FRW wormhole instantons in the non-Abelian Born-Infeld theory

P. Vargas Moniz*

Departamento de Fisica, Universidade da Beira Interior, Rua Marquês d'Avila e Bolama, 6200 Covilhã, Portugal

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A family of wormhole instanton solutions is obtained in the non-Abelian Born-Infeld theory in the presence of a positive cosmological constant. These wormholes have a closed Friedmann-Robertson-Walker geometry and a nontrivial SO(4)-symmetric gauge sector. It is found that when the matter sector in the action is dominated by Born-Infeld terms inducing a string gas behavior, then no wormhole solutions are admissible. However, when the gauge fields asymptotically lead to a radiation scenario, wormhole solutions become possible. The presence of Born-Infeld perturbations in this stage determines specific modifications for the wormhole dynamics. Some of these physical implications are consequently discussed. In particular, we comment on how they can modify the quantization in energy levels of the wormhole solution. We also mention how this may affect the quantization of topological charges and conservation of fermion number.

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I. INTRODUCTION

Actions of the Born-Infeld type have been the subject of wide interest in the context of M and string theory [1]. This comes from the result that the effective action for the open string ending on D-branes can be written in a Born-Infeld form. Moreover, the consistency of the σ model for the world sheet of the string is shown to require that the brane be described by a Born-Infeld action type, just as the general curved background requiring consistency of string theory leads to the Einstein-Hilbert action [2].

The inclusion of a non-Abelian gauge sector is not unique. In fact, different definitions of non-Abelian Born-Infeld Lagrangians are possible, regarding the way of tracing the group indices. A relevant suggestion was proposed by Tseytlin [3], afterward called the symmetrized trace. Other proposals exist [3,4]. A useful trace operation was proposed by Gal'tsov and Kerner [4,5], where the trace is done over the group indices but under a square root sign. This useful version amounts to defining the gauge field Lagrangian in the form

$$L_{BI}^{tr} \sim \beta^2 \mathrm{Tr} \times \left[1 - \sqrt{1 + \frac{1}{2\beta^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{16\beta^4} (F_{\mu\nu}\tilde{F}^{\mu\nu})^2} \right], \tag{1}$$

where β denotes the maximal field strength and $\tilde{F}^{\mu\nu}$ the dual of $F^{\mu\nu}$. Hereafter, we choose the square root ordinary trace Lagrangian for its analytical simplicity.

Classical solutions in the Born-Infeld theory are important in the understanding of brane dynamics regarding the gravitational interaction. In this respect, dyons, monopoles, and other solitonlike solutions have recently attracted much attention [6]. In the existing literature there are several papers discussing cosmological models with U(1) Born-Infeld matter [7]. Such models are necessarily anisotropic (or inhomogeneous) since there is no homogeneous and isotropic configuration of the classical U(1) field. Non-Abelian Born-Infeld cosmologies were recently investigated by Dyadichev *et al.* for Friedmann-Robertson-Walker (FRW) models [8]. For an SU(2) gauge field and using a spherically symmetric ansatz (extended from Ref. [9]), the authors obtained a complete description of the space of solutions. The effective equations of motion interpolate between $p = -\rho/3$ in the regime of strong field (i.e., large energy densities $\rho \gg \rho_c \equiv \beta/4\pi$, $\rho \simeq a^{-2}$ near the singularity) and $p = \rho/3$ in the regime of weak field (i.e., small energy densities $\rho \ll \rho_c$, $\rho \simeq a^{-4}$ near large times).

Among other cosmological scenarios where the Born-Infeld theory could determine interesting physical effects are wormhole instanton solutions [10–13]. Wormholes constitute nonsingular solutions of the Euclidean Einstein equations of motion. They correspond to manifolds with $\mathrm{Tr}K = 0$ on the boundary (where K is the trace of the extrinsic curvature). Wormhole solutions do not exist for any type of geometry and/or matter. In fact, it seems that, for such solutions to exist, the eigenvalues of the effective Ricci tensor or the traceless energy-momentum tensor

$$R_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \equiv \overline{T}_{\mu\nu}$$

must be negative somewhere (see Refs. [14,15]) in the Euclidean regime. These wormhole solutions connect two asymptotically flat regions of Euclidean space between two Lorentzian solutions [10–13,16–20]. In some conditions, wormhole type solutions describe transitions from S^3 universes with small radius r_{min} to universes with larger radius r_{max}. This corresponds to tunneling through a potential barrier in a finite conformal time with an arbitrary number of bounces in the process. Wormhole solutions have received quite some attention since the Coleman mechanism for the vanishing of the cosmological constant was described [21].

