

Quadrupole moments of N and Δ in the $1/N_c$ expansion

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We calculate expressions for the quadrupole moments of nonstrange baryons in which the number of QCD color charges is N_c . Using only the assumption of single-photon exchange, we obtain 4 relations among the 6 moments, and show how all of them may be obtained from Q_{Δ^+p} up to $O(1/N_c^2)$ corrections. We compare to the $N_c=3$ case, and obtain relations between the neutron charge radius and quadrupole moments. We also discuss prospects for the measurement of these moments.

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I. INTRODUCTION

The most interesting observable properties of hadrons, such as masses, electromagnetic moments, and scattering amplitudes, fall squarely in the nonperturbative regime of QCD and thus resist first-principles analytic calculation. While lattice simulations for static observables continue to improve numerically, they do not typically address questions regarding which features of QCD explain which various aspects of hadronic observables. Other techniques, such as the operator product expansion, relate perturbative quantities, hadronic observables, and nonperturbative matrix elements pertaining to vacuum structure. The $1/N_c$ expansion [1], with N_c the number of color charges, is alone among QCD-inspired techniques in providing a perturbative parameter at all energy scales. While the notion of varying the fixed value of $N_c=3$ may seem peculiar on first glance, the $1/N_c$ expansion is made phenomenologically useful by the observation that many key features of the physical universe would not be qualitatively very different were the precise value of $N_c=5, 11, \text{ or } 113$ [2].

The $1/N_c$ expansion has proved quite useful in describing the properties of baryons, such as ground-state masses [3], magnetic moments [4,5], and axial vector couplings [5,6], as well as excited baryon masses [7–9], axial vector couplings [9–11], and pion-production [12] and electromagnetic decay and production [13] amplitudes. Similar studies have investigated baryons containing a heavy quark [14,15], and the nature of the nucleon-nucleon interaction [16], as well as a variety of other properties.

The analysis proceeds by studying interaction Hamiltonian operators of definite spin-flavor transformation properties acting upon the baryon states. Each such operator is characterized by a coefficient proportional to a particular power of $1/N_c$. The latter may be obtained by counting the minimum number of gluon exchanges necessary to connect the requisite number of quark lines in order for the operator

to have a nonvanishing expectation value for the baryon state, since the QCD coupling $\alpha_s \propto 1/N_c$. Of course, the possibility that contributions from a given operator may add coherently over the N_c quarks, giving an additional enhancement of $O(N_c)$ to the full matrix element, must be taken into account. To this scheme one must add the observation that not all apparently different spin-flavor operators are linearly independent on a given multiplet of baryons. Often there appear relations between seemingly distinct operators due to the structure of the spin-flavor group [SU(6) for three quark flavors] or due to the fact that one considers operators acting upon a particular representation of that group (the ground-state **56** for $N_c=3$, for example). Application of such *operator reduction rules* [7,17] are then necessary to remove redundant degrees of freedom; see Sec. III for explicit examples. Together, these effects provide a power-counting scheme for deciding which Hamiltonian operators are most significant in determining contributions to the physical properties listed above.

While we have been discussing the baryon as an N_c -quark state, of course the physical baryon is a much more complicated object, containing a complex soup of gluons and sea quark-antiquark pairs. The simple-minded picture may be justified by noting that physical baryons for $N_c=3$ have precisely the same quantum numbers as those predicted by the simple quark model. Therefore, one can invent interpolating fields whose effect is to subsume the entire complicated substructure into N_c precisely defined constituent quark fields [18]. This identification allows the old constituent quark model to be placed on a rigorous footing and elucidates the nature of the physical picture used in the studies noted above.

Once the relevant operators are ordered in a $1/N_c$ expansion and any operator coefficients due to known suppressions [such as SU(3) flavor or isospin breaking] are taken into account, the matrix elements of the spin-flavor structures are computed by standard group-theoretical methods. Then, only an unknown dimensionless coefficient determined by strong interaction effects remains uncomputed. This approach is none other than the application of the Wigner-Eckart theorem in a novel context: The matrix elements of the spin-flavor structures constitute Clebsch-Gordan coefficients, and the

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leftover coefficients are reduced matrix elements. Inasmuch as the expansion is natural, i.e., takes into account all known suppressions or enhancements, these reduced matrix elements should be of order unity.

