Perturbations on a moving D3-brane and mirage cosmology

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(Received 18 June 2002; published 25 September 2002)

We study the evolution of perturbations on a moving probe D3-brane coupled to a four-form field in an AdS_5-Schwarzschild bulk. The unperturbed dynamics are parametrized by a conserved energy E and lead to a Friedmann-Robertson-Walker (FRW) ''mirage'' cosmology on the brane with a scale factor $a(\tau)$. The fluctuations about the unperturbed worldsheet are then described by a scalar field $\phi(\tau,\vec{x})$. We derive an equation of motion for ϕ, and find that in certain regimes of a the effective mass squared is negative. On an expanding Bogomol'nyi-Prasad-Sommerfield (BPS) brane with $E=0$ superhorizon modes grow as a^4 while subhorizon modes are stable. When the brane contracts, all modes grow. We also briefly discuss the case when $E>0$, BPS antibranes as well as non-BPS branes. Finally, the perturbed brane embedding gives rise to scalar perturbations in the FRW universe. We show that ϕ is proportional to the gauge invariant Bardeen potentials on the brane.

DOI: 10.1103/PhysRevD.66.063510 PACS number(s): 98.80.Cq, 04.25.Nx

I. INTRODUCTION

The idea that our universe may be a three-brane embedded in a higher dimensional space-time is strongly motivated by string and M theory, and it has recently received a great deal of attention. Much work has focused on the case in which the universe 3-brane is of codimension 1 [1–3] and the resulting cosmology (see, e.g., [4–6]) and cosmological perturbation theory (e.g., [7–13]) have been studied in depth. When there is more than one extra dimension the Israel junction conditions, which are central to the 5D studies, do not apply and other approaches must be used [14–16]. In the ''mirage'' cosmology approach [15,17] the bulk is taken to be a given supergravity solution, and our universe is a *test* D3-brane which moves in this background spacetime so that its back-reactions onto the bulk are neglected. If the bulk metric has certain symmetry properties, the unperturbed brane motion leads to a Friedmann-Robertson-Walker (FRW) cosmology with a scale factor $a(\tau)$ on the brane [15,18]. Our aim in this paper is to study the evolution of perturbations on such a moving brane. Given the probe nature of the brane, this question has many similarities with the study of the dynamics and perturbations of cosmic topological defects such as cosmic strings [19–22].

Though we derive the perturbation equations in a more general case, we consider in the end a bulk with AdS_5-Schwarzschild$\times S_5$ geometry which is the near horizon limit of the ten-dimensional black D3-brane solution. In this limit [using the AdS conformal field theory (CFT) correspondence] black-hole thermodynamics can be studied via the probe D3-brane dynamics [23,24]. As discussed in Sec. II A, we make the assumption that the D3-brane has no dynamics around the S_5 so that the bulk geometry is effectively AdS_5-Schwarzschild geometry. Because of the generalized Birkhoff theorem [25], this 5D geometry plays an important role in work on codimension 1 brane cosmology. Hence links can be made between the unperturbed probe brane FRW cosmology discussed here and exact brane cosmology based on the junction conditions [18]. Similarly, the perturbation theory we study here is just one limit of the full, self-interacting and non-Z_2-symmetric brane perturbation theory which has been studied elsewhere [10]. Comments will be made in the conclusions regarding generalizations of this work to the full 10D case.

Regarding the universe brane, the zeroth order (or background) solution is taken to be an infinitely straight brane whose motion is now constrained to be along the single extra dimension labeled by coordinate r. The brane motion is parametrized by a conserved positive energy E [15]. In AdS_5-Schwarzschild geometry and to an observer on the brane, the motion appears to be FRW expansion or contraction with a scale factor given by $a\propto r$. Both the perturbed and unperturbed brane dynamics will be obtained from the Dirac-Born-Infeld action for type IIB superstring theory (see, e.g., [26]),

$$S_{D3}=-T_3\int d^4\sigma\sqrt{-\det(\hat{\gamma}_{ab}+2\pi\alpha' F_{ab}+\hat{B}_{ab})}-\rho_3\int d^4\sigma\hat{C}_4. \tag{1.1}$$

Here σ^a $(a=0,1,2,3)$ are coordinates on the brane worldsheet, T_3 is the brane tension, and in the second Wess-Zumino term ρ_3 is the brane charge under a Ramond-Ramond (RR) four-form field living in the bulk. We will write

$$\rho_3=qT_3 \tag{1.2}$$

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so that $q=(-)1$ for Bogomol'nyi-Prasad-Sommerfield (BPS) (anti)branes. In Eq. (1.1) $\hat{\gamma}_{ab}$ is the induced metric and F_{ab} the field strength tensor of the gauge fields on the brane. The quantities $\hat{B}_{ab}$ and $\hat{C}_4$ are the pullbacks of the Neveu-Schwarz (NS) two-form, and the Ramond-Ramond four-form field in the bulk. In the background we consider, the dilaton is a constant and we set it to zero. In general the brane will not move slowly, and hence the square root in the Dirac-Born-Infeld DBI part of Eq. (1.1) may not be expanded: we will consider the full nonlinear action. Finally, notice that since the 4D Riemann scalar does not appear in Eq. (1.1) (and it is not inherited from the background in this probe brane approach) there is no brane self-gravity. Hence the "mirage" cosmology we discuss here is solely sourced by the brane motion, and it leads to effects which are not present in four-dimensional Einstein gravity. The lack of brane self-gravity is a serious limitation. However, in certain cases it may be included, for instance by compactifying the background space-time as discussed in [27] (see also [14]). Generally this leads to bimetric theories. Even in that case, the mirage cosmology scale factor $a(\tau)$ which we discuss below plays an important role and hence we believe it is of interest to study perturbations in this "probe brane" approach.

Deviations from the infinitely straight moving brane give rise to perturbations around the FRW solution. Are these "wiggles" stretched away by the expansion, or on the contrary do they grow, leading to instabilities? To answer this question, we exploit the similarity with uncharged cosmic topological defects and make use of the work developed in that context by Garriga and Vilenkin [20], Guven [21], and Battye and Carter [22]. The perturbation dynamics are studied through a scalar field $\phi(\sigma)$ whose equation of motion is derived from the action (1.1). We find that, for an observer comoving with the brane, ϕ has a tachyonic mass in certain ranges of r which depend on the conserved energy E characterizing the unperturbed brane dynamics. We discuss the evolution of the modes ϕ_k for different E and show that in many cases the brane is unstable. In particular, both sub- and superhorizon modes grow for a brane falling into the black hole. It remains an open question to see if brane self-gravity, neglected in this approach, can stabilize the system.

Finally, we also relate ϕ to the standard 4D gauge invariant scalar Bardeen potentials $\underline{\Phi}$ and $\underline{\Psi}$ on the brane. We find that $\underline{\Phi}\propto\underline{\Psi}\propto\phi$ (no derivatives of ϕ enter into the Bardeen potentials).

The work presented here has some overlap with that of Carter *et al.* [28] who also studied perturbations on moving charged branes in the limit of negligible self-gravity. Their emphasis was on trying to mimic gravity on the brane, and in addition they included matter on the brane. Here we consider the simplest case in which there is no matter on the brane: namely, $F_{ab}=0$ in Eq. (1.1). Our focus is on studying the evolution of perturbations solely due to motion of the brane: we expect the contribution of these perturbations to be important also when matter is included. Moreover, we hope that this study may more generally be of interest for the dynamics and perturbations of moving D-branes in non-BPS backgrounds.

The outline of the paper is as follows. In Sec. II we link our five-dimensional metric to the ten-dimensional black D3-brane solution and specify the unperturbed embedding of the probe brane. To determine its dynamics from the action (1.1) the bulk four-form RR field must be specified. We discuss the normalization of this field. At the end of the section we summarize the motion of the probe brane by means of an effective potential. Comments are made regarding the Friedmann equation for an observer on the brane. In Sec. III we consider small deviations from the background brane trajectory and investigate their evolution. The equation of motion for ϕ is derived, and we solve it in various regimes, commenting on the resulting instabilities. In Sec. IV we link ϕ to the scalar Bardeen potentials on the brane. Finally, in Sec. V we summarize our results.

II. UNPERTURBED DYNAMICS OF THE D3-BRANE

In this section we discuss the background metric, briefly review the unperturbed D3-brane dynamics, and comment on the cosmology as seen by an observer on the brane. The reader is referred to [15,29] for a more detailed analysis on which part of this section is based.

A. Background metric and brane scale factor

For the reasons mentioned in the Introduction, we focus mainly on an $\mathrm{AdS}_5\text{-S}\times\mathrm{S}^5$ bulk space-time. This is closely linked to the 10D black three-brane supergravity solution [30–32] which describes N coincident D3-branes carrying RR charge $Q=NT_3$ and which is given by

$$ds_{10}^2=H_3^{-1/2}(-F dt^2+d\vec{x}\cdot d\vec{x}) \\ +H_3^{1/2}\left(\frac{dr^2}{F}+r^2 d\Omega_5^2\right) \tag{2.1}$$

where the coordinates $(t,\vec{x})$ are parallel to the N D3-branes, $d\Omega_5^2$ is the line element on a five-sphere, and

$$H_3(r)=1+\frac{\ell^4}{r^4}, \quad F=1-\frac{r_H^4}{r^4}. \tag{2.2}$$

The quantity ℓ is the AdS_5 curvature radius and the horizon r_H vanishes when the Arnowitt-Deser-Misner (ADM) mass equals Q. The link between the metric parameters ℓ, r_H and the string parameters N, T_3 is given, e.g., in [32]. The corresponding bulk RR field may also be found in [32].

