Primordial perturbations in a nonsingular bouncing universe model

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We construct a simple nonsingular cosmological model in which the currently observed expansion phase was preceded by a contraction. This is achieved, in the framework of pure general relativity, by means of a radiation fluid and a free scalar field having negative energy. We calculate the power spectrum of the scalar perturbations that are produced in such a bouncing model under the assumption of initial vacuum state for the quantum field associated with the hydrodynamical perturbation. The matching conditions applying to this bouncing model are derived and shown to be different from those in the case of a sharp transition. We show that if our bounce transition is smoothly connected to a slowly contracting phase, this provides a new way to generate a scale invariant power spectrum.

I. INTRODUCTION

For more than two decades, inflation [1] has been the only available paradigm to solve the standard cosmological problems of flatness, homogeneity, and monopole excess. It also predicts, as a bonus, that primordial fluctuations, assumed to be of quantum origin, could be enhanced to the level required to trigger large scale structure formation, with an almost scale-invariant spectrum. To date, no model has ever come close to challenging this impressive list of successes.

Inflation cosmology suffers, however, from a few problems of its own, whose seriousness is largely a matter of opinion. For instance, in a typical realization, the underlying parameters (mass and coupling constants of the inflaton field) must be assigned ''un-natural'' values in order to reproduce the observed temperature fluctuations in the Cosmic Microwave Background Radiation (CMBR) which the mechanism seeds. However, such a fine tuning can be accounted for in various realistic models.

The inflation paradigm is also endowed with two specific problems, conceptually much more serious, that may ultimately be related, namely the meaning of the trans-Planckian [2] perturbations and the existence of a past singularity [3]. Concerning the latter, many ideas were discussed, among them the Tolman Phoenix universe [4] and many others in the 1970s [5], and recently revived under the name ''ekpyrotic'' [6] in the somewhat different context of superstring [7] inspired brane cosmology [8]. This model, however, was the subject of many criticisms, both from the string [9] and cosmological [10] points of view. In its latest version [11], moreover, the model also contains a supposedly actual singularity [12]. The singularity may also be avoided in other brane models [13].

Quantum cosmology, in the framework of the Wheeler-DeWitt (WDW) equation, exhibits bouncing solutions [14]

which can be interpreted as truly avoiding the singularity, even for flat ($\mathcal{K}=0$) spatial sections, a possibility strictly forbidden in classical general relativity (GR) unless [15] some exotic material, with negative energy density violating the null energy condition (NEC) [16], is introduced, which most cosmologists are reluctant to do. Such a bouncing universe model provides a solution to the horizon problem by geodesically completing the manifold in the past, and avoids the monopole formation if the bounce takes place at a temperature below that of the grand unification theory (GUT); this class of models does not, however, address the question of flatness and one must assume $\mathcal{K}=0$ from the outset. Moreover, in such a context, the trans-Planckian issue simply does not exist because the initial conditions for the perturbations can be imposed during a phase where the universe is as close to the Minkowski spacetime as one wishes, without ever passing through a Planck phase. Thus, it could be a natural competitor to the inflationary paradigm, and it is therefore of interest to estimate the primordial perturbation spectrum that it can produce.

In a previous work [17] we examined the stability of a bouncing universe dominated, at the bounce, by a single exotic hydrodynamical perfect fluid. We showed, by computing scalar perturbations using the gauge invariant Bardeen [18] potential, that its adiabatic perturbations grow unboundedly either at the bouncing point, or at the time when the NEC was violated or restored, thereby contradicting the hypothesis of low amplitude first order perturbation theory [19,20]. Such models are thus incompatible with observational data, e.g., CMBR data [21] according to which these first order effects indeed still dominate (over nonlinear effects) at large scales.

The next to simple possibility consists of combining two components, i.e., to allow for entropy modes. The purpose of this paper is to exhibit such a toy model in which a radiation fluid is coupled to a negative energy free scalar field that is supposed to be important in the universe for a limited amount of time, during which the bounce occurs [22]. Our universe thus comes from a low-density, radiation-

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dominated, contracting state, passes through a bounce, and connects again back to the usual hot big-bang phase [23]. In this context, we shall be concerned with the scalar perturbations induced during the transition between the collapsing and expanding phases.

In Sec. II we set the various constraints our model needs to satisfy, and we explain how it can be made phenomenologically reasonable. Then we calculate, in Sec. III, the power spectrum of the perturbations by matching the relevant solutions in the various regions of interest, and we compare the results with numerical calculations. Contrary to what one would naively expect from a fluid analysis [17], we find that scalar perturbations are perfectly well behaved all along.

Setting vacuum initial conditions for the quantized hydrodynamical perturbations deep in the low-density radiation-dominated phase, we find that the relevant spectrum of perturbations, at last horizon crossing during the expanding radiation-dominated era, has a spectral index $n_s = -1$. It is thus incompatible with observational data. This result is directly related to the fact that, in order to calculate explicitly the power spectrum of scalar perturbations, we have assumed the algebraic form of the scale factor to be valid for all times, and in particular when the relevant scales of perturbation leave the horizon. Changing this assumption, as is discussed below, changes the spectrum as the perturbation leaves the horizon, and therefore the power spectrum induced in the expanding phase. As it is a model-dependent result, further investigations of more realistic models [24,25], from the point of view of particle physics, need to be done [26]. We show, by means of the example of an ekpyrotic-inspired slow collapse [6], how one can obtain a scale invariant spectrum in a bouncing universe model.

While other models yield a scale invariant spectrum by making use of various assumptions [27], the present calculations are made with a specific model where the transition through the bounce is made with an exact solution. This allows one to obtain, qualitatively and numerically, the transitions in the Bardeen potential and its derivative through the bounce, yielding indications on what kind of matching conditions [28] should be proposed for perturbations passing through a general bounce. We obtain the perhaps not so surprising result that the Bardeen potential changes sign through the bounce, even though its derivative is continuous, contrary to the case of a sharp transition [29,30]. This result is discussed in more detail in Sec. IV, in which we explicitly indicate how to obtain a scale invariant spectrum for the scalar perturbation by connecting our bounce and radiation-dominated model with a slowly contracting phase. If such a four-dimensional and singularity-free model can be constructed, it will be able to reproduce all the available observational data while avoiding most of the questions raised by the inflation solution.

This paper only deals with the scalar part of the perturbations, which we show does not yield a spectrum compatible with the data. The tensor part was already calculated in Ref. [12], where the spectral index $n_T = n_s - 1 = 2$ was obtained, and is therefore unable to reproduce the data. Finally, we will

not be concerned with vectorial (rotational) perturbations even though, contrary to the usual inflationary case, one could think that those have no reason to be *a priori* negligible with a time symmetric scale factor. However, the Universe is torque-free [31] since at least the nucleosynthesis epoch that occurred at a redshift of $z_{nucl} \sim 3 \times 10^8$. Hence, the present relative contribution δ_v for the vectorial perturbation, which scales as a^{-2} [19], is expected to be $\delta_v \lesssim 10^{-17}$, independent of the scale k at which it is evaluated, and hence observationally irrelevant.

II. THE MODEL

We shall consider a very simple toy model for which we demand the following conditions to hold. First of all, we want general relativity to be valid for all times. We also impose that at late times, the model should reproduce the standard hot big bang case, i.e. there should exist a time in which radiation dominates. This implies in particular that we assume some amount of radiation to be present in our model. We also restrict our attention to the spatially flat situation. Finally, the model should have a bouncing phase. This means, given that there is already some radiation present, that, in the context of GR there must exist some other fluid having negative energy. In particular, for the special case at hand for which the spatial curvature $\mathcal{K}=0$, this means that the null energy condition (NEC) must be violated at some time near the bounce [15].