*Also at CENTRA, IST, Rua Rovisco Pais, 1049 Lisboa Codex, Portugal. Email address: pmoniz@dfisica.ubi.pt; URL: http://www.dfis.ubi.pt/~pmoniz

The study of these solutions in the framework of the Born-Infeld theory would advance our knowledge of their dynamics and gravitational consequences, in particular, if we consider the evolution of the very early universe within a more realistic perspective, through the combined influence of brane/string, gauge, and gravity effects. So far, wormhole instanton solutions have only been investigated in the framework of standard Einstein-Yang-Mills (EYM) theory with different choices for the gauge sector in Refs. [17–20] (see also Refs. [22,23]). Another particular reason to investigate wormholes in a non-Abelian Born-Infeld theory is as follows. Massless Yang-Mills dynamics are determined by a scale invariant Lagrangian which implies that Yang-Mills excitations would be diluted during the universe expansion. But Born-Infeld corrections to the Yang-Mills action would break it. The evidence for this effect was investigated in a classical setting in Ref. [8]. From a quantum cosmological perspective, similar modifications and their consequences were discussed in [25]. In addition, two papers [24] have recently appeared where instead of an ordinary scalar field, the authors considered a quantum model of gravitation interacting with a minimally coupled nonlinear scalar field, inspired by a Born-Infeld type Lagrangian.

Motivated and supported by the arguments in the previous paragraphs, we report in this paper on a family of wormhole instanton solutions in the context of non-Abelian Born-Infeld theory. These solutions have a closed FRW geometry and the gauge sector is characterized by an SO(4)-symmetric gauge field [18,22]. Our results indicate that when string effects (implied by the Born-Infeld part of the action) dominate, no wormhole solutions are possible. But when the gauge fields evolve in such a way that radiation becomes important, then we obtain wormhole solutions. These will have features similar to those found in standard EYM theory [17–20] (see also Refs. [22,23]), but will be influenced by specific modifications determined by Born-Infeld correction terms. We then discuss some of the physical implications of these perturbations, namely, those concerning the quantization of energy levels of the wormhole instanton.

II. EQUATIONS OF MOTION

Let us employ the following action where we adopt the formulation and conventions introduced in Ref. [8], following as well the framework of Refs. [18,22] for the Lie algebra structure of the gauge fields:

$$S=-\frac{1}{4\pi}\left[\int\frac{(R-2\Lambda)}{4G}\sqrt{(-g)}d^4x-\beta^2(\mathfrak{R}-1)\right] \tag{2}$$

where

$$\mathfrak{R}=\left[1+\frac{1}{2\beta^2}F^a_{\mu\nu}F^{a\mu\nu}-\frac{1}{16\beta^4}(\tilde{F}^a_{\mu\nu}F^{a\mu\nu})^2\right]^{1/2}. \tag{3}$$

We assume a closed $k=+1$ FRW geometry where the space-time M^4 is of the form $M^4=R\times S^3$, Λ is a positive cosmological constant, S^3 is the three-dimensional sphere, and SO(4) is the group of spatial homogeneity and isotropy. The most general form of a SO(4) invariant Euclidian metric for the manifold M^4 is

$$g=N^2d\tau^2+a^2(\tau)\sum_{b=1,2,3}\omega^b\otimes\omega^b, \tag{4}$$

where $\tau\in\mathbf{R}$ is the Euclidean FRW time and ω^b are left invariant one-forms satisfying

$$d\omega^b=-\tfrac{1}{2}\varepsilon_{bcd}\omega^c\wedge\omega^d. \tag{5}$$

It is further needed to find the class of gauge fields which are consistent with the above spatial homogeneity and isotropy. For theories with local internal symmetries the conditions of SO(4) invariance are too restrictive. For the physical observables to be SO(4) invariant, the fields with gauge degrees of freedom may transform under $SO(4)$ transformations *if* those changes can be compensated by a gauge transformation. This is so since the physical observables are gauge invariant quantities. A most useful class satisfying the above conditions are the so called SO(4)-symmetric fields, i.e., fields invariant up to a gauge transformation. These have been thoroughly employed before (see Refs. [18,22]). Hence, we take for the gauge field the ansatz of the SO(4) symmetric field, i.e.,

$$\mathbf{A}_\mu(t)=\sum_{1\leqslant k<i\leqslant N-3}\Lambda^{ki}(t)T^{(3)}_{k+3,i+3}dt+\tfrac{1}{4}[1+f_0(t)]\varepsilon_{acb}T^{(3)}_{ab}\omega^c, \tag{6}$$

where $f_0(t)$ and $\Lambda^{ki}(t)$ are an arbitrary function and antisymmetric matrix, respectively, and $T^{(3)}_{ab}$ are the generators of the Lie group SO(3).

