Unified description of interactions in terms of composite fiber bundles

Romualdo Tresguerres*
IMAFF, Consejo Superior de Investigaciones Científicas, Serrano 113 bis, Madrid 28006, Spain
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We claim that a composite fiber bundle satisfactorily serves as the geometrical structure underlying gauge theories of spacetime groups, such as the Poincaré gauge theory. A trivial extension of the approach to include internal symmetries also provides a unique mathematical scheme in which gravitation and the remaining forces are put together and treated in a common fiber bundle language. The result is a homogeneous characterization of interactions, gravity included, exclusively by means of connections.

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I. INTRODUCTION

Gauge theories constitute the commonly accepted approach to the description of fundamental forces. In the foundational Yang-Mills paper [1], the gauge treatment was applied to internal symmetries. But soon afterward a series of proposals appeared [2] extending the gauge principle to spacetime symmetries. In several papers by Hehl *et al.* [3] the Poincaré gauge theory (PGT), as a gauge theory of gravity, reached the standard form we will have in mind in the following. Independently of gauge theories in physics [4], a parallel mathematical development took place concerning fiber bundles [5,6]. The formal identity between gauge theories and the geometry of bundles has been recognized since the 1960s [7,8]. Nowadays, it is the standard point of view that at least nongravitational interactions are suitably described in terms of gauge potentials, the latter being interpreted as local connections in a principal fiber bundle. A single mathematical scheme thus gives an account of almost all fundamental interactions. Only the gauge status of gravity still remains problematic. Actually, its bundle structure is revealed to be not exactly that of an ordinary Yang-Mills theory, and until now there is no unique answer to the question of whether or not gravitation has to be considered as essentially different from the other forces.

Let us briefly articulate the main kinds of mathematical approaches to gravity found in the literature. As the first one, we must mention the metric tensor approach, corresponding to Einstein's original formulation. In this standard form, general relativity (GR) is conceived as the field theory of the metric tensor fixing the geometrical structure of spacetime. That makes a big contrast with the nongravitational forces. Indeed, expressed in fiber bundle language, GR seems to manifest itself as the dynamical theory of the underlying base space, gravitational interactions being mediated by the metric as the gravitational potential, while the remaining forces are characterized in terms of bundle connections (gauge potentials). If that were the correct scheme, one should accept the existence of forces of two absolutely different natures.

A different view is supplied by the tangent space approach. It originates in the extension of GR by means of tetrads, the so called *vierbeins*, necessary to make possible the coupling of gravitational fields to fundamental spinor matter. Tetrads are mixed objects possessing both a coordinate index and a Lorentz index, thus providing a bridge or *soldering* between the dynamical base space and the locally Minkowskian tangent space attached to each point. The standard bundle approach to gravity [9–14] supplies an interpretation of tetrads as local sections of an orthonormal frame bundle. Briefly, given a manifold M, the bundle $L(M)$ of linear frames of the tangent space $T(M)$ constitutes a principal bundle with structure group $GL(4,R)$, $T(M)$ being its associated vector bundle. If M admits a Riemannian metric, then the bundle of linear frames can be reduced to a bundle of orthonormal frames with the Lorentz group as structure group. If the frame bundle $L(M)$ is originally endowed with a linear connection, the reduction process transforms the latter into a Lorentz connection.

Tetrads and connections are different kinds of gravitational fields. We note this duplicity since it constitutes a main difficulty for the inclusion of gravity in a unified gauge theoretical scheme together with the remaining forces. Actually most of the current gauge theories of gravitation [15,16], contrary to ordinary gauge theories, distinguish two types of gravitational variables, namely, the gauge potentials identified with connections (reduced Lorentz connections, Ashtekar variables), and simultaneously metric or tetrad fields. Notice that gravitational potentials of such different classes are as different from each other as connections and sections are.

Ivanenko and Sardanashvily [15] proposed a somewhat different approach. They considered gravity to be a spontaneously broken $GL(4,R)$ gauge theory. The tetrads, playing the role of Goldstone-like—although nonremovable—fields, result from the contraction, induced by the equivalence principle, of the tangent bundle structure group from the general linear to the Lorentz group. Tetrads are identified as global sections of the bundle with fiber G/H, where $G=GL(4,R)$ and H is the Lorentz group.

In contrast to all the previously mentioned interpretations of gravitational potentials, either as metric tensors, or as sections of a frame bundle, as Goldstone fields, etc., to be considered in addition to Yang-Mills potentials, the gauge approach developed by Hehl *et al.* [3] admits only a single kind of potential, namely, gauge potentials. [The seeming deviation of this rule represented by the metric of metric-affine

*Email address: ceef310@imaff.cfmac.csic.es

gravity (MAG) [17] does not affect the present discussion, centered on PGT. The subject was treated in Refs. [19].] In particular, this approach regards tetrads as translational connections, that is, as true gauge potentials of the Yang-Mills type. The gauge theories of spacetime groups defended by Hehl *et al.* (PGT, MAG) constitute in fact the treatment of gravity lying closest to ordinary gauge theories. Certainly, in principle a serious difficulty exists in reconciling the gauge transformations of ordinary connections with those of tetrads, which must transform as covectors. However, the obstacle can be overcome in different ways [18,19]. It is precisely the main task of the present paper to support the correctness of one of the answers proposed to this question, namely, the one based on nonlinear realizations (NLR's) [19–21]. Our aim is to find the bundle structure underlying PGT, allowing us in particular to derive the right gauge transformations of vierbeins when considered as translational connections. Indeed, while ordinary gauge transformations are well defined on standard principal fiber bundles, we claim that nonlinear realizations rest on a different geometrical framework, namely, that of composite fiber bundles. The recognition of this fact completes the foundation of NLR's as developed in [19].

Previous attempts were made to equip such theories in the manner of Hehl with a fiber bundle interpretation [22,23]. Of particular interest is that due to Lord [24,25], who proposed an approach based on a bundle $G(G/H,H)$, as suggested by Ne'eman and Regge [26]. This bundle structure was thought to replace the standard description of interactions, namely, that in terms of the principal bundle structure $P(M,G)$, with the role of spacetime played by the base manifold M. Our own proposal retains Lord's suggestion of considering the bundle manifold split into two sectors, but instead of exploiting the structure $G(G/H,H)$, with G/H as the spacetime manifold, we propose to construct a composite fiber bundle, as will be developed in the following.

The present approach gives an answer to the criticisms enunciated in Ref. [[15], pp. 29–30], on the presumed failure of PGT to provide a well behaved tetrad with the status of a gauge potential of the Poincaré translations. Actually, the bundle structure proposed in the present paper supplies a geometrical basis for gauge transformations derived otherwise from NLR's, so that the correct transformation properties of the tetrads are obtained. Furthermore, the commutation relations of the translational generators P_μ are not violated [3,15]. Although we will concentrate on PGT [3], our results should be applicable to gravity theories based on other spacetime groups.

In summary, the aim of the present paper is to develop a fiber bundle description of forces, including gravity, in a unified scheme. It is inspired by the revision of the concept of gauge transformations when spacetime groups are considered. In fact, the new bundle structure is necessary if one wants to reconcile the standard characterization of gauge transformations with Lord's view on spacetime gauge transformations. The resulting formalism admits the inclusion of additional forces, with the tetrads (that is, with gravity) universally coupled to them. Thus, a single mathematical structure (namely, a composite fiber bundle) will suffice to describe all forces in a homogeneous way. In the present paper, we pay attention only to the geometrical (predynamical) aspects of interactions, that is, merely to the building blocks for the construction of suitable actions.

The paper is organized as follows. We begin, in Sec. II, with a review of the main concepts of fiber bundles, as constituting the mathematical structure underlying ordinary gauge theories of internal groups. Next, in Sec. III, we discuss the standard characterization of gauge transformations on ordinary bundles as vertical bundle automorphisms; extending a suggestion due to Lord [24,25], we propose a modification of the standard view in order for transformations of spacetime groups to be included. Section IV is devoted to explicitly showing the incapability of the ordinary bundle structure, when applied to the Poincaré group, to yield the right gauge transformations reproducing those of PGT. Consistently with this fact, our characterization of gauge transformations does not make sense in the framework of ordinary bundles; it needs to be realized on a different structure, namely, a composite fiber bundle, as introduced in Sec. V. Sections VI and VII are devoted to the gauge transformations in such composite bundles, concerning, respectively, bundle sections and connections. In Sec. VIII we deal with the gauge transformations induced on matter fields. Finally, in Sec. IX, we apply the composite bundle treatment to the Poincaré group, showing the identity of our bundle approach to gravity with PGT. In the Final Remarks, we point out the way to incorporate the gauge theories of the remaining ordinary forces into a single scheme with our bundle description of spacetime.

II. REVIEW OF ORDINARY PRINCIPAL FIBER BUNDLES

We will briefly review the foundations of the standard bundle approach to ordinary gauge theories [6–14] in order to fix a necessary reference, to be compared with the modified bundle structure we are going to propose later for gauge theories of spacetime groups in the manner of Hehl.

Fiber bundles appear as a further step in the successive introduction of sets endowed with increasing degree of structure (topological spaces, differentiable manifolds, etc.), necessary to formalize the concepts of continuity, smoothness, etc. Bundles are locally isomorphic to the Cartesian product of two manifolds. According to the definition of a principal fiber bundle $P(M,G)$ over the base space M, with structure group G and canonical projection $\pi: P\rightarrow M$ (see [6, p. 50]), every point $x\in M$ has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U\times G$. On the other hand, a local section is a smooth map s: $U\subset M\rightarrow P$ satisfying $\pi\circ s=(id)_M$. It assigns to each $x\in M$ a single value $s(x)$ in the fiber $\pi^{-1}(x)$ over x.

A principal fiber bundle P constitutes a particular kind of manifold [6–14]. Its tangent space, denoted as $T(P)$, may be split into the direct sum of a vertical subspace $V(P)$ plus a horizontal subspace $H(P)$, the vertical subspace consisting of the vectors tangent to the fibers, called the fundamental vector fields. In order to construct the latter, regard local fibers as orbits of the right action of the structure group G on P, that is, $\pi^{-1}(x)=u\tilde{g}_\lambda$ through $u\in P$, with $\tilde{g}_\lambda\in G$. Then, every Lie algebra element (A3) induces [in analogy with Eq. (A6)] a tangent vector to the fiber at u given by

$$G_A^{\#}[f(u)] := \left. \frac{\partial f(u\tilde{g}_\lambda)}{\partial \lambda^A} \right|_{\lambda=0}, \tag{2.1}$$

for f a differentiable function. The vector components $G_A^{\#}|_u$, tangent to the fiber through $u \in P$, are the fundamental vector fields at u, constituting $V(P)$. The mapping of the Lie algebra basis element $G_A \in \mathcal{G}$ into $G_A^{\#}|_u$ is an isomorphism of the Lie algebra $\mathcal{G}$ of G into the corresponding Lie algebra of vector fields on P. On each fiber, we can identify Eq. (2.1) with Eq. (A6), or with its explicit form Eq. (A7).

On the other hand, the definition of horizontality in $T(P)$ requires the introduction of the concept of connection [6–14]. We define the Ehresmann connection form ω to be a one-form on the cotangent space $T^*(P)$ of P with values in the Lie algebra $\mathcal{G}$ of G, satisfying the conditions

$$G_A^{\#} \rfloor \omega = G_A \in \mathcal{G} \tag{2.2}$$

and

$$R_g^* \omega = ad_{g^{-1}} \omega, \quad \forall \quad g \in G, \tag{2.3}$$

where $G_A \in \mathcal{G}$ in Eq. (2.2) is the Lie algebra element isomorphic to the fundamental vector field $G_A^{\#}$ defined by Eq. (2.1), and $\rfloor$ denotes the inner product, while $ad_{g^{-1}}\omega := g^{-1}\omega g$ in Eq. (2.3) means the adjoint representation of G in $\mathcal{G}$. In terms of ω, the horizontal subspace $H(P)$ is defined to consist of all those vectors $\tilde{X}$ of $T(P)$, with nonvanishing projection, such that $\tilde{X} \rfloor \omega = 0$.

