Gravitational instability in higher dimensions

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We explore a classical instability of spacetimes of dimension $D>4$. First, we consider static solutions: generalized black holes and brane world metrics. The dangerous mode is a tensor mode on an Einstein base manifold of dimension $D-2$. A criterion for instability is found for the generalized Schwarzschild, AdS-Schwarzschild and topological black hole spacetimes in terms of the Lichnerowicz spectrum on the base manifold. Secondly, we consider perturbations in time-dependent solutions: Generalized dS and AdS. Thirdly we show that, subject to the usual limitations of a linear analysis, any Ricci flat spacetime may be stabilized by embedding into a higher dimensional spacetime with cosmological constant. We apply our results to pure AdS black strings. Finally, we study the stability of higher dimensional ''bubbles of nothing.''

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I. INTRODUCTION

Over the last few years solutions of the Einstein equations in higher dimensions have come to play an important role as background metrics in various physical applications. These range from theories of TeV gravity, where higher dimensional black holes are predicted to be produced in the next generation of colliders [1,2], to the gravity-gauge theory correspondence [3,4].

Clearly the stability of such spacetimes is an important issue. One feature of higher dimensional spacetimes is that they often satisfy boundary conditions which differ from those encountered in four spacetime dimensions. This is because the two-sphere and two dimensional hyperbolic space are, up to discrete quotients, the unique Einstein manifolds in two dimensions with positive and negative curvature respectively. In higher dimensions there are more possibilities [5,6]. These include metrics such as the Bohm metrics [7] that exist on manifolds that are topologically S^d.

In particular, we will consider the case in which the higher dimensional spacetime includes a d-dimensional Einstein manifold, $\{B,\tilde{g}\}$, which we call the base manifold, in a common way. In these cases we shall show that part and sometimes all of the stability problem may be reduced to the solution of an ordinary differential equation of Schrödinger form. The modes we concentrate on are transverse trace-free tensor harmonics on the base manifold, $\{B,\tilde{g}\}$. The differential equation determining stability of the spacetime then depends on the spectrum of the Lichnerowicz operator on transverse traceless symmetric tensor fields of the manifold B. These modes do not exist in the stability analysis of, for example, the Schwarzschild black hole in four dimensions [8–11] because there are no suitable tensor harmonics on S^2 [12]. Thus, the instabilities we discuss are inherently higher dimensional.

Typically, the metric $\tilde{g}$ on B will be such that

$$\tilde{R}_{\alpha\beta}=\epsilon(d-1)\tilde{g}_{\alpha\beta}, \tag{1}$$

with $\epsilon=\pm 1$ or $\epsilon=0$. This is the normalization of S^d, for example. Tildes denote tensors on B.

Some examples of such spacetimes are now in order.

A. Static solutions

The spacetime is $D=(d+2)$-dimensional and of the form

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}+r^2 d\tilde{s}_d^2 \tag{2}$$

with

$$f(r)=\epsilon-\left(\frac{\alpha}{r}\right)^{d-1}-cr^2, \tag{3}$$

and $d\tilde{s}_d^2$ is the metric on B. The cosmological constant in $d+2$ dimensions is

$$R_{ab}=c(d+1)g_{ab}. \tag{4}$$

Consider first the vanishing cosmological constant, $c=0$ and with $\epsilon=1$ in Eq. (1). When $B=S^d$, these are the Schwarzschid-Tangherlini black holes [13] which are spatially asymptotically Euclidean (AE). If $B\neq S^d$ one obtains generalized higher dimensional black holes [5] which are spatially asymptotically conical (AC). These are of course not possible in four dimensions because S^2 is the only positive curvature Einstein manifold in two dimensions. If $B=S^d/\Gamma$ where $\Gamma\subset SO(d+1)$ is discrete, then the spatial metric will be asymptotically locally Euclidean (ALE). The $(d+1)$-dimensional Riemannan manifold with the metric

$$d\rho^2+\rho^2 d\tilde{s}_d^2, \tag{5}$$

is called the cone $C(B)$ with base B. It is Ricci flat precisely if $\tilde{g}$ is Einstein with the Einstein constant of Eq. (1). If $B\neq S^d$ there will typically be a singularity at the vertex of the cone, but in our case this will often be hidden inside an event horizon.

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In forming time dependent solutions below we will replace ρ^2 by $\sin^2\rho$ or $\sinh^2\rho$ in Eq. (5) to obtain $(d+1)$-dimensional Einstein metrics of positive or negative scalar curvature respectively. This will have two or one singular vertices respectively. In the latter case the Riemannian metric will be asymptotically hyperbolic (AH). Another way to get an asymptotically hyperbolic Einstein metric with negative scalar curvature is to take $\epsilon=0$ and replace ρ^2 by $e^{2\rho}$.

If c is positive and $\epsilon=1$, one gets a generalized Schwarzschild–Tangherlini–de Sitter spacetime. The static region between a cosmological event horizon and a black hole event horizon is nonsingular. If c is negative and $\epsilon=1$, one has generalized Schwarzschild–Tangherlini–anti–de Sitter, without a cosmological horizon.

Another interesting possibility with no cosmological horizon is to take c negative and $\epsilon=-1$ [14–16]. Now, if $\alpha=0$, the resulting metric will be singularity-free. If B is a hyperbolic manifold $B=H^d/\Gamma$, with $\Gamma\subset SO(d,1)$ a suitable discrete group, then we have an identification of anti–de Sitter spacetime sometimes thought of as a topological black hole. However if B is not a hyperbolic manifold then one gets a singularity-free topological black hole which is not locally isometric to anti–de Sitter spacetime.

Also of this form are solutions with a negative cosmological constant of a sort which arises in brane world scenarios [17,18]. The metric is most familiar in the form

$$ds^2=\frac{1}{z^2}(dz^2-dt^2+d\tilde{s}_d^2), \tag{6}$$

with $\epsilon=0$. If B is flat we obtain $(d+2)$-dimensional anti–de Sitter spacetime.

B. Time-dependent solutions

By suitably reinterpreting our formulas we can also discuss the stability of some time-dependent solutions. For example a generalized $D=(d+1)$-dimensional de Sitter spacetime is given by

$$ds^2=-dt^2+\frac{\cosh^2 Lt}{L^2}d\tilde{s}_d^2 \tag{7}$$

with $\epsilon=1$. This is singularity-free. Changing $\cosh Lt$ to $\sin Lt$ in Eq. (7) and letting $\epsilon=-1$ will give a generalized anti–de Sitter spacetime which will have big bang and big crunch singularities at $t=0$ and $t=\pi/L$ respectively unless B is the hyperbolic metric on H^d.

C. Ricci flat Lorentzian base and double analytic continuation

A simple generalization of this time dependent situation arises if we take B to be a d dimensional Lorentzian Ricci flat manifold whose stability properties are known.

For example we could consider the $(d+1)$-dimensional Einstein manifold with a negative scalar curvature whose metric is

$$ds^2=\frac{1}{z^2}(dz^2+d\tilde{s}_d^2), \tag{8}$$

with $\epsilon=0$, because the base is Ricci flat. If B is the flat Minkowski metric then we have the metric of $(d+1)$-dimensional anti–de Sitter spacetime. If B is a black hole metric, then we have black strings in anti–de Sitter spacetime.

Another situation in which a Lorentzian base arises is in double analytic continuation of black hole metrics. The resulting solutions describe expanding "bubbles of nothing" [19]. Double analytic continuation of the generalized Schwarzschild solution gives the metric

$$ds^2=\left[1-\left(\frac{\alpha}{r}\right)^{d-1}\right]d\psi^2+\frac{dr^2}{1-\left(\frac{\alpha}{r}\right)^{d-1}}+r^2d\tilde{s}_d^2, \tag{9}$$

where ψ is periodic and $d\tilde{s}_d^2$ is a Lorentzian metric obtained via analytic continuation of a Euclidean Einstein metric with $\epsilon=1$. If the Euclidean base is S^d, then the corresponding Lorentzian base is just de Sitter space, dS_d.

In Sec. II we relate the Lichnerowicz operator on certain modes in the spacetime with the Lichnerowicz operator on the base manifold. This will give us equations for the perturbative modes. In Sec. III we study the stability of generalized static metrics by setting up a Sturm-Liouville problem. In Sec. IV we look at perturbations in time dependent scenarios. In Sec. V we recall the Lichnerowicz spectra on some manifolds that give explicit examples for the results of Sec. III. Finally, Sec. VI contains the conclusions.

II. LICHNEROWICZ OPERATOR ON A CLASS OF SPACETIMES

A. A Lichnerowicz mode

Consider a D dimensional spacetime with metric

$$ds_D^2=-f(r)dt^2+g(r)dr^2+r^2d\tilde{s}_d^2, \tag{10}$$

where $d\tilde{s}_d^2$ is a Riemannian metric on a $d=(D-2)$-dimensional manifold B. The spacetime is taken to be Einstein.

The Lichnerowicz operator acting on a symmetric second rank tensor h is

$$(\Delta_L h)_{ab}=2R^c{}_{abd}h^d{}_c+R_{ca}h^c{}_b+R_{cb}h^c{}_a-\nabla^c\nabla_c h_{ab}. \tag{11}$$

For transverse trace-free perturbations this gives the first order change in the Ricci tensor under a small perturbation to the metric

$$g_{ab}\to g_{ab}+h_{ab}, \quad \text{such that} \quad h^a{}_a=\nabla^a h_{ab}=0$$

$$R_{ab}\to R_{ab}+\frac{1}{2}(\Delta_L h)_{ab}. \tag{12}$$

The Lichnerowicz operator is compatible with the transverse, trace-free condition [20].

We wish to study the stability of metrics of the form (10) under certain metric perturbations. It will be useful to have an expression for the Lichnerowicz operator on the spacetime in terms of the Lichnerowicz operator on the base manifold B. We shall impose the conditions

$$h_{0a}=h_{1a}=0, \tag{13}$$

where 0,1 are the t,r coordinates. These conditions are of course not a gauge choice and mean that we are restricting the modes we are looking at. More precisely, we are restricting attention to tensor modes on the base manifold and we are not considering scalar and vector modes. However, following [21] we argue in Appendix A that, at least for the manifolds in which the base is compact and Riemannian with $\epsilon=1$, the stability of the spacetime under scalar and vector perturbations is insensitive to the base manifold. Therefore, for these modes one may consider the base to be the sphere, S^d. But this leaves us with just the standard Schwarzschild-Tangherlini (-AdS) spacetimes. These standard higher dimensional black holes are expected, although to our knowledge not proven, to be stable against vector and scalar perturbations. Therefore, we expect that it is only the tensor modes which probe the base manifold sufficiently to produce instabilities. Nonetheless, it would be nice to see this from an explicit perturbation analysis. The conditions (13) and the form of the metric (10) imply that the transverse trace-free property of h_{ab} (12) is inherited by $h_{\alpha\beta}$. Here and throughout the indices $a,b,\ldots$ run from $0\ldots D$ and the indices $\alpha,\beta,\ldots$ will run from $2\ldots D$ and are the coordinates on B.

