2} dz \int_0^{2\pi} d\varphi f^2 \rho \Big|_{\rho=\rho_c}$$

$$= \frac{9\pi}{16} \mathrm{Area}(S_s), \tag{27}$$

$$\mathrm{Area}(S_{c2}) = \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi f^2 \rho^2 \sin\vartheta \Big|_{\rho=\rho_c}$$

$$= \frac{9\pi^2 r_s^2}{16L^2} \mathrm{Area}(S_s). \tag{28}$$

The length scales of circumferences C_{c1} of S_{c1} and C_{c2} of S_{c2} are determined by

$$\mathrm{Length}(C_{c1}) = \frac{3L}{2\pi r_s} \mathrm{Length}(C_s), \tag{29}$$

$$\mathrm{Length}(C_{c2}) = \frac{3\pi r_s}{4L} \mathrm{Length}(C_s), \tag{30}$$

respectively. These quantities are also depicted in Figs. 2 and 3. From these figures, we can see that length scale, area and the volume of a spindle black hole approach to corresponding black string values in the limit of $L \to \infty$.

The area of the S^2-section S of the horizon is always bounded above by the total mass:

$$\frac{\mathrm{Area}(S)}{32GM_{ADM}/3} < O(1). \tag{31}$$

C. Disk black holes

The result of the previous section shows that a horizon of arbitrarily large linear size can form in five dimensions. We here consider the possibility of disk shaped black holes. The following metric is appropriate for this problem.

$$g = f^2(x,y)(dx^2 + dy^2 + x^2 d\psi^2 + y^2 d\varphi^2). \tag{32}$$

This admits two orthogonal commuting Killing vector fields ∂_ψ, ∂_φ and $\psi \sim \psi + 2\pi$, $\varphi \sim \varphi + 2\pi$ are regarded as the coordinates on T^2.

We consider a uniform massive disk as a source,

$$f^3 \varrho = \frac{M_{ADM}}{2\pi^2 D^2 y} \delta(y) \theta(D-x). \tag{33}$$

The gravitational field outside the above source is given by

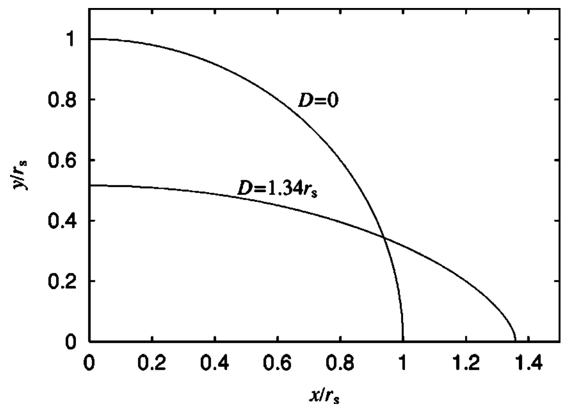

FIG. 4. Horizons of $D=0$ and $1.34r_s$ are depicted in the (x,y)-plane. The coordinate values are normalized by the radius r_s of the horizon in the spherical case $D=0$. We could not find a horizon for $D > 1.34r_s$.

$$f = 1 + \frac{2GM_{ADM}}{3\pi^2 D^2} \int_0^D x' dx' \int_0^{2\pi} d\psi' \times \frac{1}{(x - x'\cos\psi')^2 + x'^2 \sin^2\psi' + y^2}$$

$$= 1 + \frac{2GM_{ADM}}{3\pi D^2} \ln \left| \frac{1}{2y^2} \left[D^2 - x^2 + y^2 + \sqrt{(x^2+y^2)^2 - 2D^2(x^2-y^2) + D^4} \right] \right|. \tag{34}$$