Realizing such a model is in principle feasible with just another fluid, e.g., some stiff matter with negative energy, namely one for which the equation of state reads $p = \rho < 0$, and to combine this fluid with the radiation into a single one to recover the bounce at the background level. At the perturbative level, though, this is no longer feasible as it was recently shown [17] that such an approach will lead to an overproduction of large inhomogeneities at various different times, breaking the cosmological principle hypothesis long before nucleosynthesis. This is because such an approach neglects the entropy modes, which turn out to cancel exactly the adiabatic mode divergences, so that the adiabatic-entropy mode decomposition is not feasible in a bounce context (see the Appendix for a discussion of this issue). We model the second, stiff matter, fluid by a free massless scalar field to which it is equivalent both at the background and perturbative levels [32]. The action we shall start with thus reads

$$\mathcal{S} = \int \left(-\frac{1}{16\pi G} R - \epsilon - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi \right) \sqrt{-g}\, \mathrm{d}^4 x, \quad (1)$$

where R is the curvature scalar, ϵ the energy density of the radiation fluid, and ϕ the scalar field. We assume that the background metric takes the standard Friedmann-Robertson-Walker (FRW) form

$$\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j), \quad (2)$$

with η the conformal time. The cosmic time, t, is then obtained as the solution of the equation $a\,\mathrm{d}\eta = \mathrm{d}t$ once the scale factor $a(\eta)$ is known. Note that throughout this paper, we assume the background curvature to vanish, $\mathcal{K}=0$. In this

context, it has to be a particular choice: this category of models does not indeed solve the flatness problem.

Varying the action (1) with respect to the fluid and fields yields the background dynamical equations

$$\varepsilon' + 4\mathcal{H}\varepsilon = 0, \quad \varphi'' + 2\mathcal{H}\varphi' = 0, \tag{3}$$

where $\mathcal{H} \equiv a'/a$, ε and φ are the background space-independent values of the radiation energy density ϵ and scalar field ϕ, respectively, and a prime denotes a differentiation with respect to the conformal time η. These background equations imply

$$\varphi' = \frac{c}{a^2}, \quad \varepsilon = \frac{d}{a^4}, \tag{4}$$

where c and d are constant. The energy density of the scalar field is given by

$$\rho_\varphi \equiv -\frac{\varphi'^2}{2a^2} = -\frac{c^2}{2a^6}, \tag{5}$$

and as such it is large when a is small and negligible when a is very large. These solutions, together with the Friedmann equation

$$\mathcal{H}^2 = \ell_{\text{Pl}}^2 \left(a^2 \varepsilon - \frac{1}{2}\varphi'^2 \right), \quad \ell_{\text{Pl}}^2 \equiv \frac{8\pi G}{3}, \tag{6}$$

lead to the bouncing solution

$$a(\eta) = a_0 \sqrt{1 + \left(\frac{\eta}{\eta_0}\right)^2}, \tag{7}$$

where the minimum scale factor a_0 and the characteristic bouncing conformal time η_0 solely depend on the relative quantities of energy density in radiation and scalar field at some given time $a_0^2 = c^2/(2d)$ and $\eta_0^2 = c^2/(2d^2\ell_{\text{Pl}}^2)$. In what follows, these two parameters will be considered as the relevant ones.

It is interesting to note here that this solution is essentially the only possibility if one demands a late time radiation-dominated phase $a \propto \eta$ for $(\eta/\eta_0) \gg 1$ together with a bounce with quadratic behavior $(a - a_0) \propto \eta^2$ for $(\eta/\eta_0) \ll 1$ [17]. Moreover, there is only one way to get this solution, namely by means of a negative energy stiff matter fluid or, equivalently, a negative energy free scalar field, with radiation in both cases.

Before turning to first order perturbations of this background, which is the subject of the following section, we want to emphasize a point of stability of this model related to the "wrong" sign chosen in Eq. (1). Indeed, an expansion of Eq. (1) with respect to

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad \phi = \varphi + \delta\phi, \tag{8}$$

with $[g^{(0)}, \varphi]$ the classical part and $(\mathbf{h}, \delta\phi)$ interpreted, respectively, as gravitons and scalar particles in a semiclassical approach, will inevitably lead to two different kinds of instabilities, each arising at a different order in perturbation. The
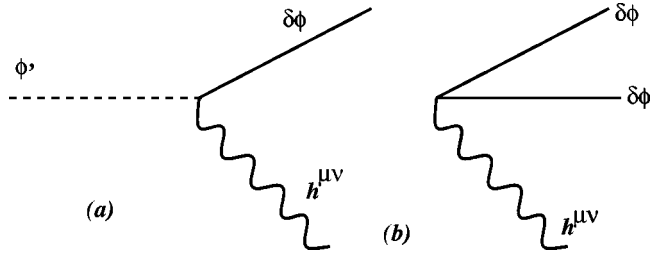

FIG. 1. Diagrams leading to instabilities in the theory (1). (a) The dynamical instability whereby the energy contained in the scalar field can be used to produce semiclassical perturbations, later to be identified with primordial fluctuations. (b) Vacuum instability. As this process in nonzero, the vacuum can spontaneously decay into a pair of negative energy scalar particles and a positive energy graviton.

first one, with which we shall deal later since it is actually the one responsible for the large scale structure formation in this model, is second order in perturbation (first order in the equations of motion) and goes essentially as $\propto h^{\mu\nu}\partial_\mu \delta\phi \partial_\nu \varphi$. This term is absent in ordinary Minkowski space, but is present in the cosmological setup we are considering because of Eq. (3) in which the classical part of the scalar field varies with time and thus behaves as a source for the production of gravitons and scalar particles. As it originates in a derivative coupling [see Fig. 1(a)], the characteristic time scale of this instability is that of the classical scalar part, in our case the typical cosmological time scale.

The second instability that must be discussed is much more serious, even though at first sight it looks innocuous because of a higher order in perturbation: it is the same term as before, but with the classical part replaced by a first order perturbation, namely $\propto h^{\mu\nu}\partial_\mu \delta\phi \partial_\nu \delta\phi$ [see Fig. 1(b)]. The presence of such a process means that the vacuum can spontaneously decay into a pair of negative energy scalar particles and a graviton, and, due to this fact, the energy levels are not bounded from below. This sounds like a catastrophe, and even more so because the only available time scale comes from the coupling constant, i.e., the Planck time. However, it is clear from the figure that the process probability amplitude $\mathcal{A}$ is $\mathcal{A} \propto p^2/M_P^2$, with $M_P \sim \ell_{\text{Pl}}^{-1} \simeq 10^{19}$ GeV the Planck mass and p the momentum at the vertex. Such an amplitude therefore becomes important when the characteristic scale p^{-1} is comparable to ℓ_{Pl}. At this point, it should be argued that the model of Eq. (1) is understood as an effective low energy theory which must be implemented with a cutoff scale much larger than the Planck one: as one reaches the Planck energy scale, the theory is expected to break down into a completely different one such as, e.g., quantum gravity or superstring theory. As a result, for cosmological purposes, one can safely ignore this instability and concentrate on the production of cosmological perturbations.

III. LINEAR PERTURBATION SPECTRUM

In what follows, we shall consider perturbations stemming from the model (1), making use of the gauge invariant formalism [18,20]. In order to perform explicit calculations,

we shall consider this model to be valid all through the history of the Universe, including at the times at which the perturbations leave the horizon. It is a simple matter to modify this assumption afterwards, as we do in the concluding section, to discuss the spectrum that a more realistic theory with many different phases would have.