Let us introduce an explicit realization of the Ehresmann connection form. We know [13] that any section $s: M \rightarrow P$ with local trivialization $(x, \tilde{g})$ can be decomposed in terms of a zero section $\tilde{\sigma}(x)$ with local trivialization (x, e_G), as the product

$$s(x) = \tilde{\sigma}(x) \cdot \tilde{g}. \tag{2.4}$$

So, at the bundle point $u = s(x)$ as given by Eq. (2.4), we can take

$$\omega = \tilde{g}^{-1}(d + \pi^* A)\tilde{g}. \tag{2.5}$$

In view of the decomposition (2.4), $\tilde{g}$ are the bundle coordinates, and the quantity

$$A = \tilde{\sigma}^* \omega \tag{2.6}$$

coincides with the usual definition of a local potential $A = A^A G_A = dx^i A_i^A G_A$ as the pullback (by means, in particular, of the zero section $\tilde{\sigma}$) of the Ehresmann connection form. One can prove (see [[13], p. 333]), that the proposed realization (2.5) of ω satisfies the defining axioms (2.2), (2.3) of connection forms. Then we can choose the horizontal vectors

$$E_i := \tilde{\sigma}_* \partial_i - A_i^A \overline{L_A}, \tag{2.7}$$

such that $E_i \rfloor \omega = 0$, with $\overline{L_A}$ given by Eq. (A10), as a basis of the horizontal tangent subspace $H(P)$.

III. DISCUSSION OF THE EXTENSION OF GAUGE TRANSFORMATIONS TO SPACETIME GROUPS

Gauge transformations are a key concept for gauge theories. In the present section we perform a critical examination of their commonly accepted [12] bundle-adapted definition due to Atiyah, Hitchin, and Singer [27], and of the proposal of Lord and Goswami [25] to modify the gauge transformation concept in order to extend it to spacetime groups. Our own approach will be developed in the next sections. We will show that the latter allows us to reconcile both the standard and Lord's points of view.

According to the standard definition [10,12,25,27], we characterize a gauge transformation on a principal fiber bundle $P(M,G)$ as a bundle automorphism $\lambda: P \rightarrow P$ satisfying two conditions. On the one hand, λ is required to commute with the right action of G,

$$\lambda \circ R_g(u) = R_g \circ \lambda(u), \tag{3.1}$$

so that fibers are mapped to fibers. With only Eq. (3.1) at hand, in general λ induces a diffeomorphism $\tilde{\lambda}: M \rightarrow M$ on the base space, given by $\tilde{\lambda} \circ \pi(u) = \pi \circ \lambda(u)$. The goal of the second defining condition is to avoid this possibility. Standard gauge transformations λ are required to satisfy

$$\pi \circ \lambda(u) = \pi(u), \tag{3.2}$$

so that they become vertical automorphisms, where both u and $\lambda(u)$ belong to the same fiber. No action is allowed to be induced on the base space M. Vertical bundle automorphisms λ are the standard gauge transformations, adapted to the principal bundle structure $P(M,G)$ describing all forces other than gravity.

However, things are less simple in the case of gauge theories of spacetime groups. Here the group action on spacetime coordinates cannot be ignored. This reason moved Lord [24,25] to relax the verticality condition (3.2), by restricting its validity to internal groups only, whereas spacetime groups and their corresponding gauge theories had to be handled in a different way. Regarding spacetime groups, Lord admitted the existence of nonvertical gauge transformations, inducing diffeomorphisms on M. Following Ne'eman and Regge [26], he suggested the principal bundle $G(G/H,H)$ as the possible fiber bundle structure of gauge theories of gravitation. Suitably choosing G, for instance, as the Poincaré group, and $H \subset G$ as the Lorentz group, the base space G/H becomes identical with the parameter space of translations, playing the role of spacetime.

Our main criticism of Lord's view consists in that, since the translational manifold G/H is not referred to a further base space, translational connections are not present. (Actually, Lord's tetrads on G/H are not connections.) We remark that only by postulating such an additional base space M will it become possible to introduce the translational connection form constituting the unavoidable requirement for tetrads to become interpretable as (nonlinear) local connections, as we will see below. The auxiliary base space introduced by us [as in ordinary principal bundles $P(M,G)$] is the main difference between Lord's approach and ours. In Secs. V–IX we will show how both verticality as defined by Eq. (3.2) and induced spacetime transformations are compatible in the context of the bundle structure provided by composite fiber bundles.

Gauge transformations of local potentials: The case of principal fiber bundles

Let us derive for λ the consequences of Eqs. (3.1) and (3.2). A theorem [24,25] establishes that a diffeomorphism (the gauge transformation λ in our case) commutes with every right translation if and only if it is a left translation. Actually, let us reexpress Eq. (3.1) as $\lambda(u\hat{g})=\lambda(u)\hat{g}$, with the caret added to $\hat{g}\in G$ in order to avoid confusion in what follows. Since $\lambda(u\hat{g})(u\hat{g})^{-1}=\lambda(u)u^{-1}$, one finds $g(x):=\lambda(u)u^{-1}$ to be the same for different points u, $u\hat{g}\in\pi^{-1}(x)$ arbitrarily chosen along a given fiber. Thus, trivially we get

$$\lambda(u)=g(x)u=:L_{g(x)}u. \tag{3.3}$$

From Eq. (3.3) we recognize the gauge transformation to be identical with the left action $\lambda=L_{g(x)}$, with the group element $g(x)\in G$ being local in the sense that it depends on points of the base space. Let us now calculate the gauge transformations induced by Eq. (3.3) on the bundle tangent space. For later computational convenience, we reexpress Eq. (3.3) in terms of the notation

$$\lambda(u)=u\,\eta(u),\quad \eta(u):=u^{-1}g(x)u, \tag{3.4}$$

where $\eta(u)$ formally behaves as a G element acting on $u\in P$ by right multiplication [12]. The differential map $\lambda_*:T_{\lambda(u)}(P)\to T_u(P)$ induced by λ on $T(P)$ is found [10,12,13] by deriving λ in Eq. (3.4), yielding

$$\lambda_*Y=R_{\eta *}Y+[Y\rfloor(\eta^{-1}\wedge d\eta)]^{\#}, \tag{3.5}$$

with $Y\in T_u(P)$, and with the last term in Eq. (3.5) being the fundamental vector field generated by $Y\rfloor(\eta^{-1}\wedge d\eta)\in T_e(G)\simeq\mathcal{G}$ at $\lambda(u_0)$ (see [13, pps. 330, 334]). Making use of Eq. (3.5), one can easily derive the corresponding induced gauge transformation of connection forms as follows. The inner product of Eq. (3.5) with an Ehresmann connection ω yields

$$(\lambda_*Y)\rfloor\omega=(R_{\eta *}Y)\rfloor\omega+Y\rfloor(\eta^{-1}\wedge d\eta), \tag{3.6}$$

where condition (2.2) has been used. Then we find

$$\begin{aligned}Y\rfloor\lambda^*\omega&=Y\rfloor R^*_\eta\omega+Y\rfloor(\eta^{-1}\wedge d\eta)\\&=Y\rfloor(\eta^{-1}\omega\eta)+Y\rfloor(\eta^{-1}\wedge d\eta),\end{aligned} \tag{3.7}$$

with condition (2.3) used in the last step. Since Eq. (3.7) is valid for arbitrary $Y\in T(P)$, we conclude, recalling $\lambda=L_g$, that

$$L^*_g\omega=\lambda^*\omega=R^*_\eta\omega+\eta^*\Theta=\eta^{-1}(d+\omega)\eta. \tag{3.8}$$

Equation (3.8) is the gauge transformation law of a connection form.

Consider now the pullback $s^*\omega$ of the connection form ω, s: $M\to P$ being a local section on an ordinary principal fiber bundle $P(M,G)$. In order to find the gauge transformations of such general local potentials [12], we apply s^* to Eq. (3.8) as

$$s^*(\lambda^*\omega)=s^*R^*_\eta\omega+s^*\eta^*\Theta. \tag{3.9}$$

Recalling the properties of pullbacks, namely, $f^*\varphi=\varphi\circ f$, $(g\circ f)^*=f^*g^*$, $f^*(\alpha\wedge\beta)=f^*\alpha\wedge f^*\beta$, valid for any function φ and differential forms α, β, we introduce the notation $s^*\eta^*=(\eta\circ s)^*=:\zeta^*$, so that Eq. (3.9) transforms into $s^*(\lambda^*\omega)=R^*_\zeta(s^*\omega)+\zeta^*\Theta$, or equivalently

$$(\lambda\circ s)^*\omega=R^*_\zeta(s^*\omega)+\zeta^*\Theta=\zeta^{-1}(d+s^*\omega)\zeta. \tag{3.10}$$

Particularizing s to the zero section $\tilde{\sigma}(x)$ in Eq. (2.4), we find

$$\zeta(x):=\eta(\tilde{\sigma}(x))=\eta(u)|_{\tilde{g}=e_G}=g(x) \tag{3.11}$$

[see Eqs. (2.4) and (3.4)], so that Eq. (3.10), with the usual replacement $\lambda=L_g$, becomes the ordinary transformation formula

$$(L_g\circ\tilde{\sigma})^*\omega=g(x)^{-1}(d+\tilde{\sigma}^*\omega)g(x). \tag{3.12}$$

Infinitesimally, using the notation $g(x)\approx I+\epsilon^A(x)G_A=:I+\epsilon$, and recalling Eq. (2.6), Eq. (3.12) gives rise to

$$\delta A:=\tilde{\sigma}^*\omega-(L_g\circ\tilde{\sigma})^*\omega\approx-(d\epsilon+[A,\epsilon]). \tag{3.13}$$

We recognize the ordinary infinitesimal gauge transformations of a gauge potential. Explicitly for $A=A^AG_A$, $\epsilon=\epsilon^AG_A$, Eq. (3.13) reads $\delta A=-(d\epsilon^A+f_{BC}{}^AA^B\epsilon^C)G_A$, where we made use of Eq. (A1).

IV. FAILURE OF THE STANDARD PRINCIPAL BUNDLE APPROACH TO THE POINCARÉ GAUGE THEORY

The result (3.13) is responsible for ordinary principal fiber bundles being unable to describe the underlying geometrical structure of gravitational theories of the PGT type. Certainly, the standard $P(M,G)$ bundle formulation is known to illuminate the geometrical background of gauge theories of internal groups. So, in principle, one could be tempted to extend this general scheme to any Lie group in order to construct the corresponding gauge theory. However, we will show explicitly, by directly applying the standard recipe to a spacetime group, that gravitational gauge theories cannot be obtained in this way. We will consider the Poincaré group, in order to display the main difficulty of this naive procedure, consisting, as it is well known, in its failure in providing a gauge theoretical derivation of well behaved tetrads.