Let us search for the apparent horizon of the form $x = r(\xi)\cos\xi$, $y = r(\xi)\sin\xi$, determined by the differential equation

$$r_{,\xi\xi} - 4\frac{(r_{,\xi})^2}{r} - 3r + \frac{(r_{,\xi})^2 + r^2}{r} \left[2\frac{r_{,\xi}}{r}\cot(2\xi) - 3(r_{,\xi}\sin\xi + r\cos\xi)\frac{f_{,x}}{f} + 3(r_{,\xi}\cos\xi - r\sin\xi)\frac{f_{,y}}{f} \right] = 0, \tag{35}$$

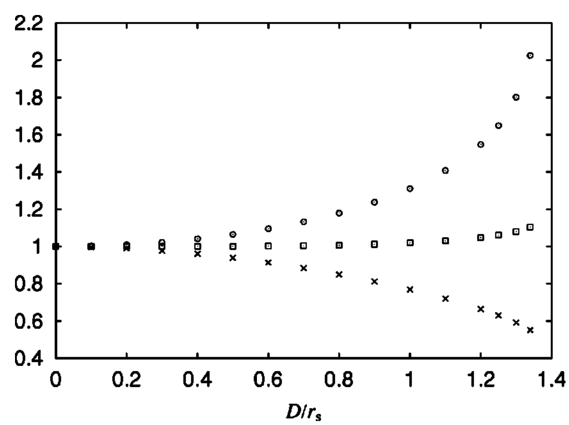

FIG. 5. The typical length scales Length(C_1) (squares), Length(C_2) (crosses) and Length(C_3) (circles) of a horizon are plotted as a function of the radius D/r_s. All quantities are normalized by those in the spherical case $D=0$.

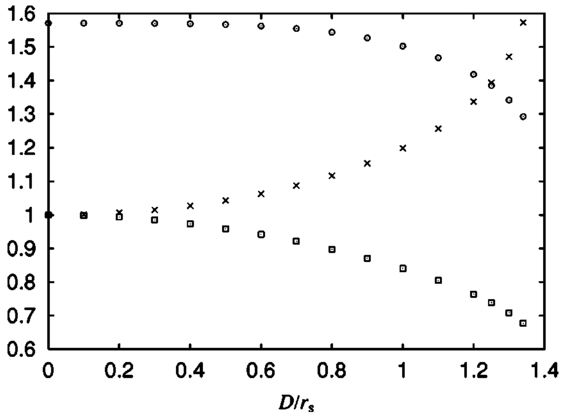

FIG. 6. The typical area scales Area(S_1) (crosses), Area(S_2) (squares) and Area(T) (circles) of a horizon are plotted as a function of the disk radius D/r_s.

subject to the boundary condition: $r_{,\xi}=0$ ($\xi=0,\pi/2$).

The results are shown in Figs. 4–7. We evaluate typical hoop length scales

$$\text{Length}(C_1)=4\int_0^{\pi/2} f\sqrt{(r_{,\xi})^2+r^2}d\xi, \tag{36}$$

$$\text{Length}(C_2)=\max\{2\pi f r\cos\xi;\xi\in[0,\pi/2]\}, \tag{37}$$

$$\text{Length}(C_3)=\max\{2\pi f r\sin\xi;\xi\in[0,\pi/2]\}, \tag{38}$$

typical area scales

$$\text{Area}(S_1)=4\pi\int_0^{\pi/2} f^2\sqrt{(r_\xi)^2+r^2}r\sin\xi d\xi, \tag{39}$$

$$\text{Area}(S_2)=4\pi\int_0^{\pi/2} f^2\sqrt{(r_\xi)^2+r^2}r\cos\xi d\xi, \tag{40}$$

$$\text{Area}(T)=\max\{4\pi^2 f^2 r^2\sin\xi\cos\xi;\xi\in[0,\pi/2]\}, \tag{41}$$