The near horizon limit of the metric (2.1) is $\mathrm{AdS}_5\text{-S}\times\mathrm{S}^5$ space time [31]. Our universe is taken to be a D3-brane moving in this background. We make the following two assumptions. First, the universe brane is a probe so that its back reaction on the bulk geometry is neglected. This may be justified if $N\gg 1$. Second, the probe is assumed to have no dynamics around S^5 so that it is constrained to move only along the radial direction r. This is a consistent solution of the *unperturbed* dynamics since the brane has a conserved angular momentum about the S^5, and this may be set to zero [15,18]. In Sec. III we assume that is also true for the perturbed dynamics. Thus in the remainder of this paper we consider an AdS_5-S bulk spacetime with metric

$$ds_5^2 = -f(r)dt^2 + g(r)d\vec{x}\cdot d\vec{x} + h(r)dr^2 \tag{2.3}$$

$$\equiv g_{\mu\nu}dx^\mu dx^\nu \tag{2.4}$$

where

$$f(r) = \frac{r^2}{\ell^2}\left(1 - \frac{r_H^4}{r^4}\right), \quad g(r) = \frac{r^2}{\ell^2},$$

$$h(r) = \frac{1}{f(r)}. \tag{2.5}$$

(In the limit $r_H \to 0$ this becomes pure AdS_5.)

More generally, by symmetry, a stack of nonrotating D3-branes generates a metric of the form $ds_{10}^2 = ds_5^2 + k(r)d\Omega_5^2$, where ds_5 is given in Eq. (2.3) [33]. In this case, since the metric coefficients are independent of the angular coordinates $(\theta^1, \ldots, \theta^5)$, the unperturbed brane dynamics are always characterized by a conserved angular momentum around the S^5 [15]. As a result of the second assumption above, we are thus effectively led to consider metrics of the form (2.3): hence for the derivation of both the unperturbed and perturbed equations of motion we keep f,g,h arbitrary and consider the specific form (2.5) only at the end.

The embedding of the probe D3-brane is given by $x^\mu = X^\mu(x^a)$. (We have used reparametrization invariance to choose the intrinsic worldsheet coordinates $\sigma^a = x^a$.) For the unperturbed trajectory we consider an infinitely straight brane parallel to the x^a hyperplane but free to move along the r direction:

$$X^a = x^a, \quad X^4 = R(t). \tag{2.6}$$

Later, in Sec. III, we will consider a perturbed brane for which $X^4 = R(t) + \delta R(t,\vec{x})$.

The induced metric on the brane is given by

$$\hat{\gamma}_{ab} = g_{\mu\nu}(X)\frac{\partial X^\mu}{\partial x^a}\frac{\partial X^\nu}{\partial x^b} \tag{2.7}$$

(where the caret denotes a pullback), so that the line element on the unperturbed brane worldsheet is

$$\begin{aligned} ds_4^2 &= \hat{\gamma}_{ab}dx^a dx^b \\ &= -[f(R) - h(R)\dot{R}^2]dt^2 + g(R)d\vec{x}\cdot d\vec{x} \\ &\equiv -d\tau^2 + a^2(\tau)d\vec{x}\cdot d\vec{x}. \end{aligned} \tag{2.8}$$

An observer on the brane therefore sees a homogeneous and isotropic universe in which the time τ and the scale factor $a(\tau)$ are given by

$$\tau = \int \sqrt{(f - h\dot{R}^2)}dt, \quad a(\tau) = \sqrt{g(R(\tau))}. \tag{2.9}$$

The properties of the resulting Friedmann equation depend on $f(R), g(R), h(R)$ (i.e., the bulk geometry) as well as $\dot{R}$ (the brane dynamics) as discussed in [15,18] and summarized briefly below.

B. Brane action and bulk four-form field

In AdS_5-S geometry, $B_{\mu\nu}$ vanishes, and we do not consider the gauge field F_{ab} on the brane. (For a detailed discussion of the unperturbed brane dynamics with and without F_{ab}, which essentially corresponds to radiation on the brane, see [15,18]. Nonzero $B_{\mu\nu}$ has been discussed in [34].) Thus the brane action (1.1) reduces to

$$S_{D3} = -T_3\int d^4x\sqrt{-\hat{\gamma}} - \rho_3\int d^4x\hat{C}_4 \tag{2.10}$$

where

$$\hat{\gamma} = \det(\hat{\gamma}_{ab}), \quad \hat{C}_4 = C_{\mu\nu\sigma\rho}\frac{\partial X^\mu}{\partial x^0}\frac{\partial X^\nu}{\partial x^1}\frac{\partial X^\sigma}{\partial x^2}\frac{\partial X^\rho}{\partial x^3}, \tag{2.11}$$

and $C_{\mu\nu\sigma\rho}$ are components of the bulk RR four-form field. The first term in Eq. (2.10) is just the Nambu-Goto action.

In the gauge (2.6), $\hat{\gamma}$ and $\hat{C}_4$ depend on t only through R. Thus rather than varying Eq. (2.10) with respect to X^μ and then integrating the equations of motion, it is more straightforward to obtain the equations of motion from the Lagrangian

$$\mathcal{L} = -\sqrt{-\hat{\gamma}} - C = -\sqrt{fg^3 - g^3h\dot{R}^2} - C \tag{2.12}$$

where $C = C(R) = (\rho_3/T_3)\hat{C}_4 = q\hat{C}_4$. Since $\mathcal{L}$ does not explicitly depend on time, the brane dynamics are parametrized by a (positive) conserved energy $E = (\partial\mathcal{L}/\partial\dot{R})\dot{R} - \mathcal{L}$, from which

$$\dot{R}^2 = \frac{f}{h}\left(1 - \frac{fg^3}{(E-C)^2}\right). \tag{2.13}$$

Transforming to brane time τ defined in Eq. (2.9) yields

$$R_\tau^2 = \frac{(E-C)^2}{fg^3h} - \frac{1}{h} \tag{2.14}$$

where the subscript denotes a derivative with respect to τ.

In order to analyze the brane dynamics in AdS_5-S spacetime where f,g and h are given in Eq. (2.5), one must finally specify $C(R)$ or equivalently the four-form potential $C_{\mu\nu\sigma\rho}$. To that end[1] recall that the 5D bulk action is

[1]For the 10D $\mathrm{AdS}_5\text{-S}\times S_5$ geometry the solution for the four-form field is given, for example, in [32]. For completeness, we rederive the result starting directly from the 5D metric (2.5).

$$S=\frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}(R-2\Lambda)-\frac{1}{4\kappa_5^2}\int F_5\wedge{}^*F_5 \tag{2.15}$$

where Λ is the bulk cosmological constant and $F_5=dC_4$ is the five-form field strength associated with the four-form C_4. The resulting equations of motion are

$$R_{\mu\nu}=\frac{2}{3}\Lambda g_{\mu\nu}+\frac{1}{2\cdot 4!}\bigg(F_{\mu\beta\gamma\delta\epsilon}F_{\nu}^{\beta\gamma\delta\epsilon}-\frac{4}{3\cdot 5}F_{\alpha\beta\gamma\delta\epsilon}F^{\alpha\beta\gamma\delta\epsilon}g_{\mu\nu}\bigg), \tag{2.16}$$

$$d^*F_5=\frac{1}{2}\frac{1}{\sqrt{fg^3h}}\bigg[\bigg(\frac{f'}{f}+3\frac{g'}{g}+\frac{h'}{h}\bigg)F_{01234}-2F'_{01234}\bigg]dr=0, \tag{2.17}$$

where the prime denotes a derivative with respect to r. In AdS$_5$-S geometry, $R_{\mu\nu}=-(4/\ell^2)g_{\mu\nu}$ and Eq. (2.17) gives

$$\frac{\ell^3}{r^3}\bigg(\frac{3}{r}F_{01234}-F'_{01234}\bigg)=0\Rightarrow F_{01234}=c\frac{r^3}{\ell^4} \tag{2.18}$$

where c is a dimensionless constant (see, for example, [35]). (Note that this solution satisfies $dF_5=0$ since the only nonzero derivative is ∂_4F_{01234} which vanishes on antisymmetrizing.) Integration gives

$$C_{0123}=v\frac{r^4}{\ell^4}+w \tag{2.19}$$

where $v=c/4$ and w are again dimensionless constants. Hence the function $C(r)$ appearing in Eq. (2.12) is

$$C(r)=qC_{0123}=qv\frac{r^4}{\ell^4}+qw. \tag{2.20}$$

In ten dimensions the constant c (and hence v) is fixed by $\int{}^*F=Q$, and w may be determined by imposing [before taking the near horizon limit—hence with metric (2.1)] that the four-form potential should die off at infinity [32]. This second argument is not applicable here. Instead, we fix v and w in the following way. Consider the motion of the unperturbed brane seen by a bulk observer with time coordinate t. One can define an effective potential $V_{\rm eff}^t$ through

$$\frac{1}{2}\dot{R}^2+V_{\rm eff}^t\equiv E \tag{2.21}$$

so that on using equation (2.13),

$$V_{\rm eff}^t(E,q,R)=E-\frac{1}{2}\bigg(\frac{R}{\ell}\bigg)^4\alpha^2\bigg[1-\bigg(\frac{R}{\ell}\bigg)^8\frac{\alpha}{(E-C)^2}\bigg] \tag{2.22}$$