The reduced action or Lagrangian will be

$$L=\frac{1}{4\pi}\left[\frac{1}{4G}(-6)\frac{\dot{a}a^2}{N}-\frac{1}{4G}\frac{3}{2}aN+\frac{1}{4G}2\Lambda Na^3+\beta^2Na^3(\mathfrak{R}-1)\right] \tag{7}$$

with

$$\mathfrak{R}=\left[1+\frac{3\dot{f}_0^2}{2\beta^2a^2N^2}+\frac{3V_1}{\beta^2a^4}-\frac{9\dot{f}_0^2V_1}{2\beta^4a^6N^2}\right]^{1/2}. \tag{8}$$

Following the conventions and definitions introduced in Ref. [8], we can further write

$$L=\frac{1}{4\pi Ge}\left[-\frac{3}{2}\frac{\dot{a}a^2}{N}-\frac{3}{8}aN+Na^3\frac{\Lambda}{2}+gNa^3(\mathfrak{R}-1)\right], \tag{9}$$

with $g\equiv G\beta$, after having redefined $t\rightarrow\beta^{-1/2}t$ and $a\rightarrow\beta^{-1/2}a$. In addition, we now have

$$\mathfrak{R}=\left[1+\frac{3\dot{f}_0^2}{2a^2N^2}+\frac{3V_1}{a^4}+\frac{9\dot{f}_0^2V_1}{2a^6N^2}\right]^{1/2} \tag{10}$$

$$=[(1+K^2)(1+V^2)]^{1/2}, \tag{11}$$

where

$$K^2\equiv\frac{3\dot{f}_0^2}{2a^2N^2}, \tag{12}$$

$$V^2=\frac{3V_1}{a^4}, \tag{13}$$

with

$$V_1=V_{gauge\ field}=\tfrac{1}{8}(1-f_0^2)^2. \tag{14}$$

The Friedmann equation is thus given by

$$\tfrac{3}{2}\dot{a}^2-\tfrac{3}{8}+\tfrac{1}{2}\Lambda a^2+ga^2(\mathsf{P}-1)=0, \tag{15}$$

with the definition

$$\mathsf{P}=\frac{\sqrt{1+V^2}}{\sqrt{1-K^2}}. \tag{16}$$

The Friedmann equation can be presented in a more suitable form as

$$\frac{\dot{a}^2}{a^2}-\frac{1}{4a^2}+\frac{\Lambda}{3}=-\beta G\frac{2}{3}(\mathsf{P}-1)\equiv-\frac{8\pi G}{3}\varepsilon, \tag{17}$$

where we further define

$$\varepsilon=\varepsilon_c(\mathsf{P}-1),\quad \varepsilon_c=\frac{\beta}{4\pi}. \tag{18}$$

The Einstein equation is given by

$$\ddot{a}+\frac{\dot{a}^2}{2a}-\frac{1}{8a}+\frac{\Lambda a}{2}+\frac{ga}{3}\mathsf{P}+\frac{ga}{3}\mathsf{P}^{-1}-ga=0. \tag{19}$$

If we use the relation

$$\mathsf{P}=\frac{\varepsilon}{\varepsilon_c}+1, \tag{20}$$

together with the Friedmann equation, then Eq. (19) can be further presented as

$$\begin{aligned}\frac{\ddot{a}}{a}&=-\frac{\Lambda}{3}+\frac{4\pi G}{3}\varepsilon-\frac{g}{3}\left(\frac{\varepsilon}{\varepsilon_c}+1\right)\\&\quad-\frac{2g}{3}\left(\frac{\varepsilon}{\varepsilon_c+\varepsilon}\right)+g\\&=-\frac{\Lambda}{3}+\frac{4\pi G}{3}(\varepsilon+3P),\end{aligned} \tag{21}$$

with P being the pressure

$$P=\tfrac{1}{3}\varepsilon_c(3-\mathsf{P}-2\mathsf{P}^{-1}), \tag{22}$$

satisfying the interesting relation

$$P=\frac{\varepsilon}{3}\left(\frac{\varepsilon_c-\varepsilon}{\varepsilon_c+\varepsilon}\right). \tag{23}$$

As far as the gauge field is concerned, its corresponding equation takes the form

$$\ddot{f}_0=\dot{f}_0\frac{\dot{a}}{a}\left[-2\frac{(1+K^2)}{(1+V^2)}+1\right]+\frac{1}{a^2}\frac{dV_1}{df_0}\frac{(1+K^2)}{(1+V^2)}. \tag{24}$$

We also derive the equation

$$\dot{\varepsilon}=-2\frac{\dot{a}}{a}\varepsilon\frac{2\varepsilon_c+\varepsilon}{\varepsilon_c+\varepsilon}, \tag{25}$$

followed by

$$\dot{\mathsf{P}}=2\frac{\dot{a}}{a}(\mathsf{P}^{-1}-\mathsf{P})\Rightarrow\mathsf{P}=\pm\sqrt{1+\frac{\mathsf{C}}{a^4}}. \tag{26}$$