In the most general application of the method, one first generates a complete set of linearly independent operators, which is guaranteed to exist since the baryons belong to a finite representation for any particular N_c . Since ultimately one sets $N_c \rightarrow 3$, it is sufficient to consider only up to 3-body operators, i.e., irreducible operators involving 3 distinct quark lines. While this would seem to offer no predictivity—after all, then one would have an equal number of observables and operator coefficients—the hierarchy specified by the $1/N_c$ expansion determines which operators are more or less phenomenologically significant.

In fact, the method can easily be adapted to various model assumptions. Suppose, for example, that one works in a model in which isospin violation arises from only one source, say, quark charges. The most general expansion allows for every possible source of isospin violation; therefore, a number of possible operators can be neglected in this model. Since each independent operator is sensitive to a particular pattern of spin-flavor symmetry breaking that contributes only to certain combinations of baryon observables, the absence of each such operator in the model indicates a relation obeyed by the baryon states. This is precisely the same as the approach that was adopted for studying baryon charge radii in Ref. [18], and for the $N_c=3$ case for quadrupole moments in Ref. [19].

In this work we consider the quadrupole moments of the non-strange ground-state baryons. The quadrupole moments hold interest in providing vital information on the charge distributions of the baryons, which are inherently non-perturbative quantities. Only the non-strange baryons are considered in this work, as explained below, because the tricks of group theory for the 2-flavor case are different from those necessary to study the 3-flavor case. The latter case for quadrupole moments as well as charge radii will be considered in a subsequent work [20].

But are baryon quadrupole moments measurable quantities? We address this central issue in Sec. II. In Sec. III we generate the operator basis for a very general model: The only assumption is the one-photon exchange approximation, by which we mean that an arbitrary number of quarks can be involved in the quadrupole operator, but the photon itself couples to only one of them. The results of the analysis are presented in Sec. IV and conclusions in Sec. V.

II. PROSPECTS FOR MEASUREMENT

The dearth of literature on baryon quadrupole moments compared to that on masses or magnetic moments points to the basic difficulty of measuring these quantities. The quadrupole tensor is rank-2, and therefore by the Wigner-Eckart theorem its expectation value, called the *spectroscopic* quadrupole moment Q , vanishes for the spin-1/2 nucleons. However, it should be noted [21] that Q , measured in space-fixed coordinates, is not identical to the second moment of the charge distribution,

$$eQ_0 \equiv \left\langle \int d^3r \rho(\mathbf{r})(3z^2 - r^2) \right\rangle, \quad (2.1)$$

measured in body-fixed coordinates, with e the proton charge. Q_0 is called the *intrinsic* quadrupole moment, which would describe the shape of the charge distribution were it possible to take a snapshot of the particle at a fixed time. The most obvious manifestation of the intrinsic quadrupole moment of a spheroidal particle is whether it is prolate (football-shaped) or oblate (pancake-shaped). The relation between the two definitions for a particle of ground-state spin J is given by [21]

$$Q = \frac{J(2J-1)}{(J+1)(2J+3)} Q_0. \quad (2.2)$$

For $J=0$ or $1/2$, the particle body-fixed axis points with equal probability in all directions by rotational invariance, leading to the necessary vanishing of Q . However, Q_0 can be nonzero for such particles, and it has been shown in a variety of models [22] that the proton is prolate and the Δ^+ is oblate. It has also been shown [23] that, given a sufficient number of measured electric quadrupole ($E2$) transition matrix elements, Q_0 can be extracted in a model-independent way. Nevertheless, the quadrupole moments discussed below are of the spectroscopic variety and are obtainable through quadrupole transitions.