and the volume of the horizon

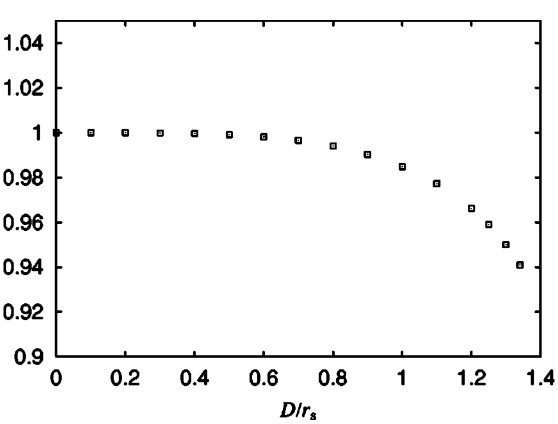

FIG. 7. The volume of a horizon is plotted as a function of the radius D/r_s. The values are normalized by those in the spherical case $D=0$.

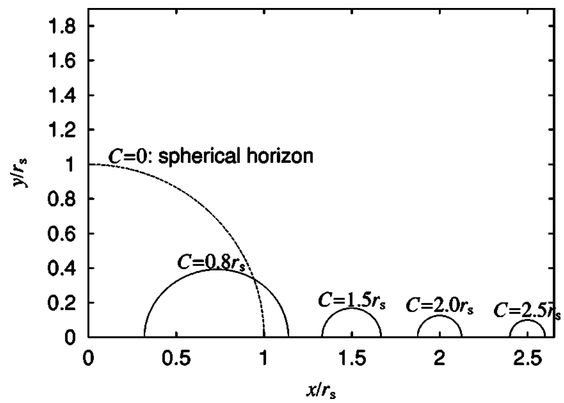

FIG. 8. Black rings for $C=0.8r_s$, $1.5r_s$, $2.0r_s$, $2.5r_s$ are depicted in (x,y)-plane. The coordinate values are normalized by r_s. A black ring can be found for $C=0.79r_s$, but not for $C=0.78r_s$.

$$\text{Vol}(H)=2\pi^2\int_0^{\pi/2} r^2 f^3\sqrt{(r_{,\xi})^2+r^2}\sin 2\xi d\xi. \tag{42}$$

It can be seen that the inequality (31) still holds in this case. Remarkably, a horizon does not form for large disks; we have not found a minimal surface for $D>1.34r_s$.

D. Black rings

In five-dimensional space-time, a black hole may have nontrivial topology [20], while in four dimensions, the apparent horizon must be homeomorphic to sphere [21]. In particular, black rings homeomorphic to $S^2\times S^1$ are possible; Emparan and Reall have found explicitly a stationary black ring solution [8]. We here show the validity of the inequality (31) for this black ring case. The metric used here is the same as the disk case Eq. (32). The black ring will form if we put simply a uniform massive circle,

$$f^3\varrho=\frac{M_{ADM}}{4\pi^2 Cy}\delta(x-C)\delta(y). \tag{43}$$

The conformal factor is then given by

$$f=1+\frac{GM_{ADM}}{3\pi^2}\times\int_0^{2\pi}\frac{d\psi'}{(x-C\cos\psi')^2+C^2\sin^2\psi'+y^2}=1+\frac{2GM_{ADM}}{3\pi\sqrt{(x+C)^2+y^2}\sqrt{(x-C)^2+y^2}}. \tag{44}$$

Let us search for the apparent horizon in the form: $x=C+r(\xi)\cos\xi$, $y=r(\xi)\sin\xi$. This surface is governed by the differential equation

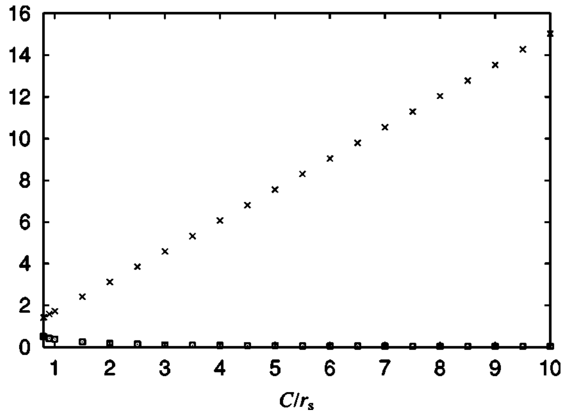

FIG. 9. Typical length scales Length(C_1) (squares), Length(C_2) (crosses) and Length(C_3) (circles) of black rings are plotted as a function of the circle radius C/r_s. All quantities are normalized by those in the spherical case $C=0$.