These enable us to retrieve the expression

$$a^4(2\varepsilon_c+\varepsilon)\varepsilon=C, \tag{27}$$

where C and C are constants of integration. Furthermore, we can write

$$\varepsilon^2+2\varepsilon\varepsilon_c=\frac{C}{a^4}\Rightarrow\varepsilon=-\varepsilon_c\pm\sqrt{\varepsilon_c^2+\frac{C}{a^4}}, \tag{28}$$

with $\varepsilon_c^2\mathsf{C}=C$. Finally, after some calculations, the following decoupled equation for the scale factor is obtained:

$$\ddot{a}+\frac{a}{3}(-2g+\Lambda)-\frac{4g^2a^3/3}{3\left(\dot{a}^2-\frac{1}{4}\right)+a^2(-2g+\Lambda)}=0. \tag{29}$$

This expression will play an important role regarding the identification of wormhole solutions under the influence of a string gas induced by the Born-Infeld matter sector in the action.

Before proceeding to the next section, let us point out that expressions (7)–(29) correspond to a physical system *different* from that recently investigated in Ref. [8]. There are of course some similarities, but these are a consequence of the following. We intentionally employed the framework and definitions introduced in [8] in order that the Euclidean dynamics of FRW instantons in the non-Abelian Born-Infeld theory adequately complements and joins the Lorentzian analysis. Consequently, there should be some similarities with the Lorentzian expressions presented in Ref. [8]. There the authors have thoroughly described the phase space of solutions by means of a dynamical system analysis, as well as providing approximate solutions (power series expansion or numerical) for the scale factor and exact solutions for the gauge field. In our paper, we will follow a similar presentation but without providing a dynamical system analysis. Let us just stress that we employ an Euclidean framework with a SO(3) gauge sector satisfying the SO(4)-symmetric algebra formulation defined in Refs. [18,22]. This framework implies important differences, namely, concerning signs in, e.g., Eqs. (15), (17), (19), (21), (24), and (29). As can easily be checked, it will be those differences that determine another dynamical behavior, allowing for the existence of wormhole solutions bounded between values of $a_{\min}$ and $a_{\max}$.

III. ANALYSIS OF THE FRIEDMANN EQUATION

In this section we consider the Friedmann equation to which we can identify two limiting cases: Yang-Mills radiation or a gas of strings derived from the Born-Infeld terms in the action (9)

$$\dot{a}^2-\frac{1}{4}+\frac{\Lambda}{3}a^2=-\frac{8\pi G}{3}\begin{cases} C\leftarrow\text{string effects}\\ \dfrac{C}{a^2}\leftarrow\text{YM radiation.}\end{cases} \tag{30}$$

The latter limit is consistent with Refs. [18,22] for the standard Einstein-Yang-Mills (radiation) scenario. For this YM radiation limit, Eq. (30) can be expressed as

$$a'^2-\frac{a^2}{4}+\frac{\Lambda}{3}a^4=-\frac{8\pi G}{3}C, \tag{31}$$

using the Euclidean conformal time $d\eta=a^{-1}(\tau)d\tau$. This equation represents the motion of a unit mass particle with energy $-(8\pi G/3)C$. We then conclude that wormholes are possible in this limit. However, for string/brane domination (corresponding to a gas of strings), this procedure shows that we obtain solutions which always meet the singularity. Given this physical context, the subsequent and appropriate direction should therefore be to investigate if and how the presence of string gas terms (considered then as perturbations) can modify the wormhole dynamics that could eventually form.

To be consistent and strengthen our analysis, we need first to establish if such wormholes (influenced by correction terms corresponding to the perturbative presence of a string gas) can indeed be formed. In order to address this issue, we will analyze in detail the dynamics retrieved from the Friedmann equation. Taking Eq. (17) and using Eq. (18) for the energy density ε, we write it as

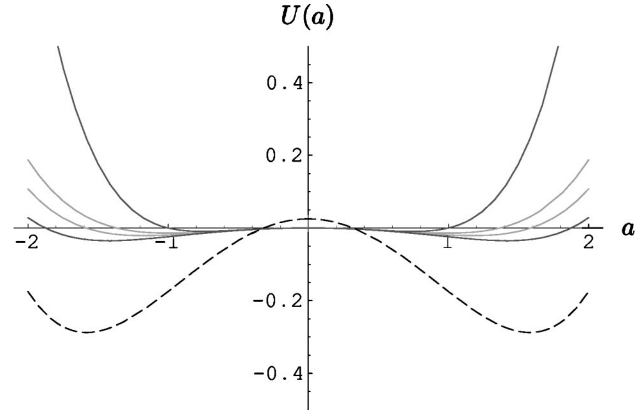

FIG. 1. A plot of U is presented for different values of the cosmological constant. If the cosmological constant is negligible, the Born-Infeld square root term seems to reduce the dynamical influence of the extrema.