A calculation then gives

$$(\Delta_L h)_{\alpha\beta}=\frac{1}{r^2}(\tilde{\Delta}_L h)_{\alpha\beta}+\frac{1}{f}\frac{d^2}{dt^2}h_{\alpha\beta}-\frac{1}{g}\frac{d^2}{dr^2}h_{\alpha\beta}+\left[\frac{-f'}{2fg}+\frac{g'}{2g^2}+\frac{4-d}{gr}\right]\frac{d}{dr}h_{\alpha\beta}-\frac{4}{gr^2}h_{\alpha\beta}, \tag{14}$$

where $(\tilde{\Delta}_L h)_{\alpha\beta}$ is the Lichnerowicz operator on B. All the other components of $(\Delta_L h)_{ab}$ are zero because of the transverse trace-free property. This expression is the backbone of all the calculations in this paper.

It should be noted that Eq. (13) is strong conditions in low dimensions. For the four dimensional Schwarzschild solution, for example, there are no perturbations of this form because S^2 does not admit any tensor harmonics [12]. *We are looking at a potentially unstable mode that is specific to higher dimensional spacetimes.*

Our strategy in applying this to static spacetimes in the next section will be to calculate first a criterion for instability in terms of the minimum Lichnerowicz eigenvalue, λ_{min}, on the base manifold B. In a later section we will then find this minimum for several relevant manifolds.

B. Gauge freedom

Diffeomorphism invariance of the Einstein equations implies gauge invariance of the linear theory under

$$h_{ab}\rightarrow h_{ab}+\nabla_a\xi_b+\nabla_b\xi_a. \tag{15}$$

The invariance can be used to set

$$\nabla^a\overline{h_{ab}}\equiv\nabla^a(h_{ab}-\tfrac{1}{2}g_{ab}h^c{}_c)=0. \tag{16}$$

This is the transverse gauge condition which may always be imposed. The residual gauge freedom is given by vectors ξ satisfying

$$\Box\xi_a+R_a{}^b\xi_b=0. \tag{17}$$

Recall that the trace transforms as $h^a{}_a\rightarrow h^a{}_a+2\nabla^a\xi_a$; we would like to find a ξ satisfying Eq. (17) and such that $h^a{}_a\rightarrow 0$.

We show in Appendix B that if the background spacetime is vacuum, possibly with a cosmological constant, then one may impose the transverse trace-free condition for perturbations as a gauge choice. This is slightly more subtle than the standard argument in which the cosmological constant is zero.

Therefore the transverse trace-free choice made in the previous subsection is merely a gauge choice if the background spacetime is vaccum, with or without a cosmological constant. However, this will not fix all the gauge freedom and we need to check that any solutions we find are not pure gauge. A pure gauge solution would be of the form

$$h_{ab}=\nabla_a\xi_b+\nabla_b\xi_a. \tag{18}$$

In Appendix C we show that none of the modes considered in this paper is a pure gauge.

III. APPLICATION TO STATIC METRICS

A. Sturm-Liouville problem

In this section we consider static metrics which solve the vacuum Einstein equations, possibly with a cosmological constant

$$R_{ab}=c(d+1)g_{ab}. \tag{19}$$

This requires $g=1/f$ and the metric on B will be Einstein with

$$\tilde{R}_{\alpha\beta}=\epsilon(d-1)\tilde{g}_{\alpha\beta}, \tag{20}$$

with $\epsilon=\pm1$ or $\epsilon=0$. Tildes denote tensors on B. The cases $\epsilon=\pm1$ correspond to having the same scalar curvature as S^d or H^d. The function f must be of the form

$$f(r)=\epsilon-\left(\frac{\alpha}{r}\right)^{d-1}-cr^2. \tag{21}$$

We look for unstable modes of the form

$$h_{\alpha\beta}(x)=\tilde{h}_{\alpha\beta}(\tilde{x})r^2\varphi(r)e^{\omega t}, \tag{22}$$

where $\tilde{x}$ are coordinates on B and

$$(\tilde{\Delta}_L \tilde{h})_{\alpha\beta} = \lambda \tilde{h}_{\alpha\beta}. \tag{23}$$

As discussed in the previous section and in Appendix A, we expect these tensor modes to be the dangerous modes. This is similar to the situation encountered in recent studies of stability of $AdS_p \times M_q$ metrics [21–23].

The pertubation must satisfy $\delta R_{\alpha\beta} = c(d+1) h_{\alpha\beta}$ and this gives an equation for φ that may be cast in Sturm-Liouville form

$$-\frac{d}{dr}\left(f r^d \frac{d\varphi}{dr}\right) + \left(\frac{\lambda}{r^2} - \frac{2f'}{r} - \frac{(2d-2)f}{r^2} - 2c(d+1)\right) r^d \varphi = -\omega^2 \frac{r^d}{f}\varphi. \tag{24}$$

It is convenient to rewrite this as a Schrödinger equation by changing variables to Regge-Wheeler type coordinates and rescaling

$$dr_* = \frac{dr}{f}, \quad \Phi = r^{d/2}\varphi. \tag{25}$$

Equation (24) now becomes

$$-\frac{d^2\Phi}{dr_*^2} + V(r(r_*))\Phi = -\omega^2 \Phi \equiv E\Phi, \tag{26}$$

where the potential is

$$V(r) = \frac{\lambda f}{r^2} + \frac{d-4}{2}\frac{f' f}{r} + \frac{d^2 - 10d + 8}{4}\frac{f^2}{r^2} - 2c(d+1) f. \tag{27}$$

Thus the stability problem reduces to the existence of bound states with $E<0$ of the Schrödinger equation with potential $V(r)$. If such a bound state of the Schrödinger equation exists then the spacetime (10) is unstable to modes of the form (22). That is to say, there will be an instability if the ground state eigenvalue, E_0, of Eq. (26) is negative.

The normalization of wave functions must take into account the weight function of Eq. (24), but the usual normalization is recovered for Φ:

$$1 = \int \varphi^2 \frac{r^d}{f} dr = \int \Phi^2 \frac{dr}{f} = \int \Phi^2 dr_*. \tag{28}$$

This condition of normalizability that is necessary to set up the Sturm-Liouville problem is just the condition of finite energy of the gravitational perturbation (22). The background spacetimes (10) have a timelike Killing vector $\xi^0 = 1$, up to consideration of horizons. This allows the total energy of the perturbation on a spacelike hypersurface with normal $n^0 = 1/f^{1/2}$ to be well-defined, independently of the asymptotics of the background spacetime [24]

$$E \propto \int t^{\mu\nu} n_\mu \xi_\nu \sqrt{g^{(d+1)}} d^{d+1}x = \int \frac{r^d}{f} t_{00} \sqrt{\tilde{g}^{(d)}} d^d \tilde{x} dr, \tag{29}$$

where $t_{00} = G^{(2)}_{00} = R^{(2)}_{00} - \frac{1}{2} g_{00} R^{(2)}$, the second order change in the Einstein tensor under the perturbation. Now note that the kinetic part of Eq. (29) contains terms such as

$$\int \frac{r^d}{f} h^{\alpha\beta} \partial_0 \partial_0 h_{\alpha\beta} \sqrt{\tilde{g}^{(d)}} d^d \tilde{x} dr \propto e^{2\omega t} \int \tilde{h}^{\alpha\beta} \tilde{h}_{\alpha\beta} \sqrt{\tilde{g}^{(d)}} d^d \tilde{x} \int \varphi^2 \frac{r^d}{f} dr \propto \int \varphi^2 \frac{r^d}{f} dr. \tag{30}$$

Thus requiring finite energy will recover the normalization (28).

Besides normalization, we must also consider boundedness properties. The linear approximation to the equations of motion requires that $h^a{}_b \ll 1$. For black hole spacetimes with event horizons, one should re-express solutions in Kruskal coordinates [9,25] near the horizon and check boundedness there. This is because Kruskal coordinates are well-behaved at the horizon. Fortunately, the mode we are considering has no t or r components and therefore is essentially unchanged in Kruskal coordinates. Thus it is sufficient to check boundedness in r in the original coordinates of Eq. (10). However, in Sec. III B 2 we show explicitly boundedness in Kruskal coordinates for completeness.

From Eq. (22), we see that boundedness requires $\varphi(r) = \Phi(r) r^{-d/2} \ll 1$. As $r \to \infty$ the function $\Phi(r)$ must go as $r^{d/2}$ or a lower power of r. This is a weaker constraint than is imposed by finite energy (28), with or without a cosmological constant. As $r \to 0$ we must have $\Phi(r)$ going as $r^{d/2}$ or a higher power of r. This will almost always be a stronger constraint than that required by finite energy (28). However, in many of the applications in this section there will be an event horizon at some finite r_0, where $f(r_0) = 0$. The condition of boundedness will then simply be that $\Phi(r)$ is bounded at r_0. The finite energy condition will be that $\Phi(r)$ goes to zero on the horizon, because the zero of $f(r)$ is simple. In the cases below, we need to impose the stronger condition for each limit. However, in almost all cases we encounter, the soutions either satisfy both or neither of the criteria.

B. Vanishing cosmological constant

1. Asymptotic criterion for stability

Set the cosmological constant $c=0$. Asymptotically $f \to 1$ and $r = r_*$. Alternatively, this is the massless case. We will derive first a criterion for instability by solving the asymptotic Schrödinger equation (26) with $f=1$, and hence

$$V_\infty(r)=\frac{d^2-10d+8+4\lambda}{4r^2}, \tag{31}$$

and requiring suitable behavior in the interior. Call the asymptotic solution Φ_∞. The range here is $0\leqslant r<\infty$. The argument of this subsection is in fact independent of the interior form of f.