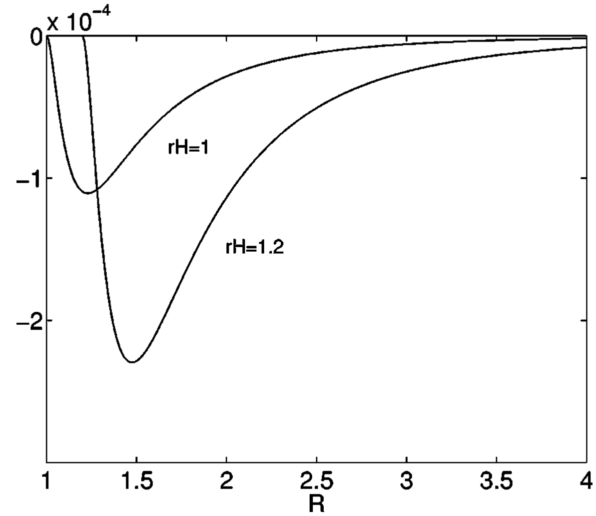

FIG. 1. $V_{\rm eff}^t(E,q,R)$ for $E=0$, $q=1$, $\ell=4$, and different values of r_H. For $R\to\infty$ the potential goes to zero according to our normalization. When $r_H=0$, the potential is exactly flat.

(see Figs. 1 and 2) where

$$\alpha=1-\frac{r_H^4}{R^4}$$

and $C=C(R)$ is given in Eq. (2.20). We now use the fact that there is no net force between static BPS objects of like charge, and hence in this case the effective potential should be identically zero. Here, such a configuration is characterized by $r_H=0$, $q=1$, $E=0$: imposing that $V_{\rm eff}^t=0$ for all R, forces $v=\pm 1$ and, in this limit, $w=0$. Second, we normalize the potential such that $V_{\rm eff}^t(E,q=1,R\to\infty)=0$ for arbitrary values of the energy E and r_H. This leads to

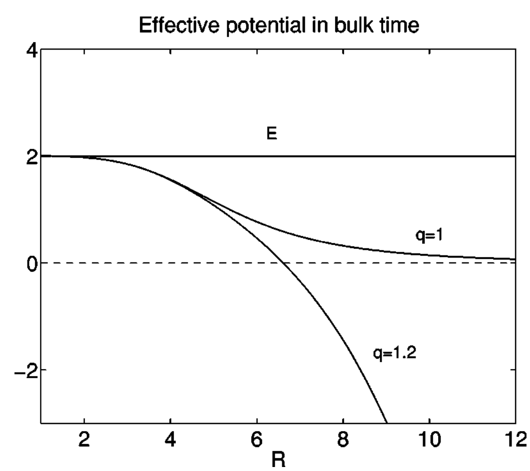

FIG. 2. $V_{\rm eff}^t(E,q,R)$ for $E=2$, $r_H=1$, $\ell=4$. For a BPS brane ($q=1$), $V_{\rm eff}^t\to 0$ as $R\to\infty$ according to our normalization. This should be contrasted with a non-BPS brane, e.g., with $q=1.2$. Note that $V_{\rm eff}^t(E,q,R=r_H)=E$. Any inwardly moving (contracting) brane takes an infinite amount of t time to reach the horizon.

$$v=-1, \quad w=+\frac{r_H^4}{2\ell^4}. \tag{2.23}$$

In particular for $E=0$, then the brane has zero kinetic energy at infinity. Even in this case the potential is not flat, unless $r_H=0$, as can be see in Fig. 1. According to this normalization

$$C(r)=-q\frac{r^4}{\ell^4}+q\frac{r_H^4}{2\ell^4} \tag{2.24}$$

as in the 10D case [32]. Notice that, since the combination appearing in the equation of motion for R is $E-C$, the constant w only acts to shift the energy. For later purposes we define the shifted energy $\tilde{E}$ by

$$\tilde{E}=E-qw=E-q\frac{r_H^4}{2\ell^4}. \tag{2.25}$$

Finally, we comment that substitution of Eq. (2.18) into Eq. (2.16) determines the bulk cosmological[2] constant to be given by $\ell^2\Lambda=-6-c^2/4=-10$.

C. Brane dynamics and Friedmann equation

We now make some comments regarding the unperturbed motion of the three-brane through the bulk, $R(\tau)$, as seen for an observer on the brane. This will be useful in Sec. III when discussing perturbations. Recall that since $a(\tau)=R(\tau)/\ell$ [see Eq. (2.9)], an "outgoing" brane leads to cosmological expansion. Contraction occurs when the brane moves inward. For the observer on the brane, one may define an effective potential by

$$\frac{1}{2}R_\tau^2+V_{\rm eff}^\tau=E \tag{2.26}$$

whence, from Eq. (2.14),

$$V_{\rm eff}^\tau(E,q,R)=E+\frac{1}{2}\left(\frac{\ell}{R}\right)^6\left[\alpha\left(\frac{R}{\ell}\right)^8-(E-C)^2\right]. \tag{2.27}$$

Consider a BPS brane $q=+1$ (see Fig. 3). As noted above, for $r_H=E=0$ one has $V_{\rm eff}^\tau=0$ so that the potential is flat. For $r_H\neq 0$, $V_{\rm eff}^\tau$ contains a term proportional to $-R^{-6}$, and the probe brane accelerates toward the horizon, which is reached in finite τ time. On the other hand, for a bulk observer with time t, it takes infinite time to reach the horizon where $V_{\rm eff}^t=E$ (see Fig. 2).

[2]Equivalently we could have started from the 10D supergravity (SUGRA) action, used the 10D solution for F [which is identical to Eq. (2.24)], and then integrated out over the five-sphere. After definition of the 5D Newton constant in terms of the 10D one, the above cosmological constant term is indeed obtained, coming from the five-sphere Ricci scalar.

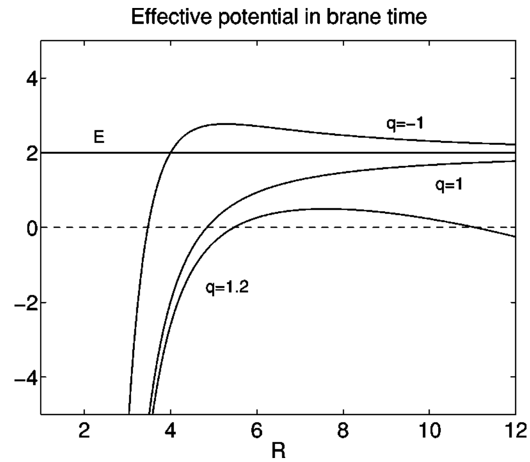

FIG. 3. $V_{\rm eff}^\tau(E,q,R)$ for the same parameters as in Fig. 2. A BPS brane has zero kinetic energy at infinity corresponding to a vanishing cosmological constant on the brane. Otherwise, the cosmological constant is proportional to q^2-1. A BPS antibrane is allowed to move only in a restricted range of R: after having reached a maximal scale factor, the universe starts contracting. Any inwardly moving brane falls into the black hole in a finite τ.

From Eqs. (2.14) and (2.20) it is straightforward to derive a Friedmann-like equation for the brane scale factor $a(\tau)$ [15,18]:

$$H^2=\left(\frac{a_\tau}{a}\right)^2 = \frac{1}{\ell^2}\left[\frac{\tilde{E}^2}{a^8}+\frac{1}{a^4}\left(2q\tilde{E}+\frac{r_H^4}{\ell^4}\right)+(q^2-1)\right]. \tag{2.28}$$

The term in $1/a^8$ (a "dark fluid" with equation of state $\tilde{p}=5/3\tilde{\rho}$) dominates at early times. The second term, in a^{-4}, is a "dark radiation" term. As discussed in [18], the part proportional to r_H corresponds to the familiar dark radiation term in conventional Z_2-symmetric (junction condition) brane cosmology, where it is associated with the projected bulk Weyl tensor. When $\tilde{E}$ is nonzero, Z_2 symmetry is broken[3] [18] and this leads to a further dark radiation term [35,36]. The last term in Eq. (2.28) defines an effective four-dimensional cosmological constant $\Lambda_4\equiv(1/\ell^2)(q^2-1)$ which vanishes if the (anti)brane is BPS (i.e., $q=\pm 1$). All these terms have previously been found in both "mirage" cosmology and conventional brane cosmology [18,35].