$$\dot{a}^2-\frac{1}{4}+\frac{\Lambda}{3}a^2=-\frac{8\pi G}{3}\left[\varepsilon_c+\sqrt{\varepsilon_c^2+\frac{C}{a^4}}\right]a^2. \tag{32}$$

In conformal time we can have Eq. (32) as

$$a'^2-\frac{1}{4}a^2+\frac{\Lambda}{3}a^4=\left(\frac{8\pi G}{3}\varepsilon_c a^2-\frac{8\pi G}{3}\sqrt{\varepsilon_c^2a^4+C}\right)a^2, \tag{33}$$

i.e.,

$$a'^2+U(a)=0, \tag{34}$$

with

$$U(a)=-\frac{1}{4}a^2+\left(\frac{\Lambda}{3}-\frac{8\pi G}{3}\varepsilon_c\right)a^4 + \frac{8\pi G}{3}\sqrt{\varepsilon_c^2a^8+Ca^4}, \tag{35}$$

noticing that $\sqrt{\varepsilon_c^2a^4+C}=\varepsilon_c\sqrt{1+(C/\varepsilon_c^2a^4)}a^4$. A plot of $U(a)$ is presented in Fig. 1 for different values of Λ, with the dashed line corresponding to the case without the square root term.

Unfortunately, in either Euclidean FRW or conformal time it is impossible to find analytical *exact* solutions for this Friedmann equation. We recall that wormholes are known to exist in the case when solely gauge fields (radiation) are present without any modifications in the action and corresponding equations. In this case (see Refs. [17–20,22,23]) wormhole solutions are found in the form of elliptc integrals [26,27]. Notice that

$$\sqrt{\varepsilon_c^2+\frac{C}{a^4}}=\varepsilon_c\sqrt{1+\frac{C}{\varepsilon_c^2a^4}}\simeq\varepsilon_c\left(1+\frac{1}{2}\frac{C}{\varepsilon_c^2a^4}\right),$$

when a is large, and that $P=\frac{1}{3}\rho$ in that limit.

Let us proceed with our twofold purpose outlined above: to determine whether wormholes can indeed form when the gauge fields induce a radiation behavior still influenced by a string like perturbation and to describe how these modifications brought in from the Born-Infeld matter sector can imply a different wormhole dynamics. We will consider two complementary approaches. On the one hand, we shall perform a numerical analysis whose results are described in the next section. On the other hand, the string gas stage corresponds to a strong field regime with large energy density. Its influence would be mostly relevant near the singularity at $a\simeq 0$. Therefore, we propose to employ an approximate simplified model whose Friedmann equation will provide an adequate alternative to integrating Eq. (32) or Eqs. (33)–(37) [see also Eqs. (30), (31)] by representing the string gas perturbative influence. To be more precise, our idea is to have on the right hand side a set of terms that will physically represent the contributions of radiation as well as of a string gas. The presence of coefficients will allow us to use these contributions either as dominant or as a perturbation as intended. The corresponding Friedmann equation in FRW Euclidean time will thus be given by

$$\dot{a}^2-\frac{1}{4}+\frac{\Lambda}{3}a^2=-\frac{E}{a^2}-F \tag{36}$$

and written in conformal time as

$$a'^2-\frac{1}{4}a^2+\frac{\Lambda}{3}a^4=-E-Fa^2. \tag{37}$$

In these expressions E,F are, respectively, constants associated with a string gas and radiation (Yang-Mills) perfect fluid contributions. We further write this equation in a simple manner as

$$a'^2-a^2G+Ha^4+E\equiv a'^2+W(a)=0, \tag{38}$$

with $\frac{1}{4}-F\equiv G$ and $H\equiv\Lambda/3$, noticing the similarities with Eqs. (31), (33)–(35). For different choices of E,G,H we can obtain scenarios where the string gas clearly dominates or where radiation is more important, possibly perturbed by Born-Infeld modifications. In particular, we can satisfactorily reproduce the behavior of $U(a)$ near the singularity and also where the *Lambda* term is dominant (see Figs. 1 and 4 as well). The advantage is to have new terms that will approximately play the effective role of the square root in Eqs. (33)–(35). The square root is the reason why Eq. (32) or Eq. (34) cannot be exactly integrated. However, Eq. (37) admits exact analytical solutions as we will describe in the next section.