The asymptotic solution which decays at infinity is

$$\Phi_\infty(r)=\mathrm{Re}[r^{1/2}K_\nu(\omega r)],$$

$$\nu=\frac{1}{2}\sqrt{(5-d)^2-4(4-\lambda)}, \tag{32}$$

where $K_\nu(\omega r)$ is the modified Bessel function that decays at infinity [26]. The behavior of Eq. (32) for small r and real, positive ν is $\Phi_\infty(r)\sim r^{-\nu+1/2}$. Three cases should be distinguished. If $\nu\geqslant 1$ the solution is divergent and not normalizable according to Eq. (28). If $1>\nu>1/2$, the solution is divergent but normalizable. If $1/2>\nu\geqslant 0$, the solution goes to zero for small r. Another possibility is that the index $\nu=i\nu_i$ is pure imaginary, in which case the Bessel function oscillates in the interior as $\sin(\nu_i\ln r)$ and the wave function Φ is then normalizable. We see that nondivergent normalizable solutions occur precisely when the potential (31) is negative, that is

$$\frac{d^2-10d+8+4\lambda}{4}=\nu^2-\frac{1}{4}\leqslant 0. \tag{33}$$

We have finite energy solutions for a continuous range of $\omega>0$. The continuous spectrum of arbitrarily low energy is a direct consequence of the asymptotic potential (31) being unbounded below. This will not be the case for the full potential and the spectrum will become discrete.

In all of the cases of the previous paragraph, $\varphi=\Phi r^{-d/2}$ is not bound at the origin and so none of these solutions give instabilities of the massless, $f=1$, metric. However, following [27] we note that in the oscillatory solutions with an imaginary index, the derivative takes all values and so one might expect to be able to match the asymptotic solution to an interior solution for which $f\neq 1$, at least for certain discrete values of ω. Thus if a λ exists such that ν is imaginary then the Schrödinger equation should have a bound state and the metric is unstable. That is

$$\lambda_{min}<\lambda_c\equiv 4-\frac{(5-d)^2}{4} \;\Leftrightarrow\; \text{instability}. \tag{34}$$

This is the criterion for instability of a massive black hole. We have also shown that the massless case is always stable. Concrete examples of Lichnerowicz spectra giving stable and unstable spacetimes are given in Sec. V.

The vacuum solution for f is of course Eq. (21) which is now just the Schwarzschild-Tangherlini [13] black hole and the asymptotically conical (AC) variants considered in [5,6]. The radial function is

$$f(r)=1-\left(\frac{\alpha}{r}\right)^{d-1}. \tag{35}$$

The higher dimensional Regge-Wheeler tortoise coordinate (25) may be given explicitly in this case as [25]

$$r_*=r+\sum_{n=1}^{d-1}\frac{e^{2\pi i n/(d-1)}}{d-1}\alpha\ln(r-e^{2\pi i n/(d-1)}\alpha). \tag{36}$$

There is an event horizon at $r=\alpha$, and so the range of r is $\alpha\leqslant r$. The potential becomes

$$V(r)=V_\infty(r)+\frac{1}{\alpha^2}\left(\frac{\alpha}{r}\right)^{d+1}\left[\frac{10d-8-4\lambda}{4}-\left(\frac{\alpha}{r}\right)^{d-1}\frac{d^2}{4}\right]. \tag{37}$$

It follows that $V(\alpha)=0$, as was clear from the initial definition (27), and that $V(r)\to 0$ as $r\to\infty$.

The potential (37) can be seen to be always positive for $\alpha\leqslant r<\infty$ if $d^2-10d+8+4\lambda\geqslant 0$. This was the condition for the asymptotic potential to be positive also and signalled the nonexistence of finite energy solutions Φ in this case. Thus, as expected, there are also no solutions to the full equations in this case.

To establish the criterion for instability (34) we still need to check that there are solutions when the asymptotic solution oscillates in the interior and that there are none for the range $1/2\geqslant\nu\geqslant 0$, where an asymptotic solution exists but does not oscillate. These statements will be supported numerically in the next section. The conclusion will be that *a generalized black hole is unstable if and only if the base manifold has a Lichnerowicz spectrum satisfying Eq. (34).*

2. Numerical support for the asymptotic criterion

The Schrödinger equation (26) with potential (37) has a regular singular point at the event horizon $r=\alpha$. Thus we may perform a Taylor expansion of the solution of the equation about this point. The leading order terms are found to be $\Phi\sim(r-\alpha)^{\pm\alpha\omega/(d-1)}$, so long as the exponent is noninteger. Typically there is thus one divergent and one convergent solution at the horizon. Further, the convergent solution vanishes on the horizon and therefore satisfies both the finite energy and boundedness requirements. We would like to see whether the solution that is well behaved at the horizon is also well behaved at infinity, giving a bound state.

Before solving the equation we can check explicitly, following [28], that the well behaved solution remains well behaved at the horizon in Kruskal coordinates. Including the time dependence and using the limit of Eq. (36) as $r\to\alpha$, we see that the mode (22) behaves near the horizon as

$$\begin{aligned} h_{\alpha\beta}&\sim(r-\alpha)^{\alpha\omega/(d-1)}e^{\omega t}\tilde{h}_{\alpha\beta}\\ &=e^{\omega[t+\alpha(r-\alpha)/(d-1)]}\tilde{h}_{\alpha\beta}\sim e^{\omega(t+r_*)}\tilde{h}_{\alpha\beta}. \end{aligned} \tag{38}$$

The Kruskal coordinates R,T are given by

$$R\pm T\sim e^{f'(\alpha)(r_*\pm t)/2}=e^{(d-1)(r_*\pm t)/2\alpha}. \tag{39}$$

TABLE I. Lowest bound states for the $\alpha=1,d=4$ $(D=6)$ generalized Schwarzschild solution.

$\lambda_c-\lambda$	Lower bound for ω_{max}	Upper bound for ω_{max}
0.5	3.63×10^{-2}	3.654×10^{-2}
0.2	2.64×10^{-3}	2.67×10^{-3}
0.1	1.41×10^{-4}	1.416×10^{-4}
0.08	4.341×10^{-5}	4.35×10^{-5}
0.05	2.22×10^{-6}	2.28×10^{-6}
<0	No solutions found	

TABLE II. Lowest bound states for the $\alpha=1,d=8$ $(D=10)$ generalized Schwarzschild solution.

$\lambda_c-\lambda$	Lower bound for ω_{max}	Upper bound for ω_{max}
0.5	2.28×10^{-2}	2.29×10^{-2}
0.2	1.59×10^{-3}	1.595×10^{-3}
0.1	8.391×10^{-5}	8.399×10^{-5}
0.08	2.58×10^{-5}	2.59×10^{-5}
0.05	1.34×10^{-6}	1.35×10^{-6}
<0	No solutions found	

Therefore the mode goes as

$$h_{\alpha\beta}\sim(R+T)^{2\omega/f'(\alpha)}\tilde{h}_{\alpha\beta}, \tag{40}$$

which is well behaved on the future horizon $R-T=0$. It is not difficult to see that this expression will remain true for other functions $f(r)$, such as that for the AdS black holes studied below. In this case f should of course be evaluated at the horizon, which would no longer be α.

To investigate the equation numerically we first find a series expansion about the horizon of the solution that is regular at the horizon. We use this to set the initial conditions away from the horizon itself. By taking sufficient terms in the series, this may be done to high accuracy. We then choose a (positive) value for ω and the equation can be numerically integrated. The solution will always diverge for large r because it is extremely unlikely that the ω we have specified corresponds precisely to a bound state. However, by varying ω we may see that the solution diverges to positive infinity for some values of ω and to negative infinity for others. Because the solutions of the differential equation depend continuously on the parameters of the equation, there must be a bound state for some intermediate value of ω. The solutions can be double checked by integrating in from infinity towards the horizon, although this is less accurate when ω is small, as for the interesting cases. Tables I and II show values of ω between which the lowest lying negative energy bound state, $\omega^2_{max}=-E_0$, is found for various small values of d and $\lambda_c-\lambda$, where λ_c is the critical value of λ of Eq. (34). Without loss of generality we take $\alpha=1$. The dependence of the eigenvalues on α is determined on dimensional grounds to be $\omega\propto1/\alpha$. Another way of seeing this is that the Schrödinger equation is invariant under $\alpha\to k\alpha$, $r\to kr$, $\omega\to\omega/k$.

The results in Tables I and II, and other similar results for different values of d, suggest that as λ approaches the critical value from below, the energy of the lowest bound state rises and tends towards 0. This would provide a nice realization of our expectation from the previous subsection that there should be no negative energy bound state if $\lambda>\lambda_c$. Thus instability is according to the criterion (34).

C. Finite cosmological constant

1. Topological black holes

Let the cosmological constant $c=-L^2$ and let the base manifold B have negative curvature, $\epsilon=-1$. The metric is seen to have no cosmological horizon. If $\alpha=0$ it has no singularity and an event horizon at $r=1/L$. These are the so-called topological black holes [14–16]. Thus f is

$$f(r)=-1+L^2r^2. \tag{41}$$

If B is a hyperbolic manifold $B=H^k/\Gamma$, with $\Gamma\subset SO(k,1)$ a suitable discrete group, then the metric is locally anti–de Sitter space.

The potential (27) is

$$V(r)=\left[\frac{d^2-10d+8-4\lambda}{4r^2}-\frac{L^2(d^2+2d)}{4}\right](1-L^2r^2). \tag{42}$$

As expected, the potential vanishes on the horizon. The potential is not necessarily positive outside the event horizon, so *a priori* there exists the possibility of bound states with negative energy if $-d^2+10d-8+4\lambda<0$. The Schrödinger equation (26) may be solved exactly in this case. Two solutions are

$$\Phi_{\pm}(r)=r^{(1\pm C)/2}(1-L^2r^2)^{-\omega/2L}\,{}_2F_1\left(-\frac{\omega}{2L}+\frac{1-d\pm C}{4},-\frac{\omega}{2L}+\frac{3+d\pm C}{4};\frac{2\pm C}{2};L^2r^2\right), \tag{43}$$

where $C=\sqrt{(d-5)^2-4(4+\lambda)}$ and ${}_2F_1(a,b;c;x)$ is the hypergeometric function. These are generically linearly independent.