Notice that the dark radiation term above has a coefficient

$$\mu\equiv 2q\tilde{E}+\frac{r_H^4}{\ell^4}=2qE-\frac{r_H^4}{\ell^4}(q^2-1) \tag{2.29}$$

[3]When making the link between mirage cosmology and the junction condition approach, $\tilde{E}\propto M_- - M_+$ where $M_\pm$ are the black-hole masses on each side of the brane [18].

which is positive for $q=+1$ (since $E \geqslant 0$). However, for BPS antibranes $q=-1$, the coefficient (2.29) is negative unless $E=0$. Thus, when $E \neq 0$ and $q=-1$ there is a regime of R for which H^2 is negative. In Fig. 3 this is represented by the forbidden region where the potential exceeds the total energy E. At $V_{\mathrm{eff}}^{\tau}=E$ the Hubble parameter is zero and an initially expanding brane starts contracting. On the contrary, we do not obtain bouncing solutions in our setup, regardless of the values of q and E. Bouncing and oscillatory universes are discussed in, e.g., [37–39].

The Friedmann equation (2.28) can be solved exactly. In the BPS case, $\Lambda_4=0$, the solution is

$$a(\tau)^4 = a_i^4 + \frac{4\mu}{\ell^2}(\tau-\tau_i)^2 \pm \frac{4}{\ell}(\tau-\tau_i)(\tilde{E}^2+\mu a_i^4)^{1/2} \tag{2.30}$$

where a_i is the value of the scale factor at the initial time τ_i, and the $\pm$ determines whether the brane is moving radially inward or outward. In the next section when we solve the perturbation equations, it will be sufficient to consider regimes in which only one of the terms in Eq. (2.28) dominates. These will be given in Sec. III.

One might wonder whether it is possible to obtain a term proportional to a^{-3} (dust) in the Friedmann equation, and also one corresponding to physical radiation on the brane (rather than dark radiation). Physical radiation comes from taking $F_{ab} \neq 0$ in Eq. (1.1) [15], and a ''dark'' dust term has been obtained in the non-BPS background studied in [27]. Finally, a curvature term a^{-2} has been obtained in [40].

III. PERTURBED EQUATIONS OF MOTION

In this section we consider perturbations of the brane position about the zeroth order solution $R(t)$ given in Eq. (2.13). Once again we work with the metric (2.3), specializing to AdS_5-S geometry only at the end. The perturbed brane embedding $X^4=R(t)+\delta R(t,\vec{x})$ leads to perturbations $\delta\hat{\gamma}_{ab}$ of the induced metric on the brane and these are discussed in Sec. IV. Note that these perturbations about the flat homogenous and isotropic solution are not sourced by matter on the brane, and their evolution will depend on the unperturbed brane dynamics and hence on E. We now derive an equation for the evolution of the perturbed brane and try to see if there are instabilities in the system.

A. The second order action

Since we consider a codimension 1 brane, the fluctuations about the unperturbed moving brane can be described by a single scalar field $\phi(x^a)$ living on the unperturbed brane worldsheet [21]. To describe the dynamics of $\phi(x^a)$ (which is defined below), we use the covariant formalism developed in [21] to study perturbed Nambu-Goto walls. (For other applications, see also [20,41].) The perturbed brane embedding is given by

$$X^{\mu}(t,\vec{x}) = \bar{X}^{\mu}(t) + \phi(t,\vec{x}) n^{\mu}(t) \tag{3.1}$$

where $\bar{X}^{\mu}(t)$ is the unperturbed embedding, and physical perturbations are only those transverse to the brane (see also Sec. IV). The unit spacelike normal to the unperturbed brane, $n^{\mu}(t)=n^{\mu}(\bar{X}^{\mu}(t))$, is defined through

$$g_{\mu\nu} n^{\mu} \frac{\partial \bar{X}^{\nu}}{\partial x^a}=0, \quad g_{\mu\nu}n^{\mu}n^{\nu}=1, \tag{3.2}$$

so that

$$n^{\mu} = \left(\dot{R}\sqrt{\frac{h}{f(f-h\dot{R}^2)}},0,0,0,\sqrt{\frac{f}{h(f-h\dot{R}^2)}} \right). \tag{3.3}$$

Thus for a 5D observer comoving with the brane, ϕ (which has dimensions of length) is the measured deviation from the background solution of the previous section [20]. For an observer living on the brane, the perturbations in the FRW metric generated by ϕ are discussed in Sec. IV in terms of the gauge invariant scalar Bardeen potentials.

An equation of motion for ϕ can be obtained by substituting Eq. (3.1) into the action (2.10) and expanding to second order in ϕ. The terms linear in ϕ give the background (unperturbed) equations of motion studied in the previous section—now we are interested in the terms quadratic in ϕ which give the linearized equations of motion. A similar analysis was carried out by Garriga and Vilenkin [20] for Nambu-Goto cosmic domain walls in Minkowski space and was generalized by Guven [21] for arbitrary backgrounds. For the action (2.10), the quadratic term is [41]

$$S_{\phi^2} = -\frac{1}{2}\int d^4x\sqrt{-\hat{\gamma}}[(\hat{\nabla}_a\phi)(\hat{\nabla}^a\phi) - (\hat{K}^a{}_b\hat{K}^b{}_a + R_{\mu\nu}n^{\mu}n^{\nu})\phi^2]. \tag{3.4}$$

Here $\hat{\nabla}$ is the covariant derivative with respect to the induced metric $\hat{\gamma}_{ab}$, and the extrinsic curvature tensor $\hat{K}_{ab}$ is given by

$$\hat{K}_{ab} = (\nabla_{\nu} n_{\mu}) \frac{\partial \bar{X}^{\mu}}{\partial x^a}\frac{\partial \bar{X}^{\nu}}{\partial x^b} \tag{3.5}$$

where ∇ is the covariant derivative with respect to the 5D metric $g_{\mu\nu}$. Finally, $R_{\mu\nu}$ is the Ricci tensor of the metric $g_{\mu\nu}$. Apart from ϕ, all the terms in Eq. (3.4) are unperturbed quantities. Note that there is no contribution to S_{ϕ^2} from the Wess-Zumino term of action (2.10): all terms quadratic in ϕ cancel since C_{0123} is the only nonzero component of the four-form field. However, C does enter into the term linear in ϕ and hence into the background equations of motion, as analyzed in the previous sections.

Variation of the action (3.4) with respect to ϕ leads to the equation of motion

$$\hat{\nabla}^a \hat{\nabla}_a \phi + [\hat{K}^a_b \hat{K}^b_a + R_{\mu\nu} n^\mu n^\nu]\phi = 0 \tag{3.6}$$

or equivalently

$$\hat{\nabla}^a \hat{\nabla}_a \phi - m^2 \phi = 0 \tag{3.7}$$

where

$$m^2 = -[\hat{K}^a_b \hat{K}^b_a + R_{\mu\nu} n^\mu n^\nu]. \tag{3.8}$$

To determine the extrinsic curvature contribution to Eq. (3.8), it is simpler to calculate first the five-dimensional extrinsic tensor defined by

$$K^\mu_\nu = \gamma^{\lambda\mu} \nabla_\lambda n_\nu \tag{3.9}$$

where $\gamma^{\lambda\mu} = g^{\lambda\mu} - n^\lambda n^\mu$ and then use

$$\hat{K}^a_b \hat{K}^b_a = K^\mu_\nu K^\nu_\mu .$$

On defining T by

$$T \equiv \left(\frac{d\tau}{dt}\right)^2 = f - h\dot{R}^2 = \frac{f^2 g^3}{(E-C)^2},$$

the nonzero components of K^μ_ν are

$$K^0_0 = \frac{1}{T^{5/2}} f^{3/2} h^{1/2} \left(\ddot{R} - \frac{f'}{f}\dot{R}^2 + \frac{1}{2}\frac{h'}{h}\dot{R}^2 + \frac{1}{2}\frac{f'}{h} \right), \tag{3.10}$$

$$K^0_4 = -\frac{h\dot{R}}{f} K^0_0, \tag{3.11}$$

$$K^1_1 = \frac{1}{T^{1/2}} \left(\frac{f}{h}\right)^{1/2} \frac{1}{2}\frac{g'}{g} = K^2_2 = K^3_3, \tag{3.12}$$

$$K^4_4 = -\frac{h\dot{R}^2}{f} K^0_0 \tag{3.13}$$

so that

$$\hat{K}^a_b \hat{K}^b_a = \frac{1}{T}\frac{f}{h}\left[3\left(\frac{g'}{g}\right)^2 + 3\frac{g'}{g}\frac{C'}{E-C} + \left(\frac{C'}{E-C}\right)^2 \right]. \tag{3.14}$$

The Ricci term is

$$R_{\mu\nu} n^\mu n^\nu = -\frac{1}{4h}\left[2\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + 3\frac{f'}{f}\frac{g'}{g} - \frac{f'}{f}\frac{h'}{h} \right] + \frac{3}{4}\frac{1}{T}\frac{f}{h}\left[\frac{f'}{f}\frac{g'}{g} - 2\frac{g''}{g} + \left(\frac{g'}{g}\right)^2 + \frac{g'}{g}\frac{h'}{h} \right]. \tag{3.15}$$

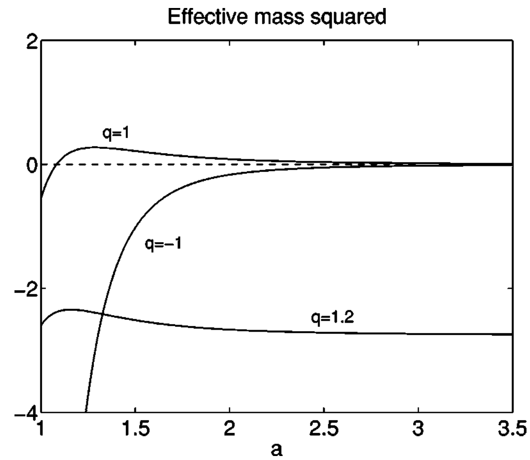

FIG. 4. The dimensionless quantity $M^2\ell^2$ as a function of a for $E=1$, $\ell=1$, $r_H=1$. Here, the effective mass squared is positive in a certain range only for the BPS brane. Note that the negative $M^2\ell^2$ region is not hidden behind the horizon.