The spectroscopic quadrupole moments for the ground-state baryons consist of diagonal transitions for the spin-3/2 decuplet baryons Δ , Σ^* , Ξ^* , and Ω , and off-diagonal transitions between the spin-3/2 decuplet and spin-1/2 octet baryons, ΔN , $\Sigma^* \Sigma$, $\Sigma^* \Lambda$, and $\Xi^* \Xi$. Since all of the decuplet baryons except the Ω decay strongly, couplings of the form decuplet-decuplet- γ can be measured only through virtual processes. Such experiments are difficult but not impossible; for example, Ref. [24] proposes the measurement of the Δ^+ magnetic moment through the process $\gamma p \rightarrow \gamma \pi^0 p$. However, the same authors note the near-impossibility of measuring the electric quadrupole moment through such experiments. The problem is that electric quadrupole ($E2$) operators are time-reversal odd. Thus, if the initial state and final state are identical (as for static diagonal quadrupole transitions), then this matrix element vanishes. There are only two ways around this restriction; the first is to use Coulomb photons rather than real radiation (the usefulness of Coulomb photons is limited by the short lifetime of the decuplet baryons), and the second is to extract the off-shell or recoil effects of the decuplet baryons, a proposition fraught with many difficulties. For our purposes, we suppose that the diagonal quadrupole transitions involving decuplet baryons will not be measured any time soon, and suggest that our calculation provides predictions of quantities that are very hard to measure.

The Ω^- is an exception to this constraint, since it has an appreciable lifetime. A number of clever experiments have been suggested to extract Q_Ω , including the measurement of energy levels of an exotic atom formed by a heavy nucleus capturing an Ω^- [25] and the precession of Ω^- spin as it traverses a crystal at small angles to the crystallographic

plane [26]. In both cases, the purpose is to enhance the effect of the electric field gradient, to which the quadrupole moment couples.

Since all the octet baryons (except the Σ^0) have appreciable lifetimes, the transition $E2$ moments are all measurable in principle, inasmuch as one possesses a suitable source of the desired long-lived baryon. In the hyperon sector, the Primakoff reaction $Y + Z \rightarrow Y^* + Z$, where $Y(Y^*)$ is the octet (decuplet) hyperon and Z is a heavy nucleus, is sensitive to both the $M1$ and $E2$ transition matrix elements; this process has been studied at SELEX [27] to obtain a bound on the radiative width of the Σ^* . Experiments at Jefferson Lab that involve kaon photoproduction ($\gamma p \rightarrow K^+ Y^* \rightarrow K^+ Y \gamma$) [28] can also provide useful information on these amplitudes.

Lastly, the $N \rightarrow \Delta$ transition amplitudes have been studied in numerous experiments, for example see [29,30]. In particular, the transition quadrupole moment $Q_{N \rightarrow \Delta}$ may be extracted from the $E2/M1$ ratio (the relative quadrupole to dipole strength). A sample recent measurement using pion photoproduction data is $[-3.07 \pm 0.26 \text{ (stat+syst)} \pm 0.24 \text{ (model)}]\%$ [29], from which one obtains the value $Q_{N \rightarrow \Delta} = -0.108 \pm 0.009 \pm 0.034 \text{ fm}^2$. However, not all researchers agree on the size of uncertainties (e.g., those induced by a particular model used to describe the nonresonant background [31]) when extracting the resonance amplitudes from the measurements.

III. THE OPERATOR METHOD

The operator method was described in the Introduction in general terms; here we lay out the specifics for this calculation. It is identical to the approach used for charge radii in [18], and is the generalization (for nonstrange states) of the calculation performed in Ref. [19]. Any spin-flavor operator can be built from a basis using no more than three 1-body operators, i.e., spin-flavor operators acting upon one quark line. The most general such operators (restricted, of course, to those that transform as quadrupole tensors) are formed from sandwiching Pauli spin matrices and Gell-Mann flavor matrices (or isospin matrices in the 2-flavor case) between quark creation and destruction operators. One then performs the operator reduction; for example, $I^2 = J^2$ on the 2-flavor states. Were we to compute this model-independent expansion for the nonstrange baryon quadrupole moments, we would find precisely 6 operators, matching the number of observables $Q_{\Delta^{++}}$, Q_{Δ^+} , Q_{Δ^0} , Q_{Δ^-} , Q_{Δ^+p} , and Q_{Δ^0n} . Some of the operators in this list are suppressed by various powers of $1/N_c$, leading to model-independent approximate relations between the quadrupole moments. Precisely this sort of expansion was carried out for nonstrange baryon charge radii in [18].