As there are no anisotropic stress perturbations in this model, the most general form of metric perturbations on the background given by Eq. (2) reads, in the longitudinal gauge,

$$ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - (1-2\Phi)\delta_{ij}dx^i dx^j], \quad (9)$$

where Φ is the gauge invariant Bardeen potential [18]. Setting also

$$\phi = \varphi(\eta) + \delta\phi(\mathbf{x}, \eta) \quad \text{and} \quad \epsilon = \varepsilon(\eta) + \delta\epsilon(\mathbf{x}, \eta), \quad (10)$$

one obtains the radiation fluid current conservation and Klein-Gordon equation, respectively, in the form

$$\delta\epsilon' + 4\mathcal{H}\delta\epsilon = \frac{4}{3}\varepsilon(3\Phi' + a^{-1}\nabla^2\widetilde{\alpha}),$$

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = 4\Phi'\varphi', \quad (11)$$

where $\widetilde{\alpha}$ is the gauge invariant fluid velocity potential, and use has been made of the relation $\delta\epsilon = \frac{1}{3}\delta p$ between the energy density $\delta\epsilon$ and pressure δp fluctuation. Einstein equations yield, after a bit of algebra [20],

$$\Phi' + \mathcal{H}\Phi = \frac{3}{2}\ell_{\mathrm{Pl}}^2\left(-\varphi'\delta\phi + \frac{4}{3}a\widetilde{\alpha}\right),$$

$$\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = \frac{3}{2}\ell_{\mathrm{Pl}}^2(-\varphi'\delta\phi' + \varphi'^2\Phi + a^2\delta\epsilon),$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi$$
$$= \frac{3}{2}\ell_{\mathrm{Pl}}^2\left(-\varphi'\delta\phi' + \varphi'^2\Phi + \frac{1}{3}a^2\delta\epsilon\right). \quad (12)$$

Simple manipulations of Eqs. (11) and (12) permit us to eliminate the radiation fluctuation in favor of the Bardeen potential through the relation

$$\Phi_k'' + 6\mathcal{H}\Phi_k' + [2(\mathcal{H}' + 2\mathcal{H}^2) + k^2]\Phi_k = -\ell_{\mathrm{Pl}}^2 a^2\delta\epsilon_k, \quad (13)$$

where, from now on, we assume a Fourier decomposition of each variable A into its components A_k defined through

$$A_k(\eta) \equiv \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i\mathbf{k}\cdot\mathbf{x}} A(|\mathbf{x}|, \eta), \quad (14)$$

where A_k only depends on the amplitude k of the wave vector $\mathbf{k}$.

The dynamical equations for the Bardeen potential and the fluctuations of the scalar field therefore decouple from the radiation fluid perturbations and are then expressible solely in terms of themselves as (making use of the background Einstein equations)

$$\Phi_k'' + 4\mathcal{H}\Phi_k' + \frac{1}{3}k^2\Phi_k = -\ell_{\mathrm{Pl}}^2\varphi'\delta\phi_k' \quad (15)$$

and

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + k^2\delta\phi_k = 4\varphi'\Phi_k'. \quad (16)$$

We shall now investigate the solution of these equations in order to get the perturbation spectrum such a bouncing model predicts.

A. The relevant phases in the perturbations evolution

In order to investigate Eqs. (15) and (16), let us write them in terms of the variables $u_k \equiv a^2\Phi_k$, and $w_k \equiv a\delta\phi_k$. Using Eq. (4), they read

$$u_k'' + \left[\frac{1}{3}k^2 - 2(\mathcal{H}' + 2\mathcal{H}^2)\right]u_k = -\ell_{\mathrm{Pl}}^2\frac{c}{a}(w_k' - \mathcal{H}w_k), \quad (17)$$

and

$$w_k'' + (k^2 - \mathcal{H}^2 - \mathcal{H}')w_k = 4\frac{c}{a^3}(u_k' - \mathcal{H}u_k). \quad (18)$$

Each of these equations can be seen as an inhomogeneous equation, i.e., one with a source term not depending on the function itself. Asymptotically, since the scale factor grows like $|\eta|$ it can be checked explicitly, by means of an expansion in powers of η^{-1} for u_k and w_k, that the source terms in both Eqs. (17) and (18) are small for[1] $\eta \gg \eta_0$. Therefore, in this limit, Eqs. (15) and (16) can be reduced to the usual parametric oscillator equations for the variables u_k and w_k, namely

$$u_k'' + \left[\frac{1}{3}k^2 - \frac{(a^2)''}{a^2}\right]u_k = 0, \quad (19)$$

and

$$w_k'' + \left(k^2 - \frac{a''}{a}\right)w_k = 0. \quad (20)$$

The potentials $(a^2)''/a^2 = 2(\mathcal{H}' + 2\mathcal{H}^2) = 2/(\eta^2 + \eta_0^2)$ for u_k and $a''/a = \mathcal{H}' + \mathcal{H}^2 = \eta_0^2/(\eta^2 + \eta_0^2)^2$ for w_k are shown in Fig. 2 on which we also define the variable $x \equiv \eta/\eta_0$, as well as the various corresponding matching values x_1 and x_2.

We shall be interested in the cosmologically relevant limit $k \ll 1$. However, as $|\eta| \to \infty$, the k-dependent terms in Eqs. (19) and (20) become important. More precisely, when

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[1]This is valid not only when $\eta \to \infty$ but also when $k^2 < 6(\mathcal{H}' + 2\mathcal{H}^2)$ as long as $\eta \gg \eta_0$. We shall return to this point in more detail below.
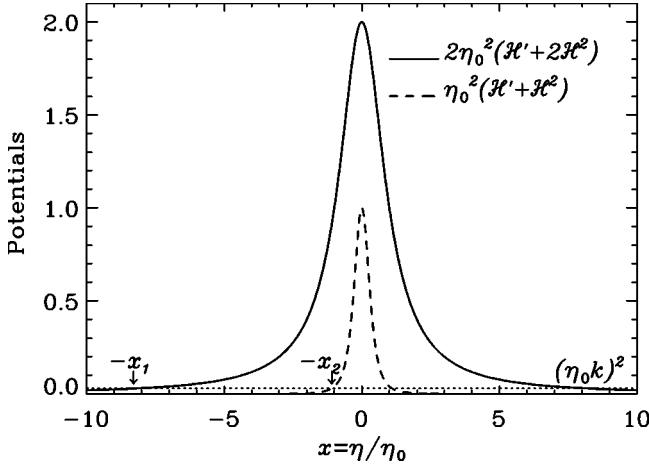
FIG. 2. Potentials for the parametric oscillator equations giving the dynamics of the Bardeen potential and the scalar field perturbations. The full line shows the potential for the variable associated with u_k [see Eq. (19)], the dashed line the potential for the scalar field w_k [Eq. (20)], and the dotted line, showing the value of $(\eta_0 k)^2$, indicates visually the different regions where the different approximations hold. The points x_1 and x_2 are the matching points for these two fields.

$k|\eta| > \sqrt{6}$, i.e., when $|x| > x_1 \equiv \sqrt{6}/(k\eta_0) \gg 1$, the solutions of the above equations in terms of Φ_k and $\delta\phi_k$ can be written in terms of Hankel functions, namely [33],

$$\Phi_k^{\mathrm{rad}} = \eta^{-3/2}[\Phi_{(1)}H_{3/2}^{(1)}(\omega\eta) + \Phi_{(2)}H_{3/2}^{(2)}(\omega\eta)],$$

$$\delta\phi_k^{\mathrm{rad}} = \eta^{-1/2}[X_{(1)}H_{1/2}^{(1)}(k\eta) + X_{(2)}H_{1/2}^{(2)}(k\eta)], \qquad (21)$$

where $\omega = k/\sqrt{3}$.