Collecting these results gives

$$m^2 = -\frac{3}{4}\frac{(E-C)^2}{fg^3h}\left[\frac{f'}{f}\frac{g'}{g} - 2\frac{g''}{g} + 5\left(\frac{g'}{g}\right)^2 + \frac{g'}{g}\frac{h'}{h} + 4\frac{g'}{g}\frac{C'}{E-C} + \frac{4}{3}\left(\frac{C'}{E-C}\right)^2 \right] + \frac{1}{4h}\left[2\frac{f''}{f} - \left(\frac{f'}{f}\right)^2 + 3\frac{f'}{f}\frac{g'}{g} - \frac{f'}{f}\frac{h'}{h} \right].$$

In the remainder of this section we try to obtain approximate solutions for ϕ from Eq. (3.7). Some aspects of this calculation are clearer in brane time τ and others in conformal time η [where $\eta = \int d\tau / a(\tau)$]. Of course the results are independent of the coordinate system. For these reasons we have decided to present both approaches, beginning with brane time.

B. Evolution of perturbations in brane time τ

On using the definition of brane time τ in Eq. (2.9), the kinetic term in Eq. (3.6) is given by

$$\hat{\nabla}^a \hat{\nabla}_a \phi = -\phi_{\tau\tau} - 3H\phi_\tau + \frac{1}{a^2}[\phi_{x^1x^1} + \phi_{x^2x^2} + \phi_{x^3x^3}].$$

(In conformal time the factor of a^{-2} multiplying the spatial derivatives disappears—see below.) We now change variables to $\varphi = a^{3/2}\phi$ so that Eq. (3.7) becomes

$$\varphi_{\tau\tau} - \frac{1}{a^2}[\varphi_{x^1x^1} + \varphi_{x^2x^2} + \varphi_{x^3x^3}] + M^2(\tau)\varphi = 0 \tag{3.16}$$

where

$$M^2(\tau)=m^2-\frac{3}{4}\left[\left(\frac{a_\tau}{a}\right)^2+2\frac{a_{\tau\tau}}{a}\right]$$

$$=m^2-\frac{3}{4}\left[\frac{g''}{g}R_\tau^2-\frac{1}{4}\left(\frac{g'}{g}\right)^2R_\tau^2+\frac{g'}{g}R_{\tau\tau}\right] \tag{3.17}$$

$$=\frac{3}{4}\frac{(E-C)^2}{fg^3h}\left[-\frac{1}{2}\frac{f'}{f}\frac{g'}{g}+\frac{g''}{g}-\frac{13}{4}\left(\frac{g'}{g}\right)^2-\frac{1}{2}\frac{g'}{g}\frac{h'}{h}-3\frac{g'}{g}\frac{C'}{E-C}\right.$$
$$\left.-\frac{4}{3}\left(\frac{C'}{E-C}\right)^2\right]+\frac{1}{4h}\left[2\frac{f''}{f}-\left(\frac{f'}{f}\right)^2+3\frac{f'}{f}\frac{g'}{g}-\frac{f'}{f}\frac{h'}{h}+3\frac{g''}{g}-\frac{3}{4}\left(\frac{g'}{g}\right)^2-\frac{3}{2}\frac{g'}{g}\frac{h'}{h}\right]. \tag{3.18}$$

This expression is valid for any f, g, and h. We now specialize to AdS$_5$-S geometry, in which case

$$M^2(\tau)=\frac{1}{\ell^2}\left[-\frac{33}{4}\frac{\tilde{E}^2}{a^8}+\frac{3}{4}\frac{1}{a^4}\left(2q\tilde{E}+\frac{r_H^4}{\ell^4}\right)-\frac{25}{4}(q^2-1)\right]$$
$$=-\frac{33}{4}H^2+\frac{9}{a^4\ell^2}\left(2q\tilde{E}+\frac{r_H^4}{\ell^4}\right)+2\frac{q^2-1}{\ell^2}. \tag{3.19}$$

Notice that there are regimes of a in which $M^2<0$—such as, for instance, for small a where the a^{-8} term dominates—and furthermore that the location of these regimes depends on the energy E of the brane. We also see that since $M^2\sim H^2$ instabilities will occur for modes with a wavelength greater than H^{-1}. Figure 4 shows the typical shape of M^2 as a function of a for fixed energy and different q. In the following, we discuss only cases with $q^2\geqslant 1$ as the 4D cosmological constant is positive.

Analysis of Eq. (3.16) is simpler in Fourier space where

$$\varphi_k(\tau)=\int d^3x\,\varphi(\tau,\vec{x})e^{-i\vec{k}\cdot\vec{x}} \tag{3.20}$$

and k is a comoving wave number related to the physical wave number k_p by $k=ak_p$. Thus Eq. (3.16) becomes

$$\varphi_{k,\tau\tau}+\frac{1}{a^2}[k^2-k_c^2(\tau)]\varphi_k=0 \tag{3.21}$$

where the time dependent critical wave number $k_c^2(\tau)$ is given by

$$k_c^2(\tau)=-M^2(\tau)a^2. \tag{3.22}$$

One might suppose that for $M^2>0$ all modes are stable. However, due to the τ-dependence of k_c this is not necessarily true [as we shall see in Eq. (3.36)].

Our aim now is to determine the a dependence of φ_k. We proceed in the following way. Notice first that the Friedmann equation (2.28) and the expression for $M^2(\tau)$ in Eq. (3.19) both contain terms in a^{-8}, a^{-4}, and a^0. We will focus on a regime in which one of these terms dominates. Then the Friedmann equation can be solved for $a(\tau)$ which, on substitution into Eq. (3.19), gives $M^2(\tau)$. A final substitution of $M^2(\tau)$ into the perturbation equation (3.21) for φ_k enables this equation to be solved in each regime. We consider the following cases: (1) $q=+1$, (2) $q=-1$, and (3) $q^2>1$.

1. BPS brane: q=+1

For a BPS brane, the Friedmann equation (2.28) and effective mass $M^2(\tau)$ are given by

$$H^2=\frac{1}{\ell^2}\left[\frac{\tilde{E}^2}{a^8}+\frac{2E}{a^4}\right], \tag{3.23}$$

$$M^2(\tau)=\frac{1}{\ell^2}\left[-\frac{33}{4}\frac{\tilde{E}^2}{a^8}+\frac{3}{2}\frac{E}{a^4}\right]. \tag{3.24}$$

The E dependence of these equations slightly complicates the analysis of these equations, and hence we begin with the simplest case in which $E=0$.

Case 1: *E=0.* When $E=0$—the static limit in which the probe has zero kinetic energy at infinity (see Fig. 1)—only the term proportional to a^{-8} survives in Eqs. (3.23) and (3.24). Recall that when r_H vanishes the potential $V_{\rm eff}^\tau$ is flat. Furthermore, since $\tilde{E}\propto r_H^4=0$, it follows from Eq. (3.24) that $M^2(\tau)=0$ in this limit: as expected, a BPS probe brane with zero energy in AdS$_5$ space-time has no dynamics and is completely stable.

When $r_H\neq 0$, $M^2(\tau)<0\ \ \forall\tau$, and the solution of Eq. (3.23) is

$$a(\tau)^4=a_i^4\pm\frac{2a_H^4}{\ell}(\tau-\tau_i). \tag{3.25}$$

Here $a_i\geqslant a_H\equiv r_H/\ell$ is the initial position of the brane at $\tau=\tau_i$, and the choice of sign determines whether the brane is moving radially inward $(-)$ or outward $(+)$: this is a question of initial conditions. Let $R_h=1/|Ha|$ denote the (comoving) Hubble radius. Then it follows from Eq. (3.24) and the definition of k_c^2 in Eq. (3.22) that

$$\frac{1}{\lambda_c}\sim|k_c(\tau)|\sim|Ha|=\frac{1}{R_h} \tag{3.26}$$

where we neglect numerical factors of order 1. Thus the critical wavelength is $\lambda_c \sim R_h$. (Notice that R_h is minimal at a_H and increases with a.)