However, we adopt the mildly model-dependent but much more physical viewpoint that any quadrupole operator involves the quark charge only once. Operators containing, e.g., extra powers of the quark charge are suppressed by additional powers of $e^2/4\pi = 1/137$ compared to the single-photon exchange approximation. This restricts the isospin structure of possible quadrupole operators considerably: Only one very particular combination of isosinglet and

isovector coupling, given by the quark charge operator, then appears.

The quadrupole spin coupling is a rank-2 tensor, and therefore clearly cannot be built with just one (rank-1) Pauli spin matrix. The diagonal rank-2 combination formed by two Pauli matrices acting on quarks labeled by i, j is just the familiar form:

$$3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j.$$

It is also possible to build a rank-2 spin tensor with 3 Pauli matrices, but such a form on any given quark pair can be divided into symmetric and antisymmetric Hermitian parts. The symmetric part turns out to be time-reversal odd (since angular momentum is T -odd), and is thus irrelevant in strong and electromagnetic matrix elements. The antisymmetric part is essentially a commutator of spin generators, and thus reduces to a single Pauli matrix; this is an example of an operator reduction rule. Thus, the only spin structure required up to the 3-body level is the operator with *two* Pauli matrices listed above.

The quark charge operator can now either act upon one of the two quark lines on which the quadrupole spin tensor acts, or upon a third line. Using the general rule that the minimum number of gluon exchanges necessary to connect the quark lines of an n -body operator is $n-1$, the former is a 2-body operator and thus has the suppression coefficient $1/N_c$, while the latter is a 3-body operator and thus is prefaced with a $1/N_c^2$. One finds that only two operators appear up to the 3-body level:

$$Q = \frac{B}{N_c} \sum_{i \neq j}^{N_c} Q_i (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) + \frac{C}{N_c^2} \sum_{i \neq j \neq k}^{N_c} Q_k (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j), \quad (3.1)$$

where B and C are unknown coefficients of order unity, times a characteristic hadronic quadrupole size (in fm^2). This implies in particular 4 all-orders relations among the 6 quadrupole moments, as we see below. The coefficient powers of $1/N_c$ are the only explicit differences between this expression and that in the $N_c = 3$ case [19].

In deriving expressions for the matrix elements of these two operators, it is of great advantage to note that the ground-state baryon representation—the analogue of the **56** of $SU(6)$ —is completely symmetric in combined spin and flavor indices. This means in particular that all quarks of a given flavor u or d are completely symmetrized, and hence carry the maximal spin $S_u = N_u/2$ and $S_d = N_d/2$, where N_i is the number of quarks of flavor i in the baryon. Furthermore, one has the constraints $N_u + N_d = N_c$ and $N_u - N_d = I_3/2$, and for nonstrange states the rule $I = J$ applies, where $\mathbf{J} = \mathbf{S}_u + \mathbf{S}_d$ is the total baryon spin. A complete list of compatible operators is thus $J^2 = I^2$, J_3 , I_3 , S_u^2 , and S_d^2 .

In the calculation one encounters operators such as $\sum_i^{N_c} Q_i \sigma_{iz}/2 = Q_u S_{uz} + Q_d S_{dz}$. The values of $Q_{u,d}$ are fixed by the anomaly cancellation conditions of the standard model with gauge group $SU(N_c) \times SU(2) \times U(1)$ to be [32]

$$Q_{u,c,t} = (N_c + 1)/2N_c, \quad Q_{d,s,b} = (-N_c + 1)/2N_c. \quad (3.2)$$

The calculation of all the necessary matrix elements for the 2-flavor case therefore requires only knowledge of two specific matrix elements for coupled angular momenta $(j_1, j_2) \rightarrow J$; of course, here j_1 and j_2 stand for S_u and S_d , the total spin angular momenta carried by the u and d quarks, respectively. These matrix elements are

$$\langle JM(j_1 j_2) | J_{1z} | JM'(j_1 j_2) \rangle = \frac{1}{2} M \delta_{M'M} \left[1 + \frac{j_1(j_1+1) - j_2(j_2+1)}{J(J+1)} \right], \quad (3.3)$$

and

$$\langle JM(j_1 j_2) | J_{1z} | J-1M'(j_1 j_2) \rangle = \frac{\delta_{M'M}}{2J} \sqrt{\frac{[J^2 - (j_1 - j_2)][(j_1 + j_2 + 1)^2 - J^2](J^2 - M^2)}{(2J+1)(2J-1)}}. \quad (3.4)$$