Let us construct the connection form (2.5), taking $\tilde{g}$ to be a Poincaré group element (B2), and

$$A = -i\, dx^i (\overset{(T)}{\Gamma}{}_i{}^{\mu} P_\mu + \Gamma_i^{\alpha\beta} \Lambda_{\alpha\beta}). \tag{4.1}$$

The explicit form of the resulting ω will be shown in Eqs. (9.2), (9.3) below. The inadequateness of the ordinary bundle approach becomes apparent merely by finding the gauge transformation (3.13) of Eq. (4.1) induced by L_g, with g an infinitesimal Poincaré group element,

$$g = e^{i\epsilon^\alpha P_\alpha} e^{i\beta^{\alpha\beta}\Lambda_{\alpha\beta}} \approx I + i(\epsilon^\alpha P_\alpha + \beta^{\alpha\beta}\Lambda_{\alpha\beta}). \tag{4.2}$$

From Eq. (4.2) we identify

$$\epsilon = i(\epsilon^\alpha P_\alpha + \beta^{\alpha\beta}\Lambda_{\alpha\beta}). \tag{4.3}$$

Replacing Eqs. (4.1) and (4.3) into Eq. (3.13), making use of the notation

$$\overset{(T)}{\Gamma}{}^{\mu} := dx^i \overset{(T)}{\Gamma}{}_i{}^{\mu}, \quad \Gamma^{\alpha\beta} := dx^i \Gamma_i^{\alpha\beta}, \tag{4.4}$$

and taking into account the Poincaré commutation relations (B1), we get

$$\delta \Gamma^{\alpha\beta} = D\beta^{\alpha\beta} \tag{4.5}$$

and

$$\delta \overset{(T)}{\Gamma}{}^{\mu} = - \overset{(T)}{\Gamma}{}^{\nu} \beta_\nu{}^{\mu} + D\epsilon^\mu, \tag{4.6}$$

with D standing for the ordinary exterior covariant derivative. From Eq. (4.6), due to the presence of the inhomogeneous term, we clearly see that the translational connection cannot be identified with the tetrad, which should transform as a covector. Obviously, ordinary principal fiber bundles $P(M,G)$ do not reflect the internal structure of PGT. Indeed, neither the geometrical meaning of $\overset{(T)}{\Gamma}{}^{\mu}$ as a vierbein, nor the universal coupling of gravity to the remaining forces receives an explanation in the standard bundle approach. We know such problems to be absent from the frame bundle treatment of gravity, where tetrads are sections rather than connections. The price one has to pay when adhering to this view is that one must accept gravitational potentials of two kinds, the tetrad potentials (sections) being different in mathematical nature from those of the other interactions (connections), such tetrads having nothing to do with the translations included in the Poincaré group. So, if we still want to supply a fiber bundle interpretation of gauge theories of spacetime groups, and in particular of PGT, with the tetrad as the translational gauge potential, then either we have to introduce auxiliary fields restoring the right transformation law expected for a tetrad [18], or we have to look for a modified bundle structure. In the spirit of the latter view, our proposal consists in introducing a composite fibered space.

V. DEFINING MAPS OF COMPOSITE BUNDLES

As discussed in Sec. III, for gravitational gauge theories Lord proposed a fiber bundle structure $G(G/H,H)$ with the base space G/H, taken to be the parameter space of translations, playing the role of spacetime. We criticized this approach, pointing out that no true connections can be ascribed to the translations in the absence of an additional base space allowing one to build up the bundle for translations themselves. In fact, only with reference to such an underlying base manifold would it become possible to treat translations as local symmetries. Accordingly, we claim the necessity of introduing a base space M in addition to the group manifold G, as in ordinary fiber bundles.

However, we saw in the previous section that, if one merely constructs the principal bundle $P(M,G)$, with G for instance the Poincaré group, then we have translational connections instead of tetrads. This makes it difficult to find a geometrical interpretation of the resulting formalism. Certainly, one can restore the right tetrad transformation rules by means of auxiliary fields [18]. More satisfactorily, nonlinear realizations provide a deductive way to obtain tetrads as nonlinear translational connections [19]. Here we will derive tetrads of the nonlinear type from a bundle structure, proposed by us as the general framework underlying gauge theories, among them those of spacetime groups.

Roughly speaking, our leading idea is that of attaching to each point of the base space a fiber with the bundle structure $G(G/H,H)$. We do it by bending each fiber of $P(M,G)$, diffeomorphic to the structure group G, as $G(G/H,H)$. Accordingly, $P(M,G)$ becomes locally isomorphic to $M\times G(G/H,H)$, which is locally homeomorphic to $M\times G/H\times H$. Thus we are interested in describing a bundle whose fibers are locally isomorphic to $\Sigma\times H$, with $\Sigma\simeq M\times G/H$ (locally). We expect, in this way, the bundle structure to become split into two sectors, both with fibered structure, namely, $P\rightarrow\Sigma$ and $\Sigma\rightarrow M$, respectively. The manifold $\Sigma\simeq M\times G/H$, called from now on the plateau, plays an intermediary role. On the one hand, it is the base space of a bundle $P(\Sigma,H)$ with typical fiber H; on the other hand, it possesses a fibered structure itself, let us say $\Sigma(M,G/H)$. Spaces of the kind discussed here are found in the literature as composite fibered spaces [28]. In our proposal, physical spacetime results from the pullback to M of quantities defined on the plateau Σ, as we will discuss later.

Let us be more explicit. Instead of the ordinary bundle structure $P(M,G)$, locally isomorphic to $M\times G$, with projection $\pi_{PM}: P\rightarrow M$, we will consider a composite fibered structure

$$\pi_{\Sigma M}\circ\pi_{P\Sigma}: P\rightarrow\Sigma\rightarrow M, \tag{5.1}$$

with the partial projections

$$\pi_{P\Sigma}: P\rightarrow\Sigma, \quad \pi_{\Sigma M}: \Sigma\rightarrow M, \tag{5.2}$$

respectively [28], whose composition gives rise to the total bundle projection

$$\pi_{PM} = \pi_{\Sigma M}\circ\pi_{P\Sigma}. \tag{5.3}$$

Partial fibers $\pi^{-1}_{\Sigma M}(x)$ of the plateau $\Sigma \rightarrow M$ are parametrized as (x,ξ), with base space coordinates $x\in M$ and group manifold coordinates ξ corresponding to G/H. The fibered manifold $P\rightarrow\Sigma$ is supposed to be a bundle; the fiber branches $\pi^{-1}_{P\Sigma}(x,\xi)$ of $P\rightarrow\Sigma$, attached to points of the plateau Σ, locally isomorphic to $M\times G/H$, are parametrized as (x,ξ,a), with a the coordinates of elements of the subgroup $H\subset G$.

Associated with the projections $\pi_{P\Sigma}$ and $\pi_{\Sigma M}$ in Eq. (5.2), one introduces the corresponding local sections defined as maps:

$$s_{M\Sigma}:U\subset M\rightarrow \pi^{-1}_{\Sigma M}(U)\subset\Sigma \tag{5.4}$$

and

$$s_{\Sigma P}:V\subset \Sigma\rightarrow \pi^{-1}_{P\Sigma}(V)\subset P, \tag{5.5}$$

respectively (with $U\subset M$ and $V\subset\Sigma$ trivializing neighborhoods), such that $\pi_{\Sigma M}\circ s_{M\Sigma}=(id)_M$ and $\pi_{P\Sigma}\circ s_{\Sigma P}=(id)_\Sigma$. We suppose the decomposition

$$s_{MP}=s_{\Sigma P}\circ s_{M\Sigma} \tag{5.6}$$

to hold, s_{MP} being a section of the composite manifold P; see the corresponding theorem in Ref. [28]. Sections (5.4), (5.5) together with projections (5.2) define the structure of the composite fibered space.

As we know [compare Eq. (2.4)], in an ordinary bundle $P(M,G)$, given a section $s_{MP}(x)\in\pi^{-1}_{PM}(x)$ with local trivialization $(x,\tilde{g})$, the decomposition

$$s_{MP}(x)=\sigma_{MP}(x)\cdot\tilde{g}=R_{\tilde{g}}\circ\sigma_{MP}(x),\quad \tilde{g}\in G, \tag{5.7}$$

is always possible in terms of the zero sections $\sigma_{MP}(x)$ locally trivializing as (x,e_G). In the composite fibered space, we proceed in the same way, decomposing, on the one hand, $s_{M\Sigma}(x)\in\pi^{-1}_{\Sigma M}(x)$ from Eq. (5.4), with local coordinates (x,ξ), as

$$s_{M\Sigma}(x)=\sigma_{M\Sigma}(x)\cdot b=R_b\circ\sigma_{M\Sigma}(x),\quad b\in G/H, \tag{5.8}$$

$b=b(\xi)\in G/H$ having the parameters ξ as its group manifold coordinates, and $\sigma_{M\Sigma}(x)$ being the zero section with local trivialization $(x,e_{G/H})$. Analogously, $s_{\Sigma P}(x,\xi)\in\pi^{-1}_{P\Sigma}(x,\xi)$ from Eq. (5.5) with coordinates (x,ξ,a) also can be decomposed as

$$s_{\Sigma P}(x,\xi)=\sigma_{\Sigma P}(x,\xi)\cdot a=R_a\circ\sigma_{\Sigma P}(x,\xi),$$
$$a\in H, \tag{5.9}$$

with the zero section $\sigma_{\Sigma P}(x,\xi)$ locally trivializing as (x,ξ,e_H). We require the sections to be related in such a way that their images coincide, that is, $s_{\Sigma P}(x,\xi)=s_{MP}(x)$, and in parallel to Eq. (5.6), we also demand the zero sections to satisfy

$$\sigma_{MP}=\sigma_{\Sigma P}\circ\sigma_{M\Sigma}. \tag{5.10}$$

This holds if $\tilde{g}=b\cdot a$ and $R_{b^{-1}}\circ\sigma_{\Sigma P}\circ R_b=\sigma_{\Sigma P}$.

Now we proceed as follows. By identifying the argument of $\sigma_{\Sigma P}(x,\xi)$ in Eq. (5.9) with the coordinatization $(x,\ \xi)$ of $s_{M\Sigma}(x)$ in Eq. (5.8), we introduce

$$\sigma_\xi(x):=\sigma_{\Sigma P}\circ s_{M\Sigma}(x)=\sigma_{\Sigma P}(x,\xi). \tag{5.11}$$

Equation (5.11) establishes the coincidence of the images of two different sections, namely, $\sigma_\xi:M\rightarrow P$ and $\sigma_{\Sigma P}:\Sigma\rightarrow P$, respectively. In the following, we will exploit the formal correspondence (5.11) using $\sigma_\xi(x)$ for $\sigma_{\Sigma P}(x,\xi)$ as a convenient notation, only distinguishing the two maps from each other when strictly necessary. On the other hand, making use of the previous assumptions, from the second equality in (5.11), with Eqs. (5.8) and (5.10), we find

$$\sigma_{\Sigma P}(x,\xi)=\sigma_{MP}(x)\cdot b,\quad b=b(\xi)\in G/H, \tag{5.12}$$

an equation which will be useful later. The relevance of the composite bundle structure for spacetime gauge theories becomes apparent when we study the particular form of the action of gauge transformations on it.

VI. GAUGE TRANSFORMATIONS IN COMPOSITE BUNDLES

The picture of a composite bundle, as resulting from the definitions of Sec. V, involves a bundle sector $P\rightarrow\Sigma$ with H-diffeomorphic fibers. Considering the particular ones $\pi^{-1}_{P\Sigma}(x,\xi)$ and $\pi^{-1}_{P\Sigma}(x,\xi')$, it is relevant to notice that they can be seen either as fibers attached to different points of the plateau Σ, or alternatively as, say, branches of a single total fiber $\pi^{-1}_{PM}(x)$ over $x\in M$. Indeed, such total fibers on $P(M,G)$ consist of the H branches together with a second portion identical with the homogeneous space G/H contained in the plateau $\Sigma\simeq M\times G/H$.

Accordingly, gauge transformations present two aspects. In the first place, when regarded as acting on the total bundle, they are defined satisfying Eq. (3.1) as much as Eq. (3.2) with respect to the base space M. On the other hand, when seen as H-branch transformations in the bundle sector $P\rightarrow\Sigma$, Eq. (3.2) does not hold with respect to the intermediate base space Σ, since the latter's ξ coordinates are affected. (That is, a transformation is induced on Σ.) So the maintenance of both defining conditions (3.1) and (3.2) of standard gauge transformations is compatible with Lord's relaxation of Eq. (3.2) in the sector $P\rightarrow\Sigma$. Briefly, in our scheme, spacetime gauge transformations affect the ξ coordinates of the homogeneous space G/H included in Σ, while $x\in M$ remains unchanged as in ordinary gauge theories of internal groups.