IV. ANALYSIS OF WORMHOLE SOLUTIONS

As previously mentioned, wormhole solutions, representing the behavior of the scale factor as a function of τ or η, interpolating between some $a_{\min}$ and $a_{\max}$, are not possible to find in a suitable analytical form. In fact, Eqs. (17), (30), (32), and (33) have no known *exact* expressions as solutions.

Wormhole solutions in terms of analytical expressions can, however, be obtained within the approximate scenario introduced with Eq. (38) and described in the previous section. With conformal time, we get the following equation to integrate:

$$\int d\eta=\frac{1}{2}\int\frac{dy}{\sqrt{G-yH-E/y}},$$

with $a^2=y$, getting solutions in the form of elliptic integrals. In the Euclidean time τ we have instead

$$\int d\tau=\pm\frac{1}{2}\int\frac{dy}{\sqrt{yG-y^2H-E}},$$

and solutions are given by

$$\tau-\tau_0=\pm\frac{1}{2}\frac{1}{\sqrt{H}}\left(\arcsin\frac{2\frac{H}{\frac{1}{4}-F}a^2-1}{\sqrt{1-4\frac{H}{\frac{1}{4}-F}\frac{E}{\frac{1}{4}-F}}}\right). \tag{39}$$

Equation (39) clearly suggests that wormhole solutions (bearing the influence of the string of gas imposed by the Born-Infeld modifications) indeed can exist in our model within the Lagrangian (9). The influence of Born-Infeld string perturbations is represented in the coefficient F. In contrast to the situation of solely radiation, the presence of string gas corrections seems to lead to either a wider or a shorter difference between $a_{\max}$ and $a_{\min}$. This may imply that the frequency of oscillations (or transitions) between the turning points can increase or not, depending on the energy of the wormhole. A discussion regarding the dynamical modifications determined by the Born-Infeld string corrections is presented in Sec. V.

As far as showing that such wormholes do exist, Eq. (39) constitutes a satisfactory insight, as argued in Sec. III. However, we can extend and strengthen it by a perhaps more adequate means for evidence of these wormholes. In fact, the discussion in this section can be further improved as follows. The fact that wormholes (influenced by Born-Infeld string corrections) do exist is retrieved from the *original* action (9). Let us consider instead Eq. (29). Employing a new time variable given by (following the notation and procedure in Ref. [8])

$$dt=3[(-2g+\Lambda)a^2+3(b^2-1)]d\tau, \tag{40}$$

with $b=\dot{a}$, Eq. (29) can be shown to be equivalent to

$$\begin{aligned}b'=[&-\Lambda(-4g+\Lambda)a^2\\&-3(b^2-1)(-2g+\Lambda)]a,\end{aligned} \tag{41}$$

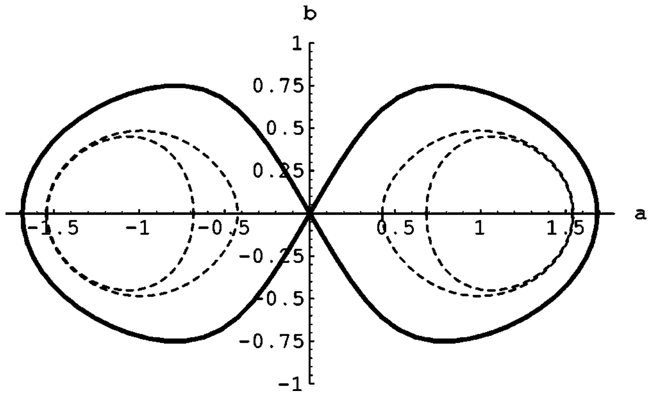

FIG. 2. Periodic regimes exist corresponding to wormhole solutions and are represented for different values of $\mathcal{C}$. The bold separatrix separates singular from nonsingular solutions (inside it) and these are the wormhole solutions.

$$a' = 3[(-2g+\Lambda)a^2 + 3(b^2-1)]b. \tag{42}$$

The integral trajectory curves are subsequently given by

$$3\left(b^2+\frac{\Lambda}{3}-1\right)^2 - 4ga^2\left(b^2+\frac{\Lambda}{3}-1\right) = \mathcal{C}. \tag{43}$$