These expressions can be derived through elementary means. The first is a special case of the Wigner-Eckart theorem often called the *projection theorem* [33]. The second may be obtained solely through the manipulation of raising and lowering operators and is proved in, e.g., Ref. [34]. Since these theorems refer to the coupling of just two states of known angular momentum, they provide a complete answer only in the 2-flavor case. Of course, the 3-flavor case requires the coupling of an additional angular momentum (\mathbf{S}_s), which introduces $6j$ and $9j$ symbols, as well as the careful consideration of additional symmetry properties of the baryon states. We do not pursue these topics here, but relegate their resolution to another paper [20].

The operators $\mathbf{S}_{u,d}$ are rank-1 tensors and therefore can connect states differing by up to one unit of total angular momentum J . Thus one obtains both diagonal and off-diagonal quadrupole transition matrix elements. Expressing Eq. (3.1) as $Q = B\mathcal{O}_B + C\mathcal{O}_C$, one finds

$$\begin{aligned} \langle JJ | \mathcal{O}_B | JJ \rangle &= + \frac{1}{N_c} \left(J - \frac{1}{2} \right) \left[\frac{(N_c + 2)(2Q - 1)}{J + 1} + \frac{4J}{N_c} \right], \\ \langle JJ - 1 | \mathcal{O}_B | J - 1J - 1 \rangle &= + \frac{3(J - 1)}{2JN_c} \\ &\quad \times \sqrt{\frac{[(2Q - 1)^2 - 4J^2][4J^2 - (N_c + 2)^2]}{2J + 1}}, \\ \langle JJ | \mathcal{O}_C | JJ \rangle &= - \frac{2}{N_c} \left(J - \frac{1}{2} \right) \left[\frac{(N_c + 2)(2Q - 1)}{J + 1} \right. \\ &\quad \left. - 4J \left(Q - \frac{1}{N_c} \right) \right], \\ \langle JJ - 1 | \mathcal{O}_C | J - 1J - 1 \rangle & \end{aligned} \quad (3.5)$$

$$\begin{aligned} &= - \frac{3(J - 1)}{JN_c} \\ &\quad \times \sqrt{\frac{[(2Q - 1)^2 - 4J^2][4J^2 - (N_c + 2)^2]}{2J + 1}}, \end{aligned}$$

where Q is the total baryon electric charge.

IV. RESULTS AND DISCUSSION

Equations (3.5) evaluated for the quantum numbers of the 6 nonstrange states give the results listed in Table I. First note that these results agree with those in the $N_c = 3$ case, Tables I and II of Ref. [19], except for our aforementioned coefficient factors of $1/N_c$ and $1/N_c^2$ for B and C , respectively, and a factor of 2 accidentally neglected in the C terms of [19]. The matrix elements in [19] were obtained by using explicit $N_c = 3$ spin-flavor baryon wave functions.

Several features are immediately obvious. The diagonal matrix elements for $N_c = 3$ are just $4Q(3B + C)/9$, while those in the $N_c \rightarrow \infty$ limit are $4I_3 B/5 = 4(Q - 1/2)B/5$, and the two off-diagonal elements are equal. These simple linear patterns arise from our model assumption that only one quark charge operator appears, and so the quadrupole operator transforms only as $I = 0$ and 1. In particular, the $I = 3/2$ Δ 's can be connected by operators transforming as $I = 0, 1, 2$, or 3, and the $I = 2$ and 3 combinations must vanish for any N_c . These combinations are

$$\begin{aligned} Q_{\Delta^{++}} - 3Q_{\Delta^+} + 3Q_{\Delta^0} - Q_{\Delta^-} &= 0 \quad (I=3), \\ Q_{\Delta^{++}} - Q_{\Delta^+} - Q_{\Delta^0} + Q_{\Delta^-} &= 0 \quad (I=2). \end{aligned} \quad (4.1)$$

Quadrupole transition operators between the $I = 3/2$ Δ and $I = 1/2$ N can transform as $I = 1$ or 2, and since the latter does not appear in our model, it leads to the relation

$$Q_{\Delta^+ p} = Q_{\Delta^0 n}, \quad (4.2)$$

TABLE I. Quadrupole moments of the nonstrange ground-state baryons for arbitrary N_c , $N_c=3$, and $N_c \rightarrow \infty$.