For *superhorizon* modes $\lambda \gg R_h$ or $|k| \ll |k_c|$, and in this limit the perturbation equation (3.21) becomes

$$\varphi_{k,\tau\tau} - \frac{k_c^2(\tau)}{a^2}\varphi_k = 0. \tag{3.27}$$

On inserting solution (3.25) into k_c^2 one obtains

$$\phi_k = \frac{\varphi_k}{a^{3/2}} = A_k a^4 + B_k a^{-3} \tag{3.28}$$

(where the constants A_k and B_k are determined by the initial conditions). Hence if the brane moves radially outward the superhorizon modes grow as $a^4 \propto \tau$. If the brane is contracting they grow as a^{-3}. In the near extremal limit ($r_H \ll l$ or) $a_H \ll 1$, the amplitude of these superhorizon modes can become very large, suggesting that they are unstable. Of course our linear analysis will break down when ϕ becomes too large.

Consider now *subhorizon* modes $\lambda \ll R_h$ or $|k| \gg |k_c|$. Then Eq. (3.21) is just $\varphi_{k,\tau\tau} + (k^2/a^2)\varphi_k = 0$. However, in this case it is much easier to solve the equation in conformal time η where the factor of a^{-2} is no longer present. We anticipate the result from Sec. III C: it is

$$\phi_k = A_k \frac{e^{ik\eta}}{a} + B_k \frac{e^{-ik\eta}}{a}. \tag{3.29}$$

For an outgoing brane a increases and subhorizon modes are stable. For an ingoing brane a decreases, and the amplitude of the perturbation becomes very large in the near extremal limit. (Note that, as the brane expands, superhorizon modes eventually become subhorizon, and similarly, on a contracting brane, subhorizon modes become superhorizon.)

To conclude, when $r_H \neq 0$, $E = 0$, and the brane expands, superhorizon modes are unstable while subhorizon modes are stable. For a contracting brane, and in the near extremal limit, both super- and subhorizon modes are unstable.

Case 2: $E \neq 0$. When the energy of the brane is nonzero the situation is more complicated. Notice first from Eq. (3.24) that $M^2(\tau)$ has one zero at $a = a_c$ given by

$$a_c^4 = \frac{11\tilde{E}^2}{2E}. \tag{3.30}$$

Hence $M^2(\tau)$ is negative when $a < a_c$ and positive for $a > a_c$ (see Fig. 5). However, since a_c is E dependent, there may be ranges of E for which the negative mass region is hidden within the black-hole horizon. Indeed, we find

$$a_c \leq a_H \Leftrightarrow E_- \leq E \leq E_+ \tag{3.31}$$

where

$$E_\pm = \frac{a_H^4}{22}(13 \pm 4\sqrt{3}). \tag{3.32}$$

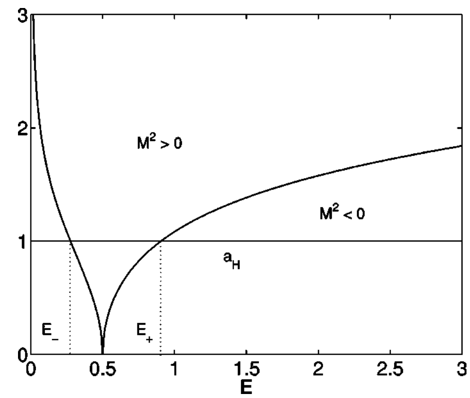

FIG. 5. The curve represents a_c, the zero of $M^2(\tau)$, as a function of the energy E as given in Eq. (3.30). Below the curve the effective mass squared is negative; above it is positive. For $E < E_-$ and $E > E_+$ the $M^2(\tau)$ already becomes negative outside the horizon, whereas for energies within the interval E_-, E_+ the $M^2(\tau) < 0$ region is hidden within the horizon. The parameters chosen are $q = 1$, $r_H = 1$, and $\ell = 1$.

The situation is shown schematically in Fig. 5.

Now consider H^2 given in Eq. (3.23). The two terms are of equal magnitude when $a = \tilde{a}_c = (\tilde{E}^2/2E)^{1/4} \sim a_c$. Thus when $a \ll a_c$ (and hence in the regions in which $M^2 < 0$ in Fig. 5), the dominant term in H^2 is the one proportional to a^{-8}. The system is therefore analogous to the one considered above when $E = 0$, and for superhorizon modes the solution is given in Eq. (3.28): for an outgoing brane $\phi_k \sim a^4$. When $E \gtrsim E_+$ or $E \lesssim E_-$, these regimes extend down to the black-hole horizon: thus in the near extremal limit the contracting brane will again be unstable since $\phi_k \sim a^{-3}$.

When $a \gg a_c$ (and hence in the regimes in which $M^2 > 0$ in Fig. 5), the dominant term in H^2 is proportional to a^{-4} so that

$$a(\tau)^2 = a_i^2 \pm 2\frac{\sqrt{2E}}{\ell}(\tau - \tau_i) \tag{3.33}$$

and

$$k_c^2(\tau) = -M^2(\tau)a^2 = -\frac{3}{2}\frac{E}{\ell^2 a^2}. \tag{3.34}$$

On *superhorizon* scales the mode equation is

$$\varphi_{k,\tau\tau} + \frac{3}{2}\frac{E}{\ell^2 a^4}\varphi_k = 0. \tag{3.35}$$

At first sight one might expect the solution to this equation to be stable since $M^2 > 0$. However, surprisingly, it is not. (Indeed, below we will see that in conformal time the effective mass is actually negative in this regime.) A change of variables to $u = a^2$ shows that the solution of Eq. (3.35) is

$$\varphi_k = A_k a^{3/2} + B_k a^{1/2} \tag{3.36}$$

which grows as $\tau^{3/4}, \tau^{1/4}$ respectively. Finally,

$$\phi_k = A_k + B_k a^{-1}. \tag{3.37}$$

For E within the band $E_- \lesssim E \lesssim E_+$, the solution (3.37) for the modes is valid for all a so that *superhorizon* modes grow as a^{-1} as the brane approaches the black hole horizon.

When $E \gtrsim E_+$ or $E \lesssim E_-$ these solutions are valid for $a \gg a_c$. Thus for an expanding brane ϕ_k tends to a constant value. For a contracting brane, the term proportional to a^{-1} could become important, although for small enough a the relevant regime is that considered above, in which case the solution is given by Eq. (3.28) and the superhorizon modes grow as a^{-3}.

For *subhorizon* modes, the solution is still as given in Eq. (3.29).

2. BPS antibranes: *q* = −1

Now the Friedmann equation (2.28) and effective mass $M^2(\tau)$ become

$$H^2 = \frac{1}{\ell^2}\left[\frac{\tilde{E}^2}{a^8} - \frac{2E}{a^4}\right], \tag{3.38}$$

$$M^2(\tau) = -\frac{1}{\ell^2}\left[\frac{33}{4}\frac{\tilde{E}^2}{a^8} + \frac{3}{2}\frac{E}{a^4}\right] \tag{3.39}$$

so that M^2 is always negative, independently of E. Note that $H^2 > 0$ for $a < \tilde{a}_c$ where $\tilde{a}_c = (\tilde{E}^2/2E)^{1/4}$. However, since $\tilde{E} = E + a_H^4/2$ for antibranes, it follows that $\tilde{a}_c \geq a_H$ for *all* E (i.e., there are no energy bands to consider in the case of antibranes). When $a \ll \tilde{a}_c$, $H^2 \propto M^2 \propto a^{-8}$ and once again this is analogous to the case studied above for $E = 0$: *superhorizon* modes grow as a^4, and in the near extremal limit the *subhorizon* modes on an ingoing brane are unstable.

3. Non-BPS branes: *q* ≠ ±1

Here we shall only briefly discuss the case $q^2 > 1$ for large a. Now, independently of E, there is a cosmological constant dominated regime [see Eq. (2.28)]. There the solution for the scale factor is

$$a(\tau) = a(\tau_i) e^{\pm\sqrt{\Lambda_4}(\tau - \tau_i)} \quad \text{where} \quad \Lambda_4 \equiv \frac{q^2 - 1}{\ell^2}. \tag{3.40}$$

In this regime, however, M^2 is negative with

$$M^2(\tau) = -\frac{25}{4}\Lambda_4 \tag{3.41}$$

and $R_h = 1/|Ha| = \Lambda_4{}^{-1/2} a^{-1}$.

For *subhorizon* modes ($\lambda \ll R_h$) the solution for φ_k is again given by Eq. (3.29). For *superhorizon* modes, and considering an outgoing brane, there is an exponentially growing unstable mode

$$\phi_k = A_k e^{\sqrt{\Lambda_4}(\tau - \tau_i)} = A_k a. \tag{3.42}$$

Hence, this non-BPS brane is unstable for large a. It is not clear to us why the acceleration due to the positive cosmological constant does not rather stretch the perturbations away.