$$r_{,\xi\xi}-3\frac{(r_{,\xi})^2}{r}-2r-\frac{(r_{,\xi})^2+r^2}{r}\left[\frac{r_{,\xi}\sin\xi+r\cos\xi}{r\cos\xi+C}-\frac{r_{,\xi}}{r}\cot\xi+3(r_{,\xi}\sin\xi+r\cos\xi)\frac{f_{,x}}{f}-3(r_{,\xi}\cos\xi-r\sin\xi)\frac{f_{,y}}{f}\right]=0, \tag{45}$$

subject to the boundary condition $r_{,\xi}=0$ ($\xi=0,\pi$).

The results are shown in Figs. 8, 9 and 10. We evaluate the typical hoop length scales

$$\mathrm{Length}(C_1)=2\int_0^\pi f\sqrt{(r_{,\xi})^2+r^2}\,d\xi, \tag{46}$$

$$\mathrm{Length}(C_2)=\max\{2\pi f r\cos\xi;\xi\in[0,\pi]\}, \tag{47}$$

$$\mathrm{Length}(C_3)=\max\{2\pi f r\sin\xi;\xi\in[0,\pi]\}, \tag{48}$$

typical area scales

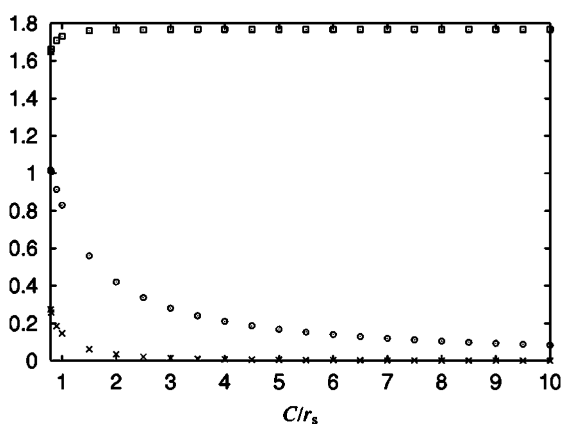

FIG. 10. The typical area scales Area(S) (crosses), Area(T) (squares) and the volume Vol(H) (circles) of a horizon are plotted as a function of the circle radius C/r_s.

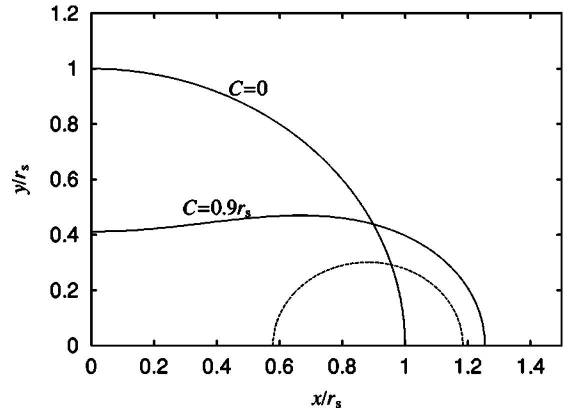

FIG. 11. Both a black hole and a black ring can be found for $0.79r_s\leq C\leq 0.90r_s$. These horizons are depicted for $C=0.9r_s$. A black hole with $C=0.91r_s$ cannot be found for a circle source.

$$\mathrm{Area}(S)=2\pi\int_0^\pi f^2\sqrt{(r_\xi)^2+r^2}\,r\sin\xi\,d\xi, \tag{49}$$

$$\mathrm{Area}(T)=\max\{4\pi^2 f^2 r^2\sin\xi\cos\xi;\xi\in[0,\pi]\}, \tag{50}$$

and the volume of the black ring

$$\mathrm{Vol}(H)=4\pi^2\int_0^\pi f^3\sqrt{(r_{,\xi})^2+r^2}\,r\sin\xi(r\cos\xi+C)\,d\xi. \tag{51}$$