A numerical analysis shows that there are singular solutions, but also periodic regimes, with limits corresponding to $a_{\rm min}^{\rm max} \neq 0$, depending on the value of $\mathcal{C}$ as shown in Fig. 2. Thus, such wormholes are indeed possible within the physical context of the original action (9). The Lorentzian geometries were described in Ref. [8]. This family of Euclidean solutions described herein interpolate and tunnel between the singular oscillating Lorentzian solution and others approaching a de Sitter–like behavior for very large a. An approximate generic solution for these wormholes can be given by

$$a \simeq \sqrt{\frac{1}{2H}\sqrt{1-4\frac{H}{\frac{1}{4}-F}\frac{E}{\frac{1}{4}-F}}\cos[2\sqrt{H}(\tau-\tau_0)]+1}, \tag{44}$$

with E,F such that

$$\frac{1-4\frac{H}{\frac{1}{4}-F}\frac{E}{\frac{1}{4}-F}}{2H} < 1. \tag{45}$$

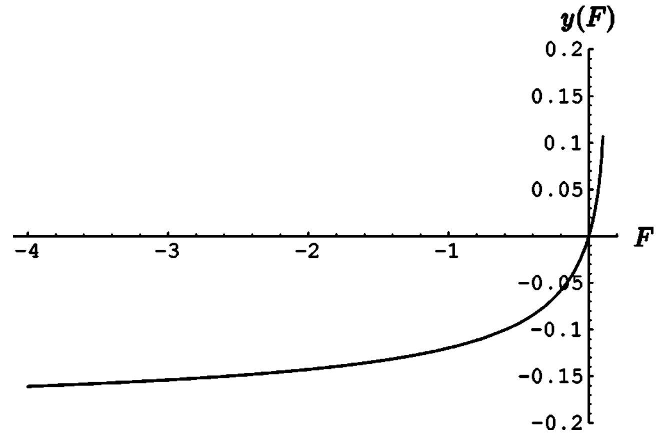

FIG. 3. The difference in the turning point close to the singularity between solely radiation and string corrections.

V. INFLUENCE OF BORN-INFELD CORRECTIONS

The influence of the terms associated with the string gas stage in the wormhole physical properties will be discussed in this section. The turning points of the potential

$$W(a) = -a^2G + Ha^4 + E \tag{46}$$

are located at

$$a_{\rm min}^{\rm max} = \frac{\sqrt{(\frac{1}{4}-F) \pm \sqrt{(\frac{1}{4}-F)^2 - 4(\Lambda/3)E}}}{\sqrt{2\Lambda/3}}. \tag{47}$$

We are still assuming (see Secs. III and IV) that it is mostly for $a_{\rm min}$ (i.e., the turning point near the singularity) that the influence of a Born-Infeld perturbative effect is important, and further we take E as a positive constant. We recall that the coefficient F represents the string gas pertubative effect. The upward right hand side of the potential is dominated by the Λa^4 term. Therefore, near the region of small values of a this model will reflect the influence of the string gas dominant terms derived from a Born-Infeld type action. The plot of $a_{\rm min}(F\neq 0) - a_{\rm min}(F=0) \equiv y(F)$ is presented in Fig. 3 for fixed values of E,Λ. The difference

$$y(F) \equiv \frac{\sqrt{(\frac{1}{4}-F) - \sqrt{(\frac{1}{4}-F)^2 - 4(\Lambda/3)E}} - \sqrt{(\frac{1}{4}) - \sqrt{(\frac{1}{4})^2 - 4(\Lambda/3)E}}}{\sqrt{(2\Lambda/3)}} \tag{48}$$

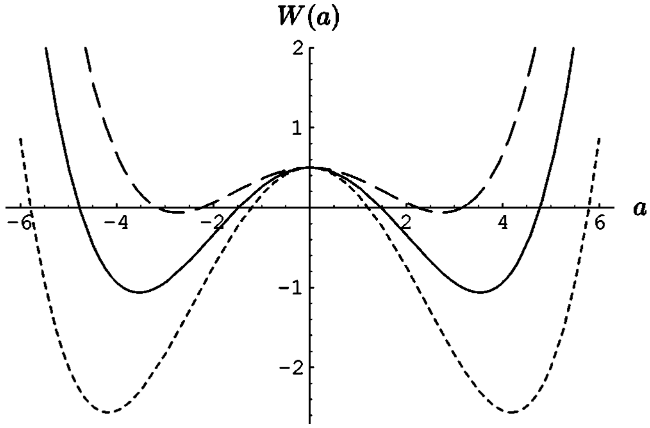

FIG. 4. A plot of W as a function of the scale factor, for different values of F and for fixed values of E and H. This exemplifies the influence of Born-Infeld string correction terms when wormholes form. The continuous line corresponds to $F=0$.