When the potential terms dominate over the k-dependent terms, which for u_k is the case so long as $k|\eta| \ll \sqrt{6}$, or $1 \ll |x| \ll x_1$, and for v_k when $k|\eta| \ll \sqrt{k\eta_0}$, or $1 \ll |x| \ll x_2$, where $x_2 \equiv 1/\sqrt{k\eta_0}$, the zeroth order solutions for Eqs. (19) and (20) read, when $x_1 \ll x \ll x_2$,

$$\Phi_k^< = A_1 + A_2 \int a^{-2}\mathrm{d}\eta + O(k\eta) \approx A_1 - A_2 \frac{\eta_0^4}{3a_0^4\eta^3}, \qquad (22)$$

$$\delta\phi_k^< = B_1 + B_2 \int a^{-1}\mathrm{d}\eta + O(k\eta) \approx B_1 - B_2 \frac{\eta_0^2}{a_0^2\eta}. \qquad (23)$$

In fact, solution (23) needs amelioration because around $|x| = x_2$ the source term of Eq. (16) becomes important. This is not the case of solution (22) because for $1 \ll |x| < x_2$, the source terms of Eq. (15) are still negligible, even taking into account the corrections of Eq. (23). Fortunately, for what follows in the next section, only the solution (22) will be needed.

Up to this point, one can assume that the two relevant perturbations, namely $\delta\epsilon$ and $\delta\phi$, are decoupled. As we have imposed vacuum normalizations for the underlying quantum variables, this leads to Φ_k and $\delta\phi_k$ behaving as two statis-
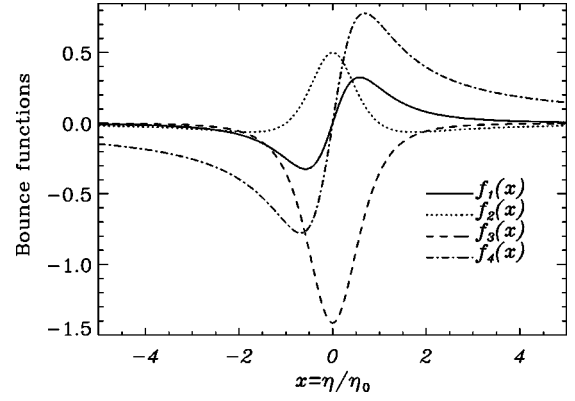
tically independent Gaussian random fields, with fixed variance. In what follows, we shall not be concerned by the spectrum of $\delta\phi$ in the late epoch at which it decouples again from Φ: its distribution should not be observable during the expanding era with which we shall ultimately be concerned.

Near the bounce, when the potentials and c/a are of order $1/\eta_0^2$ and $a_0/(\eta_0 \ell_{\mathrm{Pl}})$, the source terms become important,[2] but the terms proportional to k^2 are still negligible. In this situation, one can neglect altogether the k^2 term in Eqs. (15) and (16), yielding the solutions

$$\Phi^{\mathrm{Bounce}} = \tilde{A} + \tilde{B}f_1(x) + \tilde{C}f_2(x), \qquad (24)$$

and

$$\ell_{\mathrm{Pl}}\delta\phi^{\mathrm{Bounce}} = \tilde{D} + \tilde{B}f_3(x) + \tilde{C}f_4(x), \qquad (25)$$

with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ arbitrary constants. The bounce functions $f_i(x)$ are found to be

$$f_1(x) \equiv \frac{x}{(1+x^2)^2}, \quad f_2(x) \equiv \frac{1-x^2}{2(1+x^2)^2}, \qquad (26)$$

$$f_3(x) \equiv -\frac{\sqrt{2}}{(1+x^2)^2}, \quad f_4(x) \equiv \frac{x}{\sqrt{2}}\frac{3+x^2}{(1+x^2)^2}, \qquad (27)$$

and are displayed on Fig. 3. These solutions will be used to match the asymptotic solutions through the bounce.

B. Matching the solutions and the power spectrum

In the limit $\eta \to \pm\infty$, which is equivalent to $a \to \pm\infty$, i.e., very far from the bounce, the Universe is radiation dominated, so that the coupling term in the left-hand side of Eqs. (17) and (18) can be neglected, as it was explained in Sec.

FIG. 3. The bounce functions f_i as functions of $x \equiv \eta/\eta_0$.

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[2] The constraints on the values of a_0 and η_0 in order for the bounce to happen at a scale much larger than the Planck scale and a long time before nucleosynthesis takes place are $\ell_{\mathrm{Pl}} \ll a_0\eta_0 \ll 10^8$ cm. Hence, if one chooses η_0 of order one, the potentials are of order one, and $c/a \gg 1$ near the bounce.

III A. From Eq. (21), and for $\eta \to -\infty$, the Bardeen potential and the scalar field perturbation respectively scale as $1/a^2$ and $1/a$. From Eq. (13) one can see that the fluid perturbation $\delta \epsilon_k$ goes like $1/a^4$. Therefore, from Eq. (12) we find that the scalar field and its perturbation are irrelevant in this regime, for the evolution of the Bardeen potential with respect to the radiation fluid. For this reason, one can conclude that the appropriate quantum gauge invariant variable to be used must be the same as the one defined in Ref. [20] for the quantum treatment of hydrodynamical fluids perturbation theory, which, in the case of pure radiation, is related to Φ by

$$\Phi_k = 3 \sqrt{\frac{3}{2}} \frac{\ell_{\text{Pl}} \beta}{\mathcal{H} k^2} \left(\frac{v_k}{z} \right)', \tag{28}$$

where $\beta = \mathcal{H}^2 - \mathcal{H}'$ and $z \equiv a \sqrt{3\beta}/\mathcal{H}$. Similarly, the gauge invariant quantum variable connected to the scalar field perturbation given in Ref. [20] is given by

$$\omega_k = a[\delta\phi + (\varphi/\mathcal{H})' \Phi] \approx w_k. \tag{29}$$

It is interesting to note that the quantum field v leaves the oscillatory regime at the same conformal time as $\delta\phi$ does, and that neither of them do so at horizon crossing. This is a peculiarity of our model due to the fact that at the time at which the quantum fields leave the oscillatory regime, the space is almost radiation dominated, but not quite.

Imposing the initial vacuum state for these quantum variables implies that we can set

$$v_k = \frac{3^{1/4} e^{-ik(\eta - \eta_i)/\sqrt{3}}}{\sqrt{2k}}$$

and

$$w_k = \frac{e^{-ik(\eta - \eta_j)}}{\sqrt{2k}}$$

at $\eta \to -\infty$, with η_i and η_j two *a priori* arbitrary conformal times, having no influence on the subsequent evolution. From the solutions (21) and these initial conditions, one can write the Bardeen potential Φ_k and the scalar field perturbation $\delta\phi_k$ at $k\eta \ll -\sqrt{6}$ (or $|x| \gg x_1 \gg 1$) as

$$\Phi_k^{\text{ini}} = -\frac{\ell_{\text{Pl}} \eta_0 3^{3/4}}{2 a_0 \eta^2 k^{3/2}} \left(\frac{\sqrt{3}}{k\eta} + i \right) e^{-ik(\eta - \eta_i)/\sqrt{3}}, \tag{30}$$

and

$$\delta\phi_k^{\text{ini}} = \frac{\eta_0}{a_0 \eta \sqrt{2k}} e^{-ik(\eta - \eta_j)}. \tag{31}$$

In order to be in a position to compare the model with the cosmological data, in particular of the CMB, we are interested in calculating the primordial power spectrum of the gravitational potential Φ

$$\mathcal{P}_k \equiv k^3 |\Phi_k|^2 \equiv A_{\text{s}} k^{n_{\text{s}} - 1}, \tag{32}$$

evaluated at the time when Φ_k returns to its oscillatory regime, i.e., at $x = x_1$. As we shall see later, the values of $\delta\phi_k$ in the different phases of perturbation evolution are not necessary to calculate Φ_k at $x = x_1$. Moreover, the field $\delta\phi$ itself should not be observable during the expanding era. Hence, we will forget about $\delta\phi_k$ from now on.