$Q_{\Delta^{++}}$	$\frac{B}{N_c^2} \left(+\frac{6}{5} \right) (N_c^2 + 2N_c + 5) + \frac{C}{N_c^3} \left(-\frac{12}{5} \right) (N_c^2 - 8N_c + 5)$	$\frac{8}{3}B + \frac{8}{9}C$	$+\frac{6}{5}B$
Q_{Δ^+}	$\frac{B}{N_c^2} \left(+\frac{2}{5} \right) (N_c^2 + 2N_c + 15) + \frac{C}{N_c^3} \left(-\frac{4}{5} \right) (N_c^2 - 13N_c + 5)$	$\frac{4}{3}B + \frac{4}{9}C$	$+\frac{2}{5}B$
Q_{Δ^0}	$\frac{B}{N_c^2} \left(-\frac{2}{5} \right) (N_c + 5)(N_c - 3) + \frac{C}{N_c^3} \left(+\frac{4}{5} \right) (N_c + 5)(N_c - 3)$	0	$-\frac{2}{5}B$
Q_{Δ^-}	$\frac{B}{N_c^2} \left(-\frac{6}{5} \right) (N_c^2 + 2N_c - 5) + \frac{C}{N_c^3} \left(+\frac{12}{5} \right) (N_c^2 - 3N_c - 5)$	$-\frac{4}{3}B - \frac{4}{9}C$	$-\frac{6}{5}B$
Q_{Δ^+p}	$\left(\frac{B}{N_c} - 2\frac{C}{N_c^2} \right) \sqrt{\frac{(N_c+5)(N_c-1)}{2}}$	$\frac{2\sqrt{2}}{9}(3B-2C)$	$\frac{1}{\sqrt{2}}B$
Q_{Δ^0n}	$\left(\frac{B}{N_c} - 2\frac{C}{N_c^2} \right) \sqrt{\frac{(N_c+5)(N_c-1)}{2}}$	$\frac{2\sqrt{2}}{9}(3B-2C)$	$\frac{1}{\sqrt{2}}B$

for all N_c . The last of the 4 relations (6 degrees of freedom, 2 operators) is N_c dependent,

$$\frac{Q_{\Delta^0}}{Q_{\Delta^+p}} = -\frac{2}{5} \frac{N_c - 3}{N_c} \sqrt{\frac{2(N_c + 5)}{N_c - 1}}, \quad (4.3)$$

and in particular vanishes for $N_c=3$.

In addition to combinations that vanish identically, one may consider combinations for which the leading in $1/N_c$ terms cancel. In particular, we note that generically, the quadrupole moments for large N_c are $O(N_c^0)$, the sole leading contribution arising from the operator \mathcal{O}_B . As discussed above, the diagonal matrix elements will prove extremely difficult to measure directly, and so we now seek to express all other quadrupole moments in terms of the one most easily measurable, Q_{Δ^+p} . In terms of the combination $\mathcal{Q} = 2\sqrt{2}Q_{\Delta^+p}/5$, we find

$$\begin{aligned} Q_{\Delta^{++}} &= +3\mathcal{Q} \left[1 + \frac{1}{N_c^2} \left(\frac{19}{2} + 20\frac{C}{B} \right) + O\left(\frac{1}{N_c^3}\right) \right], \\ Q_{\Delta^+} &= +\mathcal{Q} \left[1 + \frac{1}{N_c^2} \left(\frac{39}{2} + 30\frac{C}{B} \right) + O\left(\frac{1}{N_c^3}\right) \right], \\ Q_{\Delta^0} &= -\mathcal{Q} \left[1 - \frac{21}{2N_c^2} + O\left(\frac{1}{N_c^3}\right) \right], \\ Q_{\Delta^-} &= -3\mathcal{Q} \left[1 - \frac{1}{N_c^2} \left(\frac{1}{2} - 10\frac{C}{B} \right) + O\left(\frac{1}{N_c^3}\right) \right]. \end{aligned} \quad (4.4)$$