Now we will study the effect of a gauge transformation on a composite fiber bundle making use of a decomposition of total sections analogous to Eq. (2.4). Invoking the previously postulated equality $s_{\Sigma P}(x,\xi)=s_{MP}(x)$ with Eq. (5.9), we write any arbitrary element on a total section of P as

$$u=s_{MP}(x)=R_a\circ\sigma_{\Sigma P}(x,\xi). \tag{6.1}$$

From Sec. III, we know that a gauge transformation λ satisfying Eqs. (3.1) and (3.2) can be identified with the gauged left action $L_{g(x)}$ of elements $g(x)\in G$ depending on base space coordinates $x\in M$. Since Eq. (6.1) is the general form of points on a total section, we represent the corresponding gauge transformed ones as

$$\lambda(u)=L_g u=R_{a'}\circ\sigma_{\Sigma P}(x,\xi'). \tag{6.2}$$

Comparing Eq. (6.1) with Eq. (6.2), we get

$$L_g\circ R_a\circ\sigma_{\Sigma P}(x,\xi)=R_{a'}\circ\sigma_{\Sigma P}(x,\xi'). \tag{6.3}$$

Since left and right translations commute, one only has to join the action of H elements in Eq. (6.3) as

$$R_{a^{-1}}\circ R_{a'}=R_h,\quad h:=a'a^{-1}, \tag{6.4}$$

in order to finally bring Eq. (6.3) into the form

$$L_g\circ\sigma_{\Sigma P}(x,\xi)=R_h\circ\sigma_{\Sigma P}(x,\xi'). \tag{6.5}$$

Equation (6.5) fixes the gauge action of $\lambda=L_g$ on sections $\sigma_{\Sigma P}(x,\xi)$, transforming them into sections placed at different points (x,ξ') of the intermediate base space Σ, being simultaneously "vertically" displaced along the H fiber branches by means of R_h.

Comparison of Eq. (6.5) with the definition of nonlinear transformations [20,25] shows their identity. So our derivation from the composite bundle structure provides a bundle interpretation of such transformations. It coincides with that of Lord and Goswami [25], as far as the bundle sector $P\rightarrow\Sigma$ is concerned ($P\rightarrow G/H$ in their view); however, in the approach proposed here, gauge transformations are at the same time standard ones, obeying Eqs. (3.1) and (3.2) in the framework of the composite bundle considered as a whole. Actually, when referred to $P\rightarrow M$, gauge transformations are vertical bundle automorphisms not affecting the base space M. Expressed more formally, the validity of Eq. (3.2) for the total projection π_{PM} means $\pi_{PM}\circ L_g=\pi_{PM}$. In particular, $\pi_{PM}\circ L_g\circ\sigma_{\Sigma P}(x,\xi)=\pi_{PM}\circ\sigma_{\Sigma P}(x,\xi)=x$, easily checkable with the help of Eq. (5.3). For the projection $\pi_{P\Sigma}$ of the sector $P\rightarrow\Sigma$ instead, we find by applying it to Eq. (6.5), $\pi_{P\Sigma}\circ L_g\circ\sigma_{\Sigma P}(x,\xi)=\pi_{P\Sigma}\circ R_h\circ\sigma_{\Sigma P}(x,\xi')=(x,\xi')$, whereas $\pi_{P\Sigma}\circ\sigma_{\Sigma P}(x,\xi)=(x,\xi)$, so that $\pi_{P\Sigma}\circ L_g\neq\pi_{P\Sigma}$. In this case, a gauge transformation is induced on the plateau, allowing one to transform H fibers into different H fibers (that is, permitting the kind of transformation one expects from spacetime groups in which translations are present). Equation (9.14) below significantly shows the explicit form of the spacetime gauge transformations induced on Σ by the Poincaré group.

Notice that, in view of Eq. (5.11), one could replace Eq. (6.5) by

$$L_g\circ\sigma_\xi(x)=R_h\circ\sigma_{\xi'}(x), \tag{6.6}$$

an expression which, strictly speaking, is not equivalent to Eq. (6.5), since it refers to the base space M instead of to Σ, as discussed above. However, the formal analogy to Eq. (6.5) makes Eq. (6.6) useful for the following considerations, mainly since, finally, we will be interested in the pullback of all expressions to the base space M.

VII. CONNECTIONS AND THEIR GAUGE TRANSFORMATIONS IN COMPOSITE BUNDLES

In terms of the zero sections introduced in Eqs. (5.7) and (5.9), respectively, one can pull back the Ehresmann connection either to the base space M or to the plateau Σ. Let us use the notation

$$A_M=\sigma^*_{MP}\omega\in T^*(M) \tag{7.1}$$

and

$$\Gamma:=A_\Sigma=\sigma^*_{\Sigma P}\omega\in T^*(\Sigma). \tag{7.2}$$

The former is identical with Eq. (2.6), that is, with the ordinary local connection on a principal bundle $P(M,G)$, whereas Eq. (7.2) is a local form on the intermediate base space Σ, regarding the bundle sector $P\rightarrow\Sigma$. Taking Eq. (5.10) into account, one can alternatively consider the σ^*_{MP} pullback (7.1) of ω as

$$A_M:=\sigma^*_{M\Sigma}\sigma^*_{\Sigma P}\omega=\sigma^*_{M\Sigma}\Gamma, \tag{7.3}$$

that is, as the $\sigma^*_{M\Sigma}$ pullback of Eq. (7.2) to $T^*(M)$.

The gauge transformations of Eq. (7.2) are deduced following steps formally analogous to those of ordinary bundles, as exposed in Sec. III. (Here we will deduce them pulled back to M.) Let us depart from Eq. (6.6) rewritten as

$$\lambda(\sigma_\xi(x))=\sigma_{\xi'}(x)\cdot h, \tag{7.4}$$

with $\lambda=L_g$. In view of the analogy between Eqs. (7.4) and (3.4), it is straightforward to find

$$\lambda_* Y_{\sigma_\xi}=R_{h*}Y_{\sigma_{\xi'}}+[Y_{\sigma_{\xi'}}\rfloor(h^{-1}\wedge dh)]^\#, \tag{7.5}$$

in parallel to Eq. (3.5). Observe, however, that the Y vectors in each member of Eq. (7.5) are evaluated at different points of the plateau, namely, $\sigma_\xi(x)$ and $\sigma_{\xi'}(x)$, respectively. Making use of Eq. (6.6), we write

$$\sigma_{\xi'}(x)=R_h^{-1}\circ L_g\circ\sigma_\xi(x)=:\Phi\circ\sigma_\xi(x). \tag{7.6}$$

Deriving Eq. (7.6) we find

$$\frac{d\sigma_{\xi'}}{dt}=\frac{\partial\Phi}{\partial\sigma_\xi}\frac{d\sigma_\xi}{dt}, \tag{7.7}$$

so that

$$Y_{\sigma_{\xi'}}=\Phi_* Y_{\sigma_\xi}. \tag{7.8}$$

Replacing Eq. (7.8) in Eq. (7.5), and introducing a connection form as in Eqs. (3.6), (3.7), we finally obtain

$$\lambda^*\omega=\Phi^*(R_h^*\omega+h^*\Theta)=\Phi^*[h^{-1}(d+\omega)h]. \tag{7.9}$$

That is the formula analogous to Eq. (3.8). Notice that, instead of $\eta(u)$, the element $h\in H\subset G$ defined in Eq. (6.4) appears in Eq. (7.9). Proceeding as in Eq. (3.9), we get

$$\sigma_\xi^*(\lambda^*\omega)=\sigma_\xi^*\Phi^*(R_h^*\omega+h^*\Theta). \tag{7.10}$$

In view of Eq. (7.6), we recognize in Eq. (7.10)

$$\sigma_\xi^*\Phi^*=(\Phi\circ\sigma_\xi)^*=\sigma_{\xi'}^*. \tag{7.11}$$

Thus, Eq. (7.10) transforms into

$$\begin{aligned}\sigma_\xi^*(L_g^*\omega)&=R_{h_{\xi'}}^*(\sigma_{\xi'}^*\omega)+h_{\xi'}^*\Theta\\&=h_{\xi'}^{-1}(d+\sigma_{\xi'}^*\omega)h_{\xi'},\end{aligned} \tag{7.12}$$

with

$$\begin{aligned}h_{\xi'}&:=h\circ\sigma_{\xi'}(x)=h\circ\sigma_{\Sigma P}(x,\xi')\\&=(\sigma_{\Sigma P}^*h)(x,\xi').\end{aligned} \tag{7.13}$$

The infinitesimal form of Eq. (7.12) is easily found. Let us take $h_{\xi'}=e^{\mu^A(\xi')H_A}\approx I+\mu^A(\xi)H_A=:I+\mu$, with H_A the generators of the subgroup $H\subset G$. Then we find

$$\delta\Gamma:=\sigma_{\xi'}^*\omega-(L_g\circ\sigma_\xi)^*\omega\approx-(d\mu+[\Gamma,\mu]); \tag{7.14}$$

compare with Eq. (3.13). Notice the crucial fact that, in contrast to ϵ in Eq. (3.13), defined on the Lie algebra of G, μ in Eq. (7.14) is defined on the Lie algebra of the subgroup $H\subset G$, as is characteristic for nonlinear realizations [19–21].

VIII. TENSORIAL FORMS AND MATTER FIELDS

Let us complete the result (7.14) with the deduction of the corresponding gauge transformations of matter fields. Given the principal fiber bundle $P(M,G)$, we suppose a vector space V to exist, where a representation ρ of the left action of G is established. We define [6] pseudotensorial forms of degree p on P of type (ρ,V) to be V-valued p forms φ on P such that they satisfy the equivariance condition

$$R_g^*\varphi=\rho(g^{-1})\varphi. \tag{8.1}$$

[Notice that, in view of Eq. (2.3), the connection form is a pseudotensorial one-form of type $(ad,\mathcal{G})$, with ρ the adjoint representation of G, in V the Lie algebra $\mathcal{G}$.] Tensorial forms are pseudotensorial forms which in addition are required to be horizontal, being horizontal forms φ^H defined by the condition

$$X_1\rfloor\cdots X_p\rfloor\varphi^H=X_1^H\rfloor\cdots X_p^H\rfloor\varphi,\quad X_i\in T(P), \tag{8.2}$$

where X_i^H are horizontal vectors. It follows that $X_1\rfloor\cdots X_p\rfloor\varphi^H=0$ if and only if any one of the vectors X_i is vertical. The space of equivariant V-valued forms is isomorphic [8,10] to the space of sections of the associated bundle with fiber V, usually taken to represent physical fields. So we will deal with equivariant forms instead of with the more cumbersome associated fiber bundles.