For large radius of the massive circle, there always exists a black ring. This can be expected from the result for the spindle case, since the local geometry around a large circle resembles that around a line source. On the other hand, a small circle makes a black hole homeomorphic to S^3 (see Figs. 11, 12, 13 and 14). A new aspect found here is that both a black hole and a black ring form for a cirtain range of the

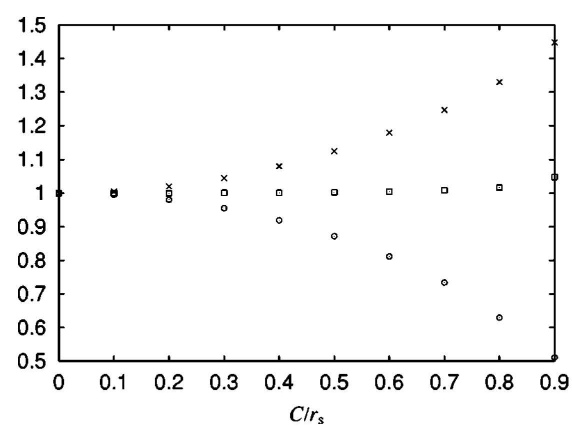

FIG. 12. The typical length scales Length(C_1) (squares), Length(C_2) (crosses) and Length(C_3) (circles) of a horizon are plotted as a function of the radius C/r_s. The definitions of these quantities are same as the disk case. All quantities are normalized by those in the spherical case $C=0$.

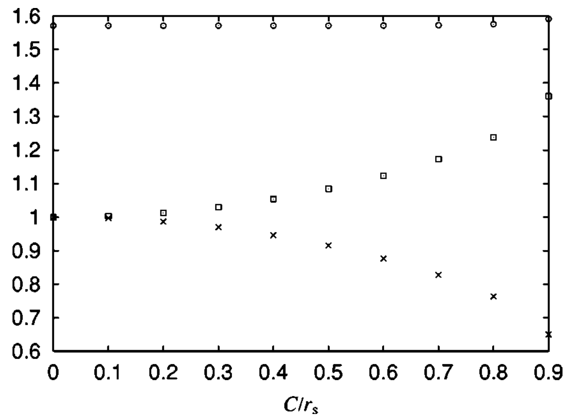

FIG. 13. The typical area scales Area(S_1) (crosses), Area(S_2) (squares) and Area(T) (circles) of a horizon are plotted as a function of the circle radius C/r_s. The definitions of these quantities are same as the disk case.

radius of the circle. The inequality (31) is anyway satisfied, where the two-section of the horizon can be characteristic sphere or torus.

III. CONCLUSION

We have investigated the momentarily static, conformally flat initial data sets for the five-dimensional Einstein equation. We consider various configurations of the gravitational source and search for the apparent horizons.

For the line source of the Euclidean length L, a black hole can be found for arbitrary L, which can be contrasted with the corresponding four-dimensional situations. In four dimensions, a black hole does not form when L is much larger than the Schwarzschild radius. The result here shows that the hoop length is not a good indicator of the horizon formation in higher dimensions. This can be interpreted as follows. For the line source of the mass M and the length $L \gg (GM)^{1/(n-2)}$ in n-dimensional space, the effective gravitational field at the symmetric hyperplane will have $(n-1)$-dimensional nature. For the line source in four dimensional space-time, the effective gravity on the hyperplane will be that of the $(2+1)$ dimensions, so that there does not form a black hole [22,23]. In $(n+1)$-dimensional space-time, there will be a black hole of the radius $R \sim (GM/L)^{1/(n-3)}$.