C. Comments on an analysis in conformal time *η*

It is instructive to carry out a similar analysis in conformal time rather than brane time, and we comment briefly on it here. In conformal time and transformed to Fourier space, Eq. (3.7) becomes

$$\phi_{k,\eta\eta} + 2\mathcal{H}\phi_{k,\eta} + (k^2 + a^2 m^2)\phi_k = 0 \tag{3.43}$$

where $\mathcal{H} = aH$. The friction term can be eliminated by a change of variables to $\psi = a\phi$, and the above equation becomes

$$\psi_{k,\eta\eta} + [k^2 - k_c^2(\eta)]\psi_k = 0 \tag{3.44}$$

where

$$k_c^2(\eta) = -\mathcal{M}^2(\eta)$$

and

$$\mathcal{M}^2(\eta) = a^2 m^2 - a_{\eta\eta}/a \tag{3.45}$$

$$= g m^2 + \frac{1}{2}\left[-\frac{g''}{g}R_\eta^2 + \frac{1}{2}\left(\frac{g'}{g}\right)^2 R_\eta^2 - \frac{g'}{g}R_{\eta\eta}\right]$$

$$= -\frac{(E-C)^2}{f g^2 h}\left[\frac{1}{2}\frac{f'}{f}\frac{g'}{g} - \frac{g''}{g} + 3\left(\frac{g'}{g}\right)^2 + \frac{1}{2}\frac{g'}{g}\frac{h'}{h} + \frac{5}{2}\frac{g'}{g}\frac{C'}{E-C} + \left(\frac{C'}{E-C}\right)^2\right]$$
$$+ \frac{g}{2h}\left[\frac{f''}{f} - \frac{1}{2}\left(\frac{f'}{f}\right)^2 + \frac{3}{2}\frac{f'}{f}\frac{g'}{g} - \frac{1}{2}\frac{f'}{f}\frac{h'}{h} + \frac{g''}{g} - \frac{1}{2}\frac{g'}{g}\frac{h'}{h}\right]. \tag{3.46}$$

Specializing to AdS$_5$-S space-time yields

$$\mathcal{M}^2(\eta) = -\frac{1}{\ell^2}\left[\frac{10\tilde{E}^2}{a^6} + 6(q^2-1)a^2\right]. \tag{3.47}$$

Notice that in conformal time and for $|q| \geqslant 1$ $\mathcal{M}^2(\eta)$ is *always* negative independently of E. From this, one can immediately see the instability for small k in Eq. (3.36), even though $M^2(\tau)$ can be positive in that case. It is clear that the results on brane (in)stability must be independent of whether or not the analysis is carried out in η or τ time. We will see that this is indeed the case: the reason is that not only the sign of the effective mass squared but also its functional dependence on time determine the stability properties. We now summarize briefly some of the aspects that differ between the τ and η analysis.

Consider the simplest case: $q=+1$ and $E=0$. The solution of the (conformal time) Friedmann equation is $a^3 = a_i^3 \pm 3a_H^2(\eta-\eta_i)/2\ell$, and $k_c(\eta) \sim |\mathcal{H}| = 1/R_h$. For *superhorizon* modes $|k| \ll |k_c|$, Eq. (3.44) reduces to $\psi_{k,\eta\eta} - k_c^2(\eta)\psi_k = 0$. Given $a(\eta)$ and hence $k_c(a(\eta))$ it is straightforward to find the solution which is, as expected, exactly that given in Eq. (3.28). For *subhorizon* modes $|k| \gg |k_c|$, the solution was given in Eq. (3.29).

Consider now $q=+1$, $E>0$. Recall that in the τ-time analysis both $M^2(\tau)$ and H^2 contained terms in a^{-4} and a^{-8} and, in particular, there was a regime in which $M^2(\tau)$ was positive and proportional to $a^{-4} \propto H^2$. In η time, however, $\mathcal{H} \propto a^{-6} + a^{-2}$ with $\mathcal{M}^2$ always being negative, proportional to $-a^{-6}$. Thus while the $a \ll a_c$ regime reduces to that discussed above for $E=0$, the $a \gg a_c$ regime is a little less clear. There $\mathcal{H}^2 \sim a(\eta)^{-2}$, but $\mathcal{M}^2 \sim -a(\eta)^{-6}$. Thus

$$a(\eta) = a_i \pm \frac{\sqrt{2E}}{\ell}(\eta - \eta_i) \tag{3.48}$$

and

$$k_c(\eta)^2 = -\mathcal{M}(\eta)^2 = \frac{10\tilde{E}^2}{\ell^2 a^6}. \tag{3.49}$$

Now $|k_c(\eta)| \sim |\mathcal{H}|^3 \ell^2 = \ell^2/R_h^3$, and so one can no longer identify the critical wavelength with the Hubble radius. For $|k| \ll |k_c|$ Eq. (3.44) reduces to $d^2\psi_k/da^2 - (5\tilde{E}^2/E)(\psi_k/a^6) = 0$. The solution is expressed in terms of Bessel functions which, however, show exactly the same behavior as Eq. (3.37): namely, $\phi_k = \psi_k/a$ tends to a constant as $a \to \infty$. The other limit $a \to 0$ is not relevant as the above equation is only valid for $a \gg a_c$.

We do not discuss further the case of $q=-1$ and $q \neq 1$ since the results obtained in this approach are exactly as discussed in Secs. III B 2 and III B 3.

IV. BARDEEN POTENTIALS

So far we have discussed the evolution of ϕ, the magnitude of the brane perturbation as seen by a 5D observer comoving with the brane. For an observer living *on* the brane, the perturbed brane embedding gives rise to perturbations about the FRW geometry. Recall [see Eq. (2.8)] that for the unperturbed brane

$$\begin{aligned}\overline{ds}_4^2 &= \overline{\hat{\gamma}}_{ab}dx^a dx^b \\ &= -[f(R) - h(R)\dot{R}^2]dt^2 + g(R)d\vec{x}\cdot d\vec{x} \\ &\equiv -n^2(t)dt^2 + a^2(t)d\vec{x}\cdot d\vec{x}\end{aligned} \tag{4.1}$$

where the overbar on $\overline{\gamma}$ denotes that it is an unperturbed quantity. Note that the scale factors $n^2(t)$ and $a^2(t)$ pick up their time dependence through $R(t)$—for instance $a^2(t) = g(R(t))$. In this section we calculate $\delta\hat{\gamma}_{ab}$ resulting from the perturbed embedding (3.1) and relate it to the Bardeen potentials.

Initially, rather than using the covariant form (3.1), let us write more generally

$$X^0(t,\vec{x}) = t + \zeta^0(t,\vec{x}), \tag{4.2}$$

$$X^i(t,\vec{x}) = x^i + \zeta^i(t,\vec{x}), \tag{4.3}$$

$$X^4(t,\vec{x}) = R(t) + \epsilon(t,\vec{x}). \tag{4.4}$$

Below we will see that the perturbations ζ^i do not enter into the two scalar Bardeen potentials which correspond to the two degrees of freedom ζ^0 and ϵ. This is expected since perturbations parallel to the brane are not physical and can be removed by a coordinate transformation [42]. Then only right at the end will we set $\zeta^0/n^0 = \epsilon/n^5 = \phi$. We will find that the two Bardeen potentials are proportional to each other and to ϕ.

By definition, the perturbed brane embedding is given by

$$\begin{aligned}\hat{\gamma}_{ab} &\equiv \overline{\hat{\gamma}}_{ab} + \delta\hat{\gamma}_{ab} \\ &= g_{\mu\nu}(\overline{X} + \delta X)\frac{\partial}{\partial x_a}(\overline{X}^\mu + \delta X^\mu)\frac{\partial}{\partial x_b}(\overline{X}^\nu + \delta X^\nu).\end{aligned} \tag{4.5}$$

Evaluating $\delta\hat{\gamma}_{ab}$ to first order for the perturbed embedding (4.2)–(4.4) and the general bulk metric (2.3), one obtains

$$\delta\hat{\gamma}_{00} = \epsilon(-f' + h'\dot{R}^2) + 2(-\dot{\zeta}^0 f + \dot{\epsilon}h\dot{R}), \tag{4.6}$$

$$\delta\hat{\gamma}_{0i} = -(\partial_i\zeta^0)f + \dot{\zeta}^i g + (\partial_i\epsilon)h\dot{R}, \tag{4.7}$$

$$\delta\hat{\gamma}_{ij} = \epsilon g'\delta_{ij} + (\partial_i\zeta_j + \partial_j\zeta_i)g. \tag{4.8}$$

Note that the terms proportional to ϵ come from the Taylor expansion of $g_{\mu\nu}(\overline{X} + \delta X)$ in Eq. (4.5) to first order.