We are interested in deducing the gauge transformations of tensorial fields in composite bundles. In order to do so, let us put Eqs. (7.5) and (7.8) together into the complete expression

$$L_{g*}Y_{\sigma_\xi}=R_{h*}\Phi_*Y_{\sigma_\xi}+[\Phi_*Y_{\sigma_\xi}\rfloor(h^{-1}\wedge dh)]^\#. \tag{8.3}$$

The last term in Eq. (8.3) is a fundamental vector, so it is purely vertical. Therefore, it vanishes when applied to a tensorial form φ, the latter being horizontal by definition, so that

$$L_{g*}Y_{\sigma_\xi}\rfloor\varphi=R_{h*}\Phi_*Y_{\sigma_\xi}\rfloor\varphi. \tag{8.4}$$

From Eq. (8.4) we get

$$Y_{\sigma_\xi}\rfloor L_g^*\varphi=Y_{\sigma_\xi}\rfloor\Phi^*R_h^*\varphi, \tag{8.5}$$

a relation which holds for arbitrary vectors Y_{σ_ξ}, so that, taking into account the equivariance condition (8.1), one concludes that

$$L_g^*\varphi=\Phi^*\rho(h^{-1})\varphi. \tag{8.6}$$

Equation (8.6) displays the gauge transformation of a tensorial form. The pullback by σ_ξ^* yields

$$\sigma_\xi^*L_g^*\varphi=\sigma_\xi^*\Phi^*\rho(h^{-1})\varphi, \tag{8.7}$$

or equivalently

$$(L_g\circ\sigma_\xi)^*\varphi=\rho(h_{\xi'}^{-1})\sigma_{\xi'}^*\varphi, \tag{8.8}$$

with $h_{\xi'}$ given by Eq. (7.13). In terms of μ as used in Eq. (7.14), we find the infinitesimal variations of local tensorial forms to be

$$\delta\sigma_\xi^*\varphi:=\sigma_{\xi'}^*\varphi-(L_g\circ\sigma_\xi)^*\varphi\approx\rho(\mu)\sigma_\xi^*\varphi. \tag{8.9}$$

In particular, for φ the zero-forms ψ, one has

$$(\sigma_\xi^*\psi)(x)=\psi(\sigma_\xi(x))=\psi(\sigma(x,\xi)). \tag{8.10}$$

These (equivariant, horizontal) fields will play the role of matter fields. Substituting Eq. (8.10) into Eq. (8.9), we find

$$\delta\psi(\sigma(x,\xi))\approx\rho(\mu)\psi(\sigma(x,\xi)), \tag{8.11}$$

showing that they transform as representation fields of the subgroup H under the action of L_g, with $g\in G$. That is a well known feature of nonlinear realizations [20], independently deduced here from the composite bundle structure.

Covariant differentials

Given an equivariant p-form φ on P, one defines [10,12] its exterior covariant derivative $D\varphi$ to be a $(p+1)$-form defined by $D\varphi:=(d\varphi)^H$, that is, by the condition

$$X_1\rfloor\cdots X_p\rfloor D\varphi := X_1\rfloor\cdots X_p\rfloor (d\varphi)^H = X_1^H\rfloor\cdots X_p^H\rfloor d\varphi, \quad X_i\in T(P); \tag{8.12}$$

compare with Eq. (8.2). This definition does not require φ to be horizontal, so it is applicable to the (equivariant) connection form ω, giving rise to the curvature form $\mathcal{F}:=D\omega$. Cartan's structure equation establishes it to read explicitly

$$\mathcal{F}:=D\omega = d\omega + \omega\wedge\omega. \tag{8.13}$$

Although constructed with the pseudotensorial form ω, the curvature $\mathcal{F}$ is tensorial. Its pullback yields the local curvature, or field strength two-form

$$F:=\sigma_{\Sigma P}^{*}\mathcal{F} = d\Gamma + \Gamma\wedge\Gamma. \tag{8.14}$$

In view of Eq. (7.14), it transforms as

$$\delta F = [\mu, F]. \tag{8.15}$$

[We invoke Eq. (5.11) as a guarantee of the formal analogy between $\sigma_{\Sigma P}^{*}$ and σ_{ξ}^{*} pullbacks. See also Eq. (9.22) below]. On the other hand, the structure equation for tensorial forms [12] provides an explicit form for $D\varphi$ as

$$D\varphi = d\varphi + \rho(\omega)\wedge\varphi, \tag{8.16}$$

where the same notation ρ is used for the representation of $\mathcal{G}$ as for g in Eq. (8.1). The exterior covariant derivative of a tensorial form is trivially horizontal, and easily checkable to be equivariant, so it is also a tensorial form. The pullback of Eq. (8.16) by σ_ξ reads

$$\sigma_\xi^{*}D\varphi = d(\sigma_\xi^{*}\varphi) + \rho(\sigma_\xi^{*}\omega)\wedge(\sigma_\xi^{*}\varphi). \tag{8.17}$$

In particular for zero-forms [see Eq. (8.10)], we rewrite Eq. (8.17) as

$$D\psi = d\psi + \rho(\Gamma)\psi. \tag{8.18}$$

Making use of Eqs. (7.14) and (8.11), it is easy to calculate the gauge transformations of Eq. (8.18), namely,

$$\delta D\psi = \rho(\mu)D\psi, \tag{8.19}$$

analogous to Eq. (8.11).

IX. COMPOSITE BUNDLE APPROACH TO THE POINCARÉ GAUGE THEORY

In ordinary principal fiber bundles $P\rightarrow M$, verticality is determined by the fibers, while it is the connection form that defines horizontality. Composite fiber bundles require a revision of these concepts, due to the fibers to be bent, so that two sectors exist, each one with its own vertical and horizontal tangent subspaces. Indeed, given the composite fibred space $P\rightarrow\Sigma\rightarrow M$, in the sector $P\rightarrow\Sigma$ locally isomorphic to $\Sigma\times H$, the vertical subspace of the tangent space is defined along the H branch of fibers; analogously, in the plateau sector $\Sigma\rightarrow M$ locally isomorphic to $M\times G/H$, verticality is defined by G/H fibers. Then, in order to characterize horizontality in the corresponding tangent spaces, a connection is required in each. Consequently, in a composite bundle one has to look for a modified connection form, decomposed into two parts, defining horizontality in $P\rightarrow\Sigma$ and $\Sigma\rightarrow M$, respectively.

Let us abandon the general abstract exposition and concentrate on the case of G as the Poincaré group, with H the Lorentz group; thus the group generators associated with G/H are those of translations. Accordingly, in Eqs. (5.8), (5.9) we take a to be elements of the Lorentz group and b to be translation group elements, parametrized as $a=e^{i\lambda^{\alpha\beta}\Lambda_{\alpha\beta}}$ and $b=e^{-i\xi^{\mu}P_{\mu}}$, respectively. So the general Poincaré group elements $\tilde g = b\cdot a$ are exactly Eq. (B2). Let us briefly return to the bundle description of the Poincaré group disregarded by us in Sec. IV, in order to compare it in the following with the composite bundle approach. Fibers of ordinary principal bundles are diffeomorphic to the structure group G taken as a whole. They allow only one kind of verticality to be present, namely, that defined by vectors tangent to the G fibers. Accordingly, the ordinary connection form Eq. (2.5) in a $P(M,G)$ bundle, that is,

$$\overset{ord}{\omega} = \tilde g^{-1}(d + \pi_{PM}^{*}A)\tilde g,$$
$$A := -i\,dx^{i}(\overset{(T)}{\Gamma}{}_{i}{}^{\mu}P_{\mu} + \Gamma_{i}^{\alpha\beta}\Lambda_{\alpha\beta}), \tag{9.1}$$

with $\tilde g$ as the Poincaré group elements (B2), defines a single horizontal subspace $H(P)$ of the tangent space. By making use of Eqs. (B1)–(B4) and (B8), (B9), we find the explicit form of $\overset{ord}{\omega}$ to be

$$\overset{ord}{\omega} = -i\vartheta_{ord}^{\mu}u_{\mu}{}^{\nu}P_{\nu} - i(i\overline{\Theta}_{(\Lambda)}^{\alpha\beta} + \pi_{PM}^{*}dx^{i}\Gamma_{i}^{\alpha\beta}) \times u_{\alpha}{}^{\mu}u_{\beta}{}^{\nu}\Lambda_{\mu\nu}, \tag{9.2}$$

where

$$\vartheta_{ord}^{\mu} := d\xi^{\mu} + \pi_{PM}^{*}dx^{i}(\Gamma_{i\nu}{}^{\mu}\xi^{\nu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu}). \tag{9.3}$$

The fundamental vectors $L_{\alpha\beta}^{(\Lambda)}$ and $L_{\mu}^{(P)}$ of the bundle, given by Eq. (B6), yield, respectively,

$$L_{\alpha\beta}^{(\Lambda)}\rfloor\overset{ord}{\omega} = \Lambda_{\alpha\beta}, \quad L_{\mu}^{(P)}\rfloor\overset{ord}{\omega} = P_{\mu}; \tag{9.4}$$

recall Eq. (2.2). On the other hand, one can choose the basis vectors of the horizontal tangent subspace $H(P)$ as

$$E_i := \sigma_{MP*}\partial_i + i(\overset{(T)}{\Gamma}{}_{i}{}^{\mu}\overline{L}_{\mu}^{(P)} + \Gamma_{i}^{\alpha\beta}\overline{L}_{\alpha\beta}^{(\Lambda)}) \tag{9.5}$$

[compare with Eq. (2.7)], satisfying the defining condition $E_i\rfloor\overset{ord}{\omega} = 0$.

In contrast with this standard decomposition of the tangent space into one vertical and one horizontal subspace, in composite fiber bundles we have to give a formal characterization of two different verticalities with their corresponding horizontalities. Two pieces of the connection will exist, oriented along the Lie algebra basis of G/H and H, respectively. In our case, G/H involves the translational generators P_μ, and H those $\Lambda_{\alpha\beta}$ of the Lorentz group. Let us pay attention to the connection

$$\omega = a^{-1}(d + \pi^*_{P\Sigma}\Gamma)a \in T^*(P) \tag{9.6}$$

on the plateau [compare with Eq. (2.5)], with $\Gamma = \sigma^*_{\Sigma P}\omega$ in $T^*(\Sigma)\times\mathcal{G}$ developed as

$$\Gamma = -i\,\vartheta^\mu_\Sigma P_\mu - i\Gamma^{\alpha\beta}\Lambda_{\alpha\beta}. \tag{9.7}$$

As required for composite bundles, we decompose Eq. (9.6) into two parts as $\omega = \overset{(P)}{\omega} + \overset{(\Lambda)}{\omega}$, identifying, respectively,

$$\overset{(P)}{\omega} = -ia^{-1}(\pi^*_{P\Sigma}\vartheta^\mu_\Sigma P_\mu)a = -i\,\overset{(P)}{\omega}{}^\mu P_\mu \tag{9.8}$$

as the connection form of the sector $\Sigma \to M$, and

$$\overset{(\Lambda)}{\omega} = a^{-1}(d - i\pi^*_{P\Sigma}\Gamma^{\alpha\beta}\Lambda_{\alpha\beta})a = -i\,\overset{(\Lambda)}{\omega}{}^{\alpha\beta}\Lambda_{\alpha\beta} \tag{9.9}$$

as the connection form of $P \to \Sigma$. The components of Eqs. (9.8) and (9.9) read more explicitly

$$\overset{(P)}{\omega}{}^\mu = \pi^*_{P\Sigma}\vartheta^\nu_\Sigma u_\nu{}^\mu,$$

$$\overset{(\Lambda)}{\omega}{}^{\alpha\beta} = (i\overline{\Theta}^{\gamma\delta}_{(\Lambda)} + \pi^*_{P\Sigma}\Gamma^{\gamma\delta})u_\gamma{}^\alpha u_\delta{}^\beta, \tag{9.10}$$

in terms of Eq. (B5); see Eqs. (B4) and (B9). We postpone to the next section the discussion on the inner structure in $T^*(\Sigma)$ of the translational local connection ϑ^μ_Σ, and of the Lorentz connection $\Gamma^{\alpha\beta}$.

In order for the splitting of the composite bundle into two fibered sectors to be well defined, a vertical (fundamental) vector $\widetilde{E}_\mu$ and a horizontal one $\widetilde{E}_i$ must exist in the sector $\Sigma \to M$ such that

$$\widetilde{E}_\mu \rfloor \overset{(P)}{\omega} = P_\mu, \quad \widetilde{E}_i \rfloor \overset{(P)}{\omega} = 0, \tag{9.11}$$

while in the sector $P \to \Sigma$, one must analogously have vectors such that

$$L^{(\Lambda)}_{\alpha\beta} \rfloor \overset{(\Lambda)}{\omega} = \Lambda_{\alpha\beta}, \quad \hat{E}_\mu \rfloor \overset{(\Lambda)}{\omega} = 0, \quad \hat{E}_i \rfloor \overset{(\Lambda)}{\omega} = 0, \tag{9.12}$$

with $L^{(\Lambda)}_{\alpha\beta}$ vertical and both $\hat{E}_\mu$ and $\hat{E}_i$ horizontal. The two kinds of horizontal vectors reflect the structure of the plateau Σ, locally isomorphic to the Cartesian product $M \times G/H$.