Looking at Eq. (17), one can see that the first matching must be imposed when $k^2/3 = 2(\mathcal{H}' + 2\mathcal{H}^2) = (a^2)''/a^2$, for $u_k = a^2 \Phi_k$. As k is very small, this happens when $|\eta| \gg 1$ (where we can ignore the source terms). Matching the solution (30) with solution (22) at the point $k\eta \approx -\sqrt{6}$ (or $x \approx -x_1$) yields

$$A_1 = \frac{\ell_{\text{Pl}} \eta_0 \sqrt{k}}{3^{5/4} 2 \sqrt{2}} e^{i(\sqrt{2} + k\eta_i)/\sqrt{3}}, \tag{33}$$

and

$$A_2 = \frac{\ell_{\text{Pl}} a_0^3 3^{5/4}}{2 \eta_0^3 k^{5/2}} (1 - 3\sqrt{2}i) e^{i(\sqrt{2} + k\eta_j)/\sqrt{3}}. \tag{34}$$

The solution (22) is valid up to the point where x is of order one, when we approach the bounce. Differentiating Eq. (15) twice and making use of Eq. (16) as well as the background equations, we obtain the following fourth order equation:

$$\Phi_k^{(\text{IV})} + 10\mathcal{H}\Phi_k''' + \left[\frac{4}{3} k^2 + 20(\mathcal{H}' + 2\mathcal{H}^2) \right] \Phi_k''$$

$$+ 6\mathcal{H} k^2 \Phi_k' + \frac{1}{3} k^2 [k^2 + 4(\mathcal{H}' + 2\mathcal{H}^2)] \Phi_k = 0, \tag{35}$$

whose analysis indicates that the solution (22) is valid for $|x| \ll x_2$, up to $|x| \approx 1$, where the source term in Eqs. (15) become relevant but k is still completely negligible. We have to match this solution with the other relevant solution in this region, namely the bounce solution (24), which have three arbitrary constants. As confirmed by the numerical analysis below, we have to set $\tilde{C} = 0$ because the function it multiplies goes like $1/x^2$, which should appear, and dominate, in Eq. (22) in a region where both solutions are valid. In this region, the perturbations in the scalar field and in the fluid are decoupled, and therefore both of them should depend only in two unrelated constants. For the contracting phase, $\tilde{A}$ and $\tilde{B}$ are well defined as they represent, respectively, the amplitudes of the decaying and growing modes (or the opposite if one considers the expanding phase) of the dominant contribution to the gravitational perturbation, so the third arbitrary constant $\tilde{C}$ must vanish.

Hence, we only need to determine $\tilde{B}$ and $\tilde{A}$. That is why we do not need to calculate the evolution of $\delta\phi_k$ in order to

determine such constants; this also explains why the initial conditions we assume for the scalar field perturbation are irrelevant for the final power spectrum. It suffices that we assume $\delta\phi_k$ not to dominate initially over Φ_k; this is in particular the case for the vacuum initial state we consider. As shown later, to assume a vanishing initial distribution $\delta\phi_k|_{ini}=0$ works equally well.

The solution (24) will propagate the Bardeen potential to the other side of the bounce, to the region where x is of order one. As we are in a region where k is negligible, the point of matching will be chosen to be $x=-N\ll -1$, where N does not depend on k but is large.[3]

The result of the matching reads

$$\widetilde{A}=A_1-\frac{8\,\eta_0}{45a_0^4 N^5}A_2 \qquad (36)$$

and

$$\widetilde{B}=-\frac{\eta_0}{3a_0^4}A_2. \qquad (37)$$

At $x=N\gg 1$, on the other side of the bounce, these solutions must be matched with a solution similar to Eq. (22), namely,

$$\Phi_k^>=C_1-C_2\frac{\eta_0^4}{3a_0^4\eta^3}, \qquad (38)$$

yielding

$$C_1=A_1-\frac{16\,\eta_0}{45a_0^4 N^5}A_2, \qquad (39)$$

and $C_2=A_2$. For the power spectrum, the important term in Eq. (38) is the constant C_1: as we are now back to a regular expanding universe, the other term is a decaying mode which rapidly becomes negligible. In C_1, the dominant term when $k\ll 1$ is the one proportional to A_2 which goes as $k^{-5/2}$, while the other is proportional to $\sqrt{k}$. Hence, we get

$$k^3|\Phi_k(x_1)|^2\propto k^3|\widetilde{A}_2|^2\frac{\eta_0^2}{a_0^8 N^{10}}\approx\frac{\ell_{Pl}^2}{a_0^2\eta_0^4 N^{10}}k^{-2}, \qquad (40)$$

yielding a spectral index $n_s=-1$.

One can then define a transfer function between "Horizon exit" and "Horizon re-entry" as the ratio of the power spectra at the corresponding two different times. It is given in the case of our bounce by the relation
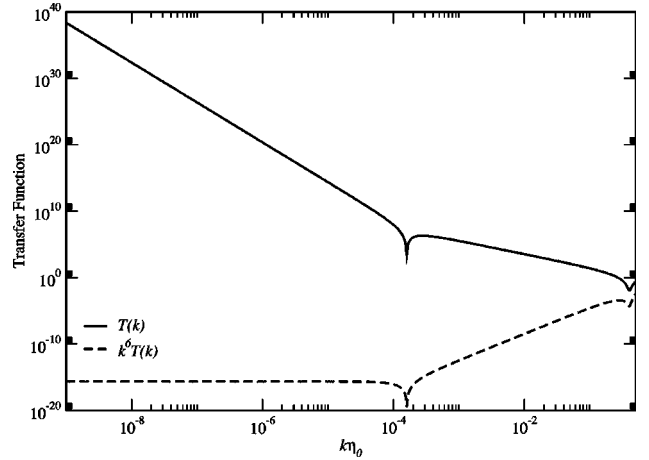

FIG. 4. Transfer function for the bounce model. Full line: ratio of the squared gravitational potential amplitude between horizon exit and re-entry. The dashed line shows the same multiplied by $\widetilde{k}^6$ to emphasize the power law behavior obtained in Eq. (41).

$$T(k)=\frac{k^3|\Phi_k(x_1)|^2}{k^3|\Phi_k(-x_1)|^2}\propto(\eta_0 k)^{-6}. \qquad (41)$$

This transfer function essentially depends on the behavior of the scale factor at both times, as well as on the nature of the bounce itself. It is represented in Fig. 4. Let us now check all these approximations through a numerical examination of Eqs. (15) and (16).