Phenomenological experience with the $1/N_c$ expansion tells us that quantities for which corrections are only $O(1/N_c^2)$ tend to agree well with their predicted central values. We

should note, however, that the numerical coefficients of the $O(1/N_c^2)$ terms in this case can be large for $N_c=3$, depending upon the precise value of the ratio C/B . An interesting difference of prediction between this work and [19] is the ratio $Q_{\Delta^+}/Q_{\Delta^+p}$, which here is $2\sqrt{2}/5 + O(1/N_c^2)$, and in Eq. (5) of [19] is $\sqrt{2}$; the latter prediction is obtained by setting $C=0$ (and of course $N_c=3$). Using the results of Eq. (4.4), the value for Q_{Δ^+p} from Sec. II, and assuming the $O(1/N_c^2)$ corrections can be neglected, one predicts $Q_{\Delta^{++}} = -0.183(60) \text{ fm}^2$ and $Q_{\Delta^+} = -0.061(20) \text{ fm}^2$. These results are consistent with the magnitudes predicted by, for example, πN bremsstrahlung models [35], but the true strength of this method is the prediction of relations among the moments. We defer additional analysis to such time as the 3-flavor predictions are also in hand [20].

Lastly, suppose that the baryon charge radii, which come from spin-spin terms in the baryon Hamiltonian, arise from the same source as the quadrupole operators; then the matrix elements for the two observables appear in fixed ratios. This occurs, for example, in one-gluon exchange picture, in which a multipole expansion of the baryon charge density operator ρ reads

$$\begin{aligned} \rho &= A \sum_i^{N_c} Q_i - \frac{B}{N_c} \sum_{i \neq j}^{N_c} Q_i [2\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)] \\ &\quad - \frac{C}{N_c^2} \sum_{i \neq j \neq k}^{N_c} Q_k [2\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j - (3\sigma_{iz}\sigma_{jz} - \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)]. \end{aligned} \quad (4.5)$$

Note that the normalization of the spin-spin operator is just -2 times that used in Ref. [18]. Thus, the results obtained there can be carried directly over for comparison. In particular, the neutron charge radius is

$$r_n^2 = \left(B - 2 \frac{C}{N_c} \right) \frac{(N_c - 1)(N_c + 3)}{N_c^2}, \quad (4.6)$$

from which we immediately see that

$$Q_{\Delta^+p} = \frac{1}{\sqrt{2}} r_n^2 \frac{N_c}{N_c + 3} \sqrt{\frac{N_c + 5}{N_c - 1}}. \quad (4.7)$$

The factor on the right-hand side (RHS) after r_n^2 is especially interesting since it equals 1 in both the $N_c=3$ and $N_c \rightarrow \infty$ cases (and in between never differs from unity by more than 1.2%). Thus, one expects that this relation should hold especially well. The $N_c=3$ version was first derived in Ref. [36] using a constituent quark model. Using the value $r_n^2 = -0.113(3)(4) \text{ fm}^2$ [37], one predicts $Q_{\Delta^+p} = -0.0799(4) \text{ fm}^2$, which agrees well with the value stated in Sec. II [29]. One could continue from here to predict all the baryon quadrupole moments using Eq. (4.4). The reader should be reminded, however, that the assumption of one-gluon exchange is more particular than the minimal assumption used to obtain Table I.

V. CONCLUSIONS

The $1/N_c$ expansion provides an additional handle on non-perturbative QCD phenomenology, through the observation that universes with odd $N_c > 3$ are in many respects very similar to our own. Using a rigorous definition of constituent quarks obtainable in the large- N_c limit and a minimal model ansatz—that quadrupole operators are proportional to the quark charge (single-photon exchange approximation)—we have shown that the 6 quadrupole moments of the nonstrange baryons can be described by just 2 distinct operators, leading to a number of relations that hold for all N_c and others that hold up to $O(1/N_c^2)$ corrections. We have also seen that in a one-gluon exchange picture, all of these can be predicted using the measured value of the neutron charge radius r_n^2 . Detailed analysis of charge radii and quadrupole moments for the 3-flavor case is forthcoming.

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