Before looking for the possible explicit realizations of these vectors, as well as of ϑ^μ_Σ and $\Gamma^{\alpha\beta}$, let us show the gauge transformations of the latter as derived from the composite bundle approach to the Poincaré group. In particular, we will see that, instead of Eq. (4.6), we obtain a gauge theoretical deduction of the correct transformation of tetrads. Transformations induced on Σ are calculated from Eq. (6.5). Taking Eq. (5.12) into account, Eq. (6.5) transforms into $g(x)\cdot\sigma_{MP}(x)\cdot b = \sigma_{MP}(x)\cdot b'\cdot h$, or equivalently

$$g(x)\cdot b = b'\cdot h. \tag{9.13}$$

We parametrize $b \in G/H$ as $b = e^{-i\xi^\mu P_\mu}$, and analogously $b' = e^{-i\xi^{\mu\prime} P_\mu}$, with $\xi^{\mu\prime} = \xi^\mu + \delta\xi^\mu$. The infinitesimal group elements $g(x)$ and h are respectively taken as Eq. (4.2) and $h = e^{i\mu^{\alpha\beta}\Lambda_{\alpha\beta}} \approx I + i\mu^{\alpha\beta}\Lambda_{\alpha\beta}$. Then, from Eq. (9.13), making use of Eq. (B1), we get

$$\delta\xi^\alpha = -\xi^\beta\beta_\beta{}^\alpha - \epsilon^\alpha, \quad \mu^{\alpha\beta} = \beta^{\alpha\beta}, \tag{9.14}$$

where we recognize the form of infinitesimal Poincaré transformations, with the particularity that, instead of coordinates, they affect the translational group parameters ξ^α. On the other hand, regarding the local connection (9.7) on Σ, in Eq. (7.14) we already deduced the corresponding gauge transformations. Directly applying this result to our case, with $\mu = i\mu^{\alpha\beta}\Lambda_{\alpha\beta}$ and $\mu^{\alpha\beta} = \beta^{\alpha\beta}$, as proved in Eq. (9.14), we find

$$\delta\Gamma^{\alpha\beta} = D\beta^{\alpha\beta} \tag{9.15}$$

and

$$\delta\vartheta^\mu_\Sigma = -\vartheta^\nu_\Sigma\beta_\nu{}^\mu; \tag{9.16}$$

compare with Eqs. (4.5), (4.6). Equation (9.16) shows that ϑ^μ_Σ transforms as a covector. This constitutes a highly relevant result. Indeed, as we will see below, it is the pullback of ϑ^μ_Σ to the base space M that will become identifiable with ordinary tetrads. In summary, as expected from a spacetime group, spacetime gauge transformations (9.14) are found to be induced on the plateau, and, furthermore, the correct gauge transformations (9.16) for tetrads (on Σ) are obtained. Thus we have established a bundle foundation for PGT.

Proposals for the explicit structure of spacetime connections

Finally, let us look for a suitable explicit form for ϑ^μ_Σ and $\Gamma^{\alpha\beta}$ in Eq. (9.7). We will consider three different possibilities. Taking Eq. (2.5) as a model, first we postulate for Eq. (9.7) the form

$$\widetilde{\Gamma} = \sigma^*_{\Sigma P}\omega = b^{-1}(d + \pi^*_{\Sigma M}A)b, \tag{9.17}$$

with $A = \sigma^*_{M\Sigma}\widetilde{\Gamma}$. (We write $\widetilde{\Gamma}$ with a tilde in order to distinguish it from the Γ we will introduce later.) Since $\widetilde{\Gamma} = \sigma^*_{\Sigma P}\omega$, making use of Eq. (5.10), we see that $A = \sigma^*_{M\Sigma}\sigma^*_{\Sigma P}\omega = \sigma^*_{MP}\omega$, and in analogy to Eq. (9.1) we take the local potentials to be

$$A = \sigma^*_{MP}\omega = -i\,dx^i(\overset{(T)}{\Gamma}{}_i{}^\mu P_\mu + \Gamma_i^{\alpha\beta}\Lambda_{\alpha\beta}). \tag{9.18}$$

Expression (9.17) then reads explicitly

$$\widetilde{\Gamma} = -i(\vartheta^{\mu}_{\Sigma}P_{\mu} + \pi^{*}_{\Sigma M}dx^{i}\Gamma_{i}^{\alpha\beta}\Lambda_{\alpha\beta}), \tag{9.19}$$

where

$$\vartheta^{\mu}_{\Sigma} := d\xi^{\mu} + \pi^{*}_{\Sigma M}dx^{i}(\Gamma_{i\nu}{}^{\mu}\xi^{\nu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu}); \tag{9.20}$$

compare Eq. (9.3). The translational parameters ξ^{μ}, being pseudocoordinate fields transforming as Eq. (9.14), may be regarded as Goldstone-like fields absorbed into the definition (9.20) of ϑ^{μ}_{Σ}. As a compatibility condition of Eqs. (9.14)–(9.16), we find

$$\delta(\pi^{*}_{\Sigma M}dx^{i}\overset{(T)}{\Gamma}{}_{i}{}^{\mu}) = -(\pi^{*}_{\Sigma M}dx^{i}\overset{(T)}{\Gamma}{}_{i}{}^{\nu})\beta_{\nu}{}^{\mu} + D\epsilon^{\mu} \tag{9.21}$$

as the infinitesimal transformation of the translational connection; compare with Eq. (4.6).

Equation (9.20), as a part of Eq. (9.17), is an object defined on the plateau Σ, so it cannot be identified as an ordinary tetrad until pulled back to the base space M. Thus we have to be careful in interpreting Eq. (5.11) rigorously, as giving rise to

$$\sigma^{*}_{\xi} = s^{*}_{M\Sigma}\sigma^{*}_{\Sigma P}, \tag{9.22}$$

that is, to a pullback in steps $\sigma^{*}_{\xi}: T^{*}(P) \to T^{*}(\Sigma) \to T^{*}(M)$, first from $T^{*}(P)$ to $T^{*}(\Sigma)$ by $\sigma^{*}_{\Sigma P}$, and then from $T^{*}(\Sigma)$ to $T^{*}(M)$ by $s^{*}_{M\Sigma}$. Having performed the first step on ω, that is, the pullback $\sigma^{*}_{\Sigma P}: T^{*}(P) \to T^{*}(\Sigma)$ [see Eq. (9.17)], now we proceed to complete the pullback $\sigma^{*}_{\xi}\omega$ to $T^{*}(M)$ as given by Eq. (9.22). Recall that we already performed such pullbacks in Secs. VII and VIII. When pulled back to M by $s^{*}_{M\Sigma}$, the plateau quantity ϑ^{μ}_{Σ} reduces to the usual tetrad as

$$\vartheta^{\mu}_{M} = s^{*}_{M\Sigma}\vartheta^{\mu}_{\Sigma} := dx^{i}e_{i}{}^{\mu}. \tag{9.23}$$

with $e_{i}{}^{\mu}$ provided with an internal structure, namely,

$$e_{i}{}^{\mu} := \partial_{i}\xi^{\mu} + \Gamma_{i\nu}{}^{\mu}\xi^{\nu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu} = D_{i}\xi^{\mu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu}, \tag{9.24}$$

expressed in terms of spacetime connections and of the Goldstone-like fields ξ^{μ}. In contrast to the usual view of tetrads as sections of a frame bundle, Eq. (9.24) supplies the necessary support of the gauge theoretical conception of tetrads as (nonlinear) translational connections [19]. For later convenience, we introduce the formal inverse of $e_{i}{}^{\mu}$ as $e_{\mu}{}^{i}$ such that $e_{\mu}{}^{i}e_{i}{}^{\nu} = \delta^{\nu}_{\mu}$ and $e_{i}{}^{\mu}e_{\mu}{}^{j} = \delta^{j}_{i}$.

In spite of the convenient features of ϑ^{μ}_{Σ}, we find the complete form (9.19) of $\widetilde{\Gamma}$ unsatisfactory due to the fact that the local Lorentz connection $dx^{i}\Gamma_{i}^{\alpha\beta}$ is defined on $T^{*}(M)$ rather than on $T^{*}(\Sigma)$, as a consequence of the particular choice (9.19) we made for $\widetilde{\Gamma}$. No reason exists to restrict $\Gamma^{\alpha\beta}$ in this way; rather one is induced to consider the general case in which, respecting the form (9.20) of ϑ^{μ}_{Σ}, the spin connection $\Gamma^{\alpha\beta}$ is defined on Σ as

$$\Gamma^{\alpha\beta} = (d\xi^{\mu}\otimes\partial_{\xi^{\mu}} + \pi^{*}_{\Sigma M}dx^{i}\otimes\sigma_{M\Sigma *}\partial_{i})\rfloor\Gamma^{\alpha\beta}. \tag{9.25}$$

Taking Eq. (9.25) into account, we begin looking for the explicit form of the vertical and horizontal vectors (9.11), (9.12) in both sectors of the composite bundle. In the first place, the conditions (9.11) of the bundle sector $\Sigma \to M$ are satisfied by

$$\widetilde{E}_{\mu} := \sigma_{\Sigma P *}L^{(P)}_{\mu}, \quad \widetilde{E}_{i} = \sigma_{\Sigma P *}\widetilde{e}_{i}, \tag{9.26}$$

with

$$\begin{aligned}\widetilde{e}_{i} &:= \sigma_{M\Sigma *}\partial_{i} + i(\overset{(T)}{\Gamma}{}_{i}{}^{\mu}\overline{L}^{(P)}_{\mu} + \Gamma_{i}^{\alpha\beta}\overline{L}^{(Orb)}_{\alpha\beta})\\ &= \sigma_{M\Sigma *}\partial_{i} - (\Gamma_{i\nu}{}^{\mu}\xi^{\nu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu})\partial_{\xi^{\mu}}.\end{aligned} \tag{9.27}$$

(Only the orbital part of $\overline{L}^{(\Lambda)}_{\alpha\beta}$ appears in $\widetilde{e}_{i}$ [see Eq. (B11)], since the intrinsic part is not defined on Σ.) The vector (9.27) is such that

$$\widetilde{e}_{i}\rfloor\vartheta^{\mu}_{\Sigma} = 0, \tag{9.28}$$

showing that the translational component (9.20) of Eq. (9.7) behaves as a sort of connection form on $\Sigma \to M$ (although pulled back from the whole bundle), with $\widetilde{e}_{i}$ as the corresponding horizontal vector. On the other hand, the conditions (9.12) of the sector $P \to \Sigma$ are satisfied, respectively, by $L^{(\Lambda)}_{\alpha\beta}$ in Eq. (B6) and by

$$\hat{E}_{\mu} = \sigma_{\Sigma P *}\partial_{\xi^{\mu}} + i(\partial_{\xi^{\mu}}\rfloor\Gamma^{\alpha\beta})\overline{L}^{(Int)}_{\alpha\beta} \tag{9.29}$$

and

$$\hat{E}_{i} = \sigma_{MP *}\partial_{i} + i(\sigma_{M\Sigma *}\partial_{i}\rfloor\Gamma^{\alpha\beta})\overline{L}^{(Int)}_{\alpha\beta}. \tag{9.30}$$

We denoted the intrinsic part of $\overline{L}_{\alpha\beta}$ as in Eq. (B11). Observe that the vector defined as the combination

$$\begin{aligned}\hat{\hat{E}}_{i} &:= \hat{E}_{i} - (\Gamma_{i\nu}{}^{\mu}\xi^{\nu} + \overset{(T)}{\Gamma}{}_{i}{}^{\mu})\hat{E}_{\mu}\\ &= \sigma_{\Sigma P *}\widetilde{e}_{i} + i(\widetilde{e}_{i}\rfloor\Gamma^{\alpha\beta})\overline{L}^{(Int)}_{\alpha\beta}\end{aligned} \tag{9.31}$$

satisfies both $\hat{\hat{E}}_{i}\rfloor\overset{(P)}{\omega} = 0$ and $\hat{\hat{E}}_{i}\rfloor\overset{(\Lambda)}{\omega} = 0$, so that it is horizontal in the total bundle $P(M,G)$, in analogy to Eq. (9.5) of the ordinary case.