C. Numerical calculations

The system (15),(16) can be solved numerically for any value of k. For that purpose, we also include the characteristic conformal timescale η_0 in the wavenumber $\widetilde{k}=k\eta_0$ (and correspondingly $\widetilde{\omega}=\omega\eta_0$), and write the system as

$$\frac{d^2\Phi_k}{dx^2}+\frac{4x}{x^2+1}\frac{d\Phi_k}{dx}+\widetilde{\omega}^2\Phi_k=-\frac{\sqrt{2}}{x^2+1}\frac{dX_k}{dx},$$

$$\frac{d^2X_k}{dx^2}+\frac{2x}{x^2+1}\frac{dX_k}{dx}+\widetilde{k}^2X_k=\frac{4\sqrt{2}}{x^2+1}\frac{d\Phi_k}{dx}, \qquad (42)$$

relations in which $X_k\equiv\ell_{Pl}\delta\phi_k$ (recall that $x\equiv\eta/\eta_0$), subject to initial conditions, far in the limit $x\to -\infty$, given by Eqs. (30) and (31) with $\eta_i=\eta_j=0$, namely

$$\Phi_k^{ini}=-\frac{3^{3/4}\alpha}{2x^2\widetilde{k}^{3/2}}\left(\frac{\sqrt{3}}{\widetilde{k}x}+i\right)e^{-i\widetilde{k}x/\sqrt{3}} \qquad (43)$$

and

$$X_k^{ini}=\frac{\alpha}{x\sqrt{2\widetilde{k}}}e^{-i\widetilde{k}x}, \qquad (44)$$
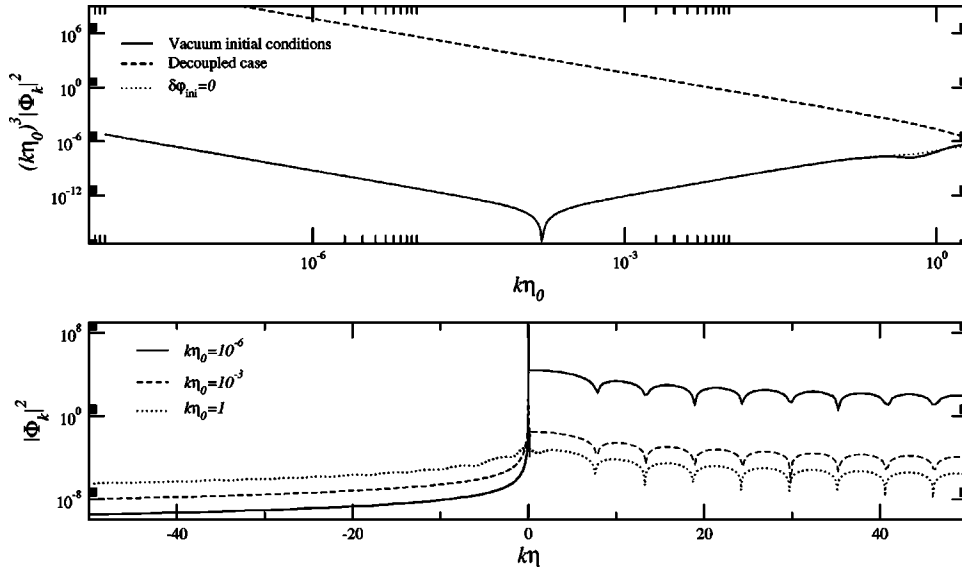
FIG. 5. Top panel: spectrum of scalar perturbations, i.e., $k^3|\Phi_k|^2$ as function of the wave number k, normalized with η_0 as indicated. The long wavelength part of the spectrum, as expected, is well fitted by a power-law with spectral index $n_s = -1$. The full line is for a vacuum initial condition for $\delta\varphi$, the dotted line is with $\delta\varphi = \delta\varphi' = 0$ at the initial time ($y_{ini} \equiv k \eta_{ini} = -100$ in the numerical calculation), and the dashed curve represents the fully decoupled situation for which φ' is assumed negligible all along. Bottom panel: Time evolution of the gravitational potential $|\Phi_k|^2$ for different wavelengths.

where we have defined the only free dimensionless parameter $\alpha = \ell_{Pl}\sqrt{\eta_0}/a_0$. In all the figures, this parameter has been arbitrarily fixed to the value $\alpha = 10^{-3}$; the conclusions do not, however, depend on this value, which acts as a simple normalization constant.

The solution of Eqs. (42) for the square of the Bardeen potential $|\Phi_k|^2$ is shown on the bottom panel of Fig. 5 for various values of the wavenumber, renormalized with the bounce characteristic conformal time scale, $\widetilde{k}$, ranging from 10^{-6} to ~ 1 on the figure as a function of the renormalized conformal time $y \equiv \widetilde{k}x = k\eta$. All calculations are started far in the radiation-dominated epoch, for $y = -100$, where the boundary conditions hold. This is verified as, indeed, for small enough values of $k\eta_0$, $|\Phi_k|^2$ behaves as η^{-4}, as expected. It can be checked that, as discussed above, in the long wavelength limit, the Bardeen potential starts with a negligible constant part and a growing, $\propto \eta^{-3}$, mode, for $-x_1 \ll x \ll 1$, which then connects to the f_1 part while crossing the bounce, and then connects back to the usual growing and decaying modes, although the new constant part has now acquired a piece from both modes of the previous epoch.

Once the system (42) is solved, one can easily compute the value of the Bardeen potential at horizon crossing, namely for $x \sim 1/\widetilde{k}$, i.e., $\eta \sim 1/k$, or $y \sim 1$. This provides the spectrum shown on the top panel of Fig. 5. It is clear on that figure that for small values of $\widetilde{k}$, the behavior of the power spectrum is indeed a power law, which we checked indeed corresponds to $n_s = -1$. Also shown is a comparison between various cases of interest, namely the vacuum case for which the initial conditions given by Eqs. (43) and (44) hold, the gravitational vacuum case for which Eq. (43) still holds, but with Eq. (44) replaced by $X_k^{ini} = 0$, and finally the decoupled case for which the coupling between Φ_k and X_k is made

to vanish, i.e. for which the left-hand side of Eqs. (42) is arbitrarily set to zero. The curves corresponding to either vacuum or gravitational vacuum initial conditions are seen to be almost undistinguishable, showing that, as expected and discussed in the preceding section, the final spectrum for the gravitational potential does not depend on the initial conditions for the scalar field perturbations. The decoupled curve shows that, for $\widetilde{k} \ll 1$, if one were to neglect the bounce duration and apply some matching conditions by brute force, one gets the same spectral index $n_s = -1$, but with a normalization that is wrong by many orders of magnitude. The situation is even worse for intermediate scales for which even the index is wrong.

In Fig. 6 is shown an enhancement of the region surrounding the bounce itself. This figure shows that the real and imaginary parts of both the Bardeen potential and the scalar field perturbation connect, respectively, with the bounce functions f_1 and f_3, thereby confirming the prediction $\widetilde{C} = 0$.

D. Junction conditions

The fact that the Bardeen potential only connects to the odd bounce function $f_1(x)$ suggests that in the limit in which the bounce duration η_0 can be neglected, one may apply the following junction conditions across the surface at which the bounce is located ($\eta = 0$):