Because there is no cosmological horizon, the perturbation extends to $r\to\infty$. To consider the asymptotics of Eq. (43), use the following result for hypergeometric functions

[26]:

$$ {}_2F_1(a,b;c;x)=k_1(-x)^{-a}{}_2F_1\left(a,a-c+1;a-b+1;\frac{1}{x}\right) + k_2(-x)^{-b}{}_2F_1\left(b,b-c+1;b-a+1;\frac{1}{x}\right), \tag{44} $$

where[1]

$$ k_1=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}, \qquad k_2=\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}. \tag{45} $$

Using the series expansion of the hypergeometric function about the origin, this implies that as $x\to\infty$

$$ {}_2F_1(a,b;c;x)=k_1(-x)^{-a}\left(1+\frac{a(a-c+1)}{a-b+1}\frac{1}{x}+\cdots\right) + k_2(-x)^{-b}\left(1+\frac{b(b-c+1)}{b-a+1}\frac{1}{x}+\cdots\right). \tag{46} $$

In particular, the power law behavior at infinity is $x^{\max(-\mathrm{Re}(a),-\mathrm{Re}(b))}$. In both the solutions of Eqs. (43) we have $-\mathrm{Re}(a)>-\mathrm{Re}(b)$, so the x^{-a} term is dominant as $x\to\infty$. It is then easy to check that the overall leading asymptotic term of both the solutions $\Phi_\pm$ at infinity is $\mathcal{O}(r^{d/2})$. These solutions are never normalizable in the sense of Eq. (28). However, because infinity is a regular singular point of this Schrödinger equation, it is possible to take a linear combination of these two solutions that gives the other allowed power law asymptotics, in this case $\mathcal{O}(r^{-(2+d)/2})$. This is precisely the linear combination of $\Phi_\pm$ in which the x^{-a} terms of Eq. (44) cancel. The result is, using the symmetry ${}_2F_1(a,b;c;x)={}_2F_1(b,a;c;x)$,

$$ \Phi_3(r)=r^{-(2+d)/2+\omega/L}(1-L^2r^2)^{-\omega/2L}{}_2F_1\left(-\frac{\omega}{2L}+\frac{3+d+C}{4},-\frac{\omega}{2L}+\frac{3+d-C}{4};\frac{6+2d}{4};\frac{1}{L^2r^2}\right). \tag{47} $$

This solution has acceptable behavior as $r\to\infty$. We now need to check the behavior as $r\to 1/L$. To do this we use another identity of hypergeometric functions [26]

$$ {}_2F_1(a,b;c;x)=h_1\,{}_2F_1(a,b;a+b+1-c;1-x)+h_2(1-x)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-x), \tag{48} $$

where

$$ h_1=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad h_2=\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{49} $$

Applying this to Eq. (47) we see that generically for the modes with $\omega>0$ that we are looking for, the leading term as $r\to 1/L$ is $\mathcal{O}[(1-L^2r^2)^{-\omega/2L}]$. Inserting this and Eq. (41) into the normalization condition (28) we see that the solution is not square integrable at the horizon. Furthermore, the solution does not satisfy the boundedness requirement because Φ, and hence φ, diverges at the horizon. However, if C is real and $C>3+d$ for some mode, which requires

$$ \lambda_{min}<-4d, \tag{50} $$

then we may set

$$ \omega=\left(\frac{C-3-d}{2}\right)L>0. \tag{51} $$

For this mode we have that $h_1=0$ and $h_2=1$ in Eq. (48) because one of the Γ functions in the denominator of h_1 diverges. Another way of seeing this is that the hypergeometric function is just a polynomial in this case. It then follows from Eq. (48) that the mode now has a better behavior at the horizon, going as $\mathcal{O}[(1-L^2r^2)^{\omega/2L}]$. In particular it is normalizable and bounded at the horizon. Therefore it gives an instability. Because the base manifold has negative curvature, the arguments of Appendix A do not apply in this case and therefore we do not know about the effect of scalar and vector modes. The statement we can make is that *if the base of a massless topological black hole has a Lichnerowicz spectrum satisfying Eq. (50) then it is unstable*. Some results on the Lichnerowicz spectrum for negative scalar curvature Einstein manifolds are collected in Sec. V.

2. Brane world metric

Here $f(r)=r^2$. The base manifold is Ricci flat because $\epsilon=0$. The metric is more familiar in terms of $z=1/r$

$$ ds^2=\frac{1}{z^2}(dz^2-dt^2+d\tilde{s}_d^2). \tag{52} $$

[1]There is a subtlety here which is that if d is odd then $a-b$ is a negative integer and one of these gamma functions diverges. However, the solution (47) that we obtain is the solution which decays at infinity even in these cases.

The r coordinate extends from the origin to infinity. This metric is of the general brane world form considered in [17,18]. More general such metrics will be studied in the section below on time dependent solutions.

The potential (27) is

$$V(r)=\lambda+\frac{(d^2+2d)r^2}{4}. \tag{53}$$

The starred coordinate (25) is just $r_*=-z$. The Schrödinger equation (26) may be solved exactly in this case. The general solution is

$$\Phi(r)=Ar^{-1/2}I_{(1+d)/2}\left(\frac{\sqrt{\lambda+\omega^2}}{r}\right)+Br^{-1/2}K_{(1+d)/2}\left(\frac{\sqrt{\lambda+\omega^2}}{r}\right), \tag{54}$$

where A,B are constants and I_ν,K_ν are the modified Bessel functions. We need to take the real or imaginary part depending on whether $\sqrt{\lambda+\omega^2}$ is real or imaginary. The term with the I_ν function is always normalizable (28) as $r\to\infty$, going as $\mathcal{O}(r^{-(2+d)/2})$. The term with the K_ν function diverges as $\mathcal{O}(r^{d/2})$ as $r\to\infty$ and hence is never normalizable because $d>1$, although φ is bounded. We must then check the I_ν solution as $r\to 0$. In terms of z, the normalizability condition, with an implied real or imaginary part being taken, is

$$\int_0^\infty z I^2_{(1+d)/2}(z\sqrt{\lambda+\omega^2})dz<\infty. \tag{55}$$

If $\lambda+\omega^2>0$ then the integrand diverges exponentially as $z\to\infty$. There was no chance of a solution in this case because the energy, $-\omega^2$, would have been lower than the minimum of the potential λ. If $\lambda+\omega^2<0$ then the integrand oscillates with constant magnitude as $z\to\infty$. In either case, normalizability cannot be enforced. Furthermore, $\varphi=\Phi r^{-d/2}$ is seen to diverge as $r\to 0$ and so the boundedness condition is not satisfied. Therefore there are no finite energy solutions and *the brane world metric (52) is* stable *under this perturbation, independent of the base manifold*.

3. Schwarzschild–Tangherlini–anti–de Sitter black holes

Set $c=-L^2$ and $\epsilon=+1$. If $\alpha=0$ then we have a generalized anti–de Sitter space that describes the asymptotics of a Schwarzschild–Tangherlini–anti–de Sitter black hole. The function $f(r)$ is

$$f(r)=1+L^2r^2. \tag{56}$$

There are no horizons. The potential is now

$$V_\infty(r)=\left[\frac{d^2-10d+8+4\lambda}{4r^2}+\frac{(d^2+2d)L^2}{4}\right](L^2r^2+1). \tag{57}$$

Two exact solutions to the Schrödinger equation are

$$\Phi_\pm(r)=r^{(1\pm C')/2}(1+L^2r^2)^{-i\omega/2L}\,{}_2F_1\left(\frac{-i\omega}{2L}+\frac{1-d\pm C'}{4},\frac{-i\omega}{2L}+\frac{3+d\pm C'}{4};\frac{2\pm C'}{2};-L^2r^2\right), \tag{58}$$

with $C'=\sqrt{(d-5)^2-4(4-\lambda)}$. This is fairly similar to the topological black hole case, but we now need to consider different limits. By exactly the same arguments as for the topological black hole, the solution that is well behaved at infinity, going as $\mathcal{O}(r^{-(2+d)/2})$, is

$$\Phi_3(r)=r^{-(2+d)/2+i\omega/L}(1+L^2r^2)^{-i\omega/2L}\,{}_2F_1\left(\frac{-i\omega}{2L}+\frac{3+d+C'}{4},\frac{-i\omega}{2L}+\frac{3+d-C'}{4};\frac{6+2d}{4};\frac{-1}{L^2r^2}\right). \tag{59}$$

By using Eq. (44) again, we see that the behavior as $r\to 0$ is $\mathcal{O}(r^{(1-C')/2})$. This both converges and is normalizable if $d^2-10d+8+4\lambda_{min}<0$ which is unsurprisingly also the condition for the potential (57) to be negative and indeed unbounded below. There is a continuum of negative energy bound states with $\omega>0$ in this case. None of these solutions satisfies the boundedness condition because $\varphi=\Phi r^{-d/2}$ always diverges at the origin. Thus the massless case is stable against the perturbation. The phenomenon of the hypergeoemtric series terminating for special values of ω to give a well behaved mode does not occur here because of the i in front of the ω in Eq. (59). More interestingly, if

$$\lambda_{min}<4-\frac{(5-d)^2}{4}, \tag{60}$$

then the solution is oscillatory in the inner regions. The oscillatory behavior suggests we can make a statement about massive black holes also. The condition is clearly the same (34) as we found before in the case of a vanishing cosmological constant. This is perhaps not surprising given that the potentials, Eqs. (31) and (57), are the same near the origin. The comments of Sec. III B should go through. However, there is now a second length scale in the problem, $1/L$,

TABLE III. Lowest bound states for the $\lambda=\lambda_c-1, d=4$ $(D=6)$ generalized Schwarzschild-AdS solution.

L	$M=\alpha L$	Lower bound for ω_{max}	Upper bound for ω_{max}
0	0	0.1331	0.1335
0.01	0.0100003 . . .	0.131	0.132
0.05	0.04004 . . .	0.0591	0.00599
0.055	0.05505 . . .	0.0264	0.0269
0.057	0.05706 . . .	0.0005	0.00051
>0.058	0.05806 . . .	No solutions found	

which could delay the onset of the oscillations until beyond the event horizon, in which case there will be no solution and no instability.

A few things may be said more concretely. In the massive case,

$$f(r)=1-\left(\frac{\alpha}{r}\right)^{d-1}+L^2r^2. \tag{61}$$

The potential becomes

$$V(r)=V_\infty(r)+\frac{1}{\alpha^2}\left(\frac{\alpha}{r}\right)^{d+1}\left[\frac{10d-8-4\lambda-2dL^2r^2}{4}-\left(\frac{\alpha}{r}\right)^{d-1}\frac{d^2}{4}\right]. \tag{62}$$

The potential has the expected property that it vanishes at the horizon, where $f(r)=0$. Furthermore, when $d^2-10d+8+4\lambda>0$ it is everywhere positive, as was the asymptotic (massless) potential (57). Thus there will be no instability in these cases. We now need to see numerically when a solution exists and what the role of the new length scale is.

4. Numerical results for Schwarzschild–Tangherlini–anti–de Sitter black holes

We wish to use the methods of Sec. III B 2 to examine the effect of the new length scale $1/L$. First note that the Schrödinger equation is now invariant under the scaling $\alpha\rightarrow k\alpha, r\rightarrow kr, \omega\rightarrow\omega/k, L\rightarrow L/k$. Previously, when there was no L we used this to set $\alpha=1$, which was the location of the horizon. Again, we want to scale the horizon to 1. This will now require scaling so that $\alpha=(1+L^2)^{1/(d-1)}$. There is a scale-invariant dimensionless mass

$$M=\alpha L, \tag{63}$$

which allows us to talk about large and small black holes independently of the scaling used. We expect the criterion for instability to be the same as for Schwarzschild black holes (34) when the AdS black hole is small $M\rightarrow 0$. As we increase M we expect the black hole to be stabilized by the cosmological constant. This behavior indeed happens and is illustrated in Table III. The numerics were done as in Sec. III B 2. For the case $d=4$ (hence spacetime dimension $D=6$) and with λ satisfying the instability criterion by $\lambda=\lambda_c-1$ we see that the unstable mode is stabilized if $M>0.058\ldots$.