Let us now return to Eq. (9.25). We look for a convenient notation, making use of the identity

$$\begin{aligned}&d\xi^{\mu}\otimes\partial_{\xi^{\mu}} + \pi_{\Sigma M}dx^{i}\otimes\sigma_{M\Sigma *}\partial_{i}\\ &\quad\equiv \vartheta^{\mu}_{\Sigma}\otimes\widetilde{e}_{\mu} + (\pi^{*}_{\Sigma M}dx^{i} - \vartheta^{\mu}_{\Sigma}e_{\mu}{}^{i})\otimes\widetilde{e}_{i},\end{aligned} \tag{9.32}$$

with the right-hand side (RHS) expressed in terms of the new vector

$$\widetilde{e}_{\mu} := \partial_{\xi^{\mu}} + e_{\mu}{}^{i}\widetilde{e}_{i} = e_{\mu}{}^{i}s_{M\Sigma *}\partial_{i} \tag{9.33}$$

such that

$$\widetilde{e}_{\mu}\rfloor\vartheta^{\nu}_{\Sigma} = \delta^{\nu}_{\mu}; \tag{9.34}$$

compare with Eq. (9.28). The vector (9.33) possesses suitable transformation properties, related to those of ϑ^{μ}_{Σ}; see Eq. (9.16). Actually

$$\delta\tilde{e}_{\mu}=\beta_{\mu}{}^{\nu}\tilde{e}_{\nu}+e_{\mu}{}^{i}(\partial_i\delta\xi^{\lambda}-\partial_i\xi^{\nu}\partial_{\xi^{\nu}}\delta\xi^{\lambda})\partial_{\xi^{\lambda}}, \quad (9.35)$$

so that, in view of Eq. (9.14), for $\epsilon^{\alpha}=\epsilon^{\alpha}(\xi(x))$, $\beta^{\alpha\beta}=\beta^{\alpha\beta}(\xi(x))$, it reduces to

$$\delta\tilde{e}_{\mu}=\beta_{\mu}{}^{\nu}\tilde{e}_{\nu}. \quad (9.36)$$

With the help of $\tilde{e}_{\mu}$, in parallel to Eq. (9.31) we introduce

$$\hat{\hat{E}}_{\mu}:=\hat{E}_{\mu}+e_{\mu}{}^{i}\hat{\hat{E}}_{i}=\sigma_{\Sigma P*}\tilde{e}_{\mu}+i(\tilde{e}_{\mu}\rfloor\Gamma^{\alpha\beta})\overline{L^{(Int)}_{\alpha\beta}}. \quad (9.37)$$

The two vectors (9.31), (9.37) may be taken instead of (9.29), (9.30) as the basis of the horizontal subspace of the bundle sector $P\rightarrow\Sigma$. Making use now of Eq. (9.32) and defining

$$\Gamma_{\mu}{}^{\alpha\beta}:=\tilde{e}_{\mu}\rfloor\Gamma^{\alpha\beta}, \quad (9.38)$$

we rewrite Eq. (9.25) as

$$\Gamma^{\alpha\beta}=\vartheta^{\mu}_{\Sigma}\Gamma_{\mu}{}^{\alpha\beta}+(\pi^{*}_{\Sigma M}dx^{i}-\vartheta^{\mu}_{\Sigma}e_{\mu}{}^{i})(\tilde{e}_{i}\rfloor\Gamma^{\alpha\beta}). \quad (9.39)$$

We emphasize that, in the pullback of $\Gamma^{\alpha\beta}$ by $s^{*}_{M\Sigma}$ analogous to Eq. (9.23), only the first term in the RHS of Eq. (9.39) gives a nonzero contribution, namely,

$$s^{*}_{M\Sigma}\Gamma^{\alpha\beta}=s^{*}_{M\Sigma}\vartheta^{\mu}_{\Sigma}\Gamma_{\mu}{}^{\alpha\beta}, \quad (9.40)$$

so that $\tilde{e}_{i}\rfloor\Gamma^{\alpha\beta}$ does not play any role on M as a base-space pulled-back object. Moreover, since the pullback of ϑ^{μ}_{Σ} consists of the ordinary tetrads on M [see Eq. (9.23)], then the same pullback (9.40) reduces $\Gamma^{\alpha\beta}$ to

$$s^{*}_{M\Sigma}\Gamma^{\alpha\beta}=dx^{i}e_{i}{}^{\mu}\Gamma_{\mu}{}^{\alpha\beta}, \quad (9.41)$$

which equals $dx^{i}\Gamma_{i}{}^{\alpha\beta}$, as in Eq. (9.19), since

$$\Gamma_{i}{}^{\alpha\beta}:=(s_{M\Sigma*}\partial_i)\rfloor\Gamma^{\alpha\beta}=e_{i}{}^{\mu}\Gamma_{\mu}{}^{\alpha\beta}; \quad (9.42)$$

compare with Eqs. (9.33) and (9.38).

Having stated that $\tilde{e}_{i}\rfloor\Gamma^{\alpha\beta}$ does not contribute to the local spin connection pulled back to *M*, let us finally consider the particularly interesting case which follows from imposing the verticality of $\Gamma^{\alpha\beta}$ on $\Sigma\rightarrow M$, that is, $\tilde{e}_{i}\rfloor\Gamma^{\alpha\beta}=0$. [Recall that $\tilde{e}_{i}$ is horizontal on the plateau; compare Eq. (9.28).] As a consequence, Eq. (9.39) reduces to

$$\Gamma^{\alpha\beta}=\vartheta^{\mu}_{\Sigma}\Gamma_{\mu}{}^{\alpha\beta}, \quad (9.43)$$

showing the nonminimal coupling of the translational connection to the spin connection. Such a composition of connections is characteristic for composite fibered spaces [28]. Replacing Eq. (9.43) into Eq. (9.7), Γ becomes the vertical form

$$\Gamma=-i\,\vartheta^{\mu}_{\Sigma}(P_{\mu}+\Gamma_{\mu}{}^{\alpha\beta}\Lambda_{\alpha\beta}), \quad (9.44)$$

satisfying $\tilde{e}_{i}\rfloor\Gamma=0$. Observe that the vectors of the sector $P\rightarrow\Sigma$ are affected by the verticality condition on Γ in such a way that Eq. (9.37) retains its form while Eq. (9.31) transforms into $\hat{\hat{E}}_{i}=\tilde{E}_{i}$.

X. FINAL REMARKS

The bundle structure proposed by us provides a general framework to deal with any possible interaction defined on spacetime in the presence of gravity, with all forces, gravitation included, described in terms of connections. Indeed, the form (9.43) in which $\Gamma^{\alpha\beta}$ couples to ϑ^{μ}_{Σ} is suitable for generalization to the coupling of ϑ^{μ}_{Σ} (that is, of gravity) to the gauge potential of any force. Suppose the gauge theory of an internal group in the presence of gravity is formulable in terms of a composite bundle, with *G* as the direct product of the Poincaré group times an internal group, say that of the standard model, and *G*/*H* as the parameter space of the translations. By imposing on the local connections *A* of the internal group on Σ the verticality condition $\tilde{e}_{i}\rfloor A=0$, in view of Eq. (9.32) we get the Yang-Mills potential

$$A=\vartheta^{\mu}_{\Sigma}A_{\mu}, \quad (10.1)$$

to be added to Eq. (9.44). Equation (10.1) reflects the coupling of the internal gauge potential to the spacetime connections $\overset{(T)}{\Gamma}{}_{i}{}^{\mu}$ and $\Gamma_{i}^{\alpha\beta}$; see Eq. (9.20). Actually Eq. (10.1) shows how, in the present approach, gravity (and thus spacetime) underlies the remaining forces. Composite fiber bundles depict an interaction space generalizing the idea of spacetime, including, in addition to ϑ^{μ}_{Σ} and $\Gamma^{\alpha\beta}$, the connections (10.1) representing *radiation*. As an example, consider electrodynamics. According to the standard general relativistic view, the deflection of light by the sun is a consequence of the metric structure of the space in which radiation is immersed. In Eq. (10.1) instead, the Lorentzian components A_{μ} of the electrodynamical potential appear multiplied by the gravitational potentials (9.20), displaying the coupling of electromagnetism to gravity as the result of a composition of connections. Certainly, seen *a posteriori*, this is a quite natural outcome in a pure gauge theoretical context, where only connections are mediators of forces.

In standard gauge theories, physical fields on spacetime are the pullbacks to the base space of equivariant fields on *P*, while spacetime itself, modelized by the base space *M*, is fixed from the beginning to be a Minkowskian metric space. In the present approach instead, even the geometrical structure of physical spacetime results from the pullback to *M* of plateau objects—namely, dynamical connections—defined on the whole bundle space. [According to Eq. (9.22), any pullback can be understood as performed in steps, first to the plateau Σ and then to *M*.] In particular, it is by pulling back the translational and spin connection forms (9.20) and (9.43) from the plateau that a definite spacetime structure becomes impressed on the base space *M*. Also, Eq. (10.1), when pulled back to *M*, yields the spacetime immersed local potential $dx^{i}e_{i}{}^{\mu}A_{\mu}$ [see Eq. (9.23)], where A_{μ}, with well defined spin, appears coupled to gravitational fields $e_{i}{}^{\mu}$ given by Eq. (9.24). The scheme is completed by pulling back to *M* matter fields, as in Eq. (8.10). All these pulled back objects are to be seen as ordinary physical fields on the base space. In particular, the pullbacks (9.23) and (9.41) represent, respectively, the usual tetrads and spin connections of PGT on *M*. Thus, we recognize such pullbacks—of quantities whose gauge transformations we know how to calculate—as the familiar building blocks commonly used to construct, in the standard way, gauge invariant physical actions.

Observe that, prior to the pullback, *M* is a structureless manifold. One can regard it as a screen waiting for the shadows of Plato's cavern or, in a different Platonic image, as a sort of *amorphous prime matter* to be stamped by certain bundle objects, mainly by connections. Accordingly, the concept of interaction (associated with that of connections) manifests itself as logically previous to spacetime with well defined geometrical properties. On *M*, a base space metric may be recovered if desired, understood not as the primary dynamical object, but as constructed from the pullback of the translational connection forms to the base space. That is, one first obtains ordinary tetrads (9.23), and then introduces

$$ds^2 = o_{\alpha\beta}\vartheta^\alpha_M \otimes \vartheta^\beta_M = g_{ij}dx^i dx^j,$$

$$g_{ij} := o_{\alpha\beta}e_i{}^\alpha e_j{}^\beta, \tag{10.2}$$

defining the GR line element. Thus, the metrized base space results as a sort of projection (a pullback in fact) of the whole bundle structure. The same holds for other possible spacetime structures, such as torsion, derived from the general Lorentz connection, or nonmetricity in the context of MAG, etc.