is mostly negative but it is also possible to be positive. This means that $a_{\min}(F\neq 0)$ can be nearer to (or sometimes further from) the singularity in $a=0$ than $a_{\min}(F=0)$ corresponding to the standard EYM radiation case (cf. Refs. [17–20,22]). We thus can have a widening (or shortening) effect concerning the turning points of the potential. From Fig. 4, for $F=0.1$ the potential is shorter between the turning points, $a_{\min}$ is farther away from the singularity at $a=0$, the area covered between the a axis and the potential line is smaller, and the minimum is higher. For $F=-0.1$, the potential is wider between the turning points, $a_{\min}$ is closer to the singularity, the minimum is lower, and the corresponding area is larger. These physical changes will definitely modify the wormhole dynamics. In particular, there are possible energy levels that will have a different separation from the case when the string gas influence is totally absent. Similarly to other bounded systems, it is known that wormholes have a discrete spectrum of energy levels (see Refs. [18,20,22]). This has been proved within EYM scenarios. But when wormholes form still subject to such string perturbative effects, their energy levels and corresponding discretized turning points will have important modifications. In fact, the energy level separation, e.g., in a potential $V(x)=kx^2$, is proportional to the frequency $\omega\sim\sqrt{k/m}$. As k increases, the potential curve is less wide, thus increasing the separation in the energy levels. It is therefore (in view of the previous sections) reasonable to expect that in our wormhole case there are scenarios where the energy level separation will be wider than with no string gas. We will additionally have different discretized turning points for each energy level. In other situations, the energy levels will be almost compactified to a continuous range and the wormhole almost negligible as $a_{\min}\simeq a_{\max}$.

Of course, a more formal comparison with the standard Einstein-Yang-Mills case in Refs. [18,22] is only possible if exact solutions are found for Eq. (33) and the period of oscillation, which will depend on $\sqrt{1-(r_{\min}/r_{\max})^2}$, is analyzed. The analysis of the the Euclidean Yang-Mills equation (24) is also required, but those solutions are expected to be in the form of Jacobi elliptic integrals (see Refs. [8,26,27]).

VI. DISCUSSION AND CONCLUSIONS

Summarizing, the issue of wormhole instanton solutions within non-Abelian Born-Infeld theory was investigated in this paper. Wormhole solutions connect two asymptotically flat regions of Euclidean space and correspond to a tunneling between two Lorentzian solutions through a potential barrier in a finite conformal time.

The Born-Infeld theory [1] has recently been widely used to analyze the physical behavior of strings and branes from a gravitational perspective. In particular, the low energy effective theory of D-branes can have a non-Abelian formulation, suggesting specific modifications to the standard Einstein-Yang-Mills action. Classical [7,8] and quantum solutions [24,25] of the Born-Infeld theory applied to cosmology may thus be useful in understanding brane dynamics and their implications toward different physical systems like, e.g., wormhole solutions.

Wormhole instanton solutions were found in this paper within the non-Abelian Born-Infeld theory, either from investigating physically reasonable approximate models or with a numerical analysis. These wormhole solutions have a closed FRW geometry and the matter content corresponds to a cosmological constant and an SO(4)-symmetric gauge sector [18,22]. Moreover, the solutions describe transitions from S^3 universes with small radius $a_{\min}$ to universes with larger radius $a_{\max}$, in the presence of radiation but influenced by Born-Infeld string-type modifications to the effective action. When the string gas ($P=-\frac{1}{3}\rho$) dominates the dynamics, wormhole solutions are effectively absent. It therefore seems that only when radiation behavior becomes dominant ($P=\frac{1}{3}\rho$) (although in the presence of Born-Infeld type perturbations) can wormholes be found. It is tempting to speculate whether an intertwined behavior of gravity and gauge fields (leading to either radiation or string gas limits) in the context of a Born-Infeld description might be responsible for creating the physical conditions (see Refs. [14,15]) where wormholes can form or not.

Our results suggest that the presence of Born-Infeld corrections modifies the possible quantization of energy levels corresponding to the wormhole solution. The effect of Born-Infeld string-type modifications in the energy level spectrum can also be discussed from the following point of view [20]. It is know that the quantization of these energy levels can be thought of as quantization conditions for the topological charge $Q\sim\int_{R\times S^3}\mathrm{Tr}(F\wedge F)$ (see also Ref. [18]), which in general is not an integer. In the standard EYM case, the topological charge Q is a function of the discretized turning points (see Ref. [20]) for both the scale factor and gauge fields. These are determined through the quantization of the energy levels. The corresponding discrete levels of Q are related to distinct vacuum states for the gauge fields. It is also known that $\mathrm{Tr}(F\wedge F)$ acts as a source for the fermion current conservation equation. Hence, it can be said that if there is a change in the energy level spectrum of the wormhole, this will modify the admissible turning points (labeled by an integer). Consequently, the topological charge will bear some modifications as well. Moreover, any such change in the topological number will imply an associated change in the fermion number.

Finally, we are aware that further research on these issues is in fact surely needed. Nevertheless, our results point to interesting wormhole aspects that are a consequence of the Born-Infeld matter action. It is hoped that that they will constitute the motivation for subsequent studies. The features characterizing the family of wormhole solutions herein described can be expected to be present also in a broader study of non-Abelian Born-Infeld wormhole dynamics.

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