This will be the generic behavior. *A given base space will have a minimum Lichnerowicz eigenvalue, λ. If this is less than the critical value λ_c then we can find a critical value for the dimensionless mass, M_c, such that if $M_c<M$ then the unstable mode is stabilized. If $M_c>M$ then the AdS black hole is unstable*. Alternatively, we could think of the mass as altering the expression for λ_c.

5. Schwarzschild–Tangherlini–de Sitter black holes

Set $c=L^2$ and $\epsilon=+1$. This gives us a generalized Schwarzschild–Tangherlini–de Sitter black hole. We have

$$f(r)=1-\left(\frac{\alpha}{r}\right)^{d-1}-L^2r^2. \tag{64}$$

There is a cosmological horizon at finite radius, and therefore we cannot discuss an asymptotic solution to the Schrödinger equation, because the mass term is not negligible near the horizon. We have all the information necessary to tackle this problem numerically, but various cases must be considered separately depending on the values of α and L. This is somewhat out of the main line of development of this work and so it will not be considered here.

IV. TIME-DEPENDENT SOLUTIONS

A. Generalized de Sitter space

The metric form (10) also covers a range of cosmological solutions. For example, a generalized de Sitter metric may be written

$$ds^2=\frac{-dr^2}{L^2r^2-1}+r^2d\tilde{s}_d^{\,2}, \tag{65}$$

where r is the time coordinate now and L is a constant. This is of the form (10) with $f=0$. Note that d would now be 3, not 2, in the usual four dimensional case. Consider a perturbation, using the same notation as in Eq. (22),

$$h_{\alpha\beta}(x)=\tilde{h}_{\alpha\beta}(\tilde{x})r^2\varphi(r), \tag{66}$$

and impose the Einstein equation, $\delta R_{\alpha\beta}=dL^2h_{\alpha\beta}$. There is now a cosmological term which is that of ordinary de Sitter space in $D=d+1$ dimensions. The equation for the perturbation φ is most familiar in terms of the coordinates

$$\cosh Lt = Lr, \tag{67}$$

where the metric becomes

$$ds^2 = -dt^2 + \frac{\cosh^2 Lt}{L^2} d\tilde{s}_d^2, \tag{68}$$

and Eq. (14) becomes the equation for a scalar field on de Sitter space

$$\frac{d^2\varphi}{dt^2} + dL\tanh Lt\frac{d\varphi}{dt} + \frac{L^2}{\cosh^2 Lt}(\lambda+2-2d)\varphi = 0. \tag{69}$$

This provides a check on our expression (14) and also shows that at late times as $t\to\infty$ the leading term in each of the two linearly independent solutions is

$$\varphi \sim A + Be^{-dLt} \tag{70}$$

with A,B constants. *Thus perturbations are frozen in, independent of the dimension and the form of the base Einstein manifold B.* This is just the behavior of such perturbations in standard four dimensional inflationary metrics [29].

B. Generalized anti–de Sitter space

Generalized anti–de Sitter space can be treated similarly. Write the metric as

$$ds^2 = \frac{-dr^2}{-L^2r^2+1} + r^2 d\tilde{s}_d^2, \tag{71}$$

where again r is the time coordinate and L a constant. The base manifold B must now have negative curvature and would be H^d for anti–de Sitter space itself. Consider the perturbation

$$h_{\alpha\beta}(x) = \tilde{h}_{\alpha\beta}(\tilde{x}) r^2 \varphi(r), \tag{72}$$

and impose the Einstein equation, $\delta R_{\alpha\beta} = -dL^2 h_{\alpha\beta}$. This is as for the de Sitter case considered previously but with a negative cosmological constant. The familiar coordinates for the space are

$$\sin Lt = Lr. \tag{73}$$

The metric in these coordinates is

$$ds^2 = -dt^2 + \frac{\sin^2 Lt}{L^2} d\tilde{s}_d^2. \tag{74}$$

These coordinates make explicit the big bang and big crunch singularities at $t=0$ and $t=\pi/L$, unless the spacetime is anti–de Sitter. The equation for the perturbation becomes

$$\frac{d^2\varphi}{dt^2} + dL\cot Lt\frac{d\varphi}{dt} + \frac{L^2}{\sin^2 Lt}(\lambda+2d-2)\varphi = 0. \tag{75}$$

The general solution to this equation is

$$\varphi = A(\sin Lt)^{(1-d)/2} P_{(d-1)/2}{}^{C/2}(\cos Lt) + B(\sin Lt)^{(1-d)/2} Q_{(d-1)/2}{}^{C/2}(\cos Lt), \tag{76}$$

where A,B are constants and as before $C = \sqrt{(d-5)^2 - 4(4+\lambda)}$. $P_\nu{}^\mu$ and $Q_\nu{}^\mu$ are Legendre functions of the first and second kind, respectively. These may be expressed in terms of hypergeometric functions as follows [30]:

$$P_\nu{}^\mu(x) \propto (x^2-1)^{\mu/2} x^{\nu-\mu} \times {}_2F_1\left(\frac{\mu-\nu}{2}, \frac{\mu-\nu+1}{2}; \frac{1}{2}-\nu; \frac{1}{x^2}\right),$$

$$Q_\nu{}^\mu(x) \propto (x^2-1)^{\mu/2} x^{-\nu-\mu-1} \times {}_2F_1\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu+2}{2}; \nu+\frac{3}{2}; \frac{1}{x^2}\right). \tag{77}$$

In the present case $x=\cos Lt$ with $0\leqslant t\leqslant \pi/L$. Thus we need to check regularity properties at $t=0,\pi/2L,\pi/L$, corresponding to $x=1,0,-1$.

It is easy to check using Eq. (44) that both solutions are finite at $t=\pi/2L$, that is $x=0$. Further, it is clear from Eqs. (77) and (76) that behavior as $t\to\pi/L$ will be the same as for $t\to 0$, up to phases in front of each Legendre function. In particular this means that regularity properties will be the same at these points. Both solutions of Eq. (76) diverge as $\mathcal{O}(t^{(1-d-C)/2})$ as $t\to 0$. Therefore they also diverge as $t\to\pi/L$. However, $t=0$ is a regular singular point of Eq. (75) and therefore there will be a linear combination of these solutions that has the other allowed power law behavior as $t\to 0$, namely $\mathcal{O}(t^{(1-d+C)/2})$. This will be oscillatory and divergent if C is pure imaginary. If C is real it will converge if $C\geqslant d-1$ and diverge otherwise. There is also the possibility that well behaved modes will exist for special values of C where the hypergeometric function becomes a polynomial.

The main conculsion of the previous paragraph is that there are always modes that if excited at some finite time will diverge in the future. *Thus this AdS cosmology is unstable, independent of the base manifold and the dimension.*

C. Ricci flat Lorentzian base (brane world metrics II)

Let the base B be a d-dimensional Ricci flat spacetime. Suppose we know the spectrum of Lichnerowicz modes on B with eigenvalues λ and such that the modes grow in time. In particular, if this spectrum includes a zero mode, then the spacetime B is unstable.

The spacetime B may be embedded in a $D=(d+1)$-dimensional Einstein manifold with negative scalar curvature

$$ds^2=\frac{1}{z^2}(dz^2+d\tilde{s}_d^{\,2})=\frac{dr^2}{r^2}+r^2d\tilde{s}_d^{\,2}. \tag{78}$$

For example, if B is a black hole metric we obtain a black string in AdS spacetime. The change of variables in Eq. (78) is of course $r=1/z$. We would like to see whether this spacetime is unstable under any of the growing modes in B. Consider the perturbation

$$h_{\alpha\beta}(x)=r^2\varphi(r)\tilde{h}_{\alpha\beta}(\tilde{x}), \tag{79}$$

where $\tilde{h}_{\alpha\beta}$ is a Lichnerowicz eigenmode on the base with eigenvalue λ. From Eq. (14), ignoring the terms with f's, the equation for $\varphi(r)$ coming from $\Delta_L h_{\alpha\beta}=-2dh_{\alpha\beta}$ is

$$\frac{d^2\varphi}{dr^2}+\frac{d+1}{r}\frac{d\varphi}{dr}-\frac{\lambda\varphi}{r^4}=0. \tag{80}$$

The general solution to this equation for $\lambda>0$ is

$$\varphi(r)=Ar^{-d/2}I_{d/2}\left(\frac{\lambda^{1/2}}{r}\right)+Br^{-d/2}K_{d/2}\left(\frac{\lambda^{1/2}}{r}\right), \tag{81}$$

where A,B are constants and I_μ,K_μ are the modified Bessel functions. If $\lambda<0$ then the expression is most transparent if we let $\lambda\to-\lambda$ and replace I_μ,K_μ by the Bessel functions J_μ,Y_μ. Finally, if $\lambda=0$ then the solution is

$$\varphi(r)=A+Br^{-d}. \tag{82}$$

To see which, if any, of these solutions are acceptable we need to find the energy of the perturbations. Assuming that B has a timelike Killing vector, this is similar to the argument in Sec. III A

$$\begin{aligned}E&\propto\int t^{\mu\nu}n_\mu\xi_\nu\sqrt{g^{(d)}}d^dx\\&=\int t^{\mu\nu}n_\mu\xi_\nu\sqrt{\tilde{g}^{(d-1)}}r^{d-2}drd^{d-1}\tilde{x}\\&\sim\int\varphi^2r^{d-3}dr,\end{aligned} \tag{83}$$

where we used the metric (78) and $n_0\propto r$, $\xi_0\propto r^2$, $t_{00}\sim\varphi(r)^2$. Note that $\tilde{g}^{(d-1)}$ is the spatial metric on B. It is easy to see that the normalization condition (83) is the same as norm of Eq. (80) cast in Sturm-Liouville form. The eigenvalue of the Sturm-Liouville problem is now λ and the weight function is seen to be r^{d-3}, consistent with Eq. (83). The boundedness condition in this context is simply that $\varphi(r)$ remains finite in the range $0\leqslant r<\infty$.

If there is a $\lambda=0$ mode, the d dimensional spacetime B is unstable. However, from Eq. (82) it is clear that none of these solutions is normalizable in the $(d+1)$-dimensional spacetime. Further, all except the constant solutions are not bounded. Therefore the unstable mode does not carry over to the full spacetime because it no longer has finite energy.