$$[\mathcal{H}\Phi]_\pm = [\Phi' + \mathcal{H}\Phi]_\pm = 0, \tag{45}$$

where $[A]_\pm = A(-\eta_B) - A(+\eta_B)$ is the jump in the geometric quantity A. In Fig. 7 are shown the time evolution across the bounce of the quantities involved in Eq. (45). For a fixed surface thickness η_B, and in a way independent of this thick-
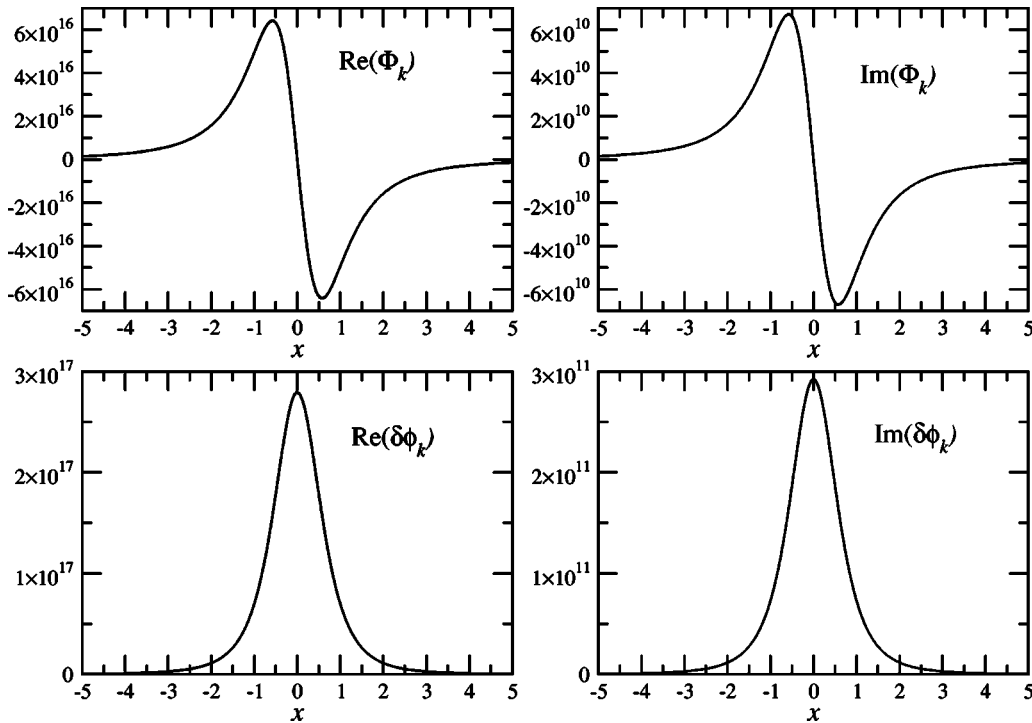
FIG. 6. Real (left) and imaginary (right) parts of the Bardeen potential (top) and the scalar field perturbation (bottom) as a function of $x = \eta/\eta_0$ for $\tilde{k} = 10^{-8}$.

ness, the relevant quantities are indeed conserved and can be safely used. This is of course true only in the particular example presented here, but it can also be conjectured to apply for a symmetric bounce in general. In fact, taking the ekpy-rotic model [6] with our matching conditions (45), one would obtain a scale invariant spectrum; the relevance of this fact to observational physics is, however, not clear as the ekpyrotic bounce, even regularized, is not symmetric.
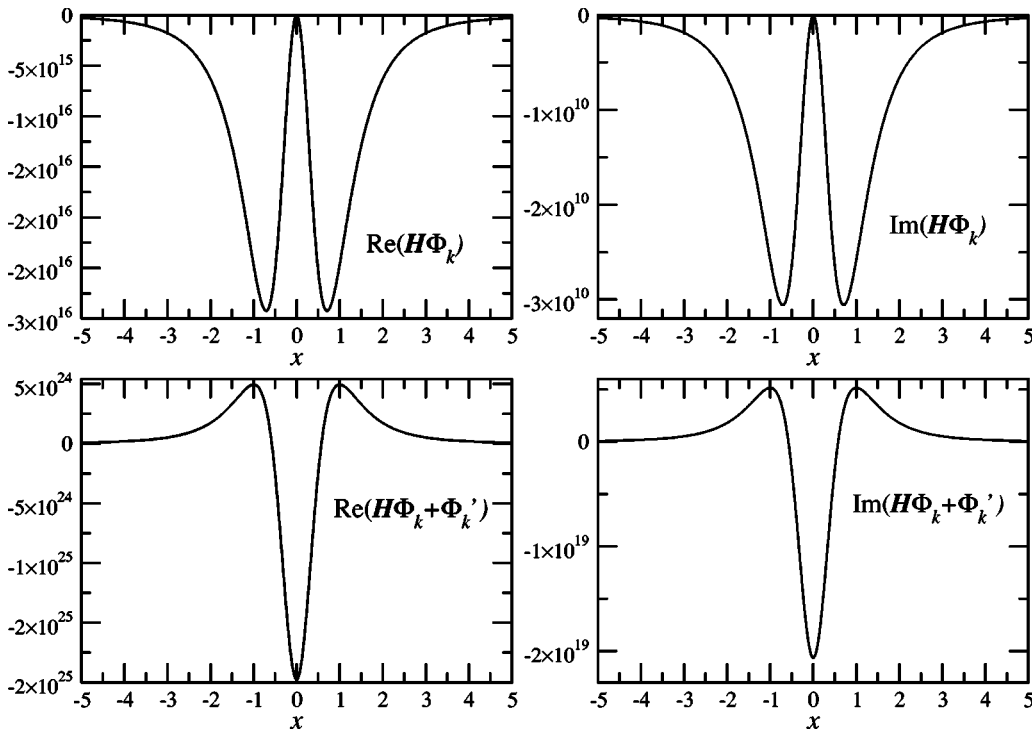

FIG. 7. Real (left) and imaginary (right) parts of the geometrical junction variables $\mathcal{H}\Phi_k$ (top) and $\mathcal{H}\Phi_k + \Phi_k'$, (bottom) as a function of $x = \eta/\eta_0$ for $\tilde{k} = 10^{-8}$.
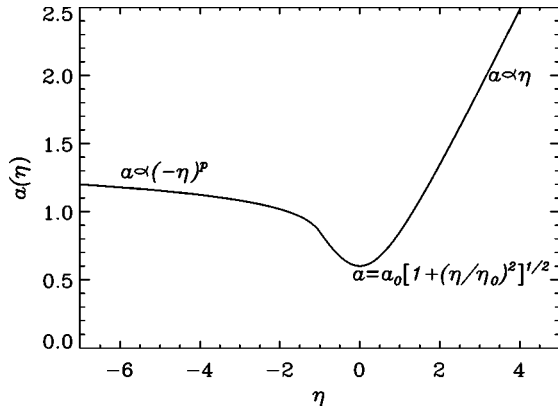
FIG. 8. Scale factor for connecting a slowly contracting phase to our model. If such a four-dimensional model was effectively constructed, it would produce a scale invariant spectrum of perturbations and would thus become a promising competitor to more usual inflationary models.

IV. CONCLUSIONS: GENERATING A SCALE-INVARIANT SPECTRUM THROUGH A BOUNCE

We have presented a cosmological model in which a bounce takes place in the framework of pure general relativity. This is achieved by assuming that, at some stage after a contracting phase, a negative energy free scalar field became important. Performing a bounce with such a scalar field, instead of an ordinary hydrodynamical fluid, permits us to regularize the perturbation, which otherwise grows unbounded near the bounce [17]. We derived the last horizon crossing spectrum and obtained, both analytically and numerically, a spectral index $n_s = -1$ in the long wavelength limit, therefore ruling out such a model as a competitor to the inflationary paradigm. However, our study of a concrete bouncing model allowed us to obtain some intuition on what happens with perturbations when they pass through a bounce. First of all, the bounce acts indeed as a ''pump field'' for perturbations. Secondly, the field which produces the bounce in the background solution, and its perturbations, is not relevant for the evolution of the Bardeen potential in almost the whole history of the model, except near the bounce itself, where it becomes very important for the power spectrum amplitude, although not for the spectral index. Finally, usual matching conditions are not valid for transitions through a bounce. In fact, even for the background metric, such conditions are not valid since the Hubble parameter $H = \mathcal{H}/a$ changes sign through the bounce, by definition. Through our bounce, the Bardeen potential also changes sign, and what happens to be continuous is the combination $\mathcal{H}\Phi$. Inspired by our concrete model, we suggested matching conditions to be applied to general models where the bounce is not specified, which are spelled out in the Sec. III C. Of course, these suggestions must be checked within other concrete examples, or through a more general formal analysis.