In the limit of absence of gravity, that is, in the global Poincaré case, the spacetime connections $\overset{(T)}{\Gamma}{}_i{}^\mu$ and $\Gamma_i^{\alpha\beta}$ vanish and Eq. (10.1) reduces to $A = d\xi^\mu A_\mu$. In other words, when gravitation is switched off, a residual tetrad $\vartheta^\mu_\Sigma = d\xi^\mu$ remains on a Minkowskian manifold, with the translational group parameters ξ^μ playing the role of coordinates. This residual coupling $d\xi^\mu A_\mu$ can be understood as the form in which Minkowskian spacetime underlies any other physical field when gravity is absent. In this case, Σ trivializes as $\Sigma = M \times G/H$. The pullback of $\vartheta^\mu_\Sigma = d\xi^\mu$ as $s^*_{M\Sigma}\vartheta^\mu_\Sigma = dx^i\partial_i\xi^\mu$ [compare with Eq. (9.24)] reduces merely to a change from Minkowskian to general coordinates. Then, instead of taking as spacetime the pullback of the plateau to *M*, one can directly identify G/H as physical spacetime, so that the bundle structure $G(G/H,H)$ suffices to describe the gauge theory of the internal group accompanying the global Poincaré group. G/H being the Minkowskian base space of the residual bundle $G(G/H,H)$ to which the composite bundle reduces when the Poincaré connections are set to zero, we claim that spacetime has a gauge theoretical origin, even in the absence of gravitational forces.

The approach presented here could be advantageous in a double sense. On the one hand, it provides a uniform treatment of forces, where gravitational potentials are the pullbacks of connections on a fiber bundle, in analogy with the other interactions. Such a homogeneous characterization of forces may perhaps be useful on the way toward their final unification. On the other hand, the present bundle description being very close to that of ordinary fiber bundles formalizing nongravitational forces, one expects that the mathematical techniques developed for the remaining interactions will be applicable to gravity.

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APPENDIX A: GEOMETRY OF LIE GROUPS

Let us consider a Lie group *G*. The left action $L_g u := gu$ of *G* on its own group manifold defines a differentiable mapping L_g: $G \to G$, inducing a differential mapping L_{g*}: $T_u(G) \to T_{gu}(G)$. In terms of the latter, we define the Lie algebra $\mathcal{G}$ to be the subset of all left invariant vector fields $(L_{g*}X_u = X_{gu})$ on *G*. Given $\{G_A\}$ as a basis for $\mathcal{G}$, there exist structure constants $f_{AB}{}^C$ such that

$$[G_A, G_B] = f_{AB}{}^C G_C, \tag{A1}$$

the commutation relations (A1) completely determining the Lie group. We will parametrize any group element $\tilde{g} \in G$ as

$$\tilde{g}_\lambda := e^{\lambda^A G_A} \tag{A2}$$

with parameters λ^A, $A = 1,...,\dim\mathcal{G}$, which is compatible with the identification of the Lie algebra basis elements as

$$G_A := \left.\frac{\partial \tilde{g}_\lambda}{\partial \lambda^A}\right|_{\lambda=0} \tag{A3}$$

belonging to $T_e(G)$. From Eq. (A2), with the help of the Hausdorff-Campbell formula, we find the adjoint representation $\mathrm{ad}_{\tilde{g}^{-1}}G_A := \tilde{g}^{-1}G_A\tilde{g} = (\tilde{g}_\lambda)_A{}^B G_B$, with the matrix

$$(\tilde{g}_\lambda)_A{}^B := [e^{\lambda^M \rho(G_M)}]_A{}^B = \delta^B_A - \lambda^C f_{CA}{}^B$$
$$+ \frac{1}{2!}\lambda^C f_{CA}{}^M \lambda^D f_{DM}{}^B - \cdots, \tag{A4}$$

where we used the representation $[\rho(G_A)]^C_B := -f_{AB}{}^C$. In terms of the same notation, the Cartan-Killing metric is defined as $\gamma_{AB} := -2\,\mathrm{tr}\rho(G_A G_B) = -2\,f_{AM}{}^L f_{BL}{}^M$. For matrix groups, any element $u \in G$ can be coordinatized by a matrix $u_A{}^B$, taken as a shorthand for an expansion of the form (A4). In terms of $u \in G$, parametrized following the pattern (A2), we define the left invariant Maurer-Cartan form of a matrix group *G* as

$$\Theta := u^{-1}du = \Theta^A G_A,$$

$$\Theta^A := -\frac{1}{2}(\gamma^{-1})^{AB}\rho(G_B)_M{}^N(u^{-1})_N{}^L du_L{}^M, \tag{A5}$$

that is, as a Lie algebra valued one-form belonging to $T^*(G)$. The left invariance of the Maurer-Cartan form means $L_g^*\Theta|_{gu} = \Theta|_u$. Taking the components Θ^A of Eq. (A5) to constitute a left invariant one-form basis of the cotangent space of G, we introduce the dual basis of left invariant vectors L_A of $T(G)$ induced by $G_A\in\mathcal{G}$ as

$$L_A[f(u)]:=\frac{\partial f(u\tilde{g}_\lambda)}{\partial\lambda^A}\bigg|_{\lambda=0}=\frac{\partial(u\tilde{g}_\lambda)}{\partial\lambda^A}\frac{\partial f}{\partial(u\tilde{g}_\lambda)}\bigg|_{\lambda=0},\tag{A6}$$

for f a differentiable function. Dealing with matrix groups, we take in particular $(u\tilde{g}_\lambda)_M{}^N=u_M{}^L(\tilde{g}_\lambda)_L{}^N$, with $(\tilde{g}_\lambda)_L{}^N$ taken from Eq. (A4). Then, Eq. (A6) gives sense to the vector

$$L_A:=u_M{}^L\rho(G_A)_L{}^N\frac{\partial}{\partial u_M{}^N}\tag{A7}$$

belonging to the tangent space $T(G)$. One can check the left invariance of Eq. (A7), namely, $L_{g*}L_A|_u=L_A|_{gu}$, and the relations

$$L_A\rfloor\Theta=G_A,\quad [L_A,L_B]=f_{AB}{}^CL_C,\tag{A8}$$

showing the duality of the bases (A5) and (A7), and the Lie algebra homomorphism induced by $G_A\in\mathcal{G}$ on left invariant vector fields of G.

In analogy to the former, one can alternatively define the right invariant bases of forms and vectors on the group manifold G. The right invariant forms read

$$\overline{\Theta}:=du\,u^{-1}=\overline{\Theta}^AG_A,$$

$$\overline{\Theta}^A:=-\frac{1}{2}(\gamma^{-1})^{AB}\rho(G_B)_M{}^Ndu_N{}^L(u^{-1})_L{}^M.\tag{A9}$$

The basis vectors analogous to (A7) and dual to (A9) are found to be

$$\overline{L}_A:=\rho(G_A)_M{}^Lu_L{}^N\frac{\partial}{\partial u_M{}^N}.\tag{A10}$$

In parallel to Eq. (A8), they satisfy

$$\overline{L}_A\rfloor\overline{\Theta}=G_A,\qquad [\overline{L}_A,\overline{L}_B]=-f_{AB}{}^C\overline{L}_C.\tag{A11}$$

Observe the change in sign in the commutation relations in Eq. (A11), as compared with Eqs. (A8) and (A1).

APPENDIX B: GEOMETRY OF THE POINCARÉ GROUP

Let us take G to be the Poincaré group, with Lorentz generators $\Lambda_{\alpha\beta}$ and translational generators P_α $(\alpha,\beta=0,...,3)$ satisfying the usual commutation relations

$$[\Lambda_{\alpha\beta},\Lambda_{\mu\nu}]=-i(o_{\alpha[\mu}\Lambda_{\nu]\beta}-o_{\beta[\mu}\Lambda_{\nu]\alpha}),$$
$$[\Lambda_{\alpha\beta},P_\mu]=io_{\mu[\alpha}P_{\beta]},\tag{B1}$$
$$[P_\alpha,P_\beta]=0$$

with the Minkowski metric $o_{\alpha\beta}:=\mathrm{diag}(-+++)$. For later convenience, we parametrize an arbitrary group element $\tilde{g}\in G$ as

$$\tilde{g}=e^{-i\xi^\mu P_\mu}e^{i\lambda^{\alpha\beta}\Lambda_{\alpha\beta}}.\tag{B2}$$

In terms of Eq. (B2), we calculate the left invariant Maurer-Cartan form (A5) as

$$\Theta_G:=\tilde{g}^{-1}d\tilde{g}=\Theta^{\alpha\beta}_{(\Lambda)}\Lambda_{\alpha\beta}+\Theta^\mu_{(P)}P_\mu,\tag{B3}$$

with

$$\Theta^{\alpha\beta}_{(\Lambda)}:=iu^{\lambda\alpha}du_\lambda{}^\beta,\quad \Theta^\mu_{(P)}:=-i\,d\xi^\lambda u_\lambda{}^\mu,\tag{B4}$$

where we used the compact matrix notation

$$u_\alpha{}^\beta:=(e^\lambda)_\alpha{}^\beta:=\delta^\beta_\alpha+\lambda_\alpha{}^\beta+\frac{1}{2!}\lambda_\alpha{}^\gamma\lambda_\gamma{}^\beta+\cdots,\tag{B5}$$

with $(u^{-1})_\alpha{}^\beta=u^\beta{}_\alpha$. The left invariant vectors (A7) dual to (B4) read

$$L^{(P)}_\mu:=iu^\nu{}_\mu\frac{\partial}{\partial\xi^\nu},$$
$$L^{(\Lambda)}_{\alpha\beta}:=-iu_{\lambda[\alpha}\frac{\partial}{\partial u_\lambda{}^{\beta]}},\tag{B6}$$

such that

$$L^{(P)}_\mu\rfloor\Theta_G=P_\mu,$$
$$L^{(\Lambda)}_{\alpha\beta}\rfloor\Theta_G=\Lambda_{\alpha\beta},\tag{B7}$$

and satisfying commutation relations formally identical to Eq. (B1). On the other hand, we introduce the (A9) right invariant forms

$$\overline{\Theta}_G:=\tilde{g}^{-1}d\tilde{g}=\overline{\Theta}^{\alpha\beta}_{(\Lambda)}\Lambda_{\alpha\beta}+\overline{\Theta}^\mu_{(P)}P_\mu,\tag{B8}$$

with

$$\overline{\Theta}^{\alpha\beta}_{(\Lambda)}:=idu^{\alpha\lambda}u^\beta{}_\lambda,$$
$$\overline{\Theta}^\mu_{(P)}:=-id(\xi^\lambda u_\lambda{}^\nu)u^\mu{}_\nu=-id\xi^\mu-\overline{\Theta}^{\nu\mu}_{(\Lambda)}\xi_\nu.\tag{B9}$$

Their dual (A10) right invariant vectors are

$$\overline{L}_{\mu}^{(P)} := i\,\frac{\partial}{\partial \xi^{\mu}},$$

$$\overline{L}_{\alpha\beta}^{(\Lambda)} := u_{\alpha}{}^{\mu} u_{\beta}{}^{\nu} L_{\mu\nu}^{(\Lambda)} + i\,\xi_{[\alpha}\frac{\partial}{\partial \xi^{\beta]}} = i\left(u_{[\alpha}{}^{\lambda}\frac{\partial}{\partial u^{\beta]\lambda}} + \xi_{[\alpha}\frac{\partial}{\partial \xi^{\beta]}}\right). \qquad \text{(B10)}$$

We recognize in Eq. (B10) the linear momentum and the angular momentum generators, respectively [26], with the total angular momentum $\overline{L}_{\alpha\beta}^{(\Lambda)}$ decomposed into intrinsic and orbital pieces as

$$\overline{L}_{\alpha\beta}^{(\Lambda)} = \overline{L}_{\alpha\beta}^{(Int)} + \overline{L}_{\alpha\beta}^{(Orb)}, \quad \overline{L}_{\alpha\beta}^{(Orb)} := i\,\xi_{[\alpha}\frac{\partial}{\partial \xi^{\beta]}}. \qquad \text{(B11)}$$

This completes the collection of formulas relevant for the present paper.

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