For $\lambda>0$ we see that the term with the I_μ Bessel function has an exponential divergence as $r\to0$ while the term with the K_μ Bessel function goes to zero exponentially and hence is normalizable at the origin. However, this term goes as $\mathcal{O}(r^0)$ as $r\to\infty$ and hence, although bounded, is not normalizable at infinity because $d\geqslant2$. Thus there are no normalizable solutions with $\lambda>0$.

For $\lambda<0$, as $r\to\infty$, the term with the Y_μ Bessel function goes as $\mathcal{O}(r^0)$, and therefore is not normalizable at infinity. The term with the J_μ Bessel function goes as $\mathcal{O}(r^{-d})$ and therefore is normalizable. As $r\to0$, the J_μ term oscillates as $\mathcal{O}(r^{(1-d)/2}\cos[(-\lambda)^{1/2}/r])$ and hence is not normalizable (83) or bounded as $r\to0$. Thus there are no normalizable modes of this type.

In conclusion, embedding a Ricci flat spacetime B into a higher dimensional spacetime with a cosmological constant as in Eq. (78) will stabilize at the linear level any unstable modes of B. Furthermore, no other unstable modes of the form we consider appear. This should be contrasted with a similar embedding into a higher dimensional Ricci flat spacetime where the stability properties get worse due to negative Lichnerowicz modes in the initial spacetime [31], such as the Gregory-Laflamme instability of nonextremal black strings [32,33,25]. It was argued in [34] that a Gregory-Laflamme instability existed also for black strings in AdS spacetime. However, the perturbed mode presented there, which agrees as a special case with the modes we have just considered, did not have $h^\alpha{}_\beta$ bounded as is required in a linearized analysis. The phenomenon of perturbations to brane world metrics diverging in the bulk has been observed before [35] and is related to the bad behavior of the curvature at the horizon. Bounded modes, and hence the instability, could reappear if one modifies the setup, such as by adding a negative tension brane at finite position.

One might worry that the instability of a very thin short black string should not be affected by immersion in an anti–de Sitter spacetime with a large radius. This may well be true. However, if one assumes that the black string runs all the way to the horizon there seems to be no way of avoiding our conclusions, although there is already a singularity near the horizon in the unperturbed metric [36]. The methods used here can say very little about what would happen for a "cigar-like" black string configuration.

D. Double analytic continuation

Higher dimensional versions of the Schwarzschild bubble solution [19] have been considered recently in a search for well-behaved time dependent backgrounds in which to study string theory [37]. Ultimately the Kerr solutions turn out to be more appropriate, but the Schwarzschild case contains many of the relevant features. The solution, obtained via double analytic continuation of the Schwarzschild-Tangherlini solution is

$$\begin{aligned}ds^2=&\left[1-\left(\frac{\alpha}{r}\right)^{d-1}\right]d\psi^2+\frac{dr^2}{1-\left(\frac{\alpha}{r}\right)^{d-1}}\\&+r^2(-d\tau^2+\cosh^2\tau d\Omega_{d-1}^2),\end{aligned} \tag{84}$$

where $r \geq \alpha$, ψ has period $4\pi\alpha/(d-1)$ and $d\Omega_{d-1}^2$ is the round metric on S^{d-1}. The coordinate $\psi = it$ is the Wick rotated time from the black hole solution and $i\tau = \theta - (\pi/2)$, where θ was the usual angular parameter on S^d and τ is now the time on dS_d. Recently, such "bubbles of nothing" have also been considered by analytically continuing AdS-Schwarzschild black holes [38,39].

A classically stable background is needed for string theory. Classical instabilities are manifested in the string worldsheet theory as a renormalization group flow arising from higher string loops with a fixed point that is not close to the original background in string units.

It was argued in [37] that the classical stability of Eq. (84) follows from the classical stability of the corresponding black hole solution. Any mode on the bubble must be periodic in the ψ variable. But $e^{-i\omega\psi} = e^{wt}$, so these correspond to growing modes of the black hole with certain frequencies ω allowed by the periodicity of ψ. Furthermore, the part of the black hole mode that is a harmonic on S^d becomes a harmonic on dS_d with the same eigenvalue. In the four dimensional case of [8–10] these modes are scalar and vector harmonics, while in our case we are considering tensor harmonics [12]. The equation for the radial dependence is the same in the black hole and bubble cases and therefore the criterion for the existence of solutions is the same. On the sphere, the harmonics are trigonometric functions of θ and these become hyperbolic functions of τ, which correspond to growing and hence unstable modes on the bubble.

The conclusion of the previous paragraph is that any unstable mode on the black hole with appropriate ω signals an unstable mode on the bubble. Conversely, any mode on the bubble signals an unstable mode on the black hole. Thus we have extended the stability arguments of the higher dimensional Schwarzschild bubble in [37] to include the tensor mode that is being considered throughout this paper.

There is a subtlety, however. After doing the double analytic continuation, we must redo the calculations of energy because the timelike Killing vector has changed, if indeed there is still a timelike Killing vector at all. The calculation is very similar to the Lorentzian base case (83). There will typically be horizons on the base manifold across which the Killing vector changes sign, as in the de Sitter base of Eq. (84). In this case, the integration over a spacelike hypersurface of the base should be restricted to the region inside the horizon. The integration over r is not changed,

$$E \propto \int t^{\mu\nu} n_\mu \xi_\nu \sqrt{g^{(d+1)}} d^{d+1}x$$
$$\sim \int t^{00} n_0 \xi_0 \sqrt{\tilde{g}^{(d-1)}} r^{d-1} dr\, d\psi\, d^{d-1}\tilde{x}$$
$$\sim \int \varphi^2 r^{d-2} dr = \int \Phi^2 r^{-2} dr, \tag{85}$$

which is indeed different from the energy of the corresponding black hole perturbation (29). As a check of this expression, we can see that this is the same normalizability condition that we get from the Sturm-Liouville problem for φ. The Sturm-Liouville equation remains (24), but the eigenvalue associated with time evolution is no longer ω, but rather λ, as was commented on for the case of the Lorentzian base. The weight function is thus seen to be r^{d-2} rather than the r^d/f that we had for the black hole. This is in agreement with Eq. (85). The condition for boundedness remains the same: φ must not diverge. We may now go through the results from the black hole spacetimes of Sec. III again and recheck for finite energy using Eq. (85). We see that the existence and nonexistence of finite energy solutions remain the same in each case.

If the Euclidean metric on the base manifold B of the generalized black hole solutions, Eqs. (10) and (21), admits an analytic continuation to a Lorentzian metric, then one may construct a generalized "bubble of nothing" with metric

$$ds^2 = f(r) d\psi^2 + \frac{dr^2}{f(r)} + r^2 d\tilde{s}_d^2, \tag{86}$$

with the notation of Eq. (84) and now $d\tilde{s}_d^2$ is a Lorentzian metric.

Assuming that the tensor harmonics on the Lorentzian base manifold are growing modes in time, the argument for the S^d above case goes through also in the generalized case. Stability is thus related to the stability of a generalized black hole solution. We could also see this directly from the relationship between the Lichnerowicz operator on the full manifold and on the base (14): perturbations of Eq. (86) that are periodic in ψ will give the same radial equation as perturbations of the generalized black hole metrics with exponential growth in t. Thus, for example, in the generalized Schwarzschild-Tangherlini case, the criterion (34) will be the same. However, there is an important caveat: the frequencies that cause instabilities for the generalized black hole will only cause an instability in the bubble spacetime if they respect the periodicity of ψ, that is

$$\omega = \frac{N(d-1)}{2\alpha}, \quad N \in \mathbb{Z}. \tag{87}$$

In practice, this is unlikely to be true for any of the bound states in the discrete spectrum. Therefore we expect that the bubble spacetimes will generally be stable against the perturbation even when they violate the criterion (34).

V. SPECTRA OF THE LICHNEROWICZ OPERATOR

This section collects and applies mostly known results about the spectrum of the Lichnerowicz operator on various Einstein manifolds. In our considerations of the static metrics above we found that in the case of Schwarzschild-Tangherlini, Schwarzschild–Tangherlini–anti–de Sitter and topological black holes, stability of the spacetime against the perturbation (12),(13) depends on the spectrum of the Lichernowicz operator on the base manifold B (34). The objective of this section is to check that the cases we expect to be stable, i.e. round spheres, are stable and also to find examples of spaces that are not stable, thus showing that the criterion for instability is not vacuous. It would be nice to have more spectra at hand, in particular for Einstein manifolds that are topological spheres such as the Bohm metrics [7]. We discuss first Einstein manifolds of positive scalar curvature, relevant for the Schwarzschild and Schwarzschild-AdS cases. We then discuss manifolds of negative scalar curvature, which are relevant for the topological black holes.

In this section, tildes will be omitted from the metric perturbations on the base manifold B and we will consider transverse trace-free perturbations

$$g_{\alpha\beta}\to g_{\alpha\beta}+h_{\alpha\beta}, \quad h^{\alpha}{}_{\alpha}=\nabla^{\alpha}h_{\alpha\beta}=0. \tag{88}$$

Define Λ by $R_{\alpha\beta}=\Lambda g_{\alpha\beta}$.

A. Round spheres

When the base manifold is a unit sphere, asymptotically conical (AC) just means asymptotically flat (AF). The unit sphere $B=S^d$ has curvature

$$R_{\alpha\beta\gamma\delta}=g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}, \tag{89}$$

and therefore the Lichnerowicz operator is

$$(\Delta_L h)_{\alpha\beta}=(-\nabla^2 h)_{\alpha\beta}+2dh_{\alpha\beta}. \tag{90}$$

However, the Laplacian on the sphere is a positive operator and therefore $\lambda_{\rm min}\geqslant 2d$. In fact, the spectrum of transverse trace-free tensor harmonics on S^d, $d\geqslant 3$, is known to be [40]

$$(-\nabla^2 h)_{\alpha\beta}=[k(k+d-1)-2]h_{\alpha\beta}. \tag{91}$$

But $2d$ is larger than the critical eigenvalue of Eq. (34) for all d, and therefore the spacetime is stable against this perturbation. In particular this means that the AF Schwarzschild-Tangherlini black holes are stable to the perturbation, as we should expect. The spectrum of quotients of the sphere by a finite group has the same lower bound and so the resulting ALF spaces also give stable black holes.