The model we have discussed is admittedly over simplistic. We may be confident in its latest part describing the radiation-dominated epoch, which we know has taken place in our Universe, and accept that it may provide a reasonable description of an immediately preceding bouncing phase.

There is, however, no reason to believe, even in the case of a bounce, that the evolution of the Universe should have been symmetric in time. One can instead set up a contracting phase with a different scale factor, assuming at some stage some form of entropy production, to end up with enough radiation before the bounce, yielding a scale factor of the form (7) that ultimately connects back to standard cosmology. Such a model should originate in a realistic underlying particle physics theory.

Let us briefly discuss an example, which is reminiscent of the ekpyrotic proposal [6], with a few differences. First, the model we have in mind would be purely four-dimensional and does not intend to address the flatness problem. Second, such a model would be effectively singularity free. Finally, we would not need to impose arbitrary [12] matching conditions across the bounce to obtain the required mixing in the growing and decaying modes before and after the bounce, since its specific form would be known.

More precisely, a model satisfying the above-mentioned requirements could consist of a bouncing model having a slowly contracting phase, $a \propto (-\eta)^p$, with $0 < p \ll 1$, connected to the phase examined in the previous sections, as the one shown in Fig. 8. When the perturbation in the Bardeen potential crosses the horizon for the first time, the dependence η^{-2} in Eq. (30), stemming from the fact that the universe is supposed to be radiation dominated at that time, would be substituted by a dependence η^{-2p}, i.e., almost independent of k when $k\eta \sim 1$. Doing calculations along the lines of those presented in Sec. III, the scale invariant spectrum follows.

APPENDIX: ADIABATIC AND ISOCURVATURE MODES

In Ref. [17] we claimed that a hydrodynamical perturbation should lead to a divergence in the second time derivative of the Bardeen potential, and hence of the Einstein tensor, for the adiabatic mode at the time at which the null energy condition ceases to be violated (referred to as the NEC transition in Ref. [17]), which in the situation at hand reads $\eta^2 = \eta_0^2/2$: this would mean the breakdown of perturbation theory at this point. As no such divergence is observed during the bounce in the model discussed in the present paper, and as a free massless scalar field is equivalent to a stiff matter fluid [32], one would then be led to conclude that the entropy modes diverge as well in such a way as to cancel exactly the adiabatic divergence. In this appendix we take the hydrodynamic representation of the present model, a two-fluid model with-

out interaction between them, with equations of state $p_r = \frac{1}{3}\epsilon_r$ and $p_s = \epsilon_s$, with $\epsilon_s < 0$, and we show that, indeed, if we take into account the perturbations on their full generality (adiabatic plus entropy terms), no such divergence appears.

In the two-fluid model, energy conservation is valid separately for both fluids, yielding $\epsilon_r = c_r/a^4$ and $\epsilon_s = -c_s/a^6$, with c_r and c_s two positive constants. The perturbed Einstein's equations in terms of the gauge invariant Bardeen potential read (we use the notations of Ref. [20] for a spatially flat, $\mathcal{K} = 0$, FRW background)

$$\nabla^2\Phi - 3\mathcal{H}\Phi' - 3\mathcal{H}^2\Phi = 4\pi G a^2 \delta\epsilon^{(gi)}, \quad \text{(A1)}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p^{(gi)}, \quad \text{(A2)}$$

where $\delta\epsilon^{(gi)} = \delta\epsilon_r^{(gi)} + \delta\epsilon_s^{(gi)}$ and $\delta p^{(gi)} = \delta p_r^{(gi)} + \delta p_s^{(gi)}$. All these quantities are defined in such a way as to be gauge invariant, as emphasized by the superscript "(gi)."

Using the linearity of the above equations, we can decompose the total gravitational perturbation Φ as the sum $\Phi = \Phi_r + \Phi_s$ and construct the two sets of decoupled equations

$$\nabla^2\Phi_r - 3\mathcal{H}\Phi_r' - 3\mathcal{H}^2\Phi_r = 4\pi G a^2 \delta\epsilon_r^{(gi)}$$

$$\Phi_r'' + 3\mathcal{H}\Phi_r' + (2\mathcal{H}' + \mathcal{H}^2)\Phi_r = 4\pi G a^2 \delta p_r^{(gi)}, \quad \text{(A3)}$$

and

$$\nabla^2\Phi_s - 3\mathcal{H}\Phi_s' - 3\mathcal{H}^2\Phi_s = 4\pi G a^2 \delta\epsilon_s^{(gi)}$$

$$\Phi_s'' + 3\mathcal{H}\Phi_s' + (2\mathcal{H}' + \mathcal{H}^2)\Phi_s = 4\pi G a^2 \delta p_s^{(gi)}. \quad \text{(A4)}$$

Substituting $\delta p_r^{(gi)} = \frac{1}{3}\delta\epsilon_r^{(gi)}$ and $\delta p_s^{(gi)} = \delta\epsilon_s^{(gi)}$ into Eqs. (A3) and (A4), we obtain the following equations for Φ_r and Φ_s:

$$\Phi_r'' + 4\mathcal{H}\Phi_r' - \frac{1}{3}\nabla^2\Phi_r + 2(\mathcal{H}' + \mathcal{H}^2)\Phi_r = 0, \quad \text{(A5)}$$

$$\Phi_s'' + 6\mathcal{H}\Phi_s' - \nabla^2\Phi_s + 2(\mathcal{H}' + 2\mathcal{H}^2)\Phi_s = 0, \quad \text{(A6)}$$

so that, plugging the scale factor given by Eq. (7) leads to (for the sake of simplicity, and to avoid an unnecessary proliferation of indices, we note simply Φ_r and Φ_s the k modes of Φ in the following equations):

$$\frac{d^2\Phi_r}{dx^2} + \frac{4x}{1+x^2}\frac{d\Phi_r}{dx} + \left[\frac{\tilde{k}^2}{3} + \frac{2}{(1+x^2)^2}\right]\Phi_r = 0, \quad \text{(A7)}$$

and

$$\frac{d^2\Phi_s}{dx^2} + \frac{6x}{1+x^2}\frac{d\Phi_s}{dx} + \left[\tilde{k}^2 + \frac{2}{(1+x^2)}\right]\Phi_s = 0. \quad \text{(A8)}$$

As the coefficient entering Eqs. (A7) and (A8) are completely regular everywhere, and thus in particular around the point of NEC transition ($x^2 = 1/2$), then, by Fuchs property [34], Φ_r, Φ_s and all their derivatives are also regular there and can be expanded in the Taylor series. Of course, the same is true for the full Bardeen potential $\Phi_k = \Phi_r + \Phi_s$. As we derived this result without performing any separation into adiabatic and entropy perturbations, we arrive at the conclusion that, in such a case, the divergences obtained in Ref. [17] for the adiabatic perturbations must be exactly compensated by divergences in pure entropy perturbations yielding a finite total result. This is in agreement with the results obtained in the present paper concerning the scalar field model. Note that, as in the scalar field case, Eqs. (A7) and (A8) have regular linear independent solutions near the NEC transition, which means that expansion into adiabatic and entropy modes is meaningless since they both diverge in such a case. In a forthcoming publication [35], we will examine in more detail the situations where such cancellations of divergences may occur for hydrodynamical perturbations near general bounces, and when expansions into adiabatic and entropy modes are meaningful in order to complement the results of Ref. [17].

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