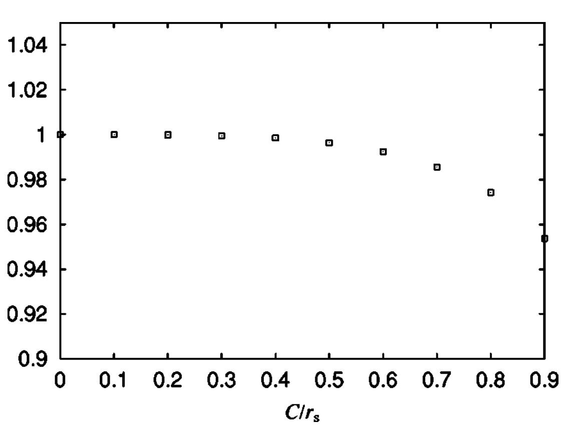

FIG. 14. The volume of a horizon is plotted as a function of the radius C/r_s. The values are normalized by those in the spherical case $C=0$.

From the above considerations, one can expect that the condition of horizon formation is determined by the effective number of codimensions of the gravitational source. The horizon will not form if the effective number of codimensions is less than three. We have confirmed this expectation by studying the horizon formation due to the disk source. Since the effective gravity produced by the disk with the radius $D \gg (GM)^{1/2}$ is that of $(2+1)$-gravity, the horizon will not form for large disk sources. In fact, the apparent horizon can be found only when D is less than or of the order of the Schwarzschild radius. We found that the good indicator of the horizon formation is the typical area scale of the system. In five-dimensional space-time, the condition for the horizon formation will be given by the inequality

$$\text{Area} \lesssim GM, \tag{52}$$

which can be regarded as the generalization of hoop conjecture for four-dimensional space-times. In other words, the scale of typical codimension-two submanifold of the horizon should be less than or of the order of the scale determined by the mass scale. This argument is independent of the space-time dimensions. The corresponding isoperimetric inequality for black holes in $(n+1)$-dimensional space-times will be

$$V_{n-2} \lesssim GM, \tag{53}$$

where V_{n-2} is the volume scale of the characteristic codimension-two submanifold of the horizon.

An interesting feature of higher-dimensional black holes is that the horizon can have nontrivial topology. In five-dimensional space-times, the horizon can be a black hole $(\simeq S^3)$, a black ring $(\simeq S^2 \times S^1)$ or their connected sums [20]. For this reason, we have also investigated the condition for the black ring formation due to the circle source. The inequality (52) still holds in this case. For large (small) circles, they form a black ring (hole). However, for appropriate ranges of the circle radius, both the black ring and the black hole can be found such that the black hole encloses the black ring. Thus we can expect that at the final stage of the gravitational collapse of the black ring, a new spherical black hole formes outside the black ring.

For large circle sources, the effective local gravity around the source will be that of $(3+1)$ dimensions. This is the physical reasoning of the possibility of the black ring in $(4+1)$ dimensions. While in $(4+1)$-dimensions, the torus horizon $(\simeq T^3)$ is forbidden. If all radii of S^1 of a torus source are large, then the effective gravity will be $(2+1)$-dimensional, while if one of S^1 is small, then the horizon will not be the torus anymore even if it forms. Though there is no topology theorem for six or higher-dimensional black holes, the torus topology might be forbidden.

For all examples studied here, the volume of the horizon is less than the spherical value:

$$\mathrm{Vol}(H) \leqslant \mathrm{Vol}(H_s) = 2\pi^2 \left(\frac{8GM_{ADM}}{3\pi} \right)^{3/2}. \quad (54)$$

This inequality resembles the Penrose inequality for black holes in $(3+1)$ dimensions. Though the higher-dimensional generalization of the Penrose inequality is not known, similar inequality might exist.

According to the brane-world scenario, the matter fields are confined to the brane, so that the gravitational collapse within our universe likely forms the black hole. If there are at least two extra dimensions, then even gravitational collapse of a disk-shaped massive object will not result in the naked singularity. Thus the cosmic censorship might work well in the higher-dimensional brane universe.

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