B. Product metrics

Let the base be metrically a product of Einstein manifolds. It is well known [41,21] that there is a Lichnerowicz transverse trace-free zero mode in which one component expands and the other shrinks, keeping the total volume constant. Specifically, if the metric decomposes as

$$d\tilde{s}_d^2=ds_n^2+ds_{d-n}^2, \tag{92}$$

then the mode

$$h_{\alpha\beta}=\begin{pmatrix} \frac{\epsilon}{n}g_n & \\ & \frac{-\epsilon}{d-n}g_{d-n}\end{pmatrix}_{\alpha\beta}, \tag{93}$$

is easily seen to be trace-free, transverse and with a zero Lichnerowicz eigenvalue. Comparing with Eq. (34), this implies that the spactime is unstable with a product base for $d<9$. It is intriguing that this is the same critical dimension for instability as was found in the context of generalized Freund-Rubin compactifications to AdS$\times B_d$ [21], where the general dependence on the dimension is different.

C. Five and seven dimensional base

In five dimensions, one has the Einstein manifolds T^{pq} which are $U(1)$ bundles over $S^2\times S^2$. See e.g. [22] for the metric and other details. We will not make their choice of normalization $a=1$ and a will appear as a parameter related to the overall scale of the metric. Instead we require that $\Lambda=4$, as this is the normalization of the unit sphere in five dimensions. We have [22]

$$\left(\frac{q}{p}\right)^2=\frac{1-8a^2}{(4a^2-1)(12a^2-1)^2}, \tag{94}$$

which is seen to imply

$$\frac{1}{8}\leqslant a^2\leqslant\frac{1}{4}. \tag{95}$$

The minimum Lichnerowicz eigenvalue is

$$\lambda_{min}=[12a^2-\sqrt{784a^4-240a^2+20}]\frac{1}{a^2}. \tag{96}$$

The critical value in four dimensions (34) is $\lambda_c=4$. For the range of a^2 in Eq. (95), we see that $\lambda_{min}\leqslant\lambda_c$, with equality occurring for the T^{11} case where $a^2=\frac{1}{6}$. Thus the T^{11} black hole is stable while all the other T^{pq} black holes are unstable. This is precisely the behavior that was observed in the context of Freund-Rubin compactifications in [22].

In seven dimensions, the Einstein manifolds M^{pqr}, which are $U(1)$ bundles over $\mathbb{CP}^2\times S^2$, and $Q^{n_1n_2n_3}$, which are $U(1)$ bundles over $S^2\times S^2\times S^2$, have been studied in the context of Freund-Rubin compactifications [42,43]. Again, results will be quoted. Interestingly enough, in seven dimensions the manifolds with the required normalization, $\Lambda=6$, turn out to be stable precisely when the corresponding Freund-Rubin supergravity compactifications are stable. Thus the bounds we obtain are familiar.

For M^{pqr}, the stability depends on p/q and Λ as follows:

$$\lambda_{min}=\Lambda\left[\frac{4+4x-2(25-48x+32x^2)^{1/2}}{1+2x}\right], \tag{97}$$

where x is defined by

$$\left(\frac{p}{q}\right)^2=\frac{2x-1}{x^2(3-2x)}. \tag{98}$$

This implies $\frac{1}{2}\leqslant x\leqslant\frac{3}{2}$. We are interested in the case $\Lambda=6$. Given that the critical value in seven dimensions is $\lambda_c=3$ (34), we have that the solution will be stable to this perturbation for

$$\frac{9}{14}\approx 0.64\ldots<x<1.15\ldots\approx\frac{39}{34}, \tag{99}$$

and unstable for x outside this interval. Translated into an interval for p/q this becomes

$$\frac{7\sqrt{6}}{27}\approx 0.63\ldots<\left|\frac{p}{q}\right|<1.18\ldots\approx\frac{17\sqrt{66}}{117}. \tag{100}$$

Thus for this range of p,q we have solutions that are stable against perturbations of the kind considered in this work, and therefore stable overall in the cases of Appendix A. This means that the Schwarzschild-Tangherlini solution with the sphere as base space is not the only classically stable solution of the form (10), contrary perhaps to initial expectations. These spaces are not necessarily simply connected, having fundamental group $\mathbb{Z}_r$.

The situation is more complicated for $Q^{n_1 n_2 n_3}$. We have $\lambda_{min}=\gamma_{min}\Lambda$, where γ_{min} is the smallest root of

$$\gamma^3-6\gamma^2+20(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)\gamma-56\alpha_1\alpha_2\alpha_3=0, \tag{101}$$

with the α_i's defined by

$$\left(\frac{n_1}{n_2}\right)^2=\frac{\alpha_1(1+\alpha_2)^2}{\alpha_2(1+\alpha_1)^2}+\text{cyclic}$$

$$\alpha_1+\alpha_2+\alpha_3=1. \tag{102}$$

We will look at the simplified case in which $\alpha_2=\alpha_3$, implying $n_2=n_3$. Further, set $\Lambda=6$. The resulting range of stability to this perturbation is found to be [43]

$$\frac{3\sqrt{2}}{5}\approx 0.85\ldots<\left|\frac{n_2}{n_1}\right|<1.18\ldots\approx\frac{17\sqrt{66}}{177}. \tag{103}$$

Again we have found a countable infinity of AC Schwarzschild-Tangherlini solutions that are stable to this perturbation. The upper bound is the same as previously. The fundamental group is $\mathbb{Z}_k$, where k is the greatest common divisor of n_1,n_2,n_3.

Finally, because the bound for stability on the Lichnerowicz operator is the same for a seven dimensional base as the Freund-Rubin bound, all the manifolds that are supersymmetric in the supergravity context give a stable spacetime. A list of such spaces can be found in [44]; examples are the squashed seven sphere and $SO(5)/SO(3)_{max}$.

D. Negative scalar curvature manifolds

Hyperbolic space and its quotients by appropriate finite groups have curvature

$$R_{\alpha\beta\gamma\delta}=-g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}, \tag{104}$$

and therefore the Lichnerowicz operator is

$$(\Delta_L h)_{\alpha\beta}=(-\nabla^2 h)_{\alpha\beta}-2dh_{\alpha\beta}. \tag{105}$$

The rough Laplacian is a positive definite operator on normalizable modes and therefore the Lichnerowicz spectrum is bounded below by $-2d$. Comparing with Eq. (50) we see that the corresponding topological black holes are always stable against the tensor perturbations.

The spectra of negative scalar curvature Einstein-Kähler manifolds were studied in [45] with the result that for these manifolds the Lichnerowicz operator on transverse trace-free tensors is bounded below by $2-2d$. Therefore these manifolds always give black holes that are stable against such perturbations.

VI. CONCLUSIONS AND OPEN ISSUES

Let us summarize the results of this paper.

(a) Spacetimes of the form (10) admit perturbations that are transverse trace-free tensors on the base Einstein manifold. The stability of these spacetimes against such perturbations depends on the spectrum of the Lichnerowicz operator on the base manifold.

(b) Higher dimensional generalized Schwarzschild black holes are stable if and only if the Lichnerowicz spectrum on the base manifold is bounded below by the critical value of Eq. (34). This statement includes all perturbations to the spacetime. The same criterion holds for Schwarzschild-AdS black holes, although in this case large black holes become stabilized. Examples of base manifolds leading to stable and unstable black holes are given in Sec. V.

(c) (Generalized) topological black holes have a criterion for instability due to tensor modes on the base (50). An Einstein-Kähler base always gives a stable black hole, as do quotients of hyperbolic space. The brane world metrics of Sec. III C 2 are always stable against these tensor perturbations. Time-dependent tensor pertubations in generalized de Sitter spacetime, considered as a cosmological spacetime, are always frozen in while in generalized anti–de Sitter spacetime they always include an unstable mode.

(d) A spacetime with no cosmological constant may be embedded in a higher dimensional spacetime with a negative cosmological constant. Lichnerowicz zero modes which correspond to instabilities of the initial spacetime become stabilized. At the linearized level, no new instabilities of the form we consider are introduced. In particular this means that the perturbative Gregory-Laflamme instability does not occur for AdS black strings in five dimensions, contrary to previous claims [34]. See the discussion in Sec. IV C.

(e) Double analytic continuation of generalized black hole spacetimes, if admissible, produces a generalized "bubble of nothing" spacetime. These spacetimes are generically more stable to the analytically continued tensor perturbation than the corresponding black hole. However, one needs to redo energy calculations because the time direction has changed.

Some issues remain to be addressed. For the perturbation analysis, these include a numerical study of the stability of generalized de Sitter black holes, an explicit calculation of the vector and scalar modes to confirm the arguments of Appendix A and to extend them to cases where the base manifold has negative or zero curvature and an extension of the arguments to nonvacuum solutions.

Regarding the Lichnerowicz spectrum, the spectrum is not known for many interesting metrics, such as the Bohm metrics. It is possible that the methods of the present work may

TABLE IV. Available tensors on the base manifold.

Free base indices	Tensors	Vectors	Scalars
Two	$\tilde{h}_{\alpha\beta}$ and $(\tilde{\Delta}_L\tilde{h})_{\alpha\beta}$	$\tilde{\nabla}_{(\alpha}\tilde{h}_{\beta)}$	$\tilde{\nabla}_{(\alpha}\tilde{\nabla}_{\beta)}\tilde{h}$ and $\tilde{g}_{\alpha\beta}\tilde{h}$
One		$\tilde{h}_\alpha$ and $(\tilde{\Delta}_V\tilde{h})_\alpha$	$\tilde{\partial}_\alpha\tilde{h}$
None			$\tilde{h}$ and $\tilde{\Delta}_S\tilde{h}$

be used to study aspects of this spectrum.

In other directions, it might be interesting to see whether rotating black holes admit a similar generalization, and to see whether these instabilities are useful in the context of gravity-gauge theory dualities.

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APPENDIX A: SCALAR AND VECTOR MODES

We argue here that if the Einstein base manifold B is compact, Riemannian and with $\epsilon=1$, then the scalar and vector modes will not produce instabilities in the spacetimes under consideration. In particular, this covers the generalized Schwarzschild-Tangherlini (-AdS) cases in which we found a criterion for instability from the tensor modes.

The argument has two steps. First, we will recall [21] that the scalar and vector second order differential operators on B are bounded below by their minimum eigenvalue on the sphere S^d. Secondly, we will show that the equations of motion for these modes depend on the base manifold only through these differential operators. We will use $\tilde{h}$ and $\tilde{h}_\alpha$ to denote generic scalar and vector modes on the base manifold.

The second order differential operator on scalars is $\tilde{\Delta}_S\tilde{h}=-\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\tilde{h}$. This is non-negative on compact manifolds without a boundary. There is always a zero mode corresponding to a constant field. In particular, zero is the minimum eigenvalue on S^d and also on any other such manifold B.