In the usual way, the perturbed line element on the brane is written as

$$\begin{aligned}ds_4^2 = &-n^2(1+2\underline{A})dt^2 - 2an\underline{B}_i dt dx^i \\ &+ a^2(\delta_{ij} + \underline{h}_{ij})dx^i dx^j\end{aligned} \tag{4.9}$$

where $n(t)$ and $a(t)$ are defined in Eq. (4.1), and as usual vectors are decomposed into a scalar part and a divergenceless vector component, e.g.,

$$\underline{B}_i=\partial_i\underline{B}+\underline{\tilde{B}}_i \tag{4.10}$$

with $\partial^i\underline{\tilde{B}}_i=0$. We will use a similar decomposition for ζ^i defined in Eq. (4.3) as well as the usual one for tensor perturbations. Thus from Eqs. (4.6)–(4.8) we have

$$\underline{A}=\frac{1}{n^2}\left[\frac{\epsilon}{2}(f'-h'\dot{R}^2)+(\dot{\zeta}^0 f-\dot{\epsilon}h\dot{R})\right],$$

$$\underline{B}=\frac{1}{an}[\zeta^0 f-\dot{\zeta}a^2-\epsilon h\dot{R}],$$

$$\underline{\tilde{B}}_i=-\frac{a}{n}\dot{\tilde{\zeta}}_i,$$

$$\underline{C}=\frac{\epsilon}{2}\left(\frac{g'}{g}\right),$$

$$\underline{E}=\zeta,$$

$$\underline{\tilde{E}}_i=\tilde{\zeta}_i,$$

$$\underline{\tilde{\tilde{E}}}_{ij}=0,$$

where we have used standard notation defined, e.g., in [10]. By considering coordinate transformations on the brane and doing standard four-dimensional perturbation theory one can define the usual two Bardeen potentials, as well as the brane vector and tensor metric perturbations. For the first Bardeen potential we find, after some algebra,

$$\begin{aligned}\underline{\Phi}&=-\underline{C}+\frac{\dot{a}}{n}\left(\underline{B}+\frac{a}{n}\dot{\underline{E}}\right)\\&=\left(\frac{\dot{a}}{a}\right)\frac{f}{n^2}\frac{1}{\dot{R}}[\zeta^0\dot{R}-\epsilon].\end{aligned} \tag{4.11}$$

Notice that all terms containing ζ^i in $\underline{B}$ and $\underline{E}$ have cancelled as expected since they are not physical degrees of freedom. Similarly,

$$\begin{aligned}\underline{\Psi}&=\underline{A}-\frac{1}{n}\partial_t\left(a\underline{B}+\frac{a^2}{n}\dot{\underline{E}}\right)\\&=\frac{1}{n^2}\frac{1}{\dot{R}}[\zeta^0\dot{R}-\epsilon]\left[f'\dot{R}-f\left(\frac{\dot{n}}{n}\right)\right].\end{aligned} \tag{4.12}$$

The important point to notice in this second case is not only the absence of ζ^i, but that all *derivatives* of the perturbations ζ^0 and ϵ (which appear in $\underline{A}$) have also cancelled. Hence we will find that the Bardeen potentials are proportional to ϕ only and not to any of its derivatives. Finally, the gauge invariant vector and tensor perturbations are identically zero.

We now set

$$\epsilon=n^4\phi,\quad \zeta^0=n^0\phi \tag{4.13}$$

(where n^ν is the normal to the brane) in order to make contact with the covariant formalism of Sec. III. Then the combination that appears in both $\underline{\Psi}$ and $\underline{\Phi}$ is

$$\zeta^0\dot{R}-\epsilon=-\left(\frac{n^2(t)}{f}n^4\right)\phi \tag{4.14}$$

where n^4 is the fourth component of the normal to the unperturbed brane. Thus

$$\underline{\Phi}=-\left(\frac{\dot{a}}{a}\right)\frac{n^4}{\dot{R}}\phi,\quad \underline{\Psi}=\left(\frac{f'}{f}\dot{R}-\frac{\dot{n}}{n}\right)\frac{n^4}{\dot{R}}\phi \tag{4.15}$$

which, on going to AdS$_5$-S space-time and using the expression for $\dot{R}^2$ in Eq. (2.13), yields

$$\underline{\Phi}=-\frac{(E-C(a))}{a^4}\left(\frac{\phi}{\ell}\right)=-\left(\frac{\tilde{E}}{a^4}+q\right)\left(\frac{\phi}{\ell}\right), \tag{4.16}$$

$$\underline{\Psi}=3\underline{\Phi}+4q\left(\frac{\phi}{\ell}\right). \tag{4.17}$$

Even though there are no anisotropic stresses, the Bardeen potentials here are not equal. We suppose that this is due to the absence of self-gravity. We see that for superhorizon modes on an expanding brane (for which, from Sec. III, $\phi_k \propto a^4$), we also have $\underline{\Phi}_k\propto a^4$. Similarly, $\underline{\Phi}_k$ also grows rapidly for a brane falling into the black-hole horizon.

To obtain a true (i.e., gauge invariant) measure of the "deviation" from the FRW case, it is useful to look at the ratio of the components of the perturbed Weyl tensor and the background Riemann tensor, which in the FRW case is roughly given by $(k\eta)^2|\underline{\Phi}_k+\underline{\Psi}_k|$ (see [43]). For $\underline{\Phi}_k\propto a^4$ this ratio grows, because $a\sim\eta^{1/3}$ when $\mathcal{H}^2\sim a^{-6}$.

V. CONCLUSIONS

In this paper we have studied the evolution of perturbations on a moving D3-brane coupled to a bulk four-form field, focusing mainly on an AdS$_5$-Schwarzschild bulk. For an observer on the unperturbed brane, this motion leads to FRW expansion or contraction with scale factor $a\propto r$. We assumed that there is no matter on the brane and ignored the back reaction of the brane onto the bulk. Instead, we aimed to investigate the growth of perturbations due only to motion, and also to study the stability of moving D3-branes. For such a probe brane, the only possible perturbations are those of the brane embedding. The fluctuations about the straight brane worldsheet are described by a scalar field ϕ which is the proper amplitude of a "wiggle" seen by an observer comoving with the unperturbed brane. Following the work of [20,21,41] we derived an equation of motion for ϕ, and investigated whether small fluctuations are stretched away by the expansion, or, on the other hand, whether they grow on a contracting brane. The equation for ϕ is characterized by an effective mass squared and we noted that if this mass is positive the system is not necessarily stable: indeed, in Sec. III we discussed a regime in which the effective mass squared is positive in brane time but negative in conformal time, and therefore the perturbations grow. Another important factor in the evolution of ϕ is the time dependence of that mass.

In Sec. III we found that on an expanding the BPS brane with total energy $E=0$, superhorizon modes grow as a^4, whereas subhorizon modes decay and hence are stable. For a contracting brane, on the contrary, both super- and subhorizon modes grow as a^{-3}, and a^{-1} respectively. These fluctuations become large in the near extremal limit, $a_H \ll 1$. We therefore concluded that the brane becomes unstable (i.e., the wiggles grow) as it falls into the black hole. We also discussed the case $E>0$ for BPS branes and BPS antibranes. Non-BPS branes were found to be unstable at late times when a positive cosmological constant dominates.

We have discussed the evolution of the fluctuations ϕ as measured by a five-dimensional observer moving with the unperturbed brane. However, for an observer at rest in the bulk, the magnitude of the perturbation is given by a Lorentz contraction factor times the proper perturbation ϕ. (For a flat bulk spacetime this was pointed out in [20].) Hence, if perturbations grow for the ''comoving'' observer, they do not necessarily grow for an observer at rest in the bulk.

Finally, the fluctuations around the unperturbed worldsheet generate perturbations in the FRW universe. In Sec. IV we discussed these perturbations from the point of view of a 4D observer now living on the perturbed brane. We calculated the Bardeen potentials $\underline{\Phi}$ and $\underline{\Psi}$ which were both found to be proportional to ϕ. Furthermore, we saw that the ratio ''Weyl to Riemann'' which, expressed in terms of $\underline{\Phi}$ and $\underline{\Psi}$, gives a gauge invariant measure for the ''deviation'' from the FRW universe, also grows.

A limitation of this work is that the back reaction of the brane onto the bulk was neglected. One may wonder whether inclusion of the back reaction could stabilize ϕ. To answer that question, recall that the setup we have analyzed here corresponds, in the junction condition approach, to one in which Z_2 symmetry across the brane is broken. Then the brane is at the interface of two AdS_5-S space-times, and its total energy is related to the difference of the respective black-hole masses: $\tilde{E} \propto M_+ - M_-$. Perturbation theory in such a non-Z_2-symmetric self-interacting case has been set up in [10], though it is technically quite complicated. However, in the future we hope to try to use that formalism to include the back reaction of the brane onto the bulk.

It would be interesting to extend this analysis to branes with n codimensions: in this case one has to consider n scalar fields—one for each normal to the brane. Formalisms to treat this problem have been developed in [22,44]. In that case the equations of motion for the scalar fields are coupled, and it becomes a complicated task to diagonalize the system.

Finally, it would also be interesting to consider nonzero F_{ab}, and hence the effect of perturbations in the radiation on the brane.

ACKNOWLEDGMENTS

We thank Ph. Brax, E. Dudas, S. Foffa, M. Maggiore, J. Mourad, M. Parry, A. Riazuelo, K. Stelle, and R. Trotta for numerous useful discussions and encouragement. We especially thank R. Durrer for her comments on the manuscript. T.B. thanks LPT Orsay for hospitality.

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