The second order differential operator on vectors in B is $(\tilde{\Delta}_V\tilde{h})_\beta=-\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha\tilde{h}_\beta+(d-1)\tilde{h}_\beta$. By considering

$$\int_B(\tilde{\nabla}^\alpha\tilde{h}^\beta+\tilde{\nabla}^\beta\tilde{h}^\alpha)(\tilde{\nabla}_\alpha\tilde{h}_\beta+\tilde{\nabla}_\beta\tilde{h}_\alpha)\geqslant 0, \tag{A1}$$

an integration by parts shows that for all eigenvalues λ of $\tilde{\Delta}_V$, we have $\lambda\geqslant 2(d-1)$. We have used the fact that the base manifold is Einstein with the same scalar curvature as the unit sphere of appropriate dimension. The inequality is saturated by modes that are Killing vectors. It is well known that the minimum eigenvalue of $\tilde{\Delta}_V$ on S^d is precisely $2(d-1)$ [12]. Therefore

$$\lambda^B_{min}\geqslant\lambda^{S^d}_{min}. \tag{A2}$$

The equations for the various modes are found by expanding the linearized equations

$$\frac{1}{2}(\tilde{\Delta}_L h)_{ab}=c(d+1)h_{ab}, \tag{A3}$$

into harmonics on the base manifold and considering each linearly independent term separately. The various components of Eq. (A3) may be classified by the number of free indices on the base manifold: two, one or none. Each equation can then be written so that it is tensorial on the base manifold. There are not many tensors available that are linear in the perturbation, they are shown in Table IV.

Multiplying these tensors in each equation will be differential expressions involving functions of r and t, but the t dependence is just an exponential. The equations corresponding to the terms along the diagonal of Table IV will be of the form

$$F[\partial_r^2\varphi^I,\partial_r\varphi^I,\varphi^I]\tilde{\Delta}\tilde{h}+G[\partial_r^2\varphi^I,\partial_r\varphi^I,\varphi^I]\tilde{h}=0, \tag{A4}$$

where F,G are functionals of the radial functions $\varphi^I(r)$. Here I indexes the scalar or vector modes because there is more than one mode of each type in the decomposition of h_{ab}. In this equation $\tilde{h}$ and $\tilde{\Delta}$ represent a harmonic mode on the base and the corresponding differential operator. But we have decomposed into harmonics of these differential operators, so $\tilde{\Delta}\tilde{h}=\lambda\tilde{h}$. This then gives a differential equation for the $\varphi^I(r)$ that only depends on the base manifold through the eigenvalue λ

$$F[\partial_r^2\varphi^I,\partial_r\varphi^I,\varphi^I]\lambda+G[\partial_r^2\varphi^I,\partial_r\varphi^I,\varphi^I]=0. \tag{A5}$$

Above the diagonal in Table IV we see that there is only one possible term in each case, after noting that $\tilde{\nabla}_{(\alpha}\tilde{\nabla}_{\beta)}\tilde{h}$ does not arise from Eq. (A3). Therefore the corresponding equations will be of the form

$$H[\partial_r^2\varphi^I,\partial_r\varphi^I,\varphi^I]\tilde{\nabla}\tilde{h}=0, \tag{A6}$$

where $\tilde{\nabla}\tilde{h}$ is one of the tensors from above the diagonal in Table IV. But this simply implies the differential equation $H=0$, with no dependence on the base manifold. Therefore the equations for the perturbations only depend on the base manifold through the eigenvalues of the second order differential operators on the base.

As we saw for the tensor mode, instabilities are associated with the eigenvalue being less than some critical value. This is the generic situation and should be thought of as being due to a sufficiently negative mass squared. However, in the first part of this appendix we saw that the eigenvalues for the scalar and vector perturbations were bounded below by the minimum eigenvalues on the sphere of the given dimension and unit radius, S^d. It is expected, but to our knowledge not proven, that the standard Schwarzschild-Tangherlini and Schwarzschild-AdS black holes are stable against these vector and scalar perturbations and therefore these eigenvalues must be larger than any critical value. This suggests that the vector and scalar modes do not contribute towards the instability of the generalized Schwarzschild-Tangherlini and Schwarzschild-AdS black holes that we have been considering in this work. It would be nice, however, to check this explicitly from a perturbation analysis.

APPENDIX B: TRANSVERSE TRACEFREE GAUGE

Here we show that one may impose the trace-free condition in addition to the transverse condition for vacuum spacetimes with a cosmological constant. This extends a familiar argument to the case of a nonzero cosmological constant. Suppose the background spacetime in $d+2$ dimensions satisfies the *vacuum* field equations, possibly with a cosmological constant,

$$R_{ab}=c(d+1)g_{ab}, \tag{B1}$$

then considering perturbations that are transverse in the sense of Eq. (16) one may take the trace of the perturbed equations

$$\begin{aligned}\frac{1}{2}(\Delta_L h)_{ab}&=c(d+1)h_{ab}\\ &\Rightarrow \Box h^a{}_a+2c(d+1)h^a{}_a\\ &=0.\end{aligned} \tag{B2}$$

Consider solving the residual gauge freedom equations (17) subject to the following initial conditions on the spacelike hypersurface, $t=t_0$:

$$\begin{aligned}2(\nabla^0\xi_0+\nabla^m\xi_m)&=-h^a{}_a,\\ 2[-\nabla_m\nabla^m\xi_0+\nabla_m\nabla_0\xi^m-2c(d+1)\xi_0]&=-\nabla_0 h^a{}_a,\end{aligned} \tag{B3}$$

where m runs over the spatial indices. These equations may always be solved for ξ and $d\xi/dt$ on the hypersurface. These initial data then define a vector field ξ in the causal future of the hypersurface using the residual gauge equation (17). We can show that this vector field gives a gauge transformation which sets the trace to zero. Define $f=h^a{}_a+2\nabla^a\xi_a$. The initial conditions (B3) and (17) are seen to imply that $f=df/dt=0$ at $t=t_0$. Now find the equation satisfied by f, using Eqs. (B2) and (17):

$$\begin{aligned}\Box f&=\Box h+2\Box\nabla^a\xi_a\\ &=-2c(d+1)h^a{}_a+2\nabla^a\Box\xi_a-2c(d+1)\nabla^a\xi_a\\ &=-2c(d+1)[h^a{}_a+2\nabla^a\xi_a]\\ &=-2c(d+1)f.\end{aligned} \tag{B4}$$

We see that, perhaps unexpectedly, the equation for f has no source term. This fact implies that the vanishing initial conditions force f to vanish identically in the causal future of the initial hypersurface. Issues of global hyperbolicity of the initial spacetime should not concern us as we are only interested in the future development of the perturbation. Thus we have shown that there exists a ξ satisfying the residual gauge equation (17) such that the trace of the perturbation may be set to 0.

APPENDIX C: NO PURE GAUGE SOLUTIONS

We show here that the tensor perturbations considered in this paper cannot be pure gauge. Suppose there were a pure gauge mode, $h_{ab}=\nabla_a\xi_b+\nabla_b\xi_a$. For the perturbations we are considering about the metric (10), consider the vanishing components

$$h_{11}=2\left(\partial_r\xi_1-\frac{g'(r)}{2g(r)}\xi_1\right)=0. \tag{C1}$$

Therefore, noting that the form of the perturbation (22) specifies the time dependence,

$$\xi_1=g(r)^{1/2}e^{\omega t}\tilde{\xi}_1(\tilde{x}). \tag{C2}$$

Now consider the vanishing component

$$h_{00}=2\left(\partial_t\xi_0-\frac{f'(r)}{2g(r)}\xi_1\right)=0. \tag{C3}$$

Together with the form we found for ξ_1 this then implies that

$$\xi_0=\frac{f'(r)}{2\omega g(r)^{1/2}}e^{\omega t}\tilde{\xi}_1(\tilde{x}). \tag{C4}$$

Next consider the vanishing component

$$h_{01}=\partial_t\xi_1+\partial_r\xi_0-\frac{f'(r)}{f(r)}\xi_0=0. \tag{C5}$$

If we now substitute the expressions we have for ξ_0,ξ_1 into this equation, we obtain a differential equation for $g(r)$ and $f(r)$ that is manifestly not satisfied by any of the functions we used in this work. This implies that $\xi_0=\xi_1=0$. Now considering the vanishing components

$$h_{0\alpha}=\partial_t\xi_\alpha+\partial_\alpha\xi_0=0, \tag{C6}$$

we see that $\xi_0=0$ implies that ξ_α is independent of t. But if ξ_α is nonzero then it must depend on t as $e^{\omega t}$ in order for the $h_{\alpha\beta}$ term to have required time dependence. The conclusion is thus that $\xi_a=0$ and therefore the perturbations we are considering can never be pure gauge.

In the section on time dependent metrics we took $f=0$ and the metric had no t components. The argument of the previous paragraph will not work in this case. The expression for ξ_1 (C2) is now just $\xi_1 = g(r)^{1/2}\tilde{\xi}_1(\tilde{x})$. We can substitute this into the component

$$h_{1\alpha} = \partial_r \xi_\alpha + \partial_\alpha \xi_1 - \frac{2}{r}\xi_\alpha = 0. \tag{C7}$$

Solving for ξ_α gives

$$\xi_\alpha = \left[-r^2 \int^r \frac{g(s)^{1/2}}{s^2} ds \right] \partial_\alpha \tilde{\xi}_1(\tilde{x}) \equiv K(r) \partial_\alpha \tilde{\xi}_1(\tilde{x}). \tag{C8}$$

Finally, substitute these results into the remaining, nonvanishing, component and recall the form of the perturbation (66) and the definition of $K(r)$ in Eq. (C8):

$$\begin{aligned} h_{\alpha\beta} &= \tilde{\nabla}_\alpha \xi_\beta + \tilde{\nabla}_\beta \xi_\alpha + \frac{2r}{g(r)} \tilde{g}_{\alpha\beta}(\tilde{x}) \xi_1 \\ &= 2K(r) \tilde{\nabla}_\alpha \partial_\beta \tilde{\xi}_1(\tilde{x}) + \frac{2r}{g(r)^{1/2}} \tilde{g}_{\alpha\beta}(\tilde{x}) \tilde{\xi}_1(\tilde{x}) \\ &= r^2 \varphi(r) \tilde{h}_{\alpha\beta}(\tilde{x}). \end{aligned} \tag{C9}$$

For the last two lines to be equal, we must have either $K(r) \propto r/g(r)^{1/2}$ or $h_{\alpha\beta} \propto \tilde{g}_{\alpha\beta}\tilde{\xi}_1 \propto \tilde{\nabla}_\alpha \partial_\beta \tilde{\xi}_1$. The former possibility is not true for the functions $g(r)$ that we have been considering while the latter is not consistent with $h^\alpha{}_\alpha = 0$. Therefore there are no pure gauge solutions. We could have used this argument in the previous case also, but the argument we gave instead did not use the trace-free property